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TECHNICAL NOTE

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THE EXCITATION OF ELECTROACOUSTIC WAVES BY ANTENNAS IN THE IONOSPHERE

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SUMMARY

Measurements of the impedance of an electrically short 7.75-Mc antenna in the ionosphere indicate that power is absorbed by some mechanism additional to electromagnetic radiation. The importance of the radiation of energy as an electron pressure (electroacoustic) wave generated near the antenna is discussed, and it is shown that the calculated power radiated by this mechanism agrees well with the observations.

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INTRODUCTION

The effects of the ionosphere on the *reactance* of an electrically short antenna have been discussed in References 1 through 6. Consideration of the *resistive* component of the impedance has been less complete. It was pointed out in Reference 5 that the observed resistive component in one rocket flight varied in a way that was not, at first sight, to be expected from simple considerations of the radiation resistance variation in a dielectric medium.

In the following discussion, measurements obtained from NASA rocket flight NASA 4.07 (September 14, 1959) are compared with a theoretical model that invokes the excitation of electron pressure waves in the ionosphere by the RF field near the antenna.

ANTENNA LOADING MEASUREMENTS

The measurements made with the RF impedance probe on NASA 4.07, which are described in Reference 5, were essentially a Q-meter type of observation. The observed relative amplitude of the resonance peak is shown as a function of the medium's electron density in Figure 1. The abscissa is in terms of X where

$$X = \frac{Ne^2}{\epsilon_0 m\omega^2} = \frac{\omega_p^2}{\omega^2} ,$$

in which

N = the density (electrons/cubic meter),

e = the electronic charge (coulombs),

ϵ_0 = $1/36\pi \times 10^{-9}$ farad/meter,

m = the electronic mass (kilograms),

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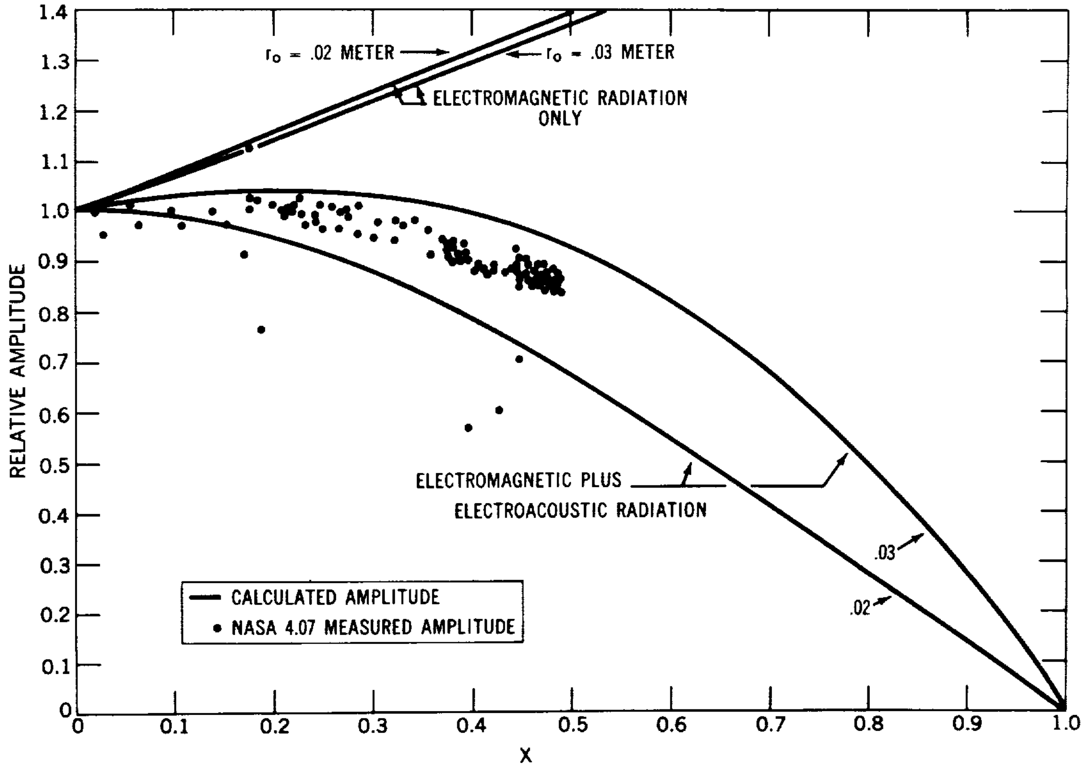

Figure 1—Comparison of the calculated variation of the amplitude of the resonance peak as a function of $X = \left(\omega_p^2/\omega^2\right)$ with the observed points. The curves are computed for two different values of ion sheath radius, 0.02 and 0.03 meter.

ω = operating frequency $\times$ 2π,

ω_p = electron plasma frequency $\times$ 2π.

A small linearly time-dependent correction was made to the results before plotting Figure 1, to compensate for the decay of battery voltage during the flight. These measurements were obtained with a peak RF signal at 7.75 Mc of approximately 2 volts on the antenna (two collinear 3-meter whips), so that the enhanced sheath effects that can occur with high RF voltages at low frequencies (described in Reference 6) are almost negligible. Since the measurements were made at altitudes above 80 km, losses arising from collisions of the electrons with heavy particles should be completely negligible. If, then, the external loss processes are negligible, the resistive component of the antenna impedance must arise from radiation of energy (plus the small losses in the internal coupling network). Two radiation processes appear to be possible: (1) the usual radiation of energy as an *electromagnetic wave;* and (2) the radiation of energy as an *electroacoustic wave,* which is described in this paper.

In the following discussion, the shape of the curve relating the amplitude of the resonance peak to the quantity X will be calculated, assuming first that *only* electromagnetic radiation occurs (it is found that this curve does not agree with that measured experimentally) and assuming second that electroacoustic radiation is *also* present. By making reasonable assumptions as to the closeness with which the electron gas approaches the antenna, the calculated curves are found to agree with the experimental results.

ELECTROMAGNETIC RADIATION

The input impedance of an electrically short antenna is a high capacitive reactance in series with a small resistive component arising from the radiation resistance and, to a less extent, from the ohmic losses in the antenna itself.

A change in the dielectric constant of the medium affects both the resistive and reactive parts of this impedance, since:

1. The radiation resistance of a short dipole antenna with a fixed dipole moment that is immersed in an incompressible medium of dielectric constant K is proportional to $\sqrt{K}$ (Reference 7, p. 437).

2. The reactance of a capacitor with a dielectric of effective dielectric constant K' is proportional to $1/K'$. A distinction is made between K, the dielectric constant of the medium, and K', the effective dielectric constant as seen by the antenna, since the latter is influenced by the presence of the positive ion sheath that normally forms around the antenna (Reference 6).

Let the peak RF voltage applied between the antenna terminals $= 2V$, and the free space capacitance between antenna halves $C_0 \approx 25$ pf in this experiment. Then the peak current in the antenna

$$I \approx j\,2\omega V K' C_0 , \tag{1}$$

where K' is the dielectric constant of the medium as seen by the antenna and $j = \sqrt{-1}$.

Let the radiation resistance of the antenna for electromagnetic radiation

$$R = \sqrt{K}\,R_0 , \tag{2}$$

where the free space radiation resistance

$$R_0 \approx 200\,(L/\lambda)^2$$

$$= 4.8 \text{ ohms (in this case),}$$

(L is the antenna length, and λ is the free space wavelength.)

Then the power radiated (as an electromagnetic wave) from the antenna is given (since I and V are peak values) by

$$P = \frac{I^2 R}{2} .$$

From Equations 1 and 2, with $\omega = 7.75 \text{ Mc} \times 2\pi$,

$$P = \left(1.43 \times 10^{-5}\right) K'^2 K^{1/2} V^2 \text{ watts.} \tag{3}$$

With small RF voltages applied to the antenna, K' is approximately proportional to K. Equation 3 shows that, with a constant voltage applied to the antenna, the power radiated decreases quite rapidly as the dielectric constant decreases (i.e., as the ambient electron density increases). The expected variation of the amplitude of the resonance peaks with electron density, considering only the electromagnetic radiation, is then as shown in Figure 1. In calculating these curves, two different values of the effective sheath radius r_0 have been assumed: 0.02 and 0.03 meter. This quantity enters in deriving K' from the K relating to the undisturbed ambient medium. The antenna radius is taken as 0.01 meter, a value obtained by averaging over the length of the antennas used. For any value of X (pertaining to the ambient medium), a value of X' (the apparent X as affecting the antenna capacitance) can be calculated for each value of r_0 (Reference 6). Then, since

$$K' = 1 - X',$$

the power radiated and hence the radiation resistance can be found from Equation 3. The unloaded parallel effective resistance of the associated tuned circuit (about 40 kΩ) has been included in computing the curves.

In the NASA 4.07 experiment, X' was measured directly. However, computing the electromagnetic radiation damping from these measured values gives curves quite close to the two shown, and does not change the experimental indication of some further mechanism operating to absorb energy from the antenna. It is postulated that this loss of energy arises, at least in part, from radiation in the form of an electroacoustic wave. There seems to be no other mechanism capable of explaining the experimental results; and, as will be seen in the following, the computed power absorbed by this means seems to be in good agreement with the observations.

ELECTROACOUSTIC WAVES

Near the surface of the cylindrical antenna the electric field is directed radially perpendicular to the antenna axis. If the antenna is electrically short, the voltage and hence the field do not vary appreciably along its length. A longer antenna may be considered by summing the effects along the length; but, for the present purpose, the antenna is considered to be electrically short. The radial field at distances from the antenna that are short compared with its length decreases as $1/r$ (where r is the distance from the antenna axis). The velocity of electrons oscillating in this field will also decrease as $1/r$, and consequently there will be velocity gradients and hence pressure gradients set up in the electron gas. In this case, the "pressure" in an electron gas arises from the mutual coulomb repulsion between the electrons. If the medium is incompressible as is assumed, for example, in deriving the usual simplified expressions for the dielectric constant,

$$K = 1 - X$$

$$= 1 - \frac{\omega_p^2}{\omega^2} \tag{4}$$

(for frequencies well above the electron gyrofrequency), then pressure variations of the above type cannot occur. (That is, in such a treatment it is assumed not only that the density of the medium is uniform but also that the field in the medium is uniform.)

It is well known, from theoretical considerations, that longitudinal electron waves (electroacoustic waves), where the electron motions are along the direction of propagation, should exist. The characteristics of such a wave may be obtained from the dispersion relation applicable to frequencies well above the electron gyrofrequency. This is given in Reference 8 as:

$$k^2 v^2 = \omega^2 - \omega_p^2 , \tag{5}$$

where

k = the propagation constant = $2\pi/\lambda_e$,

λ_e = the wavelength of the pressure wave in the electron gas,

v = $(\gamma\kappa T/m)^{1/2}$, corresponding to the sound wave phase velocity in a neutral gas,

κ = the Boltzmann constant,

T = the electron temperature.

Thus, the phase velocity of the wave,

$$u = \frac{\omega}{k}$$

$$= v/\sqrt{K}, \text{ from Equation 5, since } \omega p^2/\omega^2 = X \cdot$$

The quantity v depends on the electron temperature. As an example, for an electron temperature of T = 1500°K, we find that $v = 2.6 \times 10^5$ meters/sec.

In calculating $v = \sqrt{\gamma\kappa T/m}$, a value is required for the constant γ that is equivalent to the ratio of specific heats. We have put $\gamma = 3$ (Reference 9) as representing a gas with the minimum number of degrees of freedom since, first, the electron gas is certainly monatomic in nature and, second, the forces acting on the electrons in the wave motion are directed. This is because the field acting to accelerate a layer of electrons arises from the displacement of a whole layer of adjoining electrons; thus the field is like that from a surface charge and is directed along the wave direction.

We have taken $T = 1500°K$ as being representative of a large portion of the ionosphere. In fact, at the altitudes at which the NASA 4.07 measurements were performed, the electron temperature was probably mostly below 1500°K. This variation is of minor importance in most of what follows except in the calculation of the sheath radius. In that case a more realistic temperature variation has been included.

From Equation 5, we can now calculate the wavelength λ_e of the electron pressure wave:

$$\lambda_e = \frac{2\pi}{k}$$

$$= \frac{2\pi v}{\left(\omega^2 - \omega_p^2\right)^{1/2}}$$

$$= \frac{2\pi v}{\omega \sqrt{K}}, \text{ from Equation 4.}$$

With the further conditions of $\omega = 7.75 \text{ Mc} \times 2\pi$, and $\omega_p << \omega$,

$$\lambda_e \approx 0.034 \text{ meter.} \tag{6}$$

This wavelength is much less than the antenna lengths normally used (in this case, each half of the dipole-type antenna was 3 meters long), so that the radiated pressure wave is essentially cylindrical. From Equation 5 it is seen that k is imaginary if $\omega < \omega_p$ (i.e., the pressure waves are evanescent and do not propagate). For $\omega > \omega_p$, the frequency range of interest here, k is real so that the wave should propagate.

The question still remaining is whether the electroacoustic wave will suffer very much attenuation as it propagates. Losses arising from collisions with heavy particles are certainly small at the altitudes with which we are concerned; the other possible loss mechanism is by Landau damping (Reference 10). Theoretical considerations by Landau indicate that this mechanism, in which electrons are collected in the troughs between successive wave crests, may become important when the wavelength is comparable with the Debye length. The Debye length is given by

$$\lambda_D = \frac{1}{\omega_p} \sqrt{\frac{\kappa T}{m}}; \tag{7}$$

and

$$\lambda_e = \frac{2\pi}{\omega \sqrt{K}} \sqrt{\frac{\gamma \kappa T}{m}},$$

so that

$$\frac{\lambda_e}{\lambda_D} = 2\pi\sqrt{\gamma}\ \sqrt{\frac{X}{1-X}}\ .$$

It is shown later that the amount of energy radiated as an electroacoustic wave becomes important only for appreciable values of X. Over the range of X from 0.05 to 0.5, λ_e/λ_D varies from 2.5 to 10.9. This ratio of λ_e to λ_D is not very great, and there is a possibility that under this condition Landau damping could be appreciable. This need not be considered in the present problem however, since—once the wave leaves the vicinity of the antenna—it is of little importance whether the energy is carried away to very great distances or is dissipated in the medium *except* that, in the latter case, there could be some local heating of the medium. The degree of damping of the electroacoustic wave could be important, however, in other applications. For example, since the velocity depends on the electron temperature, it may be possible to make use of these waves in measuring such temperatures. The phenomenon is being investigated experimentally with this application in mind.

CALCULATION OF POWER RADIATED IN ELECTROACOUSTIC WAVE

Let C be the free space capacitance per unit length of the antenna. This quantity is needed to obtain the charge on the antenna and hence the field strength near to it. It varies with the position at which the element of antenna is located along its length, but a good average value for the antenna used is C = 16 pf/meter. Let $V\cos\omega t$ be the instantaneous antenna voltage with respect to the medium, so that $E\cos\omega t$ is the field strength at distance r from the antenna axis, where

$$E = \frac{VC}{2\pi\epsilon_0 r}\ \text{volts/meter.} \tag{8}$$

Electrons located at an average distance r from the antenna axis will oscillate with a peak velocity given by

$$w_0 = \frac{Ee}{m\omega}$$

$$= \frac{VCe}{2\pi m\omega r\epsilon_0}\ , \tag{9}$$

where it has been assumed that the amplitude of oscillation is small compared with the distance r.

Assume that a cylindrical pressure wave such as that described in the preceding section is generated and propagates with a propagation constant k given by Equation 5. The equation of the cylindrical pressure wave p will be of the form

$$p = \frac{A}{\sqrt{r}}\cos(kr - \omega t)\ , \tag{10}$$

where r is the distance from the antenna axis and A is a factor depending on the initial conditions. From the equation of motion,

$$\dot{r} = \frac{1}{j\rho\omega}\frac{\partial p}{\partial r}$$

$$= \frac{Ak}{\rho\omega\sqrt{r}} \cos(kr - \omega t) , \tag{11}$$

where ρ is the mass density of the medium. To find A, equate the particle velocity given by Equation 11 with that given by Equation 9 at the particular distance from the antenna axis at which the wave is generated. Denote this distance by r_0. Then

$$A = \frac{\rho eVC}{2\pi\epsilon_0 mk\sqrt{r_0}} . \tag{12}$$

The above assumes that the wave is generated by the motion of a cylindrical surface at $r = r_0$ and propagates outwards from there. We tentatively identify r_0 as the sheath radius, which was discussed previously.

In fact, of course, all the electrons in the medium will be excited by the field from the antenna, the excitation velocity decreasing as $1/r$. If all these motions are considered together, the factor A must be modified. A more complete treatment, which considers all these motions, is given in Appendix A, where it is shown that a more accurate value of w_0 (in Equation 9) is given by

$$W_0 = B(kr_0) \cdot w_0 , \tag{13}$$

where $B(kr_0)$ is plotted in Figure 2. The intensity of the pressure wave at any distance r is obtained from the average of the pressure-velocity product, Equations 10 and 11, so that

$$I = \overline{p\dot{r}}$$

$$= \frac{1}{2}\frac{A^2 k}{\rho\omega r} \quad \text{watts/meter}^2. \tag{14}$$

Using Equations 12 and 13 and noting that

$$\rho = Nm$$

and that

$$K = 1 - \frac{Ne^2}{\epsilon_0 m\omega^2} ,$$

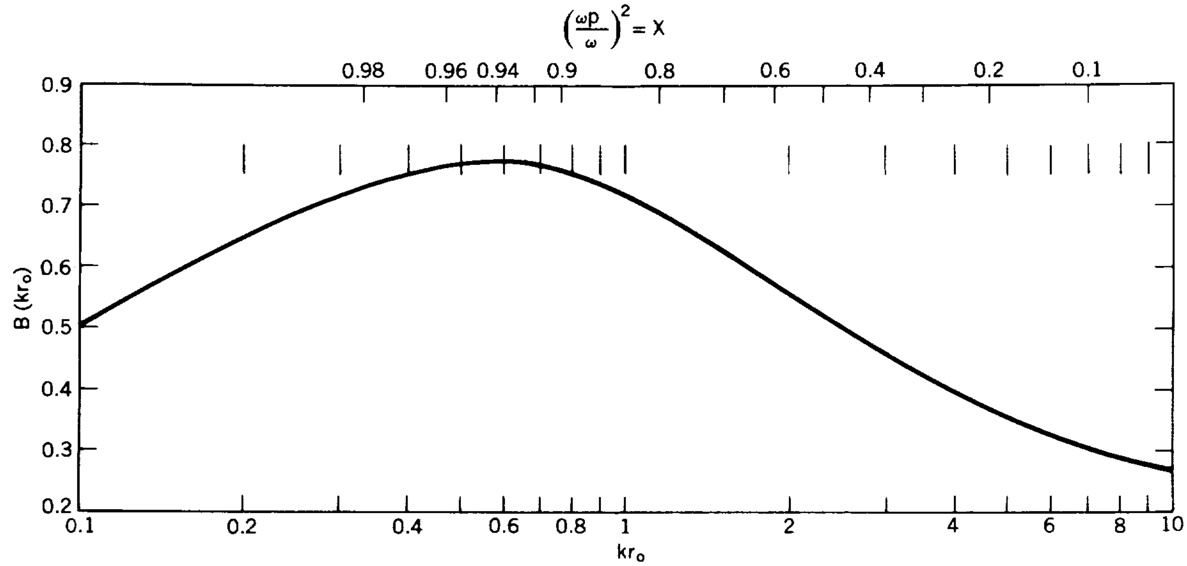

Figure 2—Function giving the effective velocity of the electrons at the sheath edge when the electric field penetrates into the medium.

the total power radiated per unit length of antenna is

$$Q_1 = 2\pi r I \text{ watts}$$

$$= \frac{(1 - K)\, v}{4\pi\epsilon_0\, r_0 \sqrt{K}}\, B^2(kr_0)\; V^2 C^2 \text{ watts.} \tag{15}$$

BOUNDARY DISTANCE

The factor r_0, which appears in Equation 15, is the radial distance to the innermost layer of electrons (i.e., the distance to the effective boundary of the ion sheath). From the measurements of the reactive part of the antenna impedance as given in Reference 5, this radius was found to be approximately 0.02 to 0.05 meter. The magnitude of r_0 is very important in determining the power radiated in the electroacoustic mode. It is difficult to assign an accurate value to r_0, but the values deduced from the different approaches to the problem are shown in Figure 3. These approaches are the following:

1. The points plotted in Figure 3 were obtained from the antenna capacitance measurements described in Reference 5. These values are in the range 0.02 to 0.05 meter. In deriving these values, no allowance was made for that portion of the capacitance near the base of the antenna which, because of the sheath around the vehicle, is unaffected by the ionosphere. In this case, the value of the unaffected capacitance is of the order of 5 pf. Including this effect reduces the values of r_0 by a small amount.

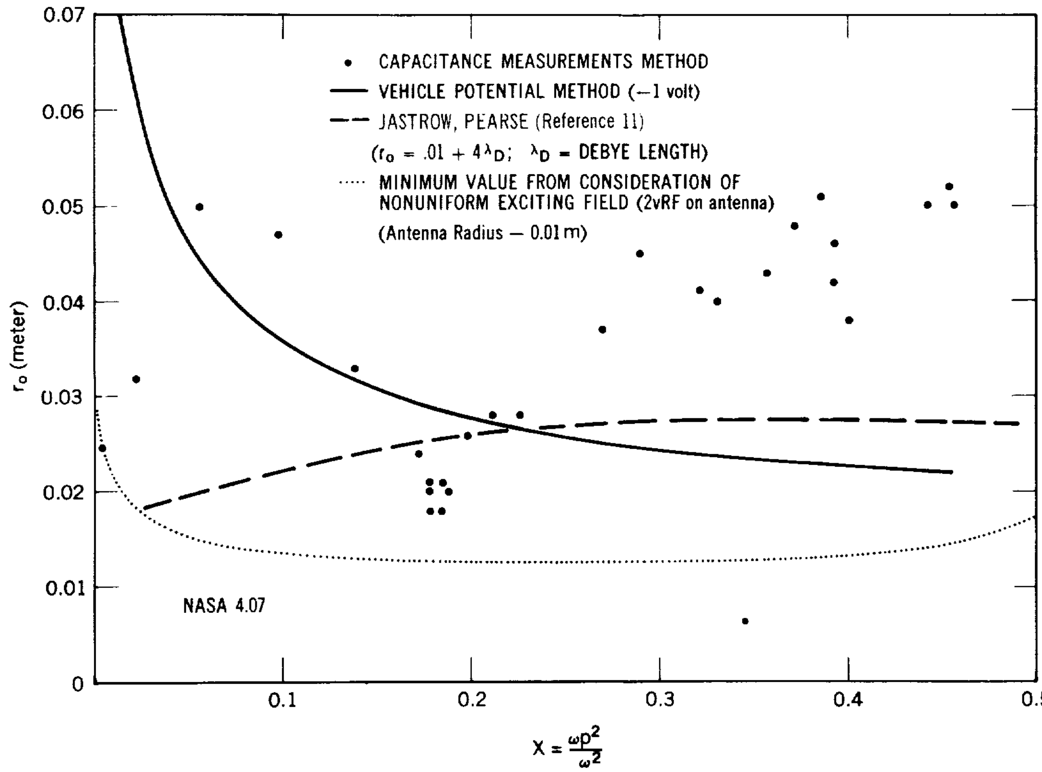

Figure 3—Value of sheath radius obtained by different methods using: capacitance measurements (dots); vehicle potential (full line); Jastrow and Pearse's formula (dashed line); minimum value obtained from consideration of the effects of the nonuniformity of the exciting field (dotted line).

2. An estimate of the vehicle potential was obtained from a Langmuir probe experiment. The indicated potential of the vehicle and antennas with respect to the medium was − 1 ±.5 volt. With − 1 volt as an average value, the sheath radius has the value shown by the full line curve in Figure 3.

3. If there are no complicating factors such as photoemission, etc., the theoretical analysis of Jastrow and Pearse (Reference 11) indicates that the sheath thickness is about $4\lambda_D$. The sheath radius (with a 0.01 meter radius antenna) is then as shown by the dashed line in Figure 3. In this curve, allowance has been made for the electron temperature variation by using the data presented in Reference 12. From these curves a linear approximation to the temperature below 200 km is employed; namely,

$$T = 200 + 12.5\,(h - 100)\ \text{deg Kelvin},$$

where h is the height in km.

4. The *minimum* value of the sheath radius with 2 volts RF (peak voltage) on the antenna is found from Reference 6 to be, over the X range of interest here, about 0.013 meter. This is shown as the dotted line in Figure 3.

Although it is expected that the value of r_0 that should be used in Equation 15 foi the electroacoustic power is close to the value of r_0 obtained from considerations of the sheath effects on the reactance of the antenna, there is no real reason why the two should be identical. The actual distribution of electrons near the antenna will be a relatively smooth function and, since the strength of the exciting field increases rapidly close to the antenna, it is likely that the nearer electrons will be more effective in producing the pressure wave than those that are more distant. This would suggest that the effective value of r_0 appropriate to the electroacoustic wave analysis would be somewhat less than that effective in the reactance calculations where the whole region between the antennas has to be considered. The experimental points in Figure 3 tend to support this conclusion.

EXPERIMENTAL RESULTS

Combining Equations 3 and 15, and using $C = 16$ pf/m and $T = 1500$°K, the total power radiated with a peak $V = 1$ volt is

$$P + Q = \frac{1.43}{10^5} K'^2 \cdot \sqrt{K} + \frac{3.6}{10^6} \frac{1 - K}{\sqrt{K}} \frac{B^2(kr_0)}{r_0} \text{ watts.} \tag{16}$$

The calculated loading on the antenna, computed with different values of the sheath radius, is shown in Figure 4 in terms of the relative amplitude of the resonance peaks. The experimental points are shown for both the ascending and descending parts of the trajectory. The square points are calculated

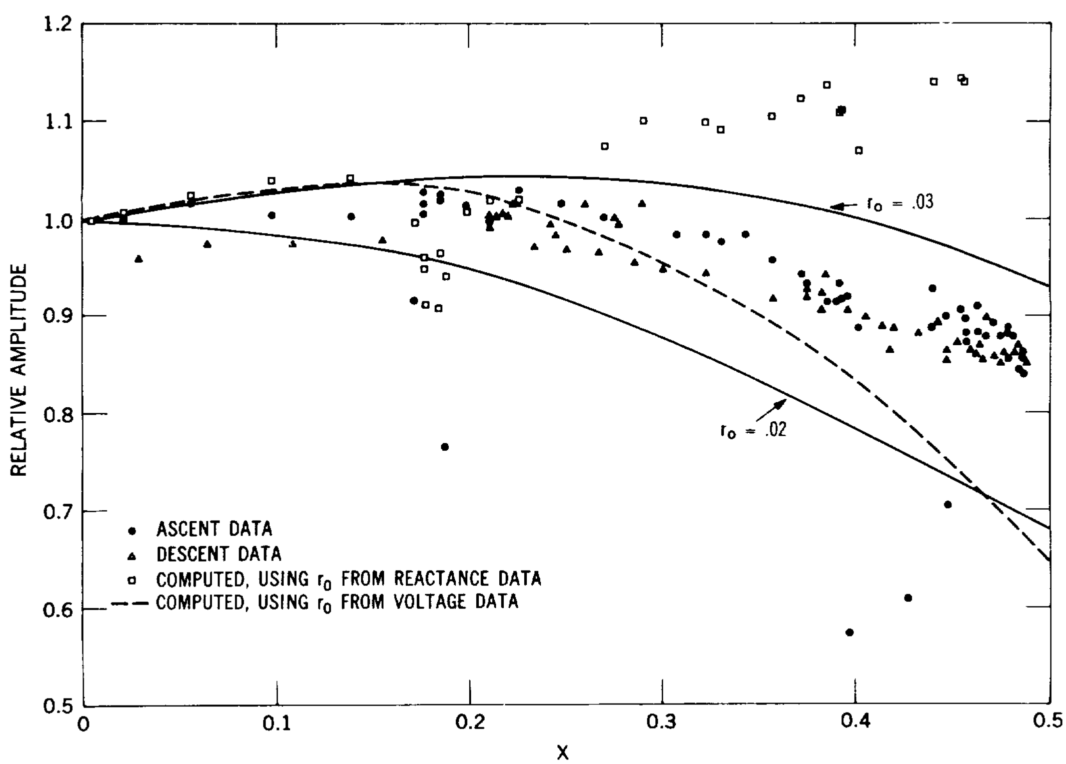

Figure 4—Detailed comparison of the calculated and experimental results indicating the effect of the ion sheath radius.

from the values of r_0 obtained from the capacitance data; the dashed line is calculated with the r_0 obtained from the vehicle voltage measurement (– 1 volt); while the two full line curves are calculated for constant values of r_0 : 0.02 and 0.03 meter. Figures 1 and 4 show that very good agreement of the experimental measurements is obtained if the sheath radius is about 0.025 meter. This value agrees with that obtained from two of the methods given above and is somewhat less than the value obtained from the capacitance data, as would be expected.

CONCLUSION

It has been shown that the observed resistive loading on an electrically short antenna suggests an absorption of energy in addition to that radiated by the electromagnetic wave. All the evidence suggests that this energy is radiated as an electroacoustic wave in the electron gas of the ionosphere, and the computed power lost by this mechanism agrees well with that observed.

The question of how well this wave propagates is left unanswered at this stage; but it has been pointed out that, if the damping is not excessive, the measurement of the phase velocity of the electroacoustic wave can provide information on the electron temperature of the medium.

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Appendix A

Derivation of Peak Electron Velocity W_0 in a Radial Field

The acceleration of an electron at any point is given by

$$\ddot{r} = \frac{Ee}{m} ,$$

where

$E = (E_0 r_0)/r$,

E_0 = the field at r_0.

If the phase velocity of the wave is u, all the accelerations may be referred back with the proper phase and amplitude as though they occurred at the inner boundary of the medium at r_0. Thus the velocity W_0 of a vibrating electron in the wave that started at r_0 at the time t_0 is, when referred back to r_0 (using the relation given in Equation 12 that $|\dot{r}| \propto r^{-1/2}$), given by

$$W_0 = \int_{r_0}^{R} \sqrt{\frac{r}{r_0}} \frac{eE_0}{mur} \cos\left(\phi_0 - \omega \frac{r - r_0}{u}\right) dr$$

$$= \frac{eE_0}{mu\sqrt{r_0}} \int_{r_0}^{R} \frac{\cos\left(\phi_0 + \frac{\omega r_0}{u} - \frac{\omega r}{u}\right) dr}{\sqrt{r}} . \qquad \text{(A1)}$$

where $\phi_0 = \omega t_0$.

The integration may be performed in terms of Fresnel integrals to give

$$W_0 = \frac{eE_0}{m_0} \sqrt{\frac{2\pi}{ur_0\omega}} \sqrt{C^2 + S^2} \cos\left(\phi_0 + kr_0 - \arctan\frac{C}{S}\right) , \qquad \text{(A2)}$$

where

$k = \omega/u$ = the propagation constant,

$$C = C(kR) - C(kr_0) , \qquad \text{(A3)}$$

$$S = S(kR) - S(kr_0) ; \qquad \text{(A4)}$$

and C(x) and S(x) are tabulated in Reference 13 by Jahnke and Emde. The cosine term in Equation A2 is time-dependent, since

$$\phi_0 = \omega t_0 ;$$

so that

$$|W_0| = \frac{eE_0}{m} \sqrt{\frac{2\pi}{ur_0 \omega}} \sqrt{C^2 + S^2} . \tag{A5}$$

As $R \to \infty$ (i.e., the integration is performed from the inner boundary out to infinity),

$$C \to 0.5 - C(kr_0) ,$$

$$S \to 0.5 - S(kr_0) .$$

The ratio

$$\frac{W_0}{w_0} = \sqrt{2\pi} \sqrt{kr_0} \sqrt{C^2 + S^2} \tag{A6}$$

$$= B(kr_0) ,$$

where w_0 is given in Equation 9, page 7. A plot of Equation A6 as a function of kr_0 is given in Figure 2. If r_0 is the ion sheath radius, which is commonly of the order of a few Debye lengths, we have (assuming $r_0 = 4\lambda_D$)

$$kr_0 = 8\pi \frac{\lambda_D}{\lambda_e}$$

$$= \frac{4\omega \sqrt{K}}{\omega_p \sqrt{\gamma}} . \tag{A7}$$

Since K is a function of ω_p/ω and $X = 1 - \left(\omega_p^2/\omega^2\right)$, B may also be plotted as a function of X as in the upper scale of Figure 2.

NASA-Langley, 1963 G-277