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TRACKING SYSTEMS, THEIR MATHEMATICAL MODELS AND THEIR ERRORS

PART I - THEORY

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SUMMARY

This paper treats the RMS-errors associated with the position and velocity of a satellite or spacecraft when tracked by all types of present day tracking systems. These errors are based on uncertainties in measurements made with the systems as well as those associated with their location.

The present paper (Part I) is principally a theoretical treatment which establishes the mathematical models necessary to solve for the errors in satellite position and velocity. It is presumed throughout the paper that these errors are to be determined for discrete points in the satellite's orbit. This approach enables one to calculate the error propagation during short time intervals (order of seconds). This is of particular interest for instance for evaluation of a guidance system during a short burning phase. A least square solution of non-simultaneous observations would diverge (matrices involved become ill-conditioned). This condition imposes a constraint on the method of solution which is, that either one tracking system can measure both position and/or velocity, or that several tracking systems observe the satellite simultaneously to produce the equivalent effect. Both these alternatives are considered in Part I .

A rigorous derivation of the "Method of Least Squares" is also presented for completeness, since it is used to a large extent.

Part II (presently in preparation) will show in detail the application of the equations derived in Part I assuming simultaneity as well as non-simultaneity. In the latter case position and velocity need not be fully determined by the tracking system or system complexes. A number ($i>6$) of range, range-rate, or angular measurements are adequate. Thus the constraint mentioned above and applicable for Part I does not exist, hence making the method more general. Numerical examples and results based upon known errors associated with the systems and systems locations will be presented for simultaneous and non-simultaneous derivations.

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INTRODUCTION

In recent years, requirements of satellite and spacecraft tracking accuracies have increased considerably; consequently more and more tracking systems and stations have been employed. From this situation, the question evolved: How accurately can the position and velocity of a space vehicle to be determined when it is tracked by a combination of tracking systems?

This paper attempts to present a detailed study of the propagation of errors in data obtained from satellite tracking systems. For establishing satellite orbits, both the position and velocity vectors of the satellite must be known. Usually, these two vectors are obtained from observations of the satellite made with various types of tracking systems. Since no tracking system can be considered completely free of errors, the observations obtained from such systems obviously contain errors, of which the most important are:

(1) Tracking system errors,

(2) Station position errors,

(3) Trajectory errors; i.e., satellite orbit errors.

These three kinds of errors will be discussed in detail here.

A separate treatment of errors due to atmospheric and ionospheric refraction (References 1-6) will not be necessary. These errors are assumed to be included in the system rms errors, that is, in the errors of the measured quantities such as range, range rate, angle, and angular rate.

To avoid too voluminous a paper, the work has been divided into two separate parts. Part I mainly treats the theory; Part II (in preparation) will treat the application of Part I by means of high speed computers, and will present concrete examples.

Part I is devoted entirely to an error analysis of tracking systems. The systems considered are complexes formed by the combination of radars; angle and angular rate systems (e.g., interferometers); and range and range rate systems. The errors sought in this analysis are those resulting from the tracking system itself as well as those due to the uncertainty in the systems location. The errors for these tracking systems have been assumed to be uncorrelated, on the basis of practical experience which showed that the slight existing correlation does not appreciably alter the result obtained under that assumption. To permit the calculation of these errors it is necessary also to assume knowledge of the orbit or trajectory. The latter assumption imposes no restriction on an analysis of this type, since the errors considered here are independent of orbit or trajectory errors. It has further been assumed that the tracking systems and complexes are capable of determining the position and velocity vectors independently of an assumed orbit. In brief, for some of the discussed systems, simultaneity is the minimal constraining condition for the solution. For instance at least three range and range rate systems, or one radar system, is required.

Nevertheless, to facilitate computation, an orbit has been assumed. (Available orbit generators permit immediate evaluation of the partial derivatives involved in the computations.) Because of the independence of the orbit — at least for a first order approximation — approximate orbits or trajectories derived from nominal injection parameters (provided in all satellite system operations plans) or from available observational tracking data will be adequate for the error analysis.

The aim of this paper is to obtain equations which permit the calculation of all errors resulting from any combination of tracking systems (such as radars, interferometers, or range and range rate systems). The coordinate systems used were so chosen that these errors could easily be transformed into orbital element.

Part II will apply the equations derived here assuming simultaneous and non-simultaneous observations. Numerical examples will be given illustrating the propagation of errors for discrete points and intervals along an orbit. The influence of the number as well as the distribution of tracking stations around the globe on the position and velocity errors will also be discussed. Treated will also be the minimum number of observations necessary for establishing an orbit of a stipulated accuracy with given tracking systems and their global distribution.

A. DEFINITION OF SYMBOLS

The following symbols will be employed throughout the development:

ϕ Geodetic latitude of the tracking station

λ Geodetic longitude of the tracking station

h Height above geoid of the tracking station

$a_\oplus$ Equatorial radius of spheroid used to represent the earth, (for Hayford Spheroid $a_\oplus$ = 6378388 meters)

e_1^2 Square of eccentricity for spheroid, (for Hayford Spheroid e_2^2 = 0.0067226700223)

θ_{G_0} Greenwich Sidereal Time at 0^h Universal Time (U.T.), obtained from the various almanacs

θ_G Greenwich Sidereal Time at U.T. of observation

t U.T. of observation

α Azimuth of the object being tracked

ϵ Elevation of the object being tracked

r Slant range of the object being tracked

a_s Semimajor axis of satellite orbit

λ' Right ascension of satellite

δ Declination of satellite

e_s Eccentricity of satellite orbit

i_s Inclination of satellite orbit

Ω_s Longitude of ascending node

ω_s Argument of perigee

E_s Eccentric anomaly

P_s Period of revolution of satellite

N Radius of curvature along Prime Vertical

ρ Magnitude of station position vector

ρ_s Magnitude of the radius vector to the satellite in the inertial coordinate system

B. MATRIX NOTATIONS

Since a large portion of this paper deals with transformation of coordinate systems, it will be useful to present a brief introduction to the transformation theory relevent to the material presented herein. A *translation* indicating a shift in coordinates is represented in matrix form as

$$\mathbf{X} = \mathbf{Y} - \mathbf{S} \qquad \text{(B-1)}$$

or, expanded in components,

$$(x_1, x_2, x_3) = (y_1 - s_1, y_2 - s_2, y_3 - s_3) . \tag{B-2}$$

A *rotation* on the other hand indicates an angular shift of two coordinate axes about a third axis — the chosen axis of rotation. In matrix notation a rotation is represented as

$$\mathbf{Z} = \mathbf{R}_i(\gamma)\,\mathbf{Y} , \tag{B-3}$$

where γ is the argument of the rotation and the index i refers to the axis of rotation in the y-coordinate system. For instance,

$$\mathbf{R}_1(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}_{(3\times 3)} . \tag{B-4}$$

The rotation illustrated in Equation (B-3) represents a rotation of a Y-coordinate system around its y_1-axis through an angle γ. This similarly holds for rotations around the y_2 or y_3-axes.

In this paper extensive use is made of rotation matrices; therefore, it is now appropriate to mention the following useful properties of such matrices:

$$\begin{aligned} &(1) \quad \mathbf{R}_i^T(\gamma) = \mathbf{R}_i^{-1}(\gamma) \\ &(2) \quad \mathbf{R}_i^T(\gamma) = \mathbf{R}_i(-\gamma) \\ &(3) \quad \mathbf{R}_i(\gamma)\,\mathbf{R}_i^T(\gamma) = \mathbf{I} \end{aligned} \tag{B-5}$$

where $i = 1, 2,$ or 3, and $\mathbf{I}$ is the identity matrix, which can easily be seen from Equation (B-4).

C. DEFINITIONS OF COORDINATE SYSTEMS

1. Coordinate Systems Used

The following coordinate systems are used:

X Inertial Cartesian coordinate system (Figure 1);

$\mathbf{X} \equiv [x_1, x_2, x_3]$,

x_1-axis directed towards vernal equinox,

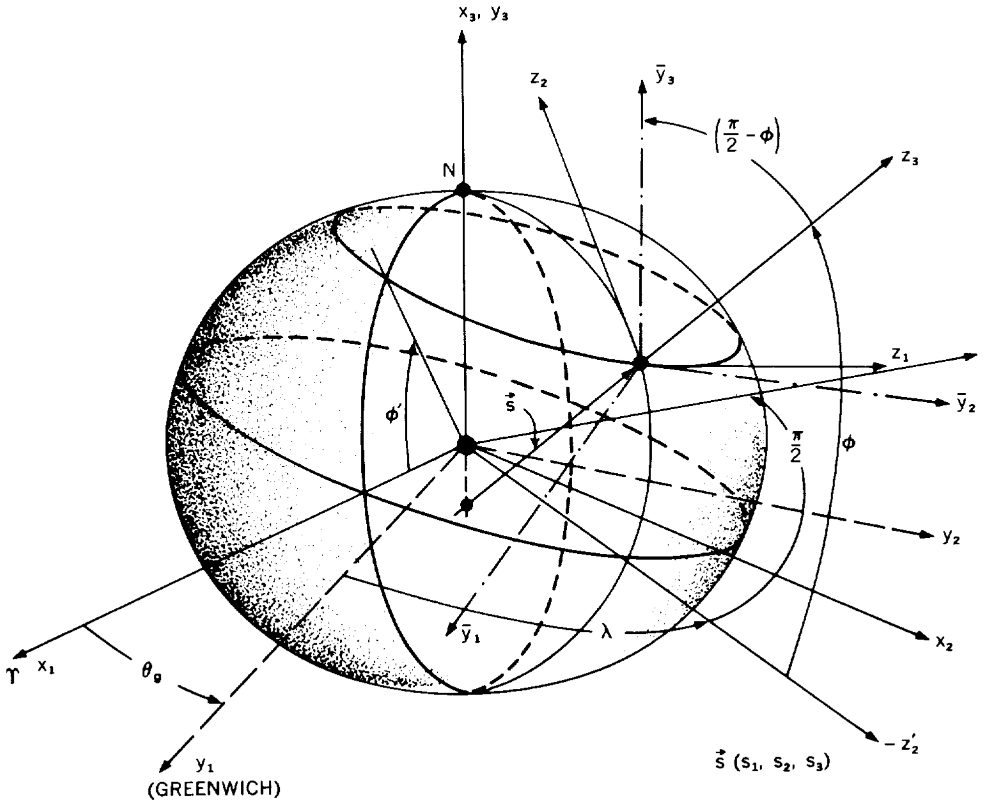

Figure 1—Geometric representation of the coordinate systems referred to the earth ellipsoid.

x_2-axis normal to both x_1 and x_3-axes,

x_3-axis directed along earth's axis of rotation.

Y Earth-centered and earth-fixed coordinate system (Figure 1);

y_1-axis directed towards Greenwich,

y_2-axis normal to (y_1, y_3) axes,

y_3-axis directed along earth's axis of rotation.

$\bar{\mathbf{Y}}$ Coordinate system parallel to the **Y**-system, centered at the observer.

Z Local Cartesian coordinate system centered at the observer (Figures 1 and 2);

z_1-axis directed towards local East,

z_2-axis directed towards local North,

z_3-axis directed along normal to local horizon plane.

S Position vector of observer with respect to **Y**-system (Figure 3 and Appendix A).

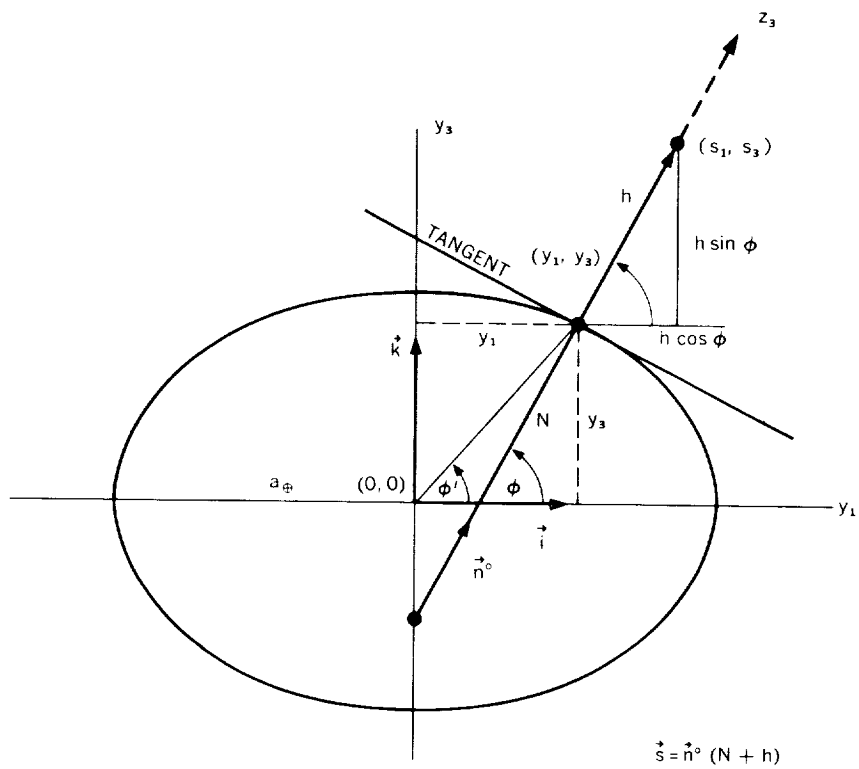

Figure 2—Cross-section of the elliptical assumed earth (Hayford ellipsoid)

2. Relationships Between Coordinate Systems

Some relationships between the various coordinate systems discussed should also be given: The transformation from the **X**-system to the **Y**-system by simple rotation is

$$\mathbf{Y} = \mathbf{R}_3 (\theta_G) \mathbf{X} \quad , \tag{C-1}$$

where

$$\theta_G \equiv \theta_{G_0} + \vartheta t \ ,$$

$$\vartheta \equiv 0.26251595 \text{ radians/hour}$$

and

$$0 \leq \theta_G < 2\pi \ .$$

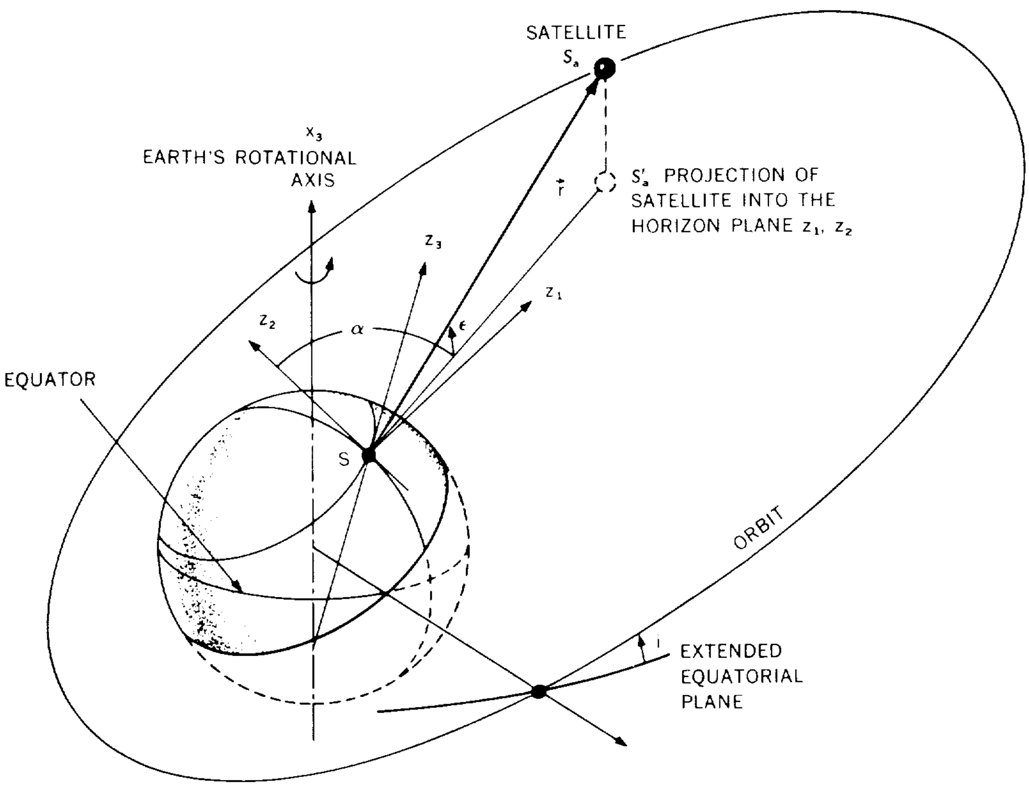

Figure 3—Artificial satellite orbit as referred to the local horizon coordination system

The transformation from the $\mathbf{Y}$-system to the $\bar{\mathbf{Y}}$-system by translation is

$$\bar{\mathbf{Y}} = \mathbf{Y} - \mathbf{S} , \tag{C-2}$$

where

$$\vec{s} = \mathbf{S} = \begin{bmatrix} (N + h) \cos\phi \cos\lambda \\ (N + h) \cos\phi \sin\lambda \\ \left[N\left(1 - e_1^2\right) + h\right] \sin\phi \end{bmatrix}_{(3\times1)} \tag{C-3}$$

and

$$N = \frac{a_\oplus}{\left[1 - e_1^2 \sin^2\phi\right]^{1/2}} .$$

The transformation (double rotation) from the $\bar{\mathbf{Y}}$-system to the $\mathbf{Z}$-system (Figure 1) can be written

$$\mathbf{Z} = \mathbf{R}_1\left(\frac{\pi}{2} - \phi\right) \mathbf{R}_3\left(\frac{\pi}{2} + \lambda\right) \bar{\mathbf{Y}} . \tag{C-4}$$

To simplify the notation, let

$$\mathbf{R} = \mathbf{R}_1\left(\frac{\pi}{2} - \phi\right)\mathbf{R}_3\left(\frac{\pi}{2} + \lambda\right) , \tag{C-5}$$

then the transformation of the $\mathbf{X}$-system into the $\mathbf{Z}$-system can be written:

$$\mathbf{Z} = \mathbf{R}\,\mathbf{R}_3(\theta_G)\,\mathbf{X} - \mathbf{R}\,\mathbf{S} , \tag{C-6}$$

or in partitioned matrix form

$$\mathbf{Z}_{(3\times1)} = \left[\mathbf{R}\,\mathbf{R}_3(\theta_G) \;\vdots\; -\mathbf{R}\right]_{(3\times6)} \left[\begin{array}{c}\mathbf{X}\\ \hline \mathbf{S}\end{array}\right]_{(6\times1)} . \tag{C-7}$$

Next the coordinates of the satellite in its orbital plane $\mathbf{U}$ will be transformed into the Cartesian coordinates of the inertial system $\mathbf{X}$. (Figure 4) In this paper it is assumed that the *earth's* center of mass and the center of the *spheroid* representing the earth are *coincident*. This, in turn, implies that the origins of $\mathbf{X}$ and $\mathbf{U}$ coincide; hence only rotations are

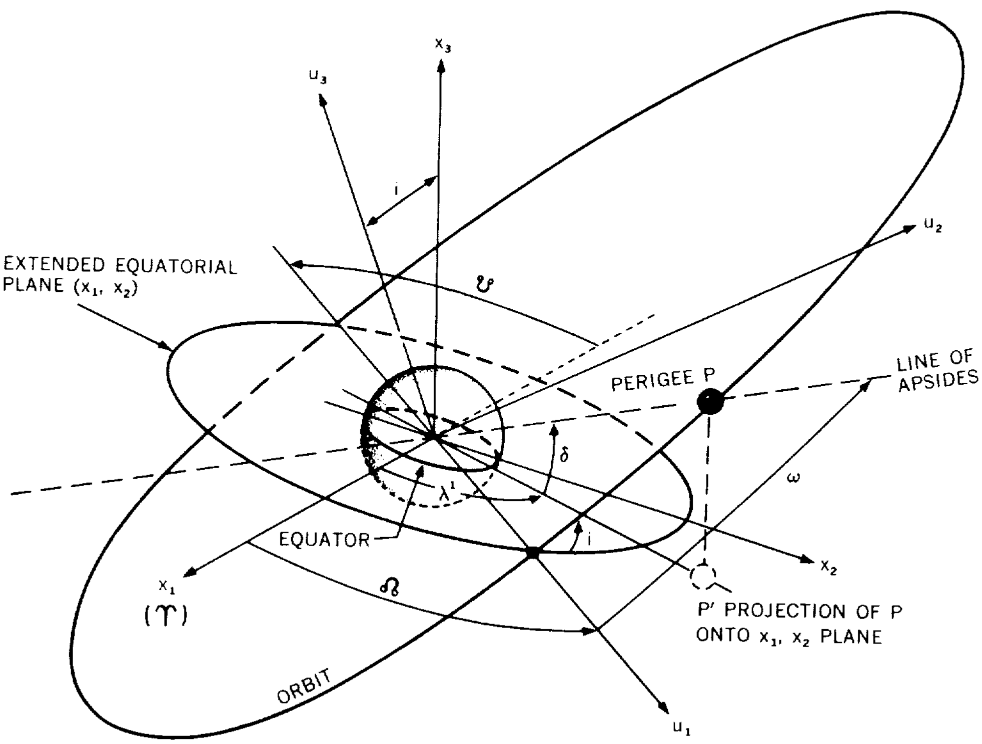

Figure 4—Orbit in the inertial frame of reference

necessary to perform the required transformation. The equation for this (via the Eulerian angles) is expressed as follows:

$$\mathbf{X} = \mathbf{R}_3(-\Omega)\,\mathbf{R}_1(-i)\,\mathbf{R}_3(-\omega)\,\mathbf{U} \tag{C-8}$$

where

$$\mathbf{U} = \begin{bmatrix} a_s\left(\cos E_s - e_s\right) \\ a_s\left(1 - e_s^2\right)^{1/2}\sin E_s \\ 0 \end{bmatrix}_{(3\times 1)} ,$$

and the elements of $\mathbf{U}$ actually represent the Cartesian coordinates of a Keplerian ellipse in the orbital plane.

D. MATHEMATICAL MODELS OF TRACKING SYSTEMS AND THEIR ERRORS

The primary purpose of any tracking system is to determine the position vector $\vec{r}(t)$ and the velocity vector $\vec{v} = d\vec{r}/dt$ of an object moving in space. The position vector can be expressed as

$$\begin{aligned} \vec{r} &= |\vec{r}|\,\vec{r}^{\,0} \\ &= r\,\vec{r}^{\,0} \quad , \end{aligned} \tag{D-1}$$

where $r = |\vec{r}|$ is the magnitude of $\vec{r}$ and $\vec{r}^{\,0}$ is the unit position vector. Differentiating Equation (D-1) directly gives the velocity vector $\vec{v}$ of the object:

$$\frac{d\vec{r}}{dt} = \vec{v} = \frac{dr}{dt}\,\vec{r}^{\,0} + r\,\frac{d\vec{r}^{\,0}}{dt} \quad . \tag{D-2}$$

The four basic tracking systems which exist and which satisfy Equations (D-1) and (D-2) will now be described.

The Radar System

Herein it is assumed that a radar measures the range r (which in actuality is a computed value based upon the travel time of the radar wave), the azimuth α, and the elevation angle ϵ. From these, the position vector $\vec{r}$ is fully determined. The time derivatives $\dot{r}$, $\dot{\alpha}$ and $\dot{\epsilon}$ are excluded in this treatment of a radar system because of the relatively large errors involved. The latter omission, however, should not give the impression that a radar system measuring r, α, ϵ, $\dot{r}$, $\dot{\alpha}$ and $\dot{\epsilon}$ cannot be treated by using the equations developed in this paper. For such a case, the equations for one range system and one angle and angular rate* system should be used.

The Angle Measuring System

An angle measuring system is best illustrated by the radio interferometer (References 7 and 8) which measures the difference in the arrival times of the wave front from a distant point source at a pair of receiving antennas separated by a known distance or "baseline". This radio path difference is measured by comparing the phase angles of the signals received at the two antennas. Two such baselines are employed (four antennas); and from the resultant differences in phase angle $\Delta\phi$ along both baselines, the direction cosines for two components of the unit position vector $\vec{r}^0$ are determined from the following relations:

$$l = \frac{\Delta\phi_1}{d_1} ,$$

$$m = \frac{\Delta\phi_2}{d_2} ,$$

$$n = (1 - l^2 - m^2)^{1/2} ,$$

where d_1 is the baseline length (in electrical degrees) directed, for example, in the east-west direction, and d_2 is the baseline length (in electrical degrees) directed in the north-south direction. From the calculated values (l, m, and n) the azimuth α and the elevation angle ϵ, can be derived, or, in the inertial coordinate system, the right ascension λ' and the declination δ.

The unit vector $\vec{r}^0$ is obtained at discrete times t_i, where $i = 1, 2, \cdots n$; therefore

$$\frac{d\vec{r}^0}{dt} \approx \frac{\Delta\vec{r}^0(t_i)}{\Delta t_i}$$

can also be obtained. This means that each observation consists of (α, $\dot{\alpha}$, ϵ, $\dot{\epsilon}$) at t_i. For a good determination of $\vec{r}$ and $\vec{v}$ which satisfies Equations (D-1) and (D-2), simultaneous observations from two stations are required to determine the three components of $\vec{r}$ and the three components of $\vec{v}$. In actuality this is an over-determined set of observation equations,

*Angular rate information, not measured by a radar system, can be calculated numerically (also see the section on smoothing) by fitting a polynominal to the measured angular information.

since the number of measurements is greater than the number of unknown parameters; that is, the least squares condition is satisfied and could be applied for determining $\vec{r}$ and $\vec{v}$.

The Range Only System

It is presumed that this system measures only the slant range r in the manner described next for the radar system. Because only one measurement r is made at time t_i ($i = 1, 2, \cdots n$) a range system is inadequate for determining the three components of the vector $\vec{r}$. To make it possible to determine $\vec{r}$ at t_i, simultaneous observations must be made from at least three tracking stations whose positions are known; it is also possible to determine roughly the vector (Reference 6):

$$\vec{v} \doteq \frac{\Delta\vec{r}(t_i)}{\Delta t_i} .$$

from discrete measurements $\vec{r}(t_i)$.

The Range Rate System

A range-rate system measures $d/dt\ |\vec{r}|$. To determine the velocity vector $\vec{v}$, at least three simultaneous observations are required from three stations in order to determine the components of $\vec{v}$ and satisfy the vector Equation (D-2) (see Reference 15). As in the foregoing determination, the quantity "determined", $\dot{r}$ in this case, is actually a calculated quantity; it is obtained from observing the Doppler shift $\Delta\nu_0$ of a frequency ν_0. For a first approximation $\dot{r} = c(\Delta\nu_0/\nu_0)$ where c is the velocity of light (299.7929×10^6 m/sec).

Advanced tracking systems are combinations of the aforementioned basic systems — for example, Azusa and Cyclops. In order to treat these various types of tracking systems, their mathematical models, derived from the $\vec{r}$ and $\vec{v}$ vectors, are now presented and the corresponding variational equations lead finally to the determination of tracking system errors.

1. The Radar System

A radar system fully determines the satellite position vector $\vec{r}$ whose components in the local Cartesian coordinate system are (z_1, z_2, z_3). We shall now derive the variational equation relating system uncertainties in range, azimuth, and elevation to the components of $\vec{r}$.

Since it is convenient, in this case, to work in spherical coordinate systems, the components of the position vector $\vec{r}$ in the local coordinate system (shown in Figure 5) can be written in matrix notation as

$$\mathbf{Z}_{(3\times1)} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}_{(3\times1)} = \begin{bmatrix} r \sin\alpha\cos\epsilon \\ r\cos\alpha\cos\epsilon \\ r\sin\epsilon \end{bmatrix}_{(3\times1)} \qquad \text{(D-3)}$$

The variations $(\delta\vec{r})$ of $\vec{r}$, which will be used later for the error calculations, are derived from the first order terms of the Taylor series as applied to Equation (D-3) and are expressed in matrix form by

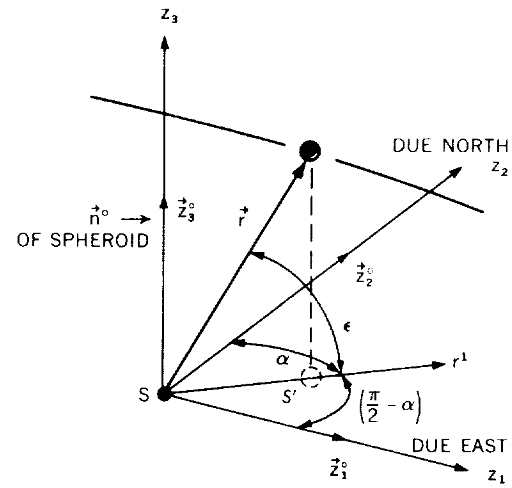

Figure 5—Satellite position with reference to local coordinate system

$$\delta\mathbf{Z}_{(3\times1)} \equiv \begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \delta z_3 \end{bmatrix}_{(3\times1)} = \begin{bmatrix} \frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial\alpha} & \frac{\partial z_1}{\partial\epsilon} \\ \frac{\partial z_2}{\partial r} & \frac{\partial z_2}{\partial\alpha} & \frac{\partial z_2}{\partial\epsilon} \\ \frac{\partial z_3}{\partial r} & \frac{\partial z_3}{\partial\alpha} & \frac{\partial z_3}{\partial\epsilon} \end{bmatrix}_{(3\times3)} \begin{bmatrix} \delta r \\ \delta\alpha \\ \delta\epsilon \end{bmatrix}_{(3\times1)} \qquad \text{(D-4)}$$

By evaluating the coefficient matrix in Equation (D-4) from Equation (D-3) we obtain

$$\delta\mathbf{Z}_{(3\times1)} = \begin{bmatrix} \cos\epsilon\sin\alpha & r\cos\epsilon\cos\alpha & -r\sin\epsilon\sin\alpha \\ \cos\epsilon\cos\alpha & -r\cos\epsilon\sin\alpha & -r\sin\epsilon\cos\alpha \\ \sin\epsilon & 0 & r\cos\epsilon \end{bmatrix}_{(3\times3)} \begin{bmatrix} \delta r \\ \delta\alpha \\ \delta\epsilon \end{bmatrix}_{(3\times1)} \qquad \text{(D-5)}$$

or

$$\delta\mathbf{Z}_{(3\times1)} = \begin{bmatrix} \cos\epsilon\sin\alpha & -\cos\alpha & -\sin\epsilon\sin\alpha \\ \cos\epsilon\cos\alpha & \sin\alpha & -\sin\epsilon\cos\alpha \\ \sin\epsilon & 0 & \cos\epsilon \end{bmatrix}_{(3\times3)} \begin{bmatrix} \delta r \\ -r\cos\epsilon\,\delta\alpha \\ r\delta\epsilon \end{bmatrix}_{(3\times1)}, \qquad \text{(D-6)}$$

where the coefficient matrix of Equation (D-6) is now an orthogonal matrix **J** which generally simplifies the necessary matrix operations. In matrix form Equation (D-6) now becomes

$$\delta\mathbf{Z} = \mathbf{J}\,\mathbf{D}\,\delta\mathbf{K}, \qquad \text{(D-7)}$$

where $\mathbf{J}$ is the orthogonal matrix in Equation (D-6) which contains only the trigonometric functions of ϵ and α, and the matrices $\mathbf{D}$ and $\delta\mathbf{K}$ are defined as

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -r\cos\epsilon & 0 \\ 0 & 0 & r \end{bmatrix}_{(3\times3)} ;$$

$$\delta\mathbf{K} = \begin{bmatrix} \delta r \\ \delta\alpha \\ \delta\epsilon \end{bmatrix}_{(3\times1)} .$$

The matrix $\mathbf{D}$ is introduced only to separate *computational* values from values based upon *observational* errors such as δr, $\delta\epsilon$, and $\delta\alpha$. The known values of Equation (D-7) are $\mathbf{J}$, $\mathbf{D}$ (from an assumed orbit), and $\delta\mathbf{K}$ (from a tracking system). Because of the orthogonality property of the $\mathbf{J}$-matrix ($\mathbf{J}^T\,\mathbf{J} = \mathbf{I}$ is the identity matrix), it follows from Equation (D-7) that

$$\mathbf{D}_{(3\times3)}\,\delta\mathbf{K}_{(3\times1)} = \mathbf{J}^T_{(3\times3)}\,\delta\mathbf{Z}_{(3\times1)} . \tag{D-8}$$

The form of Equation (D-8) will prove to be more desirable in combination with other types of tracking system equations to be used later. Only the position vector $\vec{r}$ is fully determined by Equation (D-3). The velocity vector $\vec{v} = \dot{\vec{r}}$, on the other hand, cannot be precisely determined for a radar system because of two factors: (1) The relatively poor angular rate data $\delta\dot{\alpha}$, $\delta\dot{\epsilon}$ (Reference 7); and (2) The relatively large uncertainties in the incremental range measurements which are to be used for the velocity determination (References 9, 10, and 11).

2. The Angle and Angular Rate Measuring System

Variational equations reflecting uncertainties in azimuth, azimuth rate, elevation, and elevation rate will be developed now for the purpose of relating these uncertainties to the components of the $\vec{r}$ and $\vec{v}$ vectors of the satellite in the local Cartesian coordinate system.

The unit position vector $\vec{r}^{\,0}$ can be written by using Equation (D-3) with $r = 1$:

$$\mathbf{Z}^0_{(3\times1)} = \begin{bmatrix} \sin\alpha\cos\epsilon \\ \cos\alpha\cos\epsilon \\ \sin\epsilon \end{bmatrix}_{(3\times1)} = \frac{1}{r}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}_{(3\times1)} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}_{(3\times1)} . \tag{D-9}$$

Since $n = (1 - l^2 - m^2)^{1/2}$ is a derived quantity (in Minitrack, Azusa, Mistram, etc.), Equation (D-9) should be restated as follows:

$$\begin{bmatrix} l \\ m \end{bmatrix}_{(2\times1)} = \begin{bmatrix} \sin\alpha\cos\epsilon \\ \cos\alpha\cos\epsilon \end{bmatrix}_{(2\times1)} . \tag{D-10}$$

A variation of the above equation is

$$\begin{bmatrix} \delta l \\ \delta m \end{bmatrix}_{(2\times1)} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{bmatrix}_{(2\times2)} \begin{bmatrix} \cos\epsilon\,\delta\alpha \\ \sin\epsilon\,\delta\epsilon \end{bmatrix}_{(2\times1)} . \tag{D-11}$$

From Equation (D-9) it is seen that

$$\begin{bmatrix} l \\ m \end{bmatrix}_{(2\times1)} = \frac{1}{r}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_{(2\times1)} , \qquad r^2 = z_1^2 + z_2^2 + z_3^2 ;$$

and, again, the variation equation can be written*:

$$\begin{bmatrix} \delta l \\ \delta m \end{bmatrix}_{(2\times1)} = \frac{1}{r}\begin{bmatrix} (1 - l^2) & -lm & -ln \\ -lm & 1 - m^2 & -mn \end{bmatrix}_{(2\times3)} \begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \delta z_3 \end{bmatrix}_{(3\times1)} \tag{D-12}$$

By combining Equation (D-11) with Equation (D-12) to give

$$\delta\alpha = F(\delta z_1, \ \delta z_2, \ \delta z_3) ,$$

$$\delta\epsilon = g(\delta z_1, \ \delta z_2, \ \delta z_3) ,$$

we obtain the following result:

$$\begin{bmatrix} -r\cos\epsilon\,\delta\alpha \\ r\,\delta\epsilon \end{bmatrix}_{(2\times1)} = \begin{bmatrix} -\cos\alpha & \sin\alpha & 0 \\ -\sin\epsilon\sin\alpha & -\sin\epsilon\cos\alpha & \cos\epsilon \end{bmatrix}_{(2\times3)} \begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \delta z_3 \end{bmatrix}_{(3\times1)} . \tag{D-13}$$

*The form of the variational equation requires knowledge of r which is usually obtained from nominal computed trajectory information with accuracy sufficient for Equation (D-12); for example r^{-1} or $(r + \Delta r)^{-1}$ with $\Delta r/r \leq 10^{-2}$ does not alter Equation (D-12).

Comparison of Equation (D-13) with Equation (D-8)* suggests that Equation (D-13) can also be obtained if we multiply Equation (D-8) with an operator matrix $\mathbf{F}_a$ (where the index a refers to *a*ngular data) which must be of the form:

$$\mathbf{F}_{a(2\times3)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(2\times3)} .$$

Expressing the variational Equation (D-13) by using the given parameters δK in conjunction with $\mathbf{F}_a$, we obtain*:

$$\mathbf{F}_{a(2\times3)} \mathbf{D}_{(3\times3)} \delta K_{(3\times1)} = \mathbf{F}_{a(2\times3)} \mathbf{J}^{T}_{(3\times3)} \delta \mathbf{Z}_{(3\times1)} , \tag{D-14}$$

from Equation (D-8).

For simplicity, as well as consistency with Equation (D-8) (which will be useful later), we let

$$\mathbf{D}^{0}_{a(2\times3)} \equiv \mathbf{F}_a \mathbf{D} = \begin{bmatrix} 0 & -r\cos\epsilon & 0 \\ 0 & 0 & r \end{bmatrix}_{(2\times3)} ,$$

and

$$\mathbf{J}^{0^T}_{a(2\times3)} \equiv \mathbf{F}_a \mathbf{J}^{T} = \begin{bmatrix} 0 & \sin\alpha & -\cos\alpha \\ \sin\epsilon & \cos\epsilon\cos\alpha & \cos\epsilon\sin\alpha \end{bmatrix}_{(2\times3)} . \tag{D-15}$$

Now Equation (D-14) can be written:

$$\mathbf{D}^{0}_{a(2\times3)} \delta K_{(3\times1)} = \mathbf{J}^{0^T}_{a(2\times3)} \delta \mathbf{Z}_{(3\times1)} , \tag{D-16}$$

which is the form of our basic variational equation (D-8).

The operator matrices such as $\mathbf{F}_a$ (here) and $\mathbf{F}_r$ (in succeeding sections) were introduced for notational purposes only. On the other hand this complicates the computational scheme owing to redundant computer multiplications. This difficulty will be avoided by complementing the generalized equation with an equation more suitable for computer use†.

*Since *Equation (D-8)* fully determines $\delta \bar{t}$, it becomes our *fundamental equation.*

†This was suggested by Mr. R. Sandifer.

From an interferometer type tracking system, both angular and angular rate data are obtained with great precision. Therefore, an equation describing angular rates is necessary for a complete description of the interferometer system.

Utilizing the fundamental Equation (D-7) we obtain the following by differentiation with respect to time:

$$\dot{\mathbf{Z}}_{(3\times1)} = \mathbf{J}_{(3\times3)}\,\mathbf{D}_{(3\times3)}\,\dot{\mathbf{K}}_{(3\times1)}\,, \tag{D-17}$$

which is a direct analog to (D-7) but contains the time derivative of the parameters r, α, and ϵ instead of their variations. From Equation (D-17) and using $\mathbf{J}^T\mathbf{J} = \mathbf{I}$, we can calculate

$$\mathbf{D}_{(3\times3)}\dot{\mathbf{K}}_{(3\times1)} = \mathbf{J}^T_{(3\times3)}\,\dot{\mathbf{Z}}_{(3\times1)}\,, \tag{D-18}$$

which is consistant with the fundamental variational Equation (D-8).

Varying Equation (D-18) and rearranging the form we obtain

$$\left.\begin{aligned} \mathbf{D}_{(3\times3)}\,\delta\dot{\mathbf{K}}_{(3\times1)} &= \left[\left(\delta\mathbf{J}^T\right)\dot{\mathbf{Z}} - (\delta\mathbf{D})\dot{\mathbf{K}}\right]_{(3\times1)} + \mathbf{J}^T_{(3\times3)}\delta\dot{\mathbf{Z}}_{(3\times1)}\,, \\ \text{where} \qquad & \\ \delta\dot{\mathbf{K}}_{(3\times1)} &\equiv \begin{bmatrix}\delta\dot{r}\\ \delta\dot{\alpha}\\ \delta\dot{\epsilon}\end{bmatrix}_{(3\times1)}\,. \end{aligned}\right\} \tag{D-19}$$

By substituting for $\dot{\mathbf{K}}$ from Equation (D-18), calculating $\delta\mathbf{J}^T$, $\delta\mathbf{D}$, and combining terms, we have the following equation:

$$\mathbf{D}_{(3\times3)}\,\delta\dot{\mathbf{K}}_{(3\times1)} = \mathbf{J}^T_{(3\times3)}\,\delta\dot{\mathbf{Z}}_{(3\times1)} + \mathbf{V}_{(3\times3)}\,\mathbf{J}^T_{(3\times3)}\,\delta\mathbf{Z}_{(3\times1)}\,, \tag{D-20}$$

which is the form consistent with Equations (D-8) and (D-14). The matrix $\mathbf{V} = \left[v_{ij}\right]$ in (D-20) now has the following elements:

$$v_{11} = 0\,,$$

$$v_{12} = -\frac{1}{r}\left(\dot{z}_1\cos\alpha - \dot{z}_1\sin\alpha\right)\,,$$

$$v_{13} = -\frac{1}{r}\left(\dot{z}_1 \sin\epsilon \sin\alpha + \dot{z}_2 \sin\epsilon \cos\alpha - \dot{z}_3 \cos\epsilon\right),$$

$$v_{22} = -\frac{\sec\epsilon}{r}\left(\dot{z}_1 \sin\alpha + \dot{z}_2 \cos\alpha\right),$$

$$v_{23} = -\frac{\tan\epsilon}{r}\left(\dot{z}_1 \cos\alpha - \dot{z}_2 \sin\alpha\right),$$

$$v_{33} = -\frac{1}{r}\left(\dot{z}_1 \cos\epsilon \sin\alpha + \dot{z}_2 \cos\epsilon \cos\alpha + \dot{z}_3 \sin\epsilon\right),$$

$$v_{12} = -v_{21},$$

$$v_{13} = -v_{31},$$

$$v_{23} = -v_{32},$$

which were obtained by expanding Equation (D-20) into its component form. It can be seen from above that matrix $\mathbf{V}$, is skew-symmetric; that is $v_{ij} = -v_{ji}$.

Equation (D-20) can now be modified in a manner similar to Equations (D-8) and (D-16) to contain only the angular parameters α, ϵ, and *not* r.

The range r was included in Equation (D-19) because the matrices involved are square non-singular ones, which simplifies the matrix algebra necessary to obtain Equation (D-20). By multiplying both sides of Equation (D-20) by the matrix $\mathbf{F}_{a_{(2\times3)}}$ and then using Equation (D-15) we have the variational equation, in matrix form, which relates uncertainties in azimuth rate and elevation rate to uncertainties in the components of the vectors $\vec{r}$ and $\vec{v}$ of the satellite; that is:

$$\mathbf{D}^0_{a_{(2\times3)}} \delta \mathbf{K}_{(3\times1)} = \mathbf{J}^0_{a_{(2\times3)}} \delta \dot{\mathbf{Z}}_{(3\times1)} + \mathbf{F}_{a_{(2\times3)}} \mathbf{V}_{(3\times3)} \mathbf{J}^T_{(3\times3)} \delta \mathbf{Z}_{(3\times1)} \tag{D-21}$$

where

$$\mathbf{D}^0_a \delta \dot{\mathbf{K}} = \begin{bmatrix} -r\,\delta\dot{\alpha}\cos\epsilon \\ r\,\delta\dot{\epsilon} \end{bmatrix}_{(2\times1)}$$

It should be noted that Equation (D-21) is consistent with Equations (D-8) and (D-16).

3. The Range Measuring System

A variational equation relating the uncertainty in range to the uncertainties of the components of the satellite position vector will now be derived. Since $|\vec{r}|^2 = Z^T Z$, the resulting variational equation is

$$\delta r_{(1\times1)} = \frac{1}{r} Z^T_{(1\times3)}\, \delta Z_{(3\times1)} = Z^{0^T}_{(1\times3)}\, \delta Z_{(3\times1)} \,. \tag{D-22}$$

The variation δr of r can also be expressed (as for the angle and angular rate system) by introducing another operator matrix F_r (the subscript r stands for range) which permits to use the fundamental Equation (D-8):

$$\delta r_{(1\times1)} = F_{r_{(1\times3)}}\, \delta K_{(3\times1)} = F_{r_{(1\times3)}} D_{(3\times3)} \delta K_{(3\times1)} = D^0_{r_{(1\times3)}}\, \delta K_{(3\times1)} \;; \tag{D-23}$$

where (see page 16)

$$F_r \equiv [1 \quad 0 \quad 0]_{(1\times3)} \,.$$

Similarly, the matrix $Z^{0^T}_{(1\times3)}$ in Equation (D-9) becomes, in terms of the $J_{(3\times3)}$ Equation (D-7),

$$Z^{0^T}_{(1\times3)} = F_{r_{(1\times3)}}\, J^T_{(3\times3)} = J^{0^T}_{(1\times3)} \,. \tag{D-24}$$

Equation (D-22) can be restated consistent with Equations (D-8) and (D-16) by using Equations (D-23) and (D-24):

$$F_{r_{(1\times3)}}\, \delta K_{(3\times1)} = J^{0^T}_{r_{(1\times3)}}\, \delta Z_{(3\times1)} \,. \tag{D-25}$$

4. The Range Rate System

Next, the variational equation relating the uncertainty in range rate to the components of the satellite position and velocity vectors will be derived in matrix notation. Instead of considering the variation of Equation (D-25) we shall consider its time derivative instead, so that Equation (D-24) is combined with Equation (D-25) and replaced by

$$F_{r_{(1\times3)}}\, \dot{K}_{(3\times1)} = F_{r_{(1\times3)}} J^T_{(3\times3)}\, \dot{Z}_{(3\times1)} \,. \tag{D-26}$$

The resulting variational equation can be obtained immediately by application of the operator matrix $\mathbf{F}_{r_{(1\times 3)}}$ from Equation (D-23) to Equation (D-20):

$$\mathbf{D}^{0}_{r_{(1\times 3)}} \, \delta\dot{\mathbf{K}}_{(3\times 1)} = \mathbf{J}^{0^T}_{r_{(1\times 3)}} \, \delta\dot{\mathbf{Z}}_{(3\times 1)} + \mathbf{F}_{r_{(1\times 3)}} \, \mathbf{V}_{(3\times 3)} \, \mathbf{J}^{T}_{(3\times 3)} \, \delta\mathbf{Z}_{(3\times 1)} \; . \qquad \text{(D-27)}$$

Equation (D-27) represents the desired form of the range-rate variation.

Considering only those errors (δr, $\delta\dot{r}$, $\delta\alpha$, $\delta\dot{\alpha}$, $\delta\epsilon$, $\delta\dot{\epsilon}$) pertinent to the tracking systems discussed above, we are able to calculate the error components in the local Cartesian coordinate system $\mathbf{Z}$ for each individual system.

As an analogous problem let us consider a general set of linear equations of observations which can be represented in the following matrix form:

$$\mathbf{Y}_{(n\times 1)} = \mathbf{A}_{(n\times m)} \, \mathbf{X}_{(m\times 1)} \; , \qquad \text{(D-28)}$$

where n is the number of observations, $n > m$, and m is the number of degrees of freedom in the system. It is well known that Equation (D-28) can only be solved by the Method of Least Squares (Appendix B); and to solve it for $\mathbf{X}$ we proceed in the following manner:

$$\begin{aligned} \mathbf{A}^T \mathbf{Y} &= \mathbf{A}^T \mathbf{A} \mathbf{X} \\ \mathbf{X} &= \left[\mathbf{A}^T \mathbf{A}\right]^{-1} \mathbf{A}^T \mathbf{Y} \; . \end{aligned} \qquad \text{(D-29)}$$

5a. Error Equations for Tracking Systems in the Local Coordinate System Z

From Equation (D-28), when applied to Equations (D-5) through (D-27), we are able to write the equations pertinent to the various types of tracking systems by assuming the following station complex:

(1) For α *Radars*,* the error equation (Equation D-8) becomes:

$$\mathbf{D}_{(3\alpha\times 3)} \, \delta\mathbf{K}_{(3\times 1)} = \mathbf{J}^{T}_{(3\alpha\times 3)} \, \delta\mathbf{Z}_{(3\times 1)} \; , \qquad \text{(D-30)}$$

where $\alpha \geq 1$ insures over-determination in position.

*Here α represents a row index and not an angular quantity.

(2) For β *Angle and Angular Rate measuring systems,* Equation (D-16) becomes

$$D^0_{a_{(2\beta\times3)}} \delta K_{(3\times1)} = J_{a_{(2\beta\times3)}}^{0^T} \delta Z_{(3\times1)} ; \tag{D-31}$$

and Equation (D-21) is

$$D^0_{a_{(2\beta\times3)}} \delta \dot{K}_{(3\times1)} = J_{a_{(2\beta\times3)}}^{0^T} \delta \dot{Z} + F_{a_{(2\beta\times3)}} V J^T \delta Z_{(3\times1)} , \tag{D-32}$$

where $\beta \geq 2$.

(3) For γ *Range Measuring Systems,* Equation (D-25) becomes

$$F_{r_{(\gamma\times3)}} \delta K_{(3\times1)} = J_{r_{(\gamma\times3)}}^{0^T} \delta Z_{(3\times1)} , \tag{D-33}$$

where $\gamma > 3$.

(4) For δ *Range Rate Systems,* Equation (D-27) becomes

$$F_{r_{(\delta\times3)}} \delta \dot{K}_{(3\times1)} = F_{r_{(\delta\times3)}} V_{(3\times3)} J^T_{(3\times3)} \delta Z_{(3\times1)} + J_{r_{(\delta\times3)}}^{0^T} \delta \dot{Z}_{(3\times1)} , \tag{D-34}$$

where $\delta > 6$.

The foregoing variational equations relate the uncertainties in the parameters measured by a particular tracking system to the uncertainties in the satellite position and velocity vectors. There are *two existing ways* which can be used to evaluate the unknowns in these equations – in this case, the uncertainties associated with the satellite $\vec{r}$ and $\vec{v}$ vector components:

(1) The determination of the unknowns $(\delta z_i, \delta \dot{z}_i)$ by simultaneous solution implies that the number of variational equations is equal to the number of unknown parameters; for example, consider the radar Equation (D-30) for $\alpha = 1$. This means that an exact determination of the uncertainties in the position vector components of the satellite is made without the method of least squares being required.

(2) The determination of the unknowns $(\delta z_i, \delta \dot{z}_i)$ with the method of least squares (Appendix B) implies that the number of equations exceeds the number of unknowns to be determined; that is, the system is over-determined. It is possible to meet the condition of over-determination in two ways: (1) By using n different tracking systems making one measurement; or (2) by using one tracking system making n different measurements where n is greater than the number of unknowns. From a practical point of view, however, the latter is not desirable because an orbit derived in that manner would be poorly determined in orientation and shape (Reference 12).

As an example of an over-determined system, we shall again consider Equation (D-30) for $\alpha > 1$. This type of solution for the unknown parameters reduces the uncertainties since redundancy is a well known condition for enhancing the precision of measurements. Thus an over-determined system of variational equations is always more desirable than one in which the number of equations equals the number of unknown parameters.

All the mathematical models under Sections D1 through D4 were presented in the local coordinate system Z. This follows logically from the fact that all measurements are made locally.

Since not all the stations are in close proximity, their station location uncertainties $\delta S_{(3\times1)}$ are reflected in the satellite position and velocity errors. The value $\delta S_{(3\times1)}$, on the other hand, is referred to the geocentric coordinate system Y, (as shown in Figures 1 and 2). Thus it would seem logical to use the Y-coordinate system in the entire analysis. This is not true, however, since the position and velocity errors depend upon the earth's rotation (i.e., the station location changes with the earth's rotation). Therefore all equations will be written in the inertial coordinate system X. An additional advantage is the ease with which these errors can be transformed into those of the orbital elements if desired (Part II of this analysis will treat this point in more detail).

5b. Error Equations for Tracking Systems in the Inertial Coordinate System X

The variational form $\delta Z_{(3\times1)}$ and the variational form of the time derivative $\delta\dot{Z}_{(3\times1)}$ of Equation (C-6) relating the local coordinate system Z to the inertial system X will now be used for the transformation of the error equations in Section D1 through D4:

$$\delta \mathbf{Z}_{(3\times1)} = \mathbf{R}_{(3\times3)}\mathbf{R}_3(\theta_G)_{(3\times3)}\delta\mathbf{X}_{(3\times1)} - \mathbf{R}_{(3\times3)}\,\delta \mathbf{S}_{(3\times1)} \tag{D-35}$$

and

$$\delta\dot{\mathbf{Z}}_{(3\times1)} = \mathbf{R}_{(3\times3)}\,\mathbf{R}_3(\theta_G)_{(3\times3)}\delta\dot{\mathbf{X}}_{(3\times1)} + \frac{d\theta_G}{dt}\mathbf{R}_{(3\times3)}\left[\frac{d}{d\theta_G}\mathbf{R}_3(\theta_G)\right]_{(3\times3)}\delta\mathbf{X}_{(3\times1)}\,. \tag{D-36}$$

The equations in Sections D1 through D4 are written in the inertial coordinate system X by using Equations (D-35) and (D-36) and by rearranging terms. For α radar systems, these equations are:

(1) For α *radars*,

$$\mathbf{D}_{(3\alpha\times3)}\,\delta\mathbf{K}_{(3\times1)} + \left[\mathbf{J}^T\mathbf{R}\,\delta\mathbf{S}\right]_{(3\alpha\times1)} = \mathbf{L}_{(3\alpha\times3)}\,\delta\mathbf{X}_{(3\times1)}\,, \tag{D-37}$$

where $\alpha \geq 1$ and

$$\mathbf{L}_{(3\alpha\times3)} = \left[\mathbf{J}^T \mathbf{R}\,\mathbf{R}_3(\theta_G)\right]_{(3\alpha\times3)} .$$

(2) For β *angle and angular rate systems* (interferometers),

$$\mathbf{D}^0_{a\,(2\beta\times3)}\, \delta\mathbf{K}_{(3\times1)} + \left[\mathbf{F}_a\, \mathbf{J}^T \mathbf{R}\,\delta\mathbf{S}\right]_{(2\beta\times1)} = \mathbf{L}^0_{a\,(2\beta\times3)}\, \delta\mathbf{X}_{(3\times1)} , \tag{D-38}$$

and

$$\mathbf{D}^0_{a\,(2\beta\times3)}\, \delta\dot{\mathbf{K}}_{(3\times1)} + \left[\mathbf{F}_a\, \mathbf{V}\,\mathbf{J}^T \mathbf{R}\,\delta\mathbf{S}\right]_{(2\beta\times1)} = \mathbf{M}^0_{a\,(2\beta\times3)}\, \delta\mathbf{X}_{(3\times1)} + \mathbf{L}^0_{a\,(2\beta\times3)}\, \delta\dot{\mathbf{X}}_{(3\times1)} , \tag{D-39}$$

where $\beta \geq 2$ and

$$\mathbf{L}^0_{a\,(2\beta\times3)} = \left[\mathbf{F}_a\, \mathbf{L}\right]_{(2\beta\times3)} ,$$

$$\mathbf{M}_{(3\times3)} = \left[\mathbf{V}\,\mathbf{J}^T \mathbf{R}\,\mathbf{R}_3(\theta_G) + \frac{d\theta_G}{dt}\left\{\mathbf{J}^T \mathbf{R}\left[\frac{d}{d\theta_G}\, \mathbf{R}_3(\theta_G)\right]\right\}\right]_{(3\times3)} ,$$

$$\mathbf{M}^0_{a\,(2\beta\times3)} = \left[\mathbf{F}_a\, \mathbf{M}\right]_{(2\beta\times3)} .$$

(3) For γ *range systems,*

$$\mathbf{D}^0_{r\,(\gamma\times3)}\, \delta\mathbf{K}_{(3\times1)} + \left[\mathbf{F}_r\, \mathbf{J}^T \mathbf{R}\,\delta\mathbf{S}\right]_{(\gamma\times1)} = \mathbf{L}^0_{r\,(\gamma\times3)}\, \delta\mathbf{X}_{(3\times1)} , \tag{D-40}$$

where $\gamma \geq 3$ and

$$\mathbf{L}^0_{r\,(\gamma\times3)} = \left[\mathbf{F}_r\, \mathbf{L}\right]_{(\gamma\times3)} .$$

(4) For δ *range rate systems,*

$$\mathbf{D}^0_{r\,(\delta\times3)}\, \delta\dot{\mathbf{K}}_{(3\times1)} + \left[\mathbf{F}_r\, \mathbf{V}\,\mathbf{J}^T \mathbf{R}\;\delta\mathbf{S}\right]_{(\delta\times1)} = \mathbf{M}^0_{r\,(\delta\times3)}\, \delta\mathbf{X}_{(3\times1)} + \mathbf{L}^0_{r\,(\delta\times3)}\, \delta\dot{\mathbf{X}}_{(3\times1)}, \tag{D-41}$$

where $\delta \geq 6$ and

$$\mathbf{L}^0_{r\,(\delta\times3)} = \left[\mathbf{F}_r\, \mathbf{L}\right]_{(\delta\times3)} ,$$

$$\mathbf{M}^0_{r\,(\delta\times3)} = \left[\mathbf{F}_r\, \mathbf{M}\right]_{(\delta\times3)} .$$

6a. Fundamental and Smoothing Time Intervals

Since any physical measurement requires time, here we shall assume that a basic time interval Δt_i is required to make a single measurement which can be, for instance, a range r_i, angles α_i, ϵ_i , or a range rate $\dot{r}_i$. In actuality, the measured value m_i already represents an average value over the time interval Δt_i; that is

$$m_i = \frac{1}{\Delta t_i} \int_{\Delta t_i} m_i^* \, dt \ , \qquad \text{(D-42)}$$

where m_i^* represents an instantaneous "non-measurable" quantity. The values of m_i, as seen from Equation (D-42) already represent a "smoothed" measurement.

With a time interval $\tau >> \Delta t$, modern tracking systems are able to measure a large number ($i >> 1$) of quantities m_i during τ. To reduce this number to a practical level and also reduce the errors, the concept of "smoothing" is introduced.

Now the problem arises of finding the maximum smoothing time, say $\tau = \tau_2$ (Figure 6).

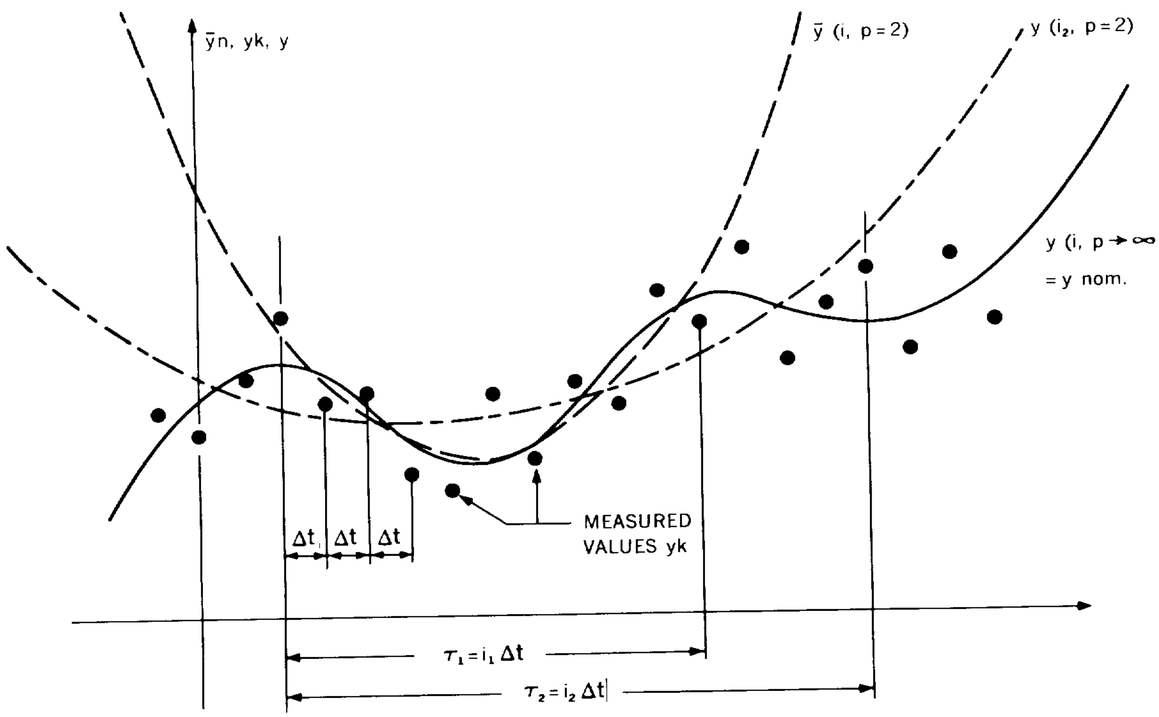

Figure 6—Geometry of the smoothing process.

For a large value of τ_2 (such as one-tenth of a period for a near earth satellite) the observations of $y = y(r, \dot{r}, \alpha, \dot{\alpha}, \epsilon, \dot{\epsilon})$ are of course nonlinear. Since the "real" values $\bar{y}(p \to \infty)$ of these quantities are never known, we are forced to solve this equation by the least squares method, since the number of measurements i far exceeds the number of unknowns n. For most practical cases a development of the values y is the polynomial

$$y_\kappa = \sum_{\gamma=1}^{p} a_\gamma x_\kappa^\gamma \quad , \tag{D-43}$$

where the number of measurements within τ is $\kappa = 1, 2, \cdots, i$; or, in matrix form,

$$\mathbf{Y}_{(i\times 1)} = \mathbf{C}_{(i\times p)} \, \Gamma_{(p\times 1)} \ ,$$

where

$$\mathbf{C}_{(i\times p)} = \left[x_\kappa^\gamma\right]_{(i\times p)} \ .$$

Solving Equation (D-43) by using the least squares method Equation (b-12), (Appendix B), we obtain for the coefficients a_γ in matrix form:

$$\Gamma_{(p\times 1)} = \left(\mathbf{C}^T \mathbf{C}\right)^{-1}_{(p\times p)} \mathbf{C}^T_{(p\times i)} \mathbf{Y}_{(i\times 1)} \ , \tag{D-44}$$

where

$$\Gamma_{(p\times 1)} = \left[a_\gamma\right]_{(p\times 1)} .$$

The measure of "goodness of fit" for a single observation is obtained from Equation (b-14) of Appendix b:

$$y = \bar{y} \pm \eta \ ,$$

where

$$\bar{y} = \frac{1}{i} \sum_{\kappa=1}^{i} y_\kappa = \frac{1}{i} \sum_{\kappa=1}^{i} \sum_{\gamma=0}^{p} a_\gamma x_\kappa^\gamma \tag{D-45}$$

and

$$\eta = \pm \left[\frac{1}{(i-n)} \sum_{\kappa=1}^{\kappa=i} \left(y_\kappa - \bar{y}\right)^2 \right]^{1/2} \ . \tag{D-46}$$

Equation (D-45) can also be utilized for an estimation of the maximum smoothing time τ_2. As can be seen from Equation (D-45), the "best" value $\bar{y}$ for an assumed order p of polynomial is a function only of the number of measurements i made during the time interval $\tau = i\Delta t$ (where Δt is the constant basic time interval necessary for making a single measurement y). Therefore an arbitrary increase of the time τ cannot be made, since the values of $\bar{y}$, calculated from a series of the order p, would deviate more and more from the nominal values (Figure 6). For example, a second order polynomial ($p = 2$) cannot fit a third order curve over *any* interval. Thus a criterion for the maximum tolerable smoothing time for a given polynomial of degree p is

$$\bar{y}(i_1, p) - \bar{y}(i_2, p) \le \bar{y}(i_1, p) - \bar{y}(i_3, p) , \tag{D-47}$$

where $i_1 < i_2$ and $i_2 < i_3$ etc. Should the above inequality reverse itself, the order of the polynomial must be increased (Figure 6).

6b. Introduction of Smoothing into the Error Equations

During the course of this error analysis it was assumed that the uncertainties δr, $\delta\alpha$, $\delta\epsilon$, $\delta\dot{r}$, $\delta\dot{\alpha}$, and $\delta\dot{\epsilon}$ for the different tracking systems were known and were represented by the systems' uncertainty matrices $\delta K_{(3\times1)}$ and $\delta\dot{K}_{(3\times1)}$ Equations (D-7) and (D-19). The term "smoothing process" means simply that the aforementioned uncertainties are to be improved (reduced) in the statistical sense. This means that each value δr, $\delta\dot{r}$ $\cdots$ must be divided by the square root of the number of measurements i made during the time interval τ. Expressed in matrix form and applied to δK and $\delta\dot{K}$, the smoothed error equations are

$$\delta K^*_{(3\times1)} = W_{(3\times3)}\delta K_{(3\times1)}$$

and

$$\delta\dot{K}^*_{(3\times1)} = W_{(3\times3)}\delta\dot{K}_{(3\times1)} , \tag{D-48}$$

where

$$W_{(3\times3)} = \frac{1}{\sqrt{i}} \mathbf{I}_{(3\times3)}$$

and $\mathbf{I}$ is the identity matrix.

The time of smoothing is then given by*

$$\tau = i \Delta t \ . \qquad \text{(D-49)}$$

It should be noted that the value of i is finite (uncertainties do not disappear completely) since, as was previously mentioned, τ and $\Delta\tau$ are finite.

7a. Error Equations in Position and Velocity Assuming Random Errors

By using the error equations presented in Section 5b it is now possible to construct two generalized equations providing rms-errors in satellite position and velocity, including smoothing. The error equation for position errors is

$$\mathbf{A}_{(p\times 1)} = \mathbf{B}_{(p\times 3)} \, \delta\mathbf{X}_{(3\times 1)} \ , \qquad \text{(D-50)}$$

where

$$\mathbf{A}_{(p\times 1)} = \begin{bmatrix} \mathbf{D}_{(3\alpha\times 3)} \, \delta K^*_{(3\times 1)} + \left[\mathbf{J}^T \mathbf{R} \, \delta \mathbf{S}\right]_{(3\alpha\times 1)} \\ \hline \mathbf{D}^0_{a\,(2\beta\times 3)} \, \delta K^*_{(3\times 1)} + \left[\mathbf{F}_a \mathbf{J}^T \mathbf{R} \, \delta \mathbf{S}\right]_{(2\beta\times 1)} \\ \hline \mathbf{D}^0_{r\,(\gamma\times 3)} \, \delta K^*_{(3\times 1)} + \left[\mathbf{F}_r \mathbf{J}^T \mathbf{R} \, \delta \mathbf{S}\right]_{(\gamma\times 1)} \end{bmatrix}_{(p\times 1)} ;$$

and

$$\mathbf{B}_{(p\times 3)} = \begin{bmatrix} \mathbf{L}_{(3\alpha\times 3)} \\ \hline \mathbf{L}^0_{a\,(2\beta\times 3)} \\ \hline \mathbf{L}^0_{r\,(\gamma\times 3)} \end{bmatrix}_{(p\times 3)} ,$$

with $p \geq 3$, where

$$p = (3\alpha + 2\beta + \gamma) \ .$$

For every case three uncertainty components of the station are always included. Thus one range system ($\gamma = 1$), for example, is sufficient to determine the position error, since an

*In this equation the quantity i is not to be confused with the inclination i of an orbit.

orbit is presumed. This statement is academic, however, since this is not a practical way to determine spacecraft position errors. Knowledge of the orbit is only necessary for evaluating the coefficient matrices $\mathbf{D}$, $\mathbf{J}$ and $\mathbf{V}$ of $\delta\mathbf{X}$, $\delta\mathbf{S}$ and $\delta\kappa$, respectively. For an error analysis of this type any nominal orbit can be used for evaluating the coefficient matrices, since an error in the assumed orbit does not influence the position and velocity error derived herein; i.e., the errors do *not* depend on the first order of the "real" position or velocity.

The solution of Equation (D-50) by the least squares method is

$$\delta\mathbf{X}_{(3\times1)} = \left(\mathbf{B}^T\,\mathbf{B}\right)^{-1}_{(3\times3)}\,\mathbf{B}^T_{(3\times p)}\,\mathbf{A}_{(p\times1)}\ . \tag{D-51}$$

which represents, in the inertial coordinate system, the total variations in each component of the satellite position vector.

As was previously mentioned, the uncorrelated errors in the components of the position vector $\vec{r}$ are desired (Figure 5). These errors can be obtained from

$$\mathbf{H}_{(3\times3)} = \delta\mathbf{X}_{(3\times1)}\,\delta\mathbf{X}^T_{(1\times3)}\ , \tag{D-52}$$

which contains the correlated terms; $\mathbf{H}'$ is formed by assuming that the correlated terms are zero, that is,

$$\mathbf{H}'_{(3\times3)} \equiv \begin{bmatrix} \eta^2_{x_1} & 0 & 0 \\ 0 & \eta^2_{x_2} & 0 \\ 0 & 0 & \eta^2_{x_3} \end{bmatrix}_{(3\times3)}\ . \tag{D-53}$$

The total rms error in satellite position is then given by (Reference 15):

$$\eta_x = \pm\left(\sum_{i=1}^{3} \eta^2_{x_i}\right)^{1/2}\ . \tag{D-54}$$

The position errors determined above can now be used in determining the velocity errors, for which the matrix equation is

$$\bar{\mathbf{A}}_{(q\times1)} = \bar{\mathbf{B}}_{(q\times3)}\,\delta\dot{\mathbf{X}}_{(3\times1)}\ , \tag{D-55}$$

where

$$\bar{A}_{(q\times 1)} = \left[\begin{array}{c} D^0_{a\,(2\beta\times 3)}\,\delta\dot{K}^*_{(3\times 1)} + \left[F_a\ V\ J^T\ R\ \delta S\right]_{(2\beta\times 1)} - M^0_{a\,(2\beta\times 3)}\,\delta X_{(3\times 1)} \\ \hdashline D^0_{r\,(\delta\times 3)}\,\delta\dot{K}^*_{(3\times 1)} + \left[F_r\ V\ J^T\ R\ \delta S\right]_{(\delta\times 1)} - M^0_{r\,(\delta\times 3)}\,\delta X_{(3\times 1)} \end{array}\right]_{(q\times 1)} ,$$

$$\bar{B}_{(q\times 3)} = \left[\begin{array}{c} L^0_{a\,(2\beta\times 3)} \\ \hdashline L^0_{r\,(\delta\times 3)} \end{array}\right]_{(q\times 3)}$$

and* with $q \geq 3$

$$q = (2\beta + \delta).$$

Solving Equation (D-55), as before, by the least squares method, we have

$$\delta\dot{X}_{(3\times 1)} = \left(\bar{B}^T\ \bar{B}\right)^{-1}_{(3\times 3)}\,\bar{B}^T_{(3\times q)}\,\bar{A}_{(q\times 1)} . \tag{D-56}$$

Thus the velocity errors are obtained:

$$\bar{H}_{(3\times 3)} = \delta\dot{X}_{(3\times 1)}\delta\dot{X}^T_{(1\times 3)} ; \tag{D-57}$$

And again assuming no correlation in the matrix:

$$\bar{H}'_{(3\times 3)} = \begin{bmatrix} \eta^2_{\dot{x}_1} & 0 & 0 \\ 0 & \eta^2_{\dot{x}_2} & 0 \\ 0 & 0 & \eta^2_{\dot{x}_3} \end{bmatrix}_{(3\times 3)} . \tag{D-58}$$

Therefore the total rms error in satellite velocity is given by

$$\eta_{\dot{x}} = \pm\left[\sum_{i=1}^{3} \eta^2_{\dot{x}_i}\right]^{1/2} . \tag{D-59}$$

*If the position error components are not computed from Equations (D-50), (D-51), and (D-52), then, of course, $q \geq 6$ (e.g., six range rate stations or three angular rate stations).

The general error equations – Equations (D-50) to (D-59) – can be solved from two points of view:

(1) By letting the indices (α, β, γ, or δ) represent number of stations which observe the satellite; or

(2) By letting (α, β, γ, or δ) represent the total number of observations made by each type of tracking station.

The former approach will require less computer storage, since the evaluation of the partial derivatives – see Equation (D-4) – comprising the coefficient matrices of δK, δS, and δX is kept at a minimum. For example, if the latter method were chosen for α radars, 30 sets of partial derivatives would have to be evaluated per second instead of the 3 per second for the first approach. It was presupposed in this example that the radar sampling rate is 1 measurement every 1/10 second. By introducing the concept of smoothing into the first approach a result equivalent to the second is obtained.

In the case of a tracking system not capable of measuring $\vec{r}$ and $\vec{v}$ independently of knowledge of the orbit, then the second method is more desirable. For example, consider the evaluation of the position and velocity error components of the satellite using one range and range rate system. In this case the error is no longer independent of the orbit, since knowledge of the orbit is required; and independence of the orbit can be presumed only in the case of tracking systems or tracking system complexes (e.g., three range and range-rate stations) which are capable of determining $\vec{r}$ and $\vec{v}$ independently of the satellite orbit. In some cases, therefore, the precision of the orbit must be incorporated in the position and velocity error components of the satellite.

7b. Bias Errors

Every measurement no matter how carefully it is made, is subject to errors. Experience has shown that repeated measurements of the same quantity do not give the same result. The two kinds of errors influencing the results are: (1) constant or bias errors whose magnitudes depend on parameters such as boresight errors etc. and (2) accidental errors referred to here as "uncertainties" which have no fixed cause and hence follow no fixed physical law. For example, a measurement of some quantity can be represented in mathematical form as follows:

$$m = m_0 + \Delta m \pm \delta m \tag{D-60}$$

If m_0 represents the "most probable value" then $(\Delta m \pm \delta m)$ must represent the errors in the measurement. Those errors characterized by Δm are defined as "bias errors", whereas the accidental errors or "uncertainties" are those represented by $\pm \delta m$.

One possible way to determine the bias errors is to use a precise orbit (determined by other means, for instance optical observations) which is to be compared with the tracking systems measurements. The arithmetic average of the resulting residuals will then give a good estimate of the bias errors. A more detailed discussion on the determination of bias errors combined with numerical examples will be presented as a separate paper.

8. Comments on Tracking System Uncertainties

Proper precautions must be exercised in using the error equations derived herein. For combinations of tracking systems simultaneously tracking a satellite or a spacecraft, care must be taken in selecting the uncertainties (δr, $\delta \dot{r}$, $\delta \alpha$, $\delta \dot{\alpha}$, $\delta \epsilon$, $\delta \dot{\epsilon}$) related to the particular systems. Consider the case of one, two or three radars used to determine the position vector of a spacecraft. For one radar, the uncertainties (δr, $\delta \alpha$, $\delta \epsilon$) are used. In the case of three radar stations the position of the spacecraft is fully determined by the three range measurements alone (Reference 6). Therefore only δr_1, δr_2 and δr_3 need be considered, and the angular uncertainties $\delta \alpha_i$, $\delta \epsilon_i$ can be assumed to be zero since they are ~~not~~ required for this case: $\delta r << (r\, \delta \alpha \cos \epsilon)$, or $\delta r << (r\, \delta \epsilon)$.

In the case of two radars, a very simple "assumption" can be made – multiply the angular errors by the factor 1/2. A weighting of the angular errors of 1, 1/2 and 0 seems reasonable for these three cases. Similar weighting procedures will have to be considered for other tracking system combinations.

In brief, when the solution is over-determined the uncertainties contributing the most to the overall error should be eliminated.

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Appendix a

Transformation of Cartesian Coordinates into Spheroidal Coordinates

From Figure 2 (page 7) we can write the equation for the elliptical cross section of the earth as (Reference a1):

$$\frac{y_1^2}{a_\oplus^2} + \frac{y_3^2}{a_\oplus^2\left(1 - e_1^2\right)} = 1 \; , \tag{a-1}$$

where the semimajor axis is

$$a_\oplus = \left(y_1^2 + y_3^2\right)^{1/2} \tag{a-2}$$

and

$$\begin{aligned} y_1 &= a_\oplus \cos\phi' \; , \\ y_3 &= a_\oplus \sin\phi' \; . \end{aligned} \tag{a-3}$$

The unit normal to the spheroid in the y_1y_3-plane is

$$\vec{n}^0 = \frac{1}{m}\left[\frac{y_1}{a_\oplus^2}\,\vec{i} + \frac{y_3}{a_\oplus^2\left(1 - e^2\right)}\,\vec{k}\right] \; , \tag{a-4}$$

where

$$m \equiv \frac{y_1}{a_\oplus^2}\left[1 + \left(\frac{\frac{y_3}{y_1}}{1 - e_1^2}\right)^2\right]^{1/2}$$

From Figure 2 it follows that

$$\vec{n}^0 \cdot \vec{i} = \cos\phi = \frac{1}{m}\left(\frac{y_1}{a_\oplus^2}\right) . \tag{a-5}$$

And by using Equation (a-3) with Equation (a-5) we see that

$$\cos\phi\left[1+\frac{\tan^2\phi'}{(1-e_1^2)^2}\right]^{1/2} = 1 \ . \tag{a-6}$$

or, simplifying,

$$\tan\phi' = (1-e_1^2)\tan\phi \ . \tag{a-7}$$

We also find from Figure 2 that

$$N\cos\phi = a_\oplus \cos\phi' \ ; \tag{a-8}$$

and from Equations (a-1), (a-4), and (a-8), we have

$$N = \frac{a_\oplus}{\sqrt{1-e_1^2\sin^2\phi}} \tag{a-9}$$

from which it follows that

$$\left.\begin{aligned} y_1 &= N\cos\phi \ , \\ y_3 &= N(1-e_1^2)\sin\phi \ . \end{aligned}\right\} \tag{a-10}$$

But

$$s_1 = y_1 + h\cos\phi \ ,$$

$$s_3 = y_3 + h\sin\phi \ ;$$

and thus

$$\left.\begin{aligned} s_1 &= (N+h)\cos\phi \ , \\ s_3 &= \left(N(1-e_1^2)+h\right)\sin\phi \ , \end{aligned}\right\} \tag{a-11}$$

Since in Euclidian space

$$\vec{s} = \vec{s}_1 + \vec{s}_3 \tag{a-12}$$

where

$$\vec{s}_1 = s_1\left[\cos\lambda\,\vec{i} + \sin\lambda\,\vec{j}\right],$$
$$\vec{s}_3 = s_3\,\vec{k}\ , \tag{a-13}$$

we have

$$\vec{s} = \mathbf{S}_{(3\times 1)} = \begin{bmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ \left[N\left(1-e_1^2\right) + h\right]\sin\phi \end{bmatrix}_{(3\times 1)} . \tag{a-14}$$

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Appendix b

The Method of Least Squares

The discussion which follows is a brief exposition on the method of least squares, neglecting the conditions of constraint.

In order to treat the generalized case of the method of least squares, we assume a set of n-observation equations in the following form:

$$\left.\begin{aligned} y_1 &= f_1(x_1, x_2, \cdots, x_k) , \\ y_2 &= f_2(x_1, x_2, \cdots, x_k) , \\ &\vdots \\ y_n &= f_n(x_1, x_2, \cdots, x_k) , \end{aligned}\right\} \qquad \text{(b-1)}$$

where the equations are subject to the condition that $n > k$, which implies that the system of equations must be over-determined.

Since any observation is subject to errors, we shall rewrite Equations (b-1) as

$$\begin{aligned} y_1 + \epsilon_1 &= f_1(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_k + \Delta x_k) , \\ y_2 + \epsilon_2 &= f_2(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_k + \Delta x_k) , \\ &\vdots \\ y_n + \epsilon_n &= f_n(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_k + \Delta x_k) , \end{aligned} \qquad \text{(b-2)}$$

where the values ϵ_i and Δx_j, with $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$, represent the errors in the observations y_i and the corrections to the independent variables x_j.

In order to determine the quantities of interest Δx_j, we assume that the relative errors are small, i.e., $\Delta x_j / x_j << 1$; then Equation (b-2) can be developed in a Taylor's series by using only the linear terms. The following result is obtained:

$$y_i + \epsilon_i = f_i(x_1, x_2, \cdots, x_k) + \sum_{j=1}^{k} \frac{\partial f_i}{\partial x_j} \Delta x_j + \text{(higher order terms)} ,$$

or

$$\epsilon_i \approx \sum_{j=1}^{k} \frac{\partial f_i}{\partial x_j} \Delta x_j , \tag{b-3}$$

where $i = 1, 2, \cdots, n$.

From Equation (b-3), we obtain:

$$\left.\begin{aligned} \epsilon_1 &\approx \sum_{j=1}^{k} \frac{\partial f_1}{\partial x_j} \Delta x_j , \\ \epsilon_2 &\approx \sum_{j=1}^{k} \frac{\partial f_2}{\partial x_j} \Delta x_j , \\ &\vdots \\ \epsilon_n &\approx \sum_{j=1}^{k} \frac{\partial f_n}{\partial x_j} \Delta x_j ; \end{aligned}\right\} \tag{b-4}$$

and

$$\left.\begin{aligned} v_1 &= \epsilon_1 - \sum_{j=1}^{k} \frac{\partial f_1}{\partial x_j} \Delta x_j , \\ v_2 &= \epsilon_2 - \sum_{j=1}^{k} \frac{\partial f_2}{\partial x_j} \Delta x_j , \\ &\vdots \\ v_n &= \epsilon_n - \sum_{j=1}^{k} \frac{\partial f_n}{\partial x_j} \Delta x_j , \end{aligned}\right\} \tag{b-5}$$

where v_i (with $i = 1, 2, \cdots, n$) represents small residuals resulting from the general equation (b-3).

Gauss (References b1-b5) made an assumption stating that the "best" values of Δx_j are those for which

$$\phi = \sum_{i=1}^{n} w_i v_i^2 \rightarrow \text{minimum} . \qquad \text{(b-6)}$$

where w_i represents a weighting factor. Equation (b-6) is known as the condition for least squares. For the following discussion we shall assume that $w_i = 1$, i.e., all measurements are to contain errors of equal importance.

In order to simplify the algebraic notation, the symbolic matrix notation will be used. Thus Equation (b-5), restated in matrix form, is

$$\mathbf{V}_{(n\times 1)} = \boldsymbol{\mathcal{E}}_{(n\times 1)} - \mathbf{P}_{(n\times k)} \, \Delta\mathbf{X}_{(k\times 1)} , \qquad \text{(b-7)}$$

where

$$\mathbf{V}_{(n\times 1)} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} ; \qquad \boldsymbol{\mathcal{E}}_{(n\times 1)} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}_{(n\times 1)} ; \qquad \Delta\mathbf{X}_{(k\times 1)} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_k \end{bmatrix}_{(k\times 1)} ;$$

and

$$\mathbf{P}_{(n\times k)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \ldots, & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1}, & \frac{\partial f_2}{\partial x_2}, & \ldots, & \frac{\partial f_2}{\partial x_k} \\ \text{- - -} & \text{- - -} & \text{- - -} & \text{- - -} \\ \text{- - -} & \text{- - -} & \text{- - -} & \text{- - -} \\ \frac{\partial f_n}{\partial x_1}, & \frac{\partial f_n}{\partial x_2}, & \ldots, & \frac{\partial f_n}{\partial x_k} \end{bmatrix}_{(n\times k)} .$$

The matrices $\mathbf{V}$ and $\boldsymbol{\mathcal{E}}$ can be considered as n-component vectors, whereas ΔX is a k-component vector.

The Gaussian assumption, therefore, may be considered in terms of vector algebra as the inner vector product which in matrix form simply becomes:

$$\phi = \mathbf{V}^T \mathbf{V} \rightarrow \text{minimum} , \qquad \text{(b-8)}$$

where

$$\mathbf{V}^T = \text{transposed matrix } \mathbf{V} \text{ of Equation (b-7)} .$$

By introducing Equation (b-7) into Equation (b-8), we have

$$\phi = \mathcal{E}^T\mathcal{E} - \mathcal{E}^T \mathbf{P}\, \Delta\mathbf{X} - \Delta\mathbf{X}^T \mathbf{P}^T\mathcal{E} + \Delta\mathbf{X}^T \mathbf{P}^T \mathbf{P}\, \Delta\mathbf{X} \rightarrow \text{minimum} . \qquad \text{(b-9)}$$

The minimum of Equation (b-9) can be determined by differentiation with respect to $\Delta\mathbf{X}$:

$$\left.\frac{\partial\phi}{\partial\Delta\mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}_0} = 0 . \qquad \text{(b-10)}$$

which gives

$$\mathcal{E}^T \mathbf{P} - \Delta\mathbf{X}_0^T \mathbf{P}^T \mathbf{P} = 0 . \qquad \text{(b-11)}$$

But the minimum condition also requires that $\partial^2\phi/\partial\Delta\mathbf{X}^2$ be positive and from Equation (b-11) it is found that

$$\left.\frac{\partial^2\phi}{\partial\Delta\mathbf{X}^2}\right|_{\mathbf{X}=\mathbf{X}_0} = \mathbf{P}^T \mathbf{P}$$

If $\mathbf{P}^T\mathbf{P}$ is positive definite, ϕ is a minimum.

Equation (b-11) is called the "normal equation", and is easily solvable by using the standard method of multiplying both sides of Equation (b-11) by the inverse matrix of $\mathbf{P}^T\mathbf{P}$, designated as $[\mathbf{P}^T\mathbf{P}]^{-1}$:

$$\Delta\mathbf{X}_{0(k\times 1)} = [\mathbf{P}^T \mathbf{P}]^{-1}_{(k\times k)} \mathbf{P}^T_{(k\times n)}\, \mathcal{E}_{(n\times 1)} . \qquad \text{(b-12)}$$

For ease of notation we let

$$\mathbf{Q}_{(k\times k)} = [\mathbf{P}^T \mathbf{P}]^{-1}_{(k\times k)}$$

and write Equation (b-12) as

$$\Delta\mathbf{X}_{0(k\times 1)} = \mathbf{Q}_{(k\times k)}\, \mathbf{P}^T_{(k\times n)}\, \mathcal{E}_{(n\times 1)} , \qquad \text{(b-13)}$$

where

$$\mathbf{P}_{(n\times k)} = \left[p_{ij}\right]_{(n\times k)} = \left[\frac{\partial f_i}{\partial x_j}\right]_{(n\times k)}, \qquad i = 1, 2, \cdots, n$$

and

$$j = 1, 2, \cdots, k$$

$$\mathbf{Q}_{(k\times k)} = \left[q_{ij}\right]_{(k\times k)} = \left[q_{ji}\right]_{(k\times k)}.$$

Equation (b-13) gives the "most probable" values, in the Gaussian sense, of $\mathbf{\Delta X}$ at $\mathbf{X}_0$.

Next, we shall determine the measure of "goodness of fit", or the root-mean-square error η. Since in most cases the observed data (for measurements such as range, range rate, angle, angular rate) will not obey the normal distribution law, the usual symbol σ is replaced by η.

By substituting Equation (b-11) into Equation (b-9), and having now evaluated $\mathbf{\Delta X}_0$, we determine η^2 in the usual manner

$$\eta^2 = \frac{1}{n-k}\mathbf{V}^T\mathbf{V} = \frac{1}{n-k}\left(\mathcal{E}^T\mathcal{E} - \mathcal{E}^T\,\mathbf{P}\,\mathbf{\Delta X}_0\right), \tag{b-14}$$

where $n > k$, n = total number of observations, k = total number of unknowns, and η is the measure of "goodness of fit" or the rms-error of a single observation.

One question still remains: how accurate are the calculated values $\mathbf{\Delta X}_0$? To answer this, we must consider $\mathbf{H}$:

$$\mathbf{\Delta X} = \mathbf{\Delta X}_0 \pm \mathbf{H}, \tag{b-15}$$

where $\mathbf{H}$ is the uncertainty associated with each component of $\mathbf{\Delta X}_0$.

To calculate the components of $\mathbf{H}$, we must first evaluate the components of $\mathbf{\Delta X}_0$ individually. This can be accomplished by expanding Equation (b-11) as follows:

$$\left.\begin{aligned}
\sum_{j=1}^{n} p_{j1}\,\epsilon_j &= \Delta x_{01}\sum_{j=1}^{n} p_{j1}^2 + \Delta x_{02}\sum_{j=1}^{n} p_{j1}p_{j2} + \cdots + \Delta x_{0k}\sum_{j=1}^{n} p_{j1}p_{jk} \\
\sum_{j=1}^{n} p_{j2}\,\epsilon_j &= \Delta x_{01}\sum_{j=1}^{n} p_{j2}p_{j1} + \Delta x_{02}\sum_{j=1}^{n} p_{j2}^2 + \cdots + \Delta x_{0k}\sum_{j=1}^{n} p_{j2}p_{jk} \\
&\vdots \\
\sum_{j=1}^{n} p_{jk}\,\epsilon_j &= \Delta x_{01}\sum_{j=1}^{n} p_{jk}p_{j1} + \Delta x_{02}\sum_{j=1}^{n} p_{jk}p_{j2} + \cdots + \Delta x_{0k}\sum_{j=1}^{n} p_{jk}^2 .
\end{aligned}\right\} \tag{b-16}$$

Now Equation (b-16) can be solved for values Δx_{0i} by using a simple method of elimination. For example, if we select element q_{1j} of Equation (b-13) we can determine Δx_{0i} as follows:

$$\left.\begin{aligned}
q_{11}\sum_{j=1}^{n} p_{ji}\epsilon_j &= \Delta x_{01}\, q_{11}\sum_{j=1}^{n} p_{j1}^2 + \cdots + q_{11}\,\Delta x_{0k}\sum_{j=1}^{n} p_{j1}p_{jk}\\
q_{12}\sum_{j=1}^{n} p_{j2}\epsilon_j &= \Delta x_{01}\, q_{12}\sum_{j=1}^{n} p_{j2}p_{j1} + \cdots + q_{12}\,\Delta x_{0k}\sum_{j=1}^{n} p_{j2}p_{jk}\\
&\vdots\\
q_{1k}\sum_{j=1}^{n} p_{jk}\epsilon_j &= \Delta x_{01}\, q_{1k}\sum_{j=1}^{n} p_{jk}p_{j1} + \cdots + q_{1k}\,\Delta x_{0k}\sum_{j=1}^{n} p_{jk}^2 .
\end{aligned}\right\} \qquad \text{(b-17)}$$

Summing Equation (b-17) by columns, gives in the first element of the vector $\mathbf{\Delta X}_0$ in Equation (b-13). Furthermore, by definition of the matrix $\mathbf{Q}$ given in Equation (b-12),

$$\mathbf{Q}_{(k\times k)}\left(\mathbf{P}^T\mathbf{P}\right)_{(k\times k)} = \mathbf{I}_{(k\times k)} .$$

The elements in the first row of this matrix multiplication are:

$$\left.\begin{aligned}
q_{11}\sum_{j=1}^{n} p_{j1}^2 + q_{12}\sum_{j=1}^{n} p_{j2}p_{j1} + \cdots + q_{1k}\sum_{j=1}^{n} p_{jk}p_{j1} &= 1\\
q_{11}\sum_{j=1}^{n} p_{j1}p_{j2} + q_{12}\sum_{j=1}^{n} p_{j2}^2 + \cdots + q_{1k}\sum_{j=1}^{n} p_{j2}p_{jk} &= 0\\
&\vdots\\
q_{11}\sum_{j=1}^{n} p_{j1}p_{jk} + q_{12}\sum_{j=1}^{n} p_{j2}p_{jk} + \cdots + q_{1k}\sum_{j=1}^{n} p_{jk}^2 &= 0 ,
\end{aligned}\right\} \qquad \text{(b-18)}$$

from which we obtain

$$\Delta x_{01} = q_{11}\sum_{j=1}^{n} p_{j1}\epsilon_j + q_{12}\sum_{j=1}^{n} p_{j2}\epsilon_j + \cdots + q_{1k}\sum_{j=1}^{n} p_{jk}\epsilon_j . \qquad \text{(b-19)}$$

Similar relationships exist for Δx_{02}, $\cdots$, Δx_{0k}.

By expanding Equation (b-19), we find an equivalent relationship for Δx_{01}

$$\Delta x_{01} = \epsilon_1 \sum_{i=1}^{k} q_{1i}p_{1i} + \epsilon_2 \sum_{i=1}^{k} q_{1i}p_{2i} + \cdots + \epsilon_n \sum_{i=1}^{k} q_{1i}p_{ni} \,. \tag{b-20}$$

For purposes of simplicity we let

$$\left.\begin{aligned} \alpha_1 &= \sum_{i=1}^{k} q_{1i}p_{1i} \\ \alpha_2 &= \sum_{i=1}^{k} q_{1i}p_{2i} \\ &\vdots \\ \alpha_n &= \sum_{i=1}^{k} q_{1i}p_{ni} \end{aligned}\right\} \tag{b-21}$$

so that Equation (b-20) may be rewritten:

$$\Delta x_{01} = \alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \cdots + \alpha_n\epsilon_n \,. \tag{b-22}$$

To use the result obtained in Equation (b-18), we multiply Equation (b-21) by the elements of the first column of the matrix $\mathbf{P}_{(n\times k)}$:

$$\left.\begin{aligned} p_{11}\alpha_1 &= q_{11}p_{11}^2 + q_{12}p_{11}p_{12} + \cdots + q_{1k}p_{11}p_{1k} \\ p_{21}\alpha_2 &= q_{11}p_{21}^2 + q_{12}p_{21}p_{22} + \cdots + q_{1k}p_{21}p_{2k} \\ &\vdots \\ p_{n1}\alpha_n &= q_{11}p_{n1}^2 + q_{12}p_{n1}p_{n2} + \cdots + q_{1k}p_{n1}p_{nk} \end{aligned}\right\} \tag{b-23}$$

By summing the terms of Equations (b-23), we obtain

$$\sum_{j=1}^{n} p_{j1}\alpha_j = q_{11}\sum_{j=1}^{n} p_{j1}^2 + q_{12}\sum_{j=1}^{n} p_{j1}p_{j2} + \cdots + q_{1k}\sum_{j=1}^{n} p_{j1}p_{jk} \,. \tag{b-24}$$

Comparing the latter with Equation (b-18), we find that

$$\left.\begin{aligned}
\sum_{j=1}^{n} p_{j1}\alpha_j &= 1 \\
\sum_{j=1}^{n} p_{j2}\alpha_j &= 0 \\
&\vdots \\
\sum_{j=1}^{n} p_{jk}\alpha_j &= 0 \ .
\end{aligned}\right\} \qquad \text{(b-25)}$$

Returning to Equation (b-21), multiplying each equation with α_j and expanding the sums, we obtain:

$$\left.\begin{aligned}
\alpha_1^2 &= q_{11}p_{11}\alpha_1 + q_{12}p_{12}\alpha_1 + \cdots + q_{1k}p_{1k}\alpha_1 \\
\alpha_2^2 &= q_{11}p_{21}\alpha_2 + q_{12}p_{22}\alpha_2 + \cdots + q_{1k}p_{1k}\alpha_2 \\
&\vdots \\
\alpha_n^2 &= q_{11}p_{n1}\alpha_n + q_{12}p_{n2}\alpha_n + \cdots + q_{1k}p_{nk}\alpha_n \ .
\end{aligned}\right\} \qquad \text{(b-26)}$$

Summing Equation (b-26), we obtain

$$\sum_{j=1}^{n} \alpha_j^2 = q_{11}\sum_{j=1}^{n} p_{j1}\alpha_j + q_{12}\sum_{j=1}^{n} p_{j2}\alpha_j + \cdots + q_{1k}\sum_{j=1}^{n} p_{jk}\alpha_j \ ; \qquad \text{(b-27)}$$

and comparing Equations (b-27) and (b-25) we immediately see that

$$\sum_{j=1}^{n} \alpha_j^2 = q_{11} \ . \qquad \text{(b-28)}$$

Similar relationships exist for the other main diagonal elements of the matrix Q.

In Equation (b-22) we have, corresponding to each ϵ_j(with $j = 1, 2, \cdots, n$), an η_j such that the total rms error η_{H_1} associated with Δx_{01} is

$$\eta_{H_1} = \pm\left[\alpha_1^2 \eta_1^2 + \cdots + \alpha_n^2 \eta_n^2\right]^{1/2} . \tag{b-29}$$

But by definition of η in Equation (b-14) we see that $\eta_1^2 = \eta_2^2 = \cdots = \eta_n^2 = \eta^2$; thus

$$\eta_{H_1} = \pm\left[\eta^2 \sum_{j=1}^{n} \alpha_j^2\right]^{1/2} , \tag{b-30}$$

which, with the use of Equation (b-28), becomes

$$\eta_{H_1} = \pm\,\eta\, q_{11}^{1/2} . \tag{b-31}$$

And likewise, we have

$$\eta_{H_2} = \pm\,\eta\, q_{22}^{1/2}$$

$$\vdots$$

$$\eta_{H_k} = \pm\,\eta\, q_{kk}^{1/2} .$$

Returning to Equation (b-15), we now can say in general that

$$\Delta x_i = \Delta x_{0i} \pm \eta\, q_{ii}^{1/2} . \tag{b-32}$$

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