A STUDY OF THE DISTRIBUTION OF MOLECULES UNDER FREE MOLECULAR FLOW CONDITIONS AFTER COLLISIONS WITH SIMPLE GEOMETRIES

By

David W. Tarbell and James O. Ballance

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ABSTRACT

The study makes a detailed examination of molecular distribution under free molecular flow conditions within surfaces of simple geometries. By considering infinite length in one dimension, the problem could be reduced to a two-dimensional analysis. Exact, closed form solutions were obtained only for circles; however, numerical solutions were obtained for ellipses and parabolas. Monte Carlo computer technique results were obtained and compared to the theoretical solutions. Derivations of (a) the distribution of molecules over the surface as a function of collision number, (b) the exit distribution across the opening of each surface configuration, and (c) the distribution across the center line of each configuration are presented.
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Any errors are, of course, the responsibility of the authors and they would be pleased to receive any comments concerning the paper.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>SECTION I. INTRODUCTION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SECTION II. DISTRIBUTION OF MOLECULES OVER A SEMICIRCULAR SURFACE AS A FUNCTION OF COLLISION NUMBER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. General Expression</td>
<td>2</td>
</tr>
<tr>
<td>B. Case I: Incident Distribution Uniform over the Surface</td>
<td>13</td>
</tr>
<tr>
<td>1. Distribution at Second Collision</td>
<td>13</td>
</tr>
<tr>
<td>2. Third Collision</td>
<td>14</td>
</tr>
<tr>
<td>3. Fourth Collision</td>
<td>16</td>
</tr>
<tr>
<td>4. Fifth Collision</td>
<td>20</td>
</tr>
<tr>
<td>C. Case II: Incident Distribution Uniform across the Y-Axis</td>
<td>25</td>
</tr>
<tr>
<td>1. Distribution at Second Collision</td>
<td>25</td>
</tr>
<tr>
<td>2. Third Collision</td>
<td>28</td>
</tr>
<tr>
<td>3. Fourth Collision</td>
<td>30</td>
</tr>
<tr>
<td>4. Fifth Collision</td>
<td>34</td>
</tr>
<tr>
<td>D. Summary of Expressions for Distribution over a Semicircular Surface at the n&lt;sup&gt;th&lt;/sup&gt; Collision</td>
<td>39</td>
</tr>
<tr>
<td>1. General Expressions</td>
<td>39</td>
</tr>
<tr>
<td>2. Incident Distribution Uniform over the Surface</td>
<td>39</td>
</tr>
<tr>
<td>3. Incident Distribution Uniform across the Y-Axis</td>
<td>41</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SECTION III. DISTRIBUTION OF MOLECULES OVER ELLIPTICAL AND PARABOLIC SURFACES AT FIRST COLLISION---INCIDENT FLUX UNIFORM ACROSS Y-AXIS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Ellipse</td>
<td>43</td>
</tr>
<tr>
<td>B. Parabola</td>
<td>46</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SECTION IV. DISTRIBUTION OF MOLECULES ACROSS THE OPENING (Y-AXIS) UPON EXIT FROM AN ARRAY AFTER ONE COLLISION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Circular Array</td>
<td>51</td>
</tr>
<tr>
<td>1. General Expression</td>
<td>51</td>
</tr>
<tr>
<td>2. Incident Distribution Uniform over the Surface</td>
<td>53</td>
</tr>
<tr>
<td>3. Incident Distribution Uniform across the Y-Axis</td>
<td>54</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS (Cont'd)

## SECTION IV. (Cont'd)

- B. Elliptical Array ........................................ 54  
- C. Parabolic Array ........................................ 64  

## SECTION V.

- DISTRIBUTION OF MOLECULES ALONG THE CENTERLINE OF ARRAYS AFTER ONE COLLISION WITH THE SURFACE .......... 66  
  - A. General Expression ................................. 66  
  - B. Ellipse ............................................. 73  
  - C. Parabola ............................................ 75  

## SECTION VI.

- NUMERICAL RESULTS AND TABLES ............................. 76  
  - A. Fraction of Molecules Making nth Collision with a Semicircular Surface .......................... 76  
    1. Incident Distribution Uniform over the Surface ........................................ 76  
    2. Incident Distribution Uniform across the Y-Axis ..................................... 77  
  - B. Distributions across Openings and Centerlines of Arrays after One Collision (Incident Distribution Uniform across the Y-Axis only) .......................... 77  

## SECTION VII.

- CONCLUSIONS AND DISCUSSION ............................... 79  
  - A. Surface Distributions ................................ 79  
  - B. Exit and Centerline Distributions .................. 80  
  - C. Monte Carlo Computer Techniques .................. 80  

## APPENDIX A:

- Useful Integrals ........................................... 81
## DEFINITION OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$x_0 - x_1$</td>
</tr>
<tr>
<td>$A$</td>
<td>variable parameter in equation for ellipse</td>
</tr>
<tr>
<td>$b$</td>
<td>distance from $P_\alpha$ to point $(0, z)$ on $Y$-axis</td>
</tr>
<tr>
<td>$c$</td>
<td>$x_0 - x_2$</td>
</tr>
<tr>
<td>$d$</td>
<td>distance from $P_\alpha$ to point $(x_1, 0)$</td>
</tr>
<tr>
<td>$F^{(n)}(\alpha, \beta)$</td>
<td>$1/2 \int N^{(n-1)}_\alpha \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha$</td>
</tr>
<tr>
<td>$n$</td>
<td>collision number</td>
</tr>
<tr>
<td>$N_\alpha$</td>
<td>number of molecules per radian reflected from the surface at $P_\alpha$</td>
</tr>
<tr>
<td>$N^{(n)}_\alpha$</td>
<td>number of molecules per radian reflected from $P_\alpha$ on nth collision</td>
</tr>
<tr>
<td>$N_{\alpha, \beta}$</td>
<td>number of molecules per radian from $P_\alpha$ that hit section of surface between $\alpha = \beta$ and $\alpha = \pi/2$</td>
</tr>
<tr>
<td>$N_{\alpha, \varphi}$</td>
<td>number of molecules per radian reflected from $P_\alpha$ at an angle $\varphi$ with the normal</td>
</tr>
<tr>
<td>$N_\beta$</td>
<td>total number of molecules that hit section of surface between $\alpha = \beta$ and $\alpha = \pi/2$</td>
</tr>
<tr>
<td>$N^{(n)}_\beta$</td>
<td>total number of molecules that hit section of surface between $\alpha = \beta$ and $\alpha = \pi/2$ on the nth collision</td>
</tr>
<tr>
<td>$N_x$</td>
<td>total number of molecules that pass through a section of the center line between $(x_1, 0)$ and $(x_2, 0)$ from all points on the surface</td>
</tr>
</tbody>
</table>

**NOTE:** Numbers in brackets refer to page on which symbol is first used.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>N&lt;sub&gt;z&lt;/sub&gt;</td>
<td>number of molecules that exit an array between (0, 0) and (0, z)</td>
</tr>
<tr>
<td>N&lt;sub&gt;z,α&lt;/sub&gt;</td>
<td>number of molecules per radian from P&lt;sub&gt;α&lt;/sub&gt; that exit the array between (0, 0) and (0, z)</td>
</tr>
<tr>
<td>N&lt;sub&gt;0&lt;/sub&gt;</td>
<td>total number of incident molecules</td>
</tr>
<tr>
<td>p</td>
<td>variable parameter in equation for parabola</td>
</tr>
<tr>
<td>P&lt;sub&gt;α&lt;/sub&gt;</td>
<td>point with coordinates (x, y) at angle α</td>
</tr>
<tr>
<td>r</td>
<td>distance from P&lt;sub&gt;α&lt;/sub&gt; to point (x&lt;sub&gt;2&lt;/sub&gt;, 0)</td>
</tr>
<tr>
<td>s</td>
<td>distance from point (0, 0) to P&lt;sub&gt;α&lt;/sub&gt;</td>
</tr>
<tr>
<td>x&lt;sub&gt;0&lt;/sub&gt;</td>
<td>intersection of normal with x-axis</td>
</tr>
<tr>
<td>x&lt;sub&gt;1&lt;/sub&gt;, x&lt;sub&gt;2&lt;/sub&gt;</td>
<td>points on the center line which define the section of interest for distribution of molecules across the center line</td>
</tr>
<tr>
<td>x, y</td>
<td>coordinates of arbitrary point P&lt;sub&gt;α&lt;/sub&gt; on surface</td>
</tr>
<tr>
<td>y&lt;sub&gt;0&lt;/sub&gt;</td>
<td>intersection of normal with y-axis</td>
</tr>
<tr>
<td>z</td>
<td>value of y which defines the area of interest for exit distribution across the y-axis</td>
</tr>
<tr>
<td>α</td>
<td>angle from negative y-axis to point (x, y) on reflecting surface</td>
</tr>
<tr>
<td>α&lt;sub&gt;n&lt;/sub&gt;</td>
<td>ratio of number of molecules that exit an array after n collisions to the number that make n collisions</td>
</tr>
<tr>
<td>β</td>
<td>angle from negative y-axis to section of interest on surface</td>
</tr>
<tr>
<td>δ, ε</td>
<td>angles from normal to P&lt;sub&gt;α&lt;/sub&gt; to limits of molecule beam to section of interest</td>
</tr>
<tr>
<td>φ</td>
<td>general angle from the normal to the surface at P&lt;sub&gt;α&lt;/sub&gt; to the path of a reflected molecule</td>
</tr>
<tr>
<td>θ</td>
<td>angle from negative x-axis to tangent at P&lt;sub&gt;α&lt;/sub&gt;</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-14</td>
<td>Geometries</td>
<td>-</td>
</tr>
<tr>
<td>15-18</td>
<td>Distribution of Particles Exiting after Single Collision (Incident Distribution Uniform across Opening) - Ellipses</td>
<td>93-96</td>
</tr>
<tr>
<td>19-22</td>
<td>Distribution of Particles Exiting after Single Collision (Incident Distribution Uniform across Opening) - Parabolas</td>
<td>97-100</td>
</tr>
<tr>
<td>23-26</td>
<td>Distribution of Particles Intercepting Centerline after Single Collision (Incident Distribution Uniform across Opening) - Ellipses</td>
<td>101-104</td>
</tr>
<tr>
<td>27-30</td>
<td>Distribution of Particles Intercepting Centerline after Single Collision (Incident Distribution Uniform across Opening) - Parabolas</td>
<td>105-108</td>
</tr>
<tr>
<td>31</td>
<td>Distribution of Particles over a Semi-Circular Surface at First Five Collisions</td>
<td>109</td>
</tr>
<tr>
<td>32</td>
<td>Sketch of Possible Cryoarray for Low Density Wind Tunnel</td>
<td>110</td>
</tr>
</tbody>
</table>

LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>Fraction of Molecules that Exit Semi-Circular Array after the nth Collision</td>
<td>84</td>
</tr>
<tr>
<td>II.</td>
<td>Ellipse Exit Distribution</td>
<td>85</td>
</tr>
<tr>
<td>III.</td>
<td>Parabola Exit Distribution</td>
<td>87</td>
</tr>
<tr>
<td>IV.</td>
<td>Ellipse Centerline Distribution</td>
<td>89</td>
</tr>
<tr>
<td>V.</td>
<td>Parabola Centerline Distribution</td>
<td>91</td>
</tr>
</tbody>
</table>

vii
A STUDY OF THE DISTRIBUTION OF MOLECULES UNDER FREE MOLECULAR FLOW CONDITIONS AFTER COLLISIONS WITH SIMPLE GEOMETRIES

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SUMMARY

The study makes a detailed examination of molecular distribution under free molecular flow conditions within surfaces of simple geometries. By considering infinite length in one dimension, the problem could be reduced to a two-dimensional analysis. Exact, closed form solutions were obtained only for circles; however, numerical solutions were obtained for ellipses and parabolas. Monte Carlo computer technique results were obtained and compared to the theoretical solutions. Derivations of (a) the distribution of molecules over the surface as a function of collision number, (b) the exit distribution across the opening of each surface configuration, and (c) the distribution across the center line of each configuration are presented.

SECTION I. INTRODUCTION

Studies of free molecular flow problems have usually been concerned with the gross effects resulting from the flow such as the conductance of a tube or the forces on a body moving at a velocity large compared to the thermal velocity of the gas molecules. The analytical solutions are usually obtained in exact forms for limited geometries. The use of high speed digital computer methods [1, 2, 3, 4] has allowed the determination of many interesting parameters such as distribution of molecules, average number of collisions within surfaces, etc., parameters which would have been difficult if not impossible to determine from exact analytical studies. It was felt by the authors that a detailed examination of molecules under free molecular flow conditions would allow much greater insight into actual physical processes and would assist in practical problems such as cryogenic pump array design, molecule-surface interaction studies, high altitude atmospheric measuring programs, etc.
Typical of most studies of free molecular flow, two assumptions have been made: (1) The molecules interact only with the surfaces and (2) molecules colliding with the wall are re-emitted diffusely. (The terms "molecules" and "particles" are used interchangeably throughout.) In all cases, the surfaces examined are considered infinitely long in the "z" direction such that the analysis can be reduced to two dimensions. Theoretical studies presented in this paper consider only elementary surfaces which, in two dimensions, are circles, ellipses, and parabolas; however, completely digital computer studies have considered hyperbolas and triangles [3]. Only for the case of the circle could solutions be made exactly in closed form. The other cases resulted in integrals, some of which were solved by computer methods. Monte Carlo methods used for this study are described in Reference 3.

This paper presents the derivation of (a) the distribution of molecules over the surface as a function of collision number, (b) the exit distribution across the opening of each surface configuration, and (c) the distribution across the center line of each configuration. The results of numerically integrating these expressions are given and compared with the results obtained by the Monte Carlo method.

SECTION II. DISTRIBUTION OF MOLECULES OVER A SEMICIRCULAR SURFACE AS A FUNCTION OF COLLISION NUMBER

A. General Expressions

\[ P_\alpha (x, y) \]

\[ \Gamma \]

**FIGURE 1**
Consider a flux of molecules entering the semicircular geometry shown in Figure 1. After being reflected at the surface, some of the molecules will exit the array, while others will collide again with the surface a number of times before exiting.

The problem to be considered here is to find expressions for the distribution of these molecules over the surface, as a function of collision number. When the initial distribution is known, the problem is simply a matter of trigonometry and calculus, as follows:

Let the initial distribution be given by

\[ N_{\alpha}^{(1)} = \text{number of molecules per radian that collide with the surface at point } P_{\alpha} \text{ on the first collision. (See Figure 1.)} \]

Then the total number of incident molecules, \( N_0 \), is given by

\[ N_0 = \int_0^\pi N_{\alpha}^{(1)} \, d\alpha. \]

According to Lambert's Cosine Law of Diffuse Reflection, the number of molecules per radian that are reflected from \( P_{\alpha} \) at an angle \( \phi \) with the normal is proportional to the cosine of this angle, i.e.,

\[ N_{\alpha, \phi} = k \cos \phi. \]

Then clearly,

\[ N_{\alpha} = \int_{-\pi/2}^{\pi/2} k \cos \phi \, d\phi, \]

\[ = \frac{\pi k}{2}, \]

where \( N_{\alpha} \) is the surface distribution function for any arbitrary collision. This yields immediately

\[ k = \frac{1}{2} N_{\alpha}, \]

so that

\[ N_{\alpha, \phi} = \frac{1}{2} N_{\alpha} \cos \phi. \]
Now the number of molecules that collide again with a section of the surface between $\beta$ and $\pi/2$ (shaded band in Figure 1) will be the sum of three contributions: (1) for $\alpha > \pi/2$, (2) for $\pi/2 \geq \alpha \geq \beta$, and (3) for $\beta > \alpha$.

**Contribution (1): $\alpha > \pi/2$**

The number of molecules per radian reflected from $P_\alpha$ that hit between $\beta$ and $\pi/2$ is

$$N_{\alpha, \beta}^1 = \frac{1}{2} N_\alpha \cos \varphi \, d\varphi = \frac{1}{2} N_\alpha \left[ \sin \varphi \right]_{\epsilon}^{\epsilon+\delta}$$

$$N_{\alpha, \beta}^1 = \frac{1}{2} N_\alpha \left[ \sin (\epsilon + \delta) - \sin \epsilon \right]$$

From Figure 1:

$$2\epsilon = \pi - (\alpha - \beta), \quad \text{or} \quad \epsilon = \frac{\pi}{2} - \left( \frac{\alpha - \beta}{2} \right),$$

also

$$2(\epsilon + \delta) = \pi - \left( \alpha - \frac{\pi}{2} \right), \quad \text{or} \quad \epsilon + \delta = \frac{\pi}{2} - \left( \frac{\alpha - \pi}{4} \right).$$

Hence,

$$\sin (\epsilon + \delta) = \sin \left[ \frac{\pi}{2} - \left( \frac{\alpha - \pi}{4} \right) \right] = \cos \left( \frac{\alpha - \pi}{4} \right),$$

and

$$\sin \epsilon = \sin \left[ \frac{\pi}{2} - \left( \frac{\alpha - \beta}{2} \right) \right] = \cos \left( \frac{\alpha - \beta}{2} \right).$$
so that

\[ N_{\alpha, \beta}^1 = \frac{1}{2} N_\alpha \left[ \cos \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) - \cos \left( \frac{\alpha - \beta}{2} \right) \right]. \]

Then the total number of molecules that hit between \( \beta \) and \( \pi/2 \), from all \( P_\alpha \), with \( \alpha \geq \pi/2 \), is

\[ N_\beta^1 = \int_{\pi/2}^{\pi} N_{\alpha, \beta}^1 \, d\alpha = \int_{\pi/2}^{\pi} \frac{1}{2} N_\alpha \left[ \cos \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) - \cos \left( \frac{\alpha - \beta}{2} \right) \right] \, d\alpha. \quad (1) \]

**Contribution (2):** \( \frac{\pi}{2} \geq \alpha \geq \beta \)

In this case there will be two contributions: (2-1), angles greater than \( \alpha \), and (2-2), angles less than \( \alpha \) (Fig. 2). For the first contribution (2-1):

![Figure 2](image-url)
\[ N_{\alpha, \beta} = \frac{1}{2} N_\alpha (1 - \sin \delta), \]

but, from Figure 2:

\[ 2\delta = \pi - \left( \frac{\pi}{2} - \alpha \right) = \frac{\pi}{2} + \alpha, \]

or

\[ \delta = \frac{\pi}{4} + \frac{\alpha}{2} \]

so that

\[ N_{\alpha, \beta}^{2-1} = \frac{1}{2} N_\alpha \left[ 1 - \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \right]. \]

For the second contribution:

\[ N_{\alpha, \beta}^{2-2} = \frac{1}{2} N_\alpha \cos \varphi d\varphi = \frac{1}{2} N_\alpha \left( \sin \frac{\pi}{2} - \sin \epsilon \right), \]

\[ N_{\alpha, \beta}^{2-2} = \frac{1}{2} N_\alpha (1 - \sin \epsilon), \]

but

\[ 2\epsilon = \pi - (\alpha - \beta) \]
or

\[ \epsilon = \frac{\pi}{2} - \left( \frac{\alpha - \beta}{2} \right), \]

so that

\[ N_{\alpha, \beta}^{2-2} = \frac{1}{2} N_{\alpha} \left[ 1 - \sin \left( \frac{\pi}{2} - \frac{\alpha - \beta}{2} \right) \right] = \frac{1}{2} N_{\alpha} \left[ 1 - \cos \left( \frac{\alpha - \beta}{2} \right) \right]. \]

Therefore,

\[ N_{\alpha, \beta}^{2} = N_{\alpha, \beta}^{2-1} + N_{\alpha, \beta}^{2-2} = \frac{1}{2} N_{\alpha} \left[ 2 - \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) - \cos \left( \frac{\alpha - \beta}{2} \right) \right]. \]

Then the total number of molecules that hit between \( \beta \) and \( \pi/2 \), from all \( \beta_{\alpha} \), such that \( \pi/2 \geq \alpha \geq \beta \) is

\[ N_{\beta}^{2} = \int_{\beta}^{\pi/2} \frac{1}{2} N_{\alpha} \left[ 2 - \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) - \cos \left( \frac{\alpha - \beta}{2} \right) \right] d\alpha, \ldots \tag{2} \]

**Contribution (3):** \( \alpha \leq \beta \)

From Figure 3:

\[ N_{\alpha, \beta}^{3} = \int_{\epsilon}^{\epsilon+\delta} \frac{1}{2} N_{\alpha} \cos \varphi \, d\varphi, \]

\[ N_{\alpha, \beta}^{3} = \frac{1}{2} N_{\alpha} \left( \sin (\epsilon+\delta) - \sin \epsilon \right). \]
Now

\[ 2(\epsilon + \delta) = \pi - (\beta - \alpha) \]

\[ \epsilon + \delta = \frac{\pi}{2} - \left( \frac{\beta - \alpha}{2} \right) \]

\[ \sin (\epsilon + \delta) = \cos \left( \frac{\beta - \alpha}{2} \right). \]

Also

\[ 2\epsilon = \pi - \left( \frac{\pi}{2} - \alpha \right) \]

\[ \epsilon = \frac{\pi}{2} - \left( \frac{\pi}{4} - \frac{\alpha}{2} \right) = \frac{\pi}{4} + \frac{\alpha}{2} \]

\[ \sin \epsilon = \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \]
so that

\[ N_{\alpha, \beta}^{(3)} = \frac{1}{2} N_\alpha \left[ \cos \left( \frac{\beta - \alpha}{2} \right) - \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \right] \]

and the total number of molecules that hit between \( \beta \) and \( \pi/2 \), from all \( P_\alpha \), such that \( \beta \geq \alpha \), is

\[ N_\beta^3 = \int_0^\beta \frac{1}{2} N_\alpha \left[ \cos \left( \frac{\alpha - \beta}{2} \right) - \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \right] d\alpha. \]  

(3)

Combining equations (1), (2) and (3), the total contribution to the number of molecules hitting between \( \beta \) and \( \pi/2 \) from all \( P_\alpha \), is:

\[ N_\beta = \int_{\pi/2}^\pi \frac{1}{2} N_\alpha \left[ \cos \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) - \cos \left( \frac{\alpha - \beta}{2} \right) \right] d\alpha + \]

(1) \hspace{2cm} (2)

\[ + \int_\beta^{\pi/2} \frac{1}{2} N_\alpha \left[ 2 - \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) - \cos \left( \frac{\alpha - \beta}{2} \right) \right] d\alpha + \]

(3) \hspace{2cm} (4)

\[ + \int_0^\beta \frac{1}{2} N_\alpha \left[ \cos \left( \frac{\alpha - \beta}{2} \right) - \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \right] d\alpha. \]

(5) \hspace{2cm} (6)

Now terms (2) and (4), and terms (3) and (6) can be combined directly. Also, since

\[ \cos \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) = \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right), \]
the expression reduces to

\[ N_{\beta} = \int_{\pi/2}^{\pi} \frac{1}{2} N_{\alpha} \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \, d\alpha - \int_{\beta}^{\pi} \frac{1}{2} N_{\alpha} \cos \left( \frac{\alpha - \beta}{2} \right) \, d\alpha + \]

\[ + \int_{\beta}^{\pi/2} N_{\alpha} \, d\alpha - \int_{0}^{\pi/2} \frac{1}{2} N_{\alpha} \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \, d\alpha + \int_{0}^{\beta} \frac{1}{2} N_{\alpha} \cos \left( \frac{\alpha - \beta}{2} \right) \, d\alpha. \]

Now, from the symmetry of the problem (Figure 1), it is clear that

\[ N_{\pi-\alpha} = N_{\alpha}. \]

Also

\[ \sin \left( \frac{\pi}{4} + \frac{\pi - \alpha}{2} \right) = \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \]

and

\[ d(\pi - \alpha) = -d\alpha, \]

so that by substituting \((\pi - \alpha)\) for \(\alpha\) in the first integral:

\[ \int_{\pi/2}^{\pi} \frac{1}{2} N_{\alpha} \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \, d\alpha = \int_{\pi/2}^{0} \frac{1}{2} N_{\pi-\alpha} \sin \left( \frac{\pi}{4} + \frac{\pi - \alpha}{2} \right) \, d(\pi - \alpha) \]

\[ = \int_{0}^{\pi/2} \frac{1}{2} N_{\alpha} \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \, d\alpha, \]
so that it cancels with the fourth integral. With this simplification, the expression becomes

\[ N_\beta = -\int_{\beta}^{\pi/2} \frac{1}{2} N_\alpha \cos \left(\frac{\alpha - \beta}{2}\right) \, d\alpha + \int_{\beta}^{\pi/2} N_\alpha \, d\alpha + \int_{0}^{\beta} \frac{1}{2} N_\alpha \cos \left(\frac{\alpha - \beta}{2}\right) \, d\alpha. \]  

(4)

This is the total number of molecules that hit between \( \beta \) and \( \pi/2 \) from all points on the surface. Since this will vary with the collision number, \( n \), let \( N_\beta^{(n)} \) = total number of molecules that hit the surface between \( \beta \) and \( \pi/2 \) on the \( n \)th collision.

Now the number of molecules per radian that hit a point, \( P_\alpha \), on the \( n \)th collision is found by calculating the limit:

\[ \lim_{\Delta \beta \to 0} \left[ \frac{N_\beta^{(n)} - N_\beta^{(n)}}{\Delta \beta} \right]. \]

But this is clearly just the derivative \( \frac{d}{d\beta} N_\beta^{(n)} \); therefore,

\[ N_\alpha^{(n)} = \left. \frac{-d}{d\beta} N_\beta^{(n)} \right|_{\beta = \alpha}, \]

so that

\[ \int_{\beta}^{\pi/2} N_\alpha^{(n-1)} \, d\alpha = \left[ -N_\beta^{(n-1)} \right]_{\beta}^{\pi/2} \]

From Figure 1 it is clear that \( N_\beta^{(n-1)} = 0 \) for \( \beta = \pi/2 \) and any \( n \). Therefore,

\[ \int_{\beta}^{\pi/2} N_\alpha^{(n-1)} \, d\alpha = N_\beta^{(n-1)}. \]  

(5)
Also,

\[ \cos \left( \frac{\alpha - \beta}{2} \right) = \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2}. \]  \hspace{1cm} (6)

Now, let

\[ F^{(n)}(\alpha, \beta) = \frac{1}{2} \int N^{(n-1)}_{\alpha} \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha. \]  \hspace{1cm} (7)

Then, using equations (5), (6), and (7) in equation (4), it follows that

\[ N^{(n)}_{\beta} = - F^{(n)}(\pi, \beta) + F^{(n)}(\beta, \beta) + N^{(n-1)}_{\beta} + F^{(n)}(\beta, \beta) - F^{(n)}(0, \beta), \]

or

\[ N^{(n)}_{\beta} = 2F^{(n)}(\beta, \beta) - F^{(n)}(0, \beta) - F^{(n)}(\pi, \beta) + N^{(n-1)}_{\beta} \hspace{1cm} (8) \]

and

\[ N^{(n)}_{\alpha} = - \frac{d}{d\beta} N^{(n)}_{\beta} \bigg|_{\beta = \alpha}, \hspace{1cm} (9) \]

which is the final expression for the distribution. Given any initial distribution, \( N^{(1)}_{\alpha} \), the distribution at the next collision can be found by first calculating \( F^{(2)}(\alpha, \beta) \) from equation (7), using this expression (with the indicated substitutions) in equation (8), and then using equation (9).

For specific cases, the elementary integrals given in Appendix A will be useful.
B. Case I: Incident Distribution Uniform over the Surface

1. Distribution at Second Collision

For an incident distribution of molecules uniform over the surface,

\[ N^{(1)}_{\alpha} = \frac{N_0}{\pi} = \text{constant}, \quad (10) \]

where \( N_0 \) is the total number of incident molecules. Then, from equation (5),

\[ N^{(1)}_{\beta} = \int_{\beta}^{\pi/2} N^{(1)}_{\alpha} \ d\alpha = \int_{\beta}^{\pi/2} \frac{N_0}{\pi} \ d\alpha, \]

\[ N^{(1)}_{\beta} = \frac{N_0}{\pi} \left( \frac{\pi}{2} - \beta \right). \quad (11) \]

From equations (7) and (10),

\[ F^{(2)}(\alpha, \beta) = \frac{1}{2} \int \frac{N_0}{\pi} \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) \ d\alpha, \]

\[ F^{(2)}(\alpha, \beta) = \frac{N_0}{2\pi} \left( 2 \sin \frac{\alpha}{2} \cos \frac{\beta}{2} - 2 \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \right). \]

Then,

\[ F^{(2)}(\beta, \beta) = \frac{N_0}{2\pi} \left( 2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - 2 \cos \frac{\beta}{2} \sin \frac{\beta}{2} \right) = 0, \]

\[ F^{(2)}(0, \beta) = \frac{N_0}{2\pi} \left( -2 \sin \frac{\beta}{2} \right) = -\frac{N_0}{\pi} \sin \frac{\beta}{2}, \]

\[ F^{(2)}(\pi, \beta) = \frac{N_0}{2\pi} \left( 2 \cos \frac{\beta}{2} \right) = \frac{N_0}{\pi} \cos \frac{\beta}{2}, \]
and, using equations (8) and (11):

\[ N^{(2)}_{\beta} = 2F^{(2)}(\beta, \beta) - F^{(2)}(0, \beta) - F^{(2)}(\pi, \beta) + N^{(1)}_{\beta} \]

\[ = \frac{N_0}{\pi} \sin \frac{\beta}{2} - \frac{N_0}{\pi} \cos \frac{\beta}{2} + \frac{N_0}{\pi} \left( \frac{\pi}{2} - \beta \right). \]

\[ N^{(2)}_{\beta} = \frac{N_0}{\pi} \left( \frac{\pi}{2} - \beta + \sin \frac{\beta}{2} - \cos \frac{\beta}{2} \right). \] (12)

From equation (9):

\[ N^{(2)}_{\alpha} = - \frac{d}{d\beta} N^{(2)}_{\beta} \bigg|_{\beta=\alpha} = - \frac{N_0}{\pi} \left( -1 + \frac{1}{2} \cos \frac{\beta}{2} + \frac{1}{2} \sin \frac{\beta}{2} \right) \bigg|_{\beta=\alpha}, \]

\[ N^{(2)}_{\alpha} = \frac{N_0}{2\pi} \left( 2 - \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \right) \] (13)

2. Third Collision

From equations (7) and (13):

\[ F^{(3)}(\alpha, \beta) = \frac{N_0}{4\pi} \int \left( 2 - \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha, \]

\[ F^{(3)}(\alpha, \beta) = \frac{N_0}{4\pi} \left[ 4 \sin \frac{\alpha}{2} \cos \frac{\beta}{2} - 4 \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \left( \frac{\alpha}{2} + \frac{1}{2} \sin \alpha \right) \cos \frac{\beta}{2} \right. \]

\[ - \sin^2 \frac{\alpha}{2} \sin \frac{\beta}{2} - \sin^2 \frac{\alpha}{2} \cos \frac{\beta}{2} - \left( \frac{\alpha}{2} - \frac{1}{2} \sin \alpha \right) \sin \frac{\beta}{2} \].
Then,

\[ F^{(3)}(\beta, \beta) = \frac{N_0}{4\pi} \left[ - \left( \frac{\beta}{2} + \frac{1}{2} \sin \beta \right) \cos \frac{\beta}{2} - \sin^2 \frac{\beta}{2} \sin \frac{\beta}{2} - \sin^2 \frac{\beta}{2} \cos \frac{\beta}{2} \right. \]

\[ - \left( \frac{\beta}{2} - \frac{1}{2} \sin \beta \right) \sin \frac{\beta}{2} \]

\[ = \frac{N_0}{4\pi} \left[ - \left( \frac{\beta}{2} + \frac{1}{2} \sin \beta \right) \cos \frac{\beta}{2} - \sin \frac{\beta}{2} + \cos^2 \frac{\beta}{2} \sin \frac{\beta}{2} \right. \]

\[ - \sin^2 \frac{\beta}{2} \cos \frac{\beta}{2} - \left( \frac{\beta}{2} - \frac{1}{2} \sin \beta \right) \sin \frac{\beta}{2} \]

\[ = \frac{N_0}{4\pi} \left[ - \left( \frac{\beta}{2} + \frac{1}{2} \sin \beta \right) \cos \frac{\beta}{2} - \sin \frac{\beta}{2} + \frac{1}{2} \cos \frac{\beta}{2} \sin \beta \right. \]

\[ - \frac{1}{2} \sin \frac{\beta}{2} \sin \beta - \left( \frac{\beta}{2} - \frac{1}{2} \sin \beta \right) \sin \frac{\beta}{2} \].

\[ F^{(3)}(\beta, \beta) = \frac{N_0}{4\pi} \left[ - \frac{\beta}{2} \left( \cos \frac{\beta}{2} + \sin \frac{\beta}{2} \right) - \sin \frac{\beta}{2} \right]. \]

\[ F^{(3)}(0, \beta) = \frac{N_0}{4\pi} \left( -4 \sin \frac{\beta}{2} \right). \]

\[ F^{(3)}(\pi, \beta) = \frac{N_0}{4\pi} \left( 4 \cos \frac{\beta}{2} - \pi \cos \frac{\beta}{2} - \sin \frac{\beta}{2} - \cos \frac{\beta}{2} - \frac{\pi}{2} \sin \frac{\beta}{2} \right). \]

Then, using equations (8) and (12):

\[ N^{(3)}_\beta = 2F^{(3)}(\beta, \beta) - F^{(3)}(0, \beta) - F^{(3)}(\pi, \beta) + N^{(2)}_\beta \]
\[ N_{\beta}^{(3)} = \frac{N_o}{4\pi} \left[ 2\pi - 4\beta + \left( \frac{\pi}{2} - \beta + 7 \right) \sin \frac{\beta}{2} + \left( \frac{\pi}{2} - \beta - 7 \right) \cos \frac{\beta}{2} \right] \]  \hspace{1cm} (14) \]

From equation (9):

\[ N_{\alpha}^{(3)} = - \frac{d}{d\beta} N_{\beta}^{(3)} \Bigg|_{\beta=\alpha} \]

\[ = - \frac{N_o}{4\pi} \left[ -4 + \frac{1}{2} \left( \frac{\pi}{2} - \beta + 7 \right) \cos \frac{\beta}{2} - \frac{1}{2} \left( \frac{\pi}{2} - \beta - 7 \right) \sin \frac{\beta}{2} - \sin \frac{\beta}{2} - \cos \frac{\beta}{2} \right] \bigg|_{\beta=\alpha} \]

\[ N_{\alpha}^{(3)} = \frac{N_o}{8\pi} \left[ 8 - \left( \frac{\pi}{2} - \alpha + 5 \right) \cos \frac{\alpha}{2} + \left( \frac{\pi}{2} - \alpha - 5 \right) \sin \frac{\alpha}{2} \right] \]  \hspace{1cm} (15) \]

3. Fourth Collision

From equations (7) and (15):

\[ f^{(4)}(\alpha, \beta) = \frac{N_o}{16\pi} \int \left[ 8 - \left( \frac{\pi}{2} - \alpha + 5 \right) \cos \frac{\alpha}{2} + \left( \frac{\pi}{2} - \alpha - 5 \right) \sin \frac{\alpha}{2} \right] \]

\[ \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha \]
\[ F(4) (\beta, \beta) = \frac{N_o}{16\pi} \left[ - \left( \frac{\pi}{2} + 5 \right) \left( \frac{\beta}{2} + \frac{1}{2} \sin \beta \right) \cos \frac{\beta}{2} - \left( \frac{\pi}{2} + 5 \right) \sin^2 \frac{\beta}{2} \sin \frac{\beta}{2} \right. \]
\[ + \frac{1}{2} \left( \frac{\beta^2}{2} + \beta \sin \beta + \cos \beta \right) \cos \frac{\beta}{2} + \frac{1}{2} \left( \sin \beta - \beta \cos \beta \right) \sin \frac{\beta}{2} \]
\[ + \left( \frac{\pi}{2} - 5 \right) \sin^2 \frac{\beta}{2} \cos \frac{\beta}{2} + \left( \frac{\pi}{2} - 5 \right) \left( \frac{\beta}{2} - \frac{1}{2} \sin \beta \right) \sin \frac{\beta}{2} - \frac{1}{2} \left( \sin \beta - \beta \cos \beta \right) \cos \frac{\beta}{2} \]
\[ - \frac{1}{2} \left( \frac{\beta^2}{2} - \beta \sin \beta - \cos \beta \right) \sin \frac{\beta}{2} \left. \right] \]

\[ = \frac{N_o}{16\pi} \left\{ \cos \frac{\beta}{2} \left[ - \frac{\beta}{2} \left( \frac{\pi}{2} + 5 \right) + \frac{1}{2} \left( \frac{\beta^2}{2} + \beta \sin \beta + \cos \beta \right) \right. \right. \]
\[ - \frac{1}{2} \left( \sin \beta - \beta \cos \beta \right) \right. \left. \right] + \sin \frac{\beta}{2} \left[ - \left( \frac{\pi}{2} + 5 \right) + \frac{1}{2} \left( \sin \beta - \beta \cos \beta \right) \right. \]}
\[
\begin{align*}
&\left(\frac{\pi}{2} - 5\right)\frac{\beta^2}{2} - \frac{1}{2}\left(\frac{\beta^2}{2} - \beta \sin \beta - \cos \beta\right)\right]\bigg] \\
= \frac{N_o}{16\pi} \left\{ \cos \frac{\beta}{2} \left[ - \frac{\beta}{2} \left(\frac{\pi}{2} + 5\right) + \frac{1}{2} \left(\frac{\beta^2}{2} + 2\beta \sin \frac{\beta}{2} \cos \frac{\beta}{2} + 1 - 2 \sin^2 \frac{\beta}{2}\right) \\
+ \frac{1}{2} \left(2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - \beta + 2\beta \sin^2 \frac{\beta}{2}\right) \right] + \sin \frac{\beta}{2} \left[ - \left(\frac{\beta}{2} + 5\right) \\
+ \frac{1}{2} \left(2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - 2\beta \cos^2 \frac{\beta}{2} + \beta\right) \right] \right\} \\
= \frac{N_o}{16\pi} \left\{ \cos \frac{\beta}{2} \left[ - \frac{\beta}{2} \left(\frac{\pi}{2} + 5\right) + \frac{1}{2} \left(\frac{\beta^2}{2} + 1\right) + \frac{\beta}{2} \right] + \sin \frac{\beta}{2} \left[ - \left(\frac{\beta}{2} + 5\right) + \frac{\beta}{2} \right. \\
\left. + \left(\frac{\pi}{2} - 5\right) \frac{\beta}{2} - \frac{1}{2} \left(\frac{\beta^2}{2} + 1\right) \right] \right\} \\
\mathcal{F}^{(4)}(\beta, \beta) = \frac{N_o}{16\pi} \left\{ \left[ - \frac{\beta}{2} \left(\frac{\pi}{2} + 4 - \frac{\beta}{2}\right) + \frac{1}{2} \right] \cos \frac{\beta}{2} + \left[ \frac{\beta}{2} \left(\frac{\pi}{2} - 4 - \frac{\beta}{2}\right) - \frac{\beta}{2} - \frac{1}{2} \right] \sin \frac{\beta}{2} \right\} \\
\mathcal{F}^{(4)}(0, \beta) = \frac{N_o}{16\pi} \left( - 16 \sin \frac{\beta}{2} + \frac{1}{2} \cos \frac{\beta}{2} + \frac{1}{2} \sin \frac{\beta}{2} \right) \\
\mathcal{F}^{(4)}(\pi, \beta) = \frac{N_o}{16\pi} \left[ 16 \cos \frac{\beta}{2} - \left(\frac{\pi}{2} + 5\right) \cos \frac{\beta}{2} - \left(\frac{\pi}{2} + 5\right) \sin \frac{\beta}{2} + \frac{1}{2} \left(\frac{\pi^2}{2} - 1\right) \cos \frac{\beta}{2} \right]
\end{align*}
\]
+ \frac{1}{2} (\pi) \sin \frac{\pi}{2} + \left( \frac{\pi}{2} - 5 \right) \cos \frac{\pi}{2} + \left( \frac{\pi}{2} - 5 \right) \sin \frac{\pi}{2} - \frac{1}{2} (\pi) \cos \frac{\pi}{2} \\
- \frac{1}{2} \left( \pi^2 + 1 \right) \sin \frac{\pi}{2} \\
F^{(4)}(\pi, \beta) = \frac{N_0}{16\pi} \left[ \left( \frac{10\frac{1}{2} - 5\pi}{2} \right) \cos \frac{\beta}{2} + \left( - \frac{5\frac{1}{2} - 5\pi}{2} \right) \sin \frac{\beta}{2} \right].

Then, using equations (8) and (14):

\[
N^{(4)}_{\beta} = 2F^{(4)}(\beta, \beta) - F^{(4)}(0, \beta) - F^{(4)}(\pi, \beta) + N^{(3)}_{\beta}
= \frac{N_0}{16\pi} \left[ \left( - \frac{\pi \beta}{2} - 4\beta + \frac{\beta^2}{2} + 1 - \frac{1}{2} - 10\frac{1}{2} + \frac{5\pi}{2} \right) \cos \frac{\beta}{2} \\
+ \left( \frac{\pi \beta}{2} - 4\beta - \frac{\beta^2}{2} - \pi - 11 + 15\frac{1}{2} + 5\frac{1}{2} + \frac{5\pi}{2} \right) \sin \frac{\beta}{2} + 8\pi - 16\beta \\
+ 4 \left( \frac{\pi}{2} - \beta + 7 \right) \sin \frac{\beta}{2} + 4 \left( \frac{\pi}{2} - \beta - 7 \right) \cos \frac{\beta}{2} \right].
\]

\[
N^{(4)}_{\beta} = \frac{N_0}{16\pi} \left[ 8\pi - 16\beta + \left( - 38 + \frac{9\pi}{2} - 8\beta - \frac{\pi \beta}{2} + \frac{\beta^2}{2} \right) \cos \frac{\beta}{2} \\
+ \left( 38 + \frac{7\pi}{2} - 8\beta + \frac{\pi \beta}{2} - \frac{\beta^2}{2} \right) \sin \frac{\beta}{2} \right].
\]  \hspace{2cm} (16)

From equation (9):

\[
N^{(4)}_{\alpha} = \frac{d}{d\beta} N^{(4)}_{\beta} \bigg|_{\beta=\alpha}
\]
\[ \frac{N}{16\pi} \left[ -16 + \left( -8 - \frac{\pi}{2} + \beta \right) \cos \frac{\beta}{2} + \left( \frac{1}{2} \right) \left( 38 + \frac{9\pi}{2} - 8\beta - \frac{\beta^2}{2} + \frac{\beta^2}{2} \right) \sin \frac{\beta}{2} \right. \\
+ \left( -8 + \frac{\pi}{2} - \beta \right) \sin \frac{\beta}{2} \left. + \frac{1}{2} \left( 38 + \frac{7\pi}{2} - 8\beta + \frac{\beta^2}{2} - \frac{\beta^2}{2} \right) \cos \frac{\beta}{2} \right] \]

\[ N_{\alpha}^{(4)} = \frac{N}{32\pi} \left[ 32 - \left( 22 + \frac{5\pi}{2} - 6\alpha + \frac{\alpha \alpha}{2} - \frac{\alpha \alpha}{2} \right) \cos \frac{\alpha}{2} - \\
- \left( 22 - \frac{7\pi}{2} + 6\alpha + \frac{\alpha \alpha}{2} - \frac{\alpha \alpha}{2} \right) \sin \frac{\alpha}{2} \right]. \quad (17) \]

4. Fifth Collision

From equations (7) and (17):

\[ F^{(5)}(\alpha, \beta) = \frac{N}{64\pi} \int \left[ 32 - \left( 22 + \frac{5\pi}{2} - 6\alpha + \frac{\alpha \alpha}{2} - \frac{\alpha \alpha}{2} \right) \cos \frac{\alpha}{2} \\
- \left( 22 - \frac{7\pi}{2} + 6\alpha + \frac{\alpha \alpha}{2} - \frac{\alpha \alpha}{2} \right) \sin \frac{\alpha}{2} \right] \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha. \]

\[ F^{(5)}(\alpha, \beta) = \frac{N}{64\pi} \left\{ \cos \frac{\beta}{2} \left[ 64 \sin \frac{\alpha}{2} - \left( 22 + \frac{5\pi}{2} + \frac{1}{2} \sin \alpha \right) \right] \\
+ \left( 6 - \frac{\pi}{2} \right) \left( \frac{1}{2} \left( \frac{\alpha \alpha}{2} + \alpha \sin \alpha + \cos \alpha \right) + \frac{1}{2} \left( \frac{\alpha \alpha}{6} + \left( \frac{\alpha \alpha}{2} - 1 \right) \sin \alpha + \alpha \cos \alpha \right) \right. \\
+ \left( \frac{7\pi}{2} - 22 \right) \sin^2 \frac{\alpha}{2} - \left( 6 + \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \sin \alpha - \alpha \cos \alpha \right) \right\}. \]
\[
+ \frac{1}{2} (\alpha \sin \alpha + \cos \alpha - \frac{1}{2} \alpha^2 \cos \alpha) + \frac{\sin \alpha}{2} \left[ -64 \cos \alpha - \left(22 + \frac{5\pi}{2}\right) \sin^2 \frac{\alpha}{2} + \frac{1}{2} \left(\alpha - \alpha \cos \alpha - \frac{\alpha^2}{2} \cos \alpha\right) + \left(6 - \frac{\pi}{2}\right) \left(\frac{1}{2}\right) \left(\sin \alpha - \alpha \cos \alpha\right) + \left(6 - \frac{\pi}{2}\right) \left(\frac{1}{2}\right) \left(\sin \alpha + \cos \alpha - \frac{\alpha^2}{2} \cos \alpha\right)
\]

Then,

\[
F(5)(\beta, \beta) = \frac{N}{64\pi} \left[ \cos \frac{\beta}{2} \left( -64 \sin \frac{\beta}{2} - \left(22 + \frac{5\pi}{2}\right) \left(\frac{1}{2} + \sin \frac{\beta}{2} \cos \frac{\beta}{2}\right) + \left(6 - \frac{\pi}{2}\right) \left(\frac{1}{2}\right) \left(\sin \alpha - \cos \alpha\right) + \left(6 - \frac{\pi}{2}\right) \left(\frac{1}{2}\right) \left(\sin \alpha + \cos \alpha - \frac{\alpha^2}{2} \cos \alpha\right) + \frac{1}{2} \left(\frac{\alpha^2}{6} - \frac{\alpha^2}{2} - 1\right) \left(2 \sin \frac{\beta}{2} \cos \frac{\alpha}{2} + \left(\frac{\pi}{2} - 22\right) \sin^2 \frac{\beta}{2}\right) - \left(6 + \frac{\pi}{2}\right) \left(\frac{1}{2}\right) \left(2 \sin \frac{\beta}{2} \cos \frac{\alpha}{2} - \beta + 2\beta \sin^2 \frac{\alpha}{2}\right) + \frac{1}{2} \left(2\beta \sin \frac{\alpha}{2} \cos \frac{\beta}{2} + 1\right) - 2 \sin^2 \frac{\beta}{2} - \frac{\beta^2}{2} + \beta^2 \sin^2 \frac{\beta}{2}\right) + \sin \frac{\beta}{2} \left[ -64 \cos \frac{\beta}{2} - \left(22 + \frac{5\pi}{2}\right) \right] + \left(22 + \frac{5\pi}{2}\right) \cos^2 \frac{\beta}{2} + \left(6 - \frac{\pi}{2}\right) \left(\frac{1}{2}\right) \left(2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - 2\beta \cos^2 \frac{\beta}{2} + \beta\right)\right].
\]
\[ + \frac{1}{2} \left( 2\beta \sin \frac{\beta}{2} \cos \frac{\beta}{2} + 2 \cos^2 \frac{\beta}{2} - 1 - \beta^2 \cos^2 \frac{\beta}{2} + \frac{\beta^2}{2} \right) \\
+ \left( \frac{7\pi}{2} - 22 \right) \left( \frac{\beta}{2} - \sin \frac{\beta}{2} \cos \frac{\beta}{2} \right) - \left( 6 + \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \frac{\beta^2}{2} - 2\beta \sin \frac{\beta}{2} \cos \frac{\beta}{2} \\
- 2 \cos^2 \frac{\beta}{2} + 1 \right) + \frac{1}{2} \left( \frac{\beta^3}{6} - \left( \frac{\beta^2}{2} - 1 \right) \cdot \sin \frac{\beta}{2} \cos \frac{\beta}{2} - 2\beta \cos^2 \frac{\beta}{2} + \beta \right) \right] \right) \\
= \frac{N}{64\pi \cdot \left( \cos \frac{\beta}{2} \right) \left[ - \left( 22 + \frac{5\pi}{2} \right) \frac{\beta}{2} + \left( 6 - \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \frac{\beta^2}{2} + 1 \right) + \frac{1}{2} \left( \frac{\beta^3}{6} + \beta \right) \\
- \left( 6 + \frac{\pi}{2} \right) \left( \frac{1}{2} \right) (- \beta) + \frac{1}{2} \left( 1 - \frac{\beta^2}{2} \right) \right] + \sin \frac{\beta}{2} \left[ - \left( 22 + \frac{5\pi}{2} \right) \\
+ \left( 6 - \frac{\pi}{2} \right) \left( \frac{1}{2} \right) (\beta) + \frac{1}{2} \left( - 1 + \frac{\beta^2}{2} \right) + \left( \frac{7\pi}{2} - 22 \right) \frac{\beta}{2} - \left( 6 + \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \frac{\beta^2}{2} + 1 \right) \\
+ \frac{1}{2} \left( \frac{\beta^3}{6} + \beta \right) \right] \right) \right) \right) \\
\]

\[ F^5(\beta, \beta) = \frac{N}{64\pi} \left[ \left( \frac{7\pi}{2} - \frac{\pi}{4} - \frac{15\beta}{2} - \pi\beta + \frac{5\beta^2}{4} - \frac{\pi\beta^2}{8} + \frac{\beta^3}{12} \right) \cos \frac{\beta}{2} \\
+ \left( - 25 \frac{1}{2} - 11 \pi \right) \frac{\beta}{4} - \frac{15\beta}{2} + \frac{3\pi\beta}{2} - \frac{5\beta^2}{4} - \frac{\pi\beta^2}{8} + \frac{\beta^3}{12} \right) \sin \frac{\beta}{2} \right] \\
\]

\[ F^5(0, \beta) = \frac{N}{64\pi} \left\{ \cos \frac{\beta}{2} \left[ \left( 6 - \frac{\pi}{2} \right) \left( \frac{1}{2} \right) + \frac{1}{2} \right] + \sin \frac{\beta}{2} \left[ - 64 + \frac{1}{2} + \\
+ \left( 6 + \frac{\pi}{2} \right) \left( \frac{1}{2} \right) (1) \right] \right\} . \]
\[ F^{(5)}(0, \beta) = \frac{N_0}{64 \pi} \left[ \left( \frac{7}{2} - \frac{\pi}{4} \right) \cos \frac{\beta}{2} + \left( -60\frac{1}{2} + \frac{\pi}{4} \right) \sin \frac{\beta}{2} \right]. \]

\[ F^{(5)}(\pi, \beta) = \frac{N_0}{64 \pi} \left\{ \cos \frac{\beta}{2} \left[ 64 - \left( 22 + \frac{5\pi}{2} \right) \left( \frac{\pi}{2} \right) + \left( 6 - \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \frac{\pi^2}{2} - 1 \right) \right. \right. \\
+ \frac{1}{2} \left( \frac{\pi^3}{6} - \pi \right) + \left( \frac{7\pi}{2} - 22 \right) - \left( 6 + \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \pi \right) + \frac{1}{2} \left( -1 + \frac{\pi^3}{2} \right) \left. \right\}. \]

\[ + \sin \frac{\beta}{2} \left\{ - \left( 22 + \frac{5\pi}{2} \right) + \left( 6 - \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \pi \right) + \frac{1}{2} \left( -1 + \frac{\pi^3}{2} \right) + \left( \frac{7\pi}{2} - 22 \right) \left( \frac{\pi}{2} \right) \right. \right. \\
- \left( 6 + \frac{\pi}{2} \right) \left( \frac{1}{2} \right) \left( \frac{\pi^2}{2} + 1 \right) + \frac{1}{2} \left( \frac{\pi^3}{6} + \pi \right) \left. \right\}. \]

\[ F^{(5)}(\pi, \beta) = \frac{N_0}{64 \pi} \left[ \cos \frac{\beta}{2} \left( 38\frac{1}{2} - 10\frac{3}{4} \pi + \frac{\pi^2}{4} + \frac{\pi^3}{24} \right) \right. \right. \\
+ \sin \frac{\beta}{2} \left( -25\frac{1}{2} - 10\frac{1}{4} \pi + \frac{\pi^2}{4} - \frac{\pi^3}{24} \right). \]

Then, using equations (8) and (16):

\[ N^{(5)}_\beta = 2F^{(5)}(\beta, \beta) - F^{(5)}(0, \beta) - F^{(5)}(\pi, \beta) + N^{(4)}_\beta \right. \right. \\
= \frac{N_0}{64 \pi} \left[ \cos \frac{\beta}{2} \left( 7 - \frac{\pi}{2} - 15\beta - 2\pi\beta + \frac{5\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} - \frac{7}{2} + \frac{\pi}{4} - 38\frac{1}{2} + 10\frac{3}{4} \pi \right. \right. \right. \\
- \left. \left. \left. 25\frac{1}{2} - 10\frac{1}{4} \pi + \frac{\pi^2}{4} - \frac{\pi^3}{24} \right) \right. \right. \right. \]
\[- \frac{\pi^2}{4} + \frac{\pi^3}{24} - 152 + 18\pi - 32\beta - 2\pi\beta + 2\beta^2 \right) + \sin \frac{\beta}{2} \left( - 51 \right) \right] - \frac{11\pi}{2} - 15\beta \\
+ 3\pi\beta - \frac{5\beta^2}{2} - \frac{\pi\beta^3}{4} + \frac{\beta^3}{6} + 60\frac{1}{2} - \frac{\pi}{4} + 25\frac{1}{4} \pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} + 152 + 14\pi \\
- 32\beta + 2\pi\beta - 2\beta^2 \right) + 32\pi - 64\beta \right] \right]. \\
N_B^{(5)} = \frac{N_o}{64\pi} \left[ 32\pi - 64\beta + \left( - 187 + 28\frac{1}{2} \pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta - 4\pi\beta + \frac{9\beta^2}{2} \\
- \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} \right) \cos \frac{\beta}{2} + \left( 187 + 18\frac{1}{2} \pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta + 5\pi\beta - \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} \\
+ \frac{\beta^3}{6} \right) \sin \frac{\beta}{2} \right] \right] \\
(18)

From equation (9):

\[ N_B^{(5)} = - \frac{d}{d\beta} N_B^{(5)} \right|_{\beta = \alpha} = - \frac{N_o}{64\pi} \left[ - 64 + \left( - \frac{1}{2} \right) \left( - 187 + 28\frac{1}{2} \pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} \\
- 47\beta - 4\pi\beta + \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} \right) \sin \frac{\beta}{2} + \left( - 47 - 4\pi + 9\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2} \right) \cos \frac{\beta}{2} \\
+ \frac{1}{2} \left( 187 + 18\frac{1}{2} \pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta + 5\pi\beta - \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} \right) \cos \frac{\beta}{2} \right] \]
\[ (+ \left( -47 + 5\pi - 9\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2}\right) \sin \frac{\beta}{2} \right)_{\beta=\alpha} \]

\[
N^{(5)}_\alpha = \frac{N_0}{128\pi} \left[ 128 + \left( -93 - \frac{101}{2}\pi + \frac{\pi^2}{4} - \frac{\pi^3}{24} + 29\alpha - 4\pi\alpha + \frac{7\alpha^2}{\alpha} - \frac{\pi\alpha^2}{4} - \frac{\alpha^3}{6} \right) \cos \frac{\alpha}{2} \right.
\]

\[
\left. + \left( -93 + 18\frac{1}{2}\pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 29\alpha - 3\pi\alpha + \frac{7\alpha^2}{\alpha} + \frac{\pi\alpha^2}{4} + \frac{\alpha^3}{6} \right) \sin \frac{\alpha}{2} \right]. \tag{19} \]

C. Case II: Incident Distribution Uniform across the Y-Axis

1. Distribution at Second Collision

For an incident distribution of molecules uniform across the y-axis:

\[
N_\alpha \equiv \frac{dN}{d\alpha} \neq \text{constant} \]

but

\[
\frac{dN}{dy} = \text{constant} = k. \]

Now

\[
N_\alpha \equiv \frac{dN}{d\alpha} = \frac{dN}{dy} \cdot \frac{dy}{d\alpha}, \]

and from Figure 1 it is clear that \( y = -\cos \alpha \) so that

\[
\frac{dy}{d\alpha} = \sin \alpha. \]
Therefore, $N_\alpha = k \sin \alpha$, but

$$N_o = \int_0^\pi k \sin \alpha \, d\alpha = 2k$$

so that

$$N_\alpha = \frac{1}{2} N_o \sin \alpha . \quad (20)$$

From equation (5).

$$N_\beta^{(1)} = \int_\beta^{\pi/2} N_\alpha^{(1)} \, d\alpha = \int_\beta^{\pi/2} \frac{1}{2} N_o \sin \alpha \, d\alpha$$

$$= \frac{1}{2} N_o (- \cos \frac{\pi}{2} + \cos \beta)$$

$$N_\beta^{(1)} = \frac{1}{2} N_o \cos \beta \quad (21)$$

From equations (7) and (20):

$$F^{(2)}(\alpha, \beta) = \frac{1}{2} \int \frac{1}{2} N_o \sin \alpha \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) \, d\alpha$$

$$= \frac{N_o}{2} \int \left( \cos \frac{\beta}{2} \sin \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} + \sin \frac{\beta}{2} \sin^2 \frac{\alpha}{2} \cos \frac{\alpha}{2} \right) \, d\alpha$$

$$F^{(2)}(\alpha, \beta) = \frac{N_o}{3} \left( \sin \frac{\beta}{2} \sin \frac{3\alpha}{2} - \cos \frac{\beta}{2} \cos^3 \frac{\alpha}{2} \right) . \quad (22)$$
Then

\[ F^{(2)}(\beta, \beta) = \frac{N_0}{3} \left( \sin^4 \frac{\beta}{2} - \cos^4 \frac{\beta}{2} \right) \]

\[ = \frac{N_0}{3} \left( \sin^2 \frac{\beta}{2} - \cos^2 \frac{\beta}{2} \right) \left( \sin^2 \frac{\beta}{2} + \cos^2 \frac{\beta}{2} \right). \]

\[ F^{(2)}(0, \beta) = \frac{N_0}{3} (- \cos \frac{\beta}{2}) \]

\[ F^{(2)}(\pi, \beta) = \frac{N_0}{3} \left( \sin \frac{\beta}{2} \right). \]

Then, using equations (8) and (21):

\[ N^{(2)}_\beta = 2F^{(2)}(\beta, \beta) - F^{(2)}(0, \beta) - F^{(2)}(\pi, \beta) + N^{(1)}_\beta, \]

\[ = \frac{N_0}{3} \left( - 2 \cos \beta + \cos \frac{\beta}{2} - \sin \frac{\beta}{2} + 3 \cos \beta \right), \]

\[ N^{(2)}_\beta = \frac{N_0}{3} \left( \cos \frac{\beta}{2} - \sin \frac{\beta}{2} - \frac{1}{2} \cos \beta \right). \quad (22) \]

From equation (9):

\[ N^{(2)}_\alpha = - \frac{d}{d\beta} N^{(2)}_\beta \bigg|_{\beta=\alpha} = - \frac{N_0}{3} \left[ - \frac{1}{2} \sin \frac{\beta}{2} - \frac{1}{2} \cos \frac{\beta}{2} + \frac{1}{2} \sin \beta \right] \bigg|_{\beta=\alpha} \]

\[ N^{(2)}_\alpha = \frac{N_0}{6} \left( \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} - \sin \alpha \right). \quad (23) \]
2. Third Collision

From equations (7) and (23):

\[
F^{(3)}(\alpha, \beta) = \frac{N_0}{12} \int \left( \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} - \sin \alpha \right) \left( \cos \frac{\beta}{2} \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha
\]

\[
= \frac{N_0}{12} \left[ \cos \frac{\beta}{2} \left( \frac{\alpha}{2} + \frac{1}{2} \sin \alpha + \sin^2 \frac{\alpha}{2} + \frac{4}{3} \cos^3 \frac{\alpha}{2} \right) \right.
\]

\[
+ \sin \frac{\beta}{2} \left( \sin^2 \frac{\alpha}{2} + \frac{\alpha}{2} - \frac{1}{2} \sin \alpha - \frac{4}{3} \sin^3 \frac{\alpha}{2} \right) \right]
\]

Then

\[
F^{(3)}(\beta, \beta) = \frac{N_0}{12} \left[ \cos \frac{\beta}{2} \left( \frac{\beta}{2} + \frac{1}{2} \sin \beta + \sin^2 \frac{\beta}{2} + \frac{4}{3} \cos^3 \frac{\beta}{2} \right) \right.
\]

\[
+ \sin \frac{\beta}{2} \left( \sin^2 \frac{\beta}{2} + \frac{\beta}{2} - \frac{1}{2} \sin \beta - \frac{4}{3} \sin^3 \frac{\beta}{2} \right) \right]
\]

\[
= \frac{N_0}{12} \left[ \cos \frac{\beta}{2} \left( \frac{\beta}{2} + \sin \frac{\beta}{2} \cos \frac{\beta}{2} + \sin^2 \frac{\beta}{2} + \frac{4}{3} \cos \frac{\beta}{2} - \frac{4}{3} \cos \frac{\beta}{2} \sin^2 \frac{\beta}{2} \right) \right.
\]

\[
+ \sin \frac{\beta}{2} \left( 1 - \cos^2 \frac{\beta}{2} + \frac{\beta}{2} - \sin \frac{\beta}{2} \cos \frac{\beta}{2} - \frac{4}{3} \sin \frac{\beta}{2} + \frac{4}{3} \sin \frac{\beta}{2} \cos^2 \frac{\beta}{2} \right) \right]
\]

\[
F^{(3)}(\beta, \beta) = \frac{N_0}{12} \left[ \frac{4}{3} \cos \beta + \frac{\beta}{2} \cos \frac{\beta}{2} + (1 + \frac{\beta}{2}) \sin \frac{\beta}{2} \right]
\]

\[
F^{(3)}(0, \beta) = \frac{N_0}{12} \left[ \cos \frac{\beta}{2} \cdot (\frac{4}{3}) \right]
\]

\[
F^{(3)}(\pi, \beta) = \frac{N_0}{12} \left[ \cos \frac{\beta}{2} \left( \frac{\pi}{2} + 1 \right) + \sin \frac{\beta}{2} \left( 1 + \frac{\pi}{2} - \frac{4}{3} \right) \right]
\]
From equation (22): 

\[ N_{\beta}^{(2)} = \frac{N_0}{12} \left( 4 \cos \frac{\beta}{2} - 4 \sin \frac{\beta}{2} - 2 \cos \beta \right). \]

Then, using equation (8):

\[ N_{\beta}^{(3)} = 2F^{(3)}(\beta, \beta) - F^{(3)}(0, \beta) - F^{(3)}(\pi, \beta) + N_{\beta}^{(2)} \]

\[ = \frac{N_0}{12} \left[ \frac{8}{3} \cos \beta + \beta \cos \frac{\beta}{2} + (2 + \beta) \sin \frac{\beta}{2} - \frac{4}{3} \cos \frac{\beta}{2} - (\frac{\pi}{2} + 1) \cos \frac{\beta}{2} \right. \]

\[ - \left. \left( \frac{\pi}{2} - \frac{1}{3} \right) \sin \frac{\beta}{2} + 4 \cos \frac{\beta}{2} - 4 \sin \frac{\beta}{2} - 2 \cos \beta \right]. \]

\[ N_{\beta}^{(3)} = \frac{N_0}{12} \left[ \frac{2}{3} \cos \beta + \left( \beta - \frac{\pi}{2} + \frac{5}{3} \right) \cos \frac{\beta}{2} + \left( \beta - \frac{\pi}{2} - \frac{5}{3} \right) \sin \frac{\beta}{2} \right]. \quad (24) \]

From equation (9):

\[ N_{\alpha}^{(3)} = -\frac{d}{d\beta} N_{\beta}^{(3)} |_{\beta=\alpha} = -\frac{N_0}{12} \left[ -\frac{2}{3} \sin \beta + \left( \beta - \frac{\pi}{2} + \frac{5}{3} \right) \left( -\sin \frac{\beta}{2} \right) \left( \frac{1}{2} \right) \right. \]

\[ + \left. \cos \frac{\beta}{2} + \left( \beta - \frac{\pi}{2} - \frac{5}{3} \right) \left( \cos \frac{\beta}{2} \right) \left( \frac{1}{2} \right) + \sin \frac{\beta}{2} \right] \quad \beta=\alpha \]

\[ = -\frac{N_0}{12} \left[ -\frac{2}{3} \sin \beta + \left( -\frac{\beta}{2} + \frac{\pi}{4} + \frac{1}{6} \right) \sin \frac{\beta}{2} + \left( \frac{\beta}{2} - \frac{\pi}{4} + \frac{1}{6} \right) \cos \frac{\beta}{2} \right] \quad \beta=\alpha \]

\[ N_{\alpha}^{(3)} = \frac{N_0}{12} \left[ -\frac{1}{6} - \frac{\pi}{4} + \frac{\alpha}{2} \right) \sin \frac{\alpha}{2} + \left( -\frac{1}{6} + \frac{\pi}{4} - \frac{\alpha}{2} \right) \cos \frac{\alpha}{2} + \frac{2}{3} \sin \alpha \right]. \quad (25) \]
3. Fourth Collision

From equations (7) and (25):

\[ F^{(4)}(\alpha, \beta) = \frac{N_o}{24} \int \left[ (- \frac{1}{6} - \frac{\pi}{4} + \frac{\alpha}{2}) \sin \frac{\alpha}{2} + \left( - \frac{1}{6} + \frac{\pi}{4} - \frac{\alpha}{2} \right) \cos \frac{\alpha}{2} + \frac{2}{3} \sin \alpha \right. \]

\[ \cdot \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) \, d\alpha. \]

\[ = \frac{N_o}{24} \left\{ \cos \frac{\beta}{2} \left[ \left( - \frac{1}{6} - \frac{\pi}{4} \right) \sin^2 \frac{\alpha}{2} + \frac{1}{4} \sin \alpha - \frac{1}{4} \alpha \cos \alpha + \left( - \frac{1}{6} + \frac{\pi}{4} \right) \right. \right. \]

\[ \cdot \left( \frac{\alpha}{2} + \frac{1}{2} \sin \alpha \right) - \frac{1}{2} \left( \frac{\alpha^2}{4} + \frac{\alpha \sin \alpha}{2} + \frac{\cos \alpha}{2} \right) - \frac{8}{9} \cos^3 \frac{\alpha}{2} \Bigg] + \sin \frac{\beta}{2} \left[ \left( - \frac{1}{6} - \frac{\pi}{4} \right) \left( \frac{\alpha}{2} - \frac{1}{2} \sin \alpha \right) + \frac{1}{2} \left( \frac{\alpha^2}{4} - \frac{\alpha \sin \alpha}{2} - \frac{\cos \alpha}{2} \right) \right. \]

\[ + \left( - \frac{1}{6} + \frac{\pi}{4} \right) \sin^2 \frac{\alpha}{2} - \frac{1}{4} \sin \alpha + \frac{1}{4} \alpha \cos \alpha + \frac{8}{9} \sin^3 \frac{\alpha}{2} \Bigg]. \]

\[ F^{(4)}(\alpha, \beta) = \frac{N_o}{24} \left\{ \cos \frac{\beta}{2} \left[ \left( - \frac{1}{6} - \frac{\pi}{4} \right) \sin^2 \frac{\alpha}{2} + \left( \frac{1}{6} + \frac{\pi}{8} - \frac{\alpha}{4} \right) \sin \alpha + \right. \right. \]

\[ \left( - \frac{1}{4} - \frac{\alpha}{4} \right) \cos \alpha + \left( - \frac{1}{12} + \frac{\pi}{8} \right) \alpha - \frac{\alpha^2}{8} - \frac{8}{9} \cos^3 \frac{\alpha}{2} \right. \]

\[ + \sin \frac{\beta}{2} \left[ \left( - \frac{1}{6} + \frac{\pi}{4} \right) \sin^2 \frac{\alpha}{2} + \left( - \frac{1}{6} + \frac{\pi}{8} - \frac{\alpha}{4} \right) \sin \alpha + \left( - \frac{1}{4} + \frac{\alpha}{4} \right) \cos \alpha \right. \]

\[ + \left( \frac{1}{12} - \frac{\pi}{8} \right) \alpha + \frac{\alpha^2}{8} + \frac{8}{9} \sin^3 \frac{\alpha}{2} \Bigg]. \]
\[
F^{(4)}(\beta, \beta) = \frac{N_0}{24} \left\{ \cos \frac{\beta}{2} \left[ \left( -\frac{1}{6} - \frac{\pi}{4} \right) \sin^2 \frac{\beta}{2} + \left( \frac{1}{6} + \frac{\pi}{8} - \frac{\beta}{4} \right) \sin \beta \right] \\
+ \left( -\frac{1}{4} - \frac{\beta}{4} \right) \cos \beta + \left( -\frac{1}{12} + \frac{\pi}{8} \right) \beta - \frac{\beta^2}{8} - \frac{8}{9} \cos^3 \frac{\beta}{2} \right] \\
+ \sin \frac{\beta}{2} \left[ \left( -\frac{1}{6} + \frac{\pi}{4} \right) \sin^2 \frac{\beta}{2} + \left( -\frac{1}{6} + \frac{\pi}{8} - \frac{\beta}{4} \right) \sin \beta + \left( -\frac{1}{4} + \frac{\beta}{4} \right) \cos \beta \\
+ \left( -\frac{1}{12} - \frac{\pi}{8} \right) \beta \right] + \frac{\beta^2}{8} + \frac{8}{9} \sin^3 \frac{\beta}{2} \right\} \\
= \frac{N_0}{24} \left\{ \left( -\frac{1}{6} - \frac{\pi}{4} \right) \cos \frac{\beta}{2} \sin^2 \frac{\beta}{2} + \left( \frac{1}{3} + \frac{\pi}{4} - \frac{\beta}{2} \right) \cos^2 \frac{\beta}{2} \sin \frac{\beta}{2} \\
+ \left( -\frac{1}{4} - \frac{\beta}{4} \right) \cos \frac{\beta}{2} - \left( -\frac{1}{2} - \frac{\beta}{2} \right) \cos \frac{\beta}{2} \sin^2 \frac{\beta}{2} + \left[ -\frac{1}{12} + \frac{\pi}{8} \right] \beta - \frac{\beta^2}{8} \right] \cos \frac{\beta}{2} \\
- \frac{8}{9} \cos^2 \frac{\beta}{2} + \frac{8}{9} \cos^2 \frac{\beta}{2} \sin^2 \frac{\beta}{2} + \left( -\frac{1}{6} + \frac{\pi}{4} \right) \sin \frac{\beta}{2} - \left( -\frac{1}{6} + \frac{\pi}{4} \right) \sin \frac{\beta}{2} \cos^2 \frac{\beta}{2} \\
+ \left( -\frac{1}{3} + \frac{\pi}{4} - \frac{\beta}{2} \right) \sin^2 \frac{\beta}{2} \cos \frac{\beta}{2} + 2 \left( -\frac{1}{4} + \frac{\beta}{4} \right) \sin \frac{\beta}{2} \cos^2 \frac{\beta}{2} - \left( -\frac{1}{4} + \frac{\beta}{4} \right) \sin \frac{\beta}{2} \\
+ \left[ -\frac{1}{12} - \frac{\pi}{8} \right] \beta + \frac{\beta^2}{8} \right] \sin \frac{\beta}{2} + \frac{8}{9} \sin \frac{\beta}{2} \cos \frac{\beta}{2} - \frac{8}{9} \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2} \right\}.
\]
\[ F^{(4)}(\beta, \beta) = \frac{N_0}{24} \left[ \cos \frac{\beta}{2} \left( -\frac{\beta}{3} + \frac{\pi \beta}{8} - \frac{1}{4} - \frac{\beta^2}{8} \right) + \sin \frac{\beta}{2} \left( \frac{1}{12} + \frac{\pi}{4} - \frac{\beta}{3} - \frac{\pi \beta}{8} + \frac{\beta^2}{8} \right) \right. \]

\[ - \frac{8}{9} \cos \beta \] \]

\[ F^{(4)}(0, \beta) = \frac{N_0}{24} \left[ \cos \frac{\beta}{2} \left( -\frac{1}{4} - \frac{8}{9} \right) + \sin \frac{\beta}{2} \left( -\frac{1}{4} \right) \right] . \]

\[ F^{(4)}(\pi, \beta) = \frac{N_0}{24} \left\{ \cos \frac{\beta}{2} \left[ \left( -\frac{1}{6} - \frac{\pi}{4} \right) - \left( -\frac{1}{4} - \frac{\pi}{4} \right) + \left( -\frac{1}{12} + \frac{\pi}{8} \right) \pi - \frac{\pi^2}{8} \right] + \right. \]

\[ \sin \frac{\beta}{2} \left[ \left( -\frac{1}{6} + \frac{\pi}{4} \right) + \left( \frac{1}{4} - \frac{\pi}{4} \right) + \left( -\frac{1}{12} - \frac{\pi}{8} \right) \pi + \frac{\pi^2}{8} + \frac{8}{9} \right] \} . \]

\[ F^{(4)}(\pi, \beta) = \frac{N_0}{24} \left[ \cos \frac{\beta}{2} \left( \frac{1}{12} - \frac{\pi}{12} \right) + \sin \frac{\beta}{2} \left( \frac{1}{12} - \frac{\pi}{12} \right) + \frac{8}{9} \sin \frac{\beta}{2} \right] . \]

From equation (24):

\[ N^{(3)}_\beta = \frac{N_0}{24} \left[ \left( -\frac{10}{3} - \pi + 2\beta \right) \sin \frac{\beta}{2} + \left( \frac{10}{3} - \pi + 2\beta \right) \cos \frac{\beta}{2} + \frac{4}{3} \cos \beta \right] . \]

Then, from equation (8):

\[ N^{(4)}_\beta = 2F^{(4)}(\beta, \beta) - F^{(4)}(0, \beta) - F^{(4)}(\pi, \beta) + N^{(3)}_\beta \]

\[ = \frac{N_0}{24} \left[ \cos \frac{\beta}{2} \left( -\frac{2\beta}{3} + \frac{\pi \beta}{4} - \frac{1}{2} - \frac{\beta^2}{4} + \frac{1}{4} + \frac{8}{9} - \frac{1}{12} + \frac{\pi}{12} + \frac{10}{3} - \pi + 2\beta \right) \right] . \]
\[ + \sin \frac{\beta}{2} \left( \frac{1}{6} + \frac{\pi}{2} - \frac{2\beta}{3} - \frac{\pi\beta}{4} + \frac{\beta^2}{4} + \frac{1}{4} - \frac{1}{12} + \frac{\pi}{12} - \frac{8}{9} - \frac{10}{3} - \pi + 2\beta \right) - \frac{4}{9} \cos \beta \]

\[ N_\beta^{(4)} = \frac{N}{24} \left[ \cos \frac{\beta}{2} \left( \frac{8}{9} + \frac{4\beta}{3} + \frac{\pi\beta}{4} - \frac{\beta^2}{4} - \frac{11\pi}{12} \right) \right. \]

\[ + \sin \frac{\beta}{2} \left( \frac{8}{9} + \frac{4\beta}{3} + \frac{\pi\beta}{4} + \frac{\beta^2}{4} - \frac{11\pi}{12} \right) - \frac{4}{9} \cos \beta \]  

(26)

From equation (9):

\[ N_\alpha^{(4)} = - \frac{d}{d\beta} N_\beta^{(4)} \bigg|_{\beta=\alpha} = - \frac{N}{24} \left[ \cos \frac{\beta}{2} \left( \frac{4}{3} + \frac{\pi}{4} - \frac{2\beta}{4} \right) \right. \]

\[ + \frac{1}{2} \cos \frac{\beta}{2} \left( \frac{8}{9} + \frac{4\beta}{3} + \frac{\pi\beta}{4} - \frac{\beta^2}{4} - \frac{5\pi}{12} \right) \]

\[ + \left( - \frac{1}{2} \sin \frac{\beta}{2} \right) \left( \frac{8}{9} + \frac{4\beta}{3} + \frac{\pi\beta}{4} - \frac{\beta^2}{4} - \frac{11\pi}{12} \right) \]

\[ + \sin \frac{\beta}{2} \left( \frac{4}{3} - \frac{\pi}{4} + \frac{\beta}{2} \right) + \frac{4}{9} \sin \beta \]  

\[ N_\alpha^{(4)} = \frac{N}{24} \left[ \left( \frac{11}{18} - \frac{11}{24} - \frac{\pi}{6} - \frac{\pi\alpha}{8} - \frac{\alpha^2}{8} \right) \cos \frac{\alpha}{2} + \left( \frac{11}{18} - \frac{5\pi}{24} + \frac{\alpha}{6} + \frac{m\alpha}{8} - \frac{\alpha^2}{8} \right) \sin \frac{\alpha}{2} \right. \]

\[ - \frac{4}{9} \sin \alpha \]  

(27)
4. Fifth Collision

From equations (7) and (27):

\[ p^{(5)}(\alpha, \beta) = - \frac{N}{48} \int \left[ \left( - \frac{11}{18} + \frac{\pi}{24} + \frac{\alpha}{6} - \frac{\pi \alpha}{8} + \frac{\alpha^2}{8} \right) \cos \frac{\alpha}{2} \right. \]

\[ + \left( - \frac{11}{18} + \frac{5\pi}{24} - \frac{\alpha}{6} - \frac{\pi \alpha}{8} + \frac{\alpha^2}{8} \right) \sin \frac{\alpha}{2} + \frac{4}{9} \sin \alpha \right] \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \right. \]

\[ \left. + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha \]

\[ = - \frac{N}{48} \left\{ \cos \frac{\beta}{2} \left[ \left( - \frac{11}{18} + \frac{\pi}{24} \right) \left( \frac{\alpha}{2} + \frac{1}{2} \sin \alpha \right) + \left( \frac{1}{6} - \frac{\pi}{8} \right) \left( \frac{\alpha^2}{4} + \frac{\alpha \sin \alpha}{2} + \cos \frac{\alpha}{2} \right) \right] \right. \]

\[ + \frac{1}{8} \left( \frac{\alpha^3}{6} + \left( \frac{\alpha^2}{2} - 1 \right) \sin \alpha + \alpha \cos \alpha \right) + \left( - \frac{11}{18} + \frac{5\pi}{24} \right) \sin^2 \frac{\alpha}{2} \]

\[ - \left( \frac{1}{6} + \frac{\pi}{8} \right) \left( \frac{1}{2} \right) \left( \sin \alpha - \alpha \cos \alpha \right) + \frac{1}{8} \left( \sin \alpha + \cos \alpha - \frac{1}{2} \alpha^2 \cos \alpha \right) \]

\[ + \frac{4}{9} \left( - \frac{4}{3} \cos^3 \frac{\alpha}{2} \right) \right] + \sin \frac{\beta}{2} \left[ \left( - \frac{11}{18} + \frac{\pi}{24} \right) \sin^2 \frac{\alpha}{2} + \left( \frac{1}{6} - \frac{\pi}{8} \right) \left( \frac{1}{2} \right) \right. \]

\[ \left. \cdot \left( \sin \alpha - \alpha \cos \alpha \right) + \frac{1}{8} \left( \alpha \sin \alpha + \cos \alpha - \frac{1}{2} \alpha^2 \cos \alpha \right) \right) \]

\[ + \left( - \frac{11}{18} + \frac{5\pi}{24} \right) \left( \frac{\alpha}{2} - \frac{1}{2} \right) \sin \alpha - \left( \frac{1}{6} + \frac{\pi}{8} \right) \left( \frac{\alpha^2}{4} - \frac{\alpha \sin \alpha}{2} - \frac{\cos \alpha}{2} \right) \]
\[ + \frac{1}{8} \left( \frac{\alpha^3}{6} - \left( \frac{\alpha^2}{2} - 1 \right) \sin \alpha - \alpha \cos \alpha \right) + \frac{4}{9} \left( \frac{4}{3} \sin^3 \frac{\alpha}{2} \right) \}\]

\[ F^{(5)}(\alpha, \beta) = -\frac{N}{48} \left\{ \cos \frac{\beta}{2} \left[ \left( -\frac{11}{36} + \frac{\pi}{48} \right) \alpha + \left( \frac{1}{24} - \frac{\pi}{32} \right) \alpha^2 \right] \right. \]

\[ + \left( -\frac{37}{72} + \frac{5\alpha}{24} - \frac{\pi \alpha}{16} - \frac{\pi}{24} + \frac{\alpha^2}{16} \right) \sin \alpha + \left( \frac{5}{24} + \frac{5\alpha}{24} + \frac{\pi \alpha}{16} - \frac{\pi}{16} - \frac{\alpha^2}{16} \right) \cos \alpha \]

\[ + \frac{\alpha^2}{48} + \left( -\frac{11}{18} + \frac{5\pi}{24} \right) \sin^2 \alpha - \frac{16}{27} \cos^3 \frac{\alpha}{2} \left[ \sin \frac{\beta}{2} \right] \left[ \left( -\frac{11}{36} + \frac{5\pi}{48} \right) \alpha \right] \]

\[ + \left( -\frac{1}{24} - \frac{\pi}{32} \right) \alpha^2 + \left( \frac{37}{72} + \frac{5\alpha}{24} + \frac{\pi \alpha}{16} - \frac{\pi}{6} - \frac{\alpha^2}{16} \right) \sin \alpha \]

\[ + \left( \frac{5}{24} - \frac{5\alpha}{24} + \frac{\pi \alpha}{16} + \frac{\pi}{16} - \frac{\alpha^2}{16} \right) \cos \alpha + \frac{\alpha^3}{48} + \left( -\frac{11}{18} + \frac{\pi}{24} \right) \sin^2 \frac{\alpha}{2} \]

\[ + \left. \frac{16}{27} \sin^3 \frac{\alpha}{2} \right] \}

\[ F^{(5)}(\beta, \beta) = -\frac{N}{48} \left\{ \cos \frac{\beta}{2} \left[ \left( -\frac{11}{36} + \frac{\pi}{48} \right) \beta + \left( \frac{1}{24} - \frac{\pi}{32} \right) \beta^2 \right] \right. \]

\[ + \left( -\frac{37}{36} + \frac{5\beta}{12} - \frac{\pi \beta}{8} - \frac{\pi}{12} + \frac{\beta^2}{8} \right) \sin \frac{\beta}{2} \cos \frac{\beta}{2} + \left( \frac{5}{24} + \frac{5\beta}{24} + \frac{\pi \beta}{16} - \frac{\pi}{16} - \frac{\beta^2}{16} \right) \]
\[
\cdot \left( 1 - 2 \sin^2 \frac{\beta}{2} \right) + \frac{\beta^3}{48} + \left( -\frac{11}{18} + \frac{5\pi}{24} \right) \sin^2 \frac{\beta}{2} - \frac{16}{27} \cos \frac{\alpha}{2} \left( 1 - \sin^2 \frac{\alpha}{2} \right)
\]

\[
+ \sin \frac{\beta}{2} \left[ \left( -\frac{11}{36} + \frac{5\pi}{48} \right) \beta + \left( -\frac{1}{24} - \frac{\pi}{32} \right) \beta^2 + \left( \frac{37}{36} + \frac{5\beta}{12} + \frac{\pi\beta}{8} - \frac{3}{8} - \frac{\beta^3}{8} \right) \right]
\]

\[
+ \sin \frac{\beta}{2} \cos \frac{\beta}{2} \left( \frac{5}{24} - \frac{5\beta}{24} + \frac{\pi\beta}{16} + \frac{\pi}{16} - \frac{\beta^2}{16} \right) \left( 2 \cos^2 \frac{\beta}{2} - 1 \right) + \frac{\beta^3}{48}
\]

\[
+ \left( -\frac{11}{18} + \frac{\pi}{24} \right) \left( 1 - \cos^2 \frac{\beta}{2} \right) + \frac{16}{27} \sin \frac{\beta}{2} \left( 1 - \cos^2 \frac{\beta}{2} \right) \right] \cdot
\]

\[
F^{(5)}(\beta, \beta) = -\frac{N}{48} \left\{ \cos \frac{\beta}{2} \left[ \frac{5}{24} - \frac{\pi}{16} + \left( \frac{1}{12} - \frac{7}{72} \right) \beta - \left( \frac{1}{48} + \frac{\pi}{32} \right) \beta^2 + \frac{1}{48} \beta^3 \right]
\]

\[
+ \sin \frac{\beta}{2} \left[ -\frac{59}{72} - \frac{\pi}{48} + \left( \frac{\pi}{24} - \frac{7}{72} \right) \beta + \left( \frac{1}{48} - \frac{\pi}{32} \right) \beta^2 + \frac{1}{48} \beta^3 \right] - \frac{16}{27} \left( \cos^2 \frac{\beta}{2} - \sin^2 \frac{\beta}{2} \right) \right\}
\]

\[
F^{(5)}(0, \beta) = -\frac{N}{48} \left\{ \cos \frac{\beta}{2} \left[ \frac{5}{24} - \frac{\pi}{16} - \frac{16}{27} \right] + \sin \frac{\beta}{2} \left[ \frac{5}{24} + \frac{\pi}{16} \right] \right\} \cdot
\]

\[
F^{(5)}(\pi, \beta) = -\frac{N}{48} \left\{ \cos \frac{\beta}{2} \left[ \left( -\frac{11}{36} + \frac{\pi}{48} \right) \pi + \left( \frac{1}{24} - \frac{\pi}{32} \right) \pi^2 \right.
\]

\[
- \frac{5}{24} + \frac{\pi}{16} + \frac{\pi^3}{48} - \frac{11}{18} \right] + \sin \frac{\beta}{2} \left[ \left( -\frac{11}{36} + \frac{5\pi}{48} \right) \pi + \right.
\]
\[
F^{(5)}(\pi, \beta) = -\frac{N_0}{48} \left\{ \cos \frac{\beta}{2} \left( -\frac{59}{72} - \frac{35\pi}{144} + \frac{\pi^2}{16} - \frac{\pi^3}{96} \right) + \sin \frac{\beta}{2} \right.
\]
\[
\left( -\frac{49}{216} - \frac{17}{144} + \frac{\pi^2}{16} - \frac{\pi^3}{96} \right) \}
\]

From equation (26):

\[
N_\beta^{(4)} = \frac{N_0}{48} \left[ \cos \frac{\beta}{2} \left( \frac{70}{9} \beta + \frac{8}{3} \beta^2 - \frac{\pi^2}{2} - \frac{11\pi}{6} \right) \right.
\]
\[
+ \sin \frac{\beta}{2} \left( -\frac{70}{9} \beta + \frac{8}{3} \beta^2 + \frac{\pi^2}{2} - \frac{5\pi}{6} \right) - \frac{8}{9} \cos \beta \}
\]

Then, from equation (8):

\[
N_\beta^{(5)} = 2F^{(5)}(\beta, \beta) - F^{(5)}(0, \beta) - F^{(5)}(\pi, \beta) + N_\beta^{(4)}
\]
\[
= \frac{N_0}{48} \left\{ -\cos \frac{\beta}{2} \left[ \frac{5}{12} - \frac{\pi}{8} + \left( \frac{\pi}{6} - \frac{7}{36} \right) \right] \beta - \left( \frac{1}{24} + \frac{\pi}{16} \right) \beta^2 + \frac{1}{24} \beta^3 \right.
\]
\[
- \frac{5}{24} + \frac{\pi}{16} + \frac{16}{27} + \frac{59}{72} + \frac{35\pi}{144} - \frac{\pi^2}{16} + \frac{\pi^3}{96} - \left( \frac{70}{9} + \frac{8}{3} \beta + \frac{\pi^2}{2} - \frac{\pi^3}{6} \right) \}
\]
- \sin \frac{\beta}{2} \left[ -\frac{59}{36} - \frac{\pi}{24} + \left( \frac{\pi}{12} - \frac{7}{36} \right) \beta + \left( \frac{1}{24} - \frac{\pi}{16} \right) \beta^2 + \frac{1}{24} \beta^3 - \frac{5}{24} - \frac{\pi}{16} \right]

+ \frac{49}{216} + \frac{17\pi}{144} - \frac{\pi^2}{16} + \frac{\pi^3}{96} - \left( -\frac{70}{9} + \frac{8}{3} \beta - \frac{\pi \beta}{2} + \frac{\beta^2}{2} - \frac{5\pi}{6} \right) + \frac{32}{27} \cos \beta - \frac{8}{9} \cos \beta \right] .

N^{(5)}_\beta = \frac{N_0}{48} \left[ \cos \frac{\beta}{2} \left( \frac{617}{108} - \frac{145\pi}{72} + \frac{\pi^2}{96} + \frac{103\beta}{36} + \frac{\pi \beta}{3} - \frac{11\beta^2}{24} + \frac{\pi \beta^2}{16} - \frac{\beta^3}{24} \right) \right]

+ \sin \frac{\beta}{2} \left( -\frac{617}{108} - \frac{61\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} - \frac{7\pi \beta}{12} + \frac{11\beta^2}{24} + \frac{\pi \beta^2}{16} - \frac{\beta^3}{24} \right) + \frac{8}{27} \cos \beta \right] . \quad (28)

From equation (9):

\[ N^{(5)}_\beta = -\frac{d}{d\beta} N^{(5)}_\beta \left. \right|_{\beta=\alpha} \]

= -\frac{N_0}{48} \left[ \cos \frac{\beta}{2} \left( \frac{103}{36} + \frac{\pi}{3} - \frac{22\beta}{24} + \frac{2\pi \beta}{16} - \frac{3\beta^2}{24} \right) \right]

+ \frac{1}{2} \cos \frac{\beta}{2} \left( -\frac{617}{108} - \frac{61\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} - \frac{7\pi \beta}{12} + \frac{11\beta^2}{24} + \frac{\pi \beta^2}{16} - \frac{\beta^3}{24} \right)

+ \frac{1}{2} \left( -\sin \frac{\beta}{2} \left( \frac{145}{72} + \frac{\pi^2}{96} + \frac{103\beta}{36} + \frac{\pi \beta}{3} - \frac{11\beta^2}{24} - \frac{\beta^3}{24} + \frac{\pi \beta^2}{16} \right) \right)

+ \sin \frac{\beta}{2} \left( \frac{103}{36} - \frac{7\pi}{12} + \frac{22\beta}{24} + \frac{2\pi \beta}{16} - \frac{3\beta^2}{24} \right) - \frac{8}{27} \sin \beta \left. \right|_{\beta=\alpha} . \quad (28)
D. Summary of Expressions for Distribution over a Semicircular Surface at the \textit{n}th Collision

1. General Expressions

\[ F^{(n)}(\alpha, \beta) = \frac{1}{2} \int_{\alpha} N^{(n-1)}(\alpha) \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha. \quad (7) \]

\[ N^{(n)}_{\beta} = 2F^{(n)}(\beta, \beta) - F^{(n)}(0, \beta) - F^{(n)}(\pi, \beta) + N^{(n-1)}_{\beta} \quad (8) \]

\[ N^{(n)}_{\alpha} = -\frac{d}{d\beta} N^{(n)}_{\beta} \quad \bigg|_{\beta=\alpha} \quad (9) \]

2. Incident Distribution Uniform over the Surface

\[ N^{(1)}_{\alpha} = \frac{N_0}{\pi} \quad (10) \]

\[ N^{(1)}_{\beta} = \frac{N_0}{\pi} \left( \frac{\pi}{2} - \beta \right). \quad (11) \]

\[ N^{(2)}_{\beta} = \frac{N_0}{\pi} \left( \frac{\pi}{2} - \beta - \cos \frac{\beta}{2} + \sin \frac{\beta}{2} \right) \quad (12) \]
\[ N^{(2)}_{\alpha} = \frac{N_0}{2\pi} \left( 2 \sin \frac{\alpha}{2} - \cos \frac{\alpha}{2} \right). \]  \hspace{1cm} (13)

\[ N^{(3)}_{\beta} = \frac{N_0}{4\pi} \left[ 2\pi - 4\beta + \left( \frac{\pi}{2} - \beta + 7 \right) \sin \frac{\beta}{2} + \left( \frac{\pi}{2} - \beta - 7 \right) \cos \frac{\beta}{2} \right]. \]  \hspace{1cm} (14)

\[ N^{(3)}_{\alpha} = \frac{N_0}{8\pi} \left[ 8 - \left( \frac{\pi}{2} - \alpha + 5 \right) \cos \frac{\alpha}{2} + \left( \frac{\pi}{2} - \alpha - 5 \right) \sin \frac{\alpha}{2} \right]. \]  \hspace{1cm} (15)

\[ N^{(4)}_{\beta} = \frac{N_0}{16\pi} \left[ 8\pi - 16\beta + \left( 38 + \frac{7\pi}{2} - 8\beta + \frac{\pi^2}{2} - \frac{\beta^2}{2} \right) \sin \frac{\beta}{2} + \left( -38 + \frac{9\pi}{2} - 8\beta - \frac{\pi^2}{2} + \frac{\beta^2}{2} \right) \cos \frac{\beta}{2} \right]. \]  \hspace{1cm} (16)

\[ N^{(4)}_{\alpha} = \frac{N_0}{32\pi} \left[ 32 - \left( 22 + \frac{5\pi}{2} - 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2} \right) \cos \frac{\alpha}{2} \right. \]
\[ - \left( 22 - \frac{7\pi}{2} + 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2} \right) \sin \frac{\alpha}{2} \right]. \]  \hspace{1cm} (17)

\[ N^{(5)}_{\beta} = \frac{N_0}{64\pi} \left[ 32\pi - 64\beta + \left( 187 + \frac{37\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta + 5\pi\beta \right. \right. \]
\[ - \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} \left] \sin \frac{\beta}{2} + \left( 187 + \frac{57\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta - 4\pi\beta \right. \right. \]
\[ + \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} \left] \cos \frac{\beta}{2} \right]. \]  \hspace{1cm} (18)
\[ N^{(5)}_{\alpha} = \frac{N_0}{128\pi} \left[ 128 + \left( -93 - \frac{21\pi}{2} + \frac{\pi^2}{4} - \frac{\pi^3}{24} + 29\alpha - 4\pi\alpha + \frac{7\alpha^2}{4} + \frac{\pi\alpha^2}{4} - \frac{\alpha^3}{6} \right) \right. \\
\left. \quad \cdot \cos \frac{\alpha}{2} + \left( -93 + \frac{37\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 29\alpha - 3\pi\alpha + \frac{7\alpha^2}{2} \right) \right. \\
\left. \quad - \frac{\pi\alpha^2}{4} + \frac{\alpha^3}{6} \right] \sin \frac{\alpha}{2}. \] (19)

3. Incident Distribution Uniform across the Y-Axis

\[ N^{(1)}_{\alpha} = \frac{1}{2} N_0 \sin \alpha. \] (20)

\[ N^{(1)}_{\beta} = \frac{1}{2} N_0 \cos \beta. \] (21)

\[ N^{(2)}_{\beta} = \frac{N_0}{3} \left( \cos \frac{\beta}{2} - \sin \frac{\beta}{2} - \frac{1}{2} \cos \beta \right) \] (22)

\[ N^{(2)}_{\alpha} = \frac{N_0}{6} \left( \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} - \sin \alpha \right) \] (23)

\[ N^{(3)}_{\beta} = \frac{N_0}{12} \left[ \left( \beta - \frac{\pi}{2} - \frac{5}{3} \right) \sin \frac{\beta}{2} + \left( \beta - \frac{\pi}{2} + \frac{5}{3} \right) \cos \frac{\beta}{2} + \frac{2}{3} \cos \beta \right] \] (24)

\[ N^{(3)}_{\alpha} = \frac{N_0}{12} \left[ \left( -\frac{1}{6} - \frac{\pi}{4} + \frac{\alpha}{2} \right) \sin \frac{\alpha}{2} + \left( -\frac{1}{6} + \frac{\pi}{4} - \frac{\alpha}{2} \right) \cos \frac{\alpha}{2} + \frac{2}{3} \sin \alpha \right] \] (25)
\[
N_{\beta}^{(4)} = \frac{N_0}{24} \left[ \left( -\frac{35}{9} - \frac{5\pi}{12} + \frac{4\beta}{3} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{4} \right) \sin \frac{\beta}{2} + \left( \frac{35}{9} - \frac{11\pi}{12} + \frac{4\beta}{3} + \frac{\pi\beta}{4} \right) \cos \frac{\beta}{2} - \frac{\beta^2}{4} \right] - \frac{4}{9} \cos \beta \right]. 
\]

(26)

\[
N_{\alpha}^{(4)} = \frac{N_0}{24} \left[ \left( \frac{11}{18} - \frac{\pi}{24} - \frac{\alpha}{6} + \frac{\pi\alpha}{8} - \frac{\alpha^2}{8} \right) \cos \frac{\alpha}{2} + \left( \frac{11}{18} - \frac{5\pi}{24} + \frac{\alpha}{6} + \frac{\pi\alpha}{8} \right) \sin \frac{\alpha}{2} - \frac{\alpha^2}{8} \sin \alpha \right].
\]

(27)

\[
N_{\beta}^{(5)} = \frac{N_0}{48} \left[ \left( \frac{665}{108} - \frac{61\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta^2}{36} - \frac{7\pi\beta}{12} + \frac{11\beta^2}{24} + \frac{\pi\beta^2}{16} - \frac{\beta^3}{24} \right) \sin \frac{\beta}{2} + \left( \frac{665}{108} - \frac{145\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta^2}{36} + \frac{\pi\beta^2}{3} - \frac{11\beta^2}{24} + \frac{\pi\beta^2}{16} - \frac{\beta^3}{24} \right) \cos \frac{\beta}{2} + \frac{8}{27} \cos \beta \right].
\]

(28)

\[
N_{\alpha}^{(5)} = \frac{N_0}{48} \left[ \left( \frac{47}{216} + \frac{13\pi}{144} - \frac{\pi^2}{32} + \frac{\pi^3}{192} - \frac{37\alpha}{72} + \frac{\pi\alpha}{6} - \frac{5\alpha^2}{48} - \frac{\pi\alpha^2}{32} + \frac{\alpha^3}{48} \right) \cos \frac{\alpha}{2} + \left( \frac{47}{216} - \frac{61\pi}{144} + \frac{\pi^2}{32} - \frac{\pi^3}{192} + \frac{37\alpha}{72} - \frac{\pi\alpha}{24} - \frac{5\alpha^2}{48} - \frac{\alpha^3}{32} \right) \sin \frac{\alpha}{2} + \frac{8}{27} \sin \alpha \right].
\]

(29)
For any given geometry, the number of molecules per radian that collide with the surface at an angle $\alpha$, measured from the negative $y$-axis, is:

$$N_\alpha \equiv \frac{dN}{d\alpha} = \frac{dN}{dy} \cdot \frac{dy}{d\alpha}.$$

Assuming a uniform incident flux across the $y$-axis: $dN/dy = \text{constant} = k$, where the constant $k$ is easily determined from the condition that a total of $N_0$ molecules enters the geometry across the opening from $y = -1$ to $y = +1$. Then

$$N_0 = \int_{-1}^{+1} kdy = 2k,$$

or

$$k = \frac{1}{2} N_0,$$

so that:

$$N_\alpha = \frac{1}{2} N_0 \frac{dy}{d\alpha}. \quad (30)$$

The derivative $dy/d\alpha$ will depend on the geometry of interest.

**A. Ellipse**

Consider now a semi-elliptical surface with its opening, two units wide, along the $y$-axis (Fig. 4). The general equation for such an ellipse is:

$$\frac{x^2}{A^2} + y^2 = 1. \quad (31)$$
(Note that when \( A = 1 \), this is the equation for the circle which was treated in the previous section). From Figure 4, for \( \alpha > \pi/2 \), \( \tan(\alpha - \pi/2) = y/x \), and, since \( \tan(\alpha - \pi/2) = -\cot \alpha \), this may be written

\[ y = x \cot \alpha. \]

For \( \alpha < \pi/2 \): \( \tan \alpha' = -\frac{x}{y} = \frac{x}{y} \), so that, in both cases;

\[ y = x \cot \alpha. \tag{32} \]

and

\[ x = y \tan \alpha. \tag{33} \]

Using equation (33) in equation (31):

\[ \frac{y^2 \tan^2 \alpha}{A^2} + y^2 = 1, \]
or

\[ y^2 \left[ \frac{\tan^2 \alpha}{A^2} + 1 \right] = 1 \]

and

\[ y = \frac{\pm A}{(\tan^2 \alpha + A^2)^{1/2}} = \frac{\pm A \cot \alpha}{(1 + A^2 \cot^2 \alpha)^{1/2}} \]  

(34)

But, when \( \cot \alpha \geq 0 \), \( y \leq 0 \) and when \( \cot \alpha \leq 0, y \geq 0 \), so:

\[ y = -A \cot \alpha(1 + A^2 \cot^2 \alpha)^{-1/2} \]

Then

\[ \frac{dy}{d\alpha} = (-A \cot \alpha) (-\frac{1}{2}) \cdot (1 + A^2 \cot^2 \alpha)^{-3/2} (2A^2 \cot \alpha) (-\csc^2 \alpha) \]

\[ + A \csc^2 \alpha (1 + A^2 \cot^2 \alpha)^{-1/2} \]

\[ = \csc^2 \alpha \left[ \frac{y^3}{\cot \alpha} - \frac{y}{\cot \alpha} \right] = \frac{y \csc^2 \alpha}{\cot \alpha} (y^2 - 1). \]

But, from the equation for the ellipse: \( y^2 - 1 = -x^2/A^2 \), so that:

\[ \frac{dy}{d\alpha} = \frac{y \csc^2 \alpha}{\cot \alpha} \left( \frac{-x^2}{A^2} \right). \]

Also

\[ y/\cot \alpha = x, \]
so that
\[
\frac{dy}{d\alpha} = \frac{-x^3}{A^2 \sin^2 \alpha}
\]
But,
\[
x^3 = y^3 \tan^3 \alpha,
\]
so that
\[
\frac{dy}{d\alpha} = \frac{-y^3 \tan^3 \alpha}{A^2 \sin^2 \alpha}
\]
and
\[
y^3 = -A^3 \cot^3 \alpha (1 + A^2 \cot^2 \alpha)^{-3/2},
\]
so that
\[
\frac{dy}{d\alpha} = A^3 \cot^3 \alpha \frac{(1 + A^2 \cot^2 \alpha)^{-3/2} \tan^3 \alpha}{A^2 \sin^2 \alpha}
\]
\[
= \frac{A}{\sin^2 \alpha (1 + A^2 \cot^2 \alpha)^{3/2}}.
\]

Multiplying numerator and denominator by \(\sin \alpha\) and using equation (30), the final expression becomes:

\[
N_\alpha = \frac{1}{2} N_0 A \sin \alpha (\sin^2 \alpha + A^2 \cos^2 \alpha)^{-3/2}.
\]  

(35)

B. Parabola

Consider next a section of a parabola with its opening, two units wide, along the y-axis. See Figure 5. The equation for this parabola is:

\[
y^2 = 4Px + 1,
\]  

(36)
with the region of interest being for \( x \leq 0 \) and \( |y| \leq 1 \). Note that \( y = \pm 1 \) when \( x = 0 \) and \( y = 0 \) when \( x = -\frac{1}{4P} \). As in the case of the ellipse:

\[
y = x \cot \alpha. \tag{37}
\]

Then, from equation (36):

\[
x^2 \cot^2 \alpha - 4Px - 1 = 0,
\]

so that

\[
x = \frac{4P \pm (16P^2 + 4 \cot^2 \alpha)^{1/2}}{2 \cot^2 \alpha} = \tan^2 \alpha \left[ 2P \pm (4P^2 + \cot^2 \alpha)^{1/2} \right],
\]

but, since \( x \leq 0 \), \( \tan^2 \alpha \geq 0 \), and \( 2P \geq 0 \) for all \( \alpha \) in the region of interest, the positive square root is discarded, and:

\[
x = \tan^2 \alpha \left[ 2P - (4P^2 + \cot^2 \alpha)^{1/2} \right] \tag{38}
\]
so that

\[ y = \tan \alpha \left[ 2P - (4P^2 + \cot^2 \alpha)^{1/2} \right]. \]

Then

\[ \frac{dy}{d\alpha} = \tan \alpha \left[ -\frac{1}{2} 4P^2 + \cot^2 \alpha - \frac{1}{2} (2 \cot \alpha \left( - \csc^2 \alpha \right) \right] \]

\[ + \sec^2 \alpha \left[ 2P - (4P^2 + \cot^2 \alpha)^{1/2} \right] \]

\[ \frac{dy}{d\alpha} = \frac{1}{\sin^2 \alpha (4P^2 + \cot^2 \alpha)^{1/2}} + \frac{2P - (4P^2 + \cot^2 \alpha)^{1/2}}{\cos^2 \alpha} \]

\[ = \frac{\cos^2 \alpha + 2P \sin^2 \alpha (4P^2 + \cot^2 \alpha)^{1/2} - \sin^2 \alpha (4P^2 + \cot^2 \alpha)}{\cos^2 \alpha \sin^2 \alpha (4P^2 + \cot^2 \alpha)^{1/2}} \]

\[ \frac{dy}{d\alpha} = \frac{2P}{\cos^2 \alpha} \left[ 1 - \frac{2P}{(4P^2 + \cot^2 \alpha)^{1/2}} \right] \]

(39)

\[ N_\alpha = \frac{dN}{d\alpha} = \frac{dN}{dy} \frac{dy}{d\alpha} = \frac{N_0 P}{\cos^2 \alpha} \left[ 1 - \frac{2P}{(4P^2 + \cot^2 \alpha)^{1/2}} \right]. \]

(40)

All these expressions are valid throughout the region of interest, except at \( \alpha = 0, \pi \) and \( \pi/2 \). At \( \alpha = 0, \pi \): \( \cot \alpha = \infty \), so that \( x \) and \( y \) are not determined from the equations given above. However, these are just the points at which \( x = 0 \) and \( y = \pm 1 \). At \( \alpha = \pi/2 \) the situation is slightly more complicated, since \( N_\alpha \to 0/0 \) as \( \alpha \to \pi/2 \). Thus to evaluate \( N_\alpha \) at this point, \( \text{L'Hôpital's Rule may be used as follows}. \)

Rewriting equation (39):

\[ \frac{dy}{d\alpha} = \frac{2P}{\cos^2 \alpha \left( 4P^2 + \cot^2 \alpha \right)^{1/2} - 2P}. \]
Now let

\[ 2P \left[ (4P^2 + \cot^2 \alpha)^{\frac{1}{2}} - 2P \right] = f(\alpha) \]

\[ \cos^2 \alpha (4P^2 + \cot^2 \alpha)^{\frac{1}{2}} \equiv g(\alpha) \]

and

\[ (4P^2 + \cot^2 \alpha)^{\frac{1}{2}} \equiv Q(\alpha). \]

Then,

\[ \frac{dy}{d\alpha} = \frac{f(\alpha)}{g(\alpha)} = \frac{2P (Q - 2P)}{Q \cos^2 \alpha} \]

Also

\[ Q'(\alpha) = \frac{1}{2} (4P^2 + \cot^2 \alpha)^{-\frac{1}{2}} (2 \cot \alpha) (- \csc^2 \alpha) = \cot \alpha Q \sin^2 \alpha \]

and

\[ Q''(\alpha) = \frac{\sin^2 \alpha (\csc^2 \alpha) + \cot \alpha (Q \cdot 2 \sin \alpha \cos \alpha + Q' \sin^2 \alpha)}{Q^2 \sin^4 \alpha} \]

\[ = \frac{Q + \cos \alpha (2Q \cos \alpha + Q' \sin \alpha)}{Q^2 \sin^4 \alpha} \]

so that

\[ Q(\pi/2) = 2P \]
\[ Q'(\pi/2) = 0 \]
\[ Q''(\pi/2) = \frac{2P}{4P^2} = \frac{1}{2P}. \]
Now

\[
\frac{dv}{d\alpha} = \lim_{\alpha \to \pi/2} \frac{f(\alpha)}{g(\alpha)} = \lim_{\alpha \to \pi/2} \frac{f'(\alpha)}{g'(\alpha)}
\]

by l'Hôpital's Rule.

\[
f'(\alpha) = 2PQ' \xrightarrow[\alpha \to \pi/2]{} 0
\]

since \(Q'(\pi/2) = 0\) and \(P < \infty\).

\[
g'(\alpha) = Q' \cos^2 \alpha + 2 \cos \alpha (-\sin \alpha) \xrightarrow[\alpha \to \pi/2]{} Q \to 0
\]

since \(Q'(\pi/2) = 0\) and \(\cos \pi/2 = 0\). Hence,

\[
\lim_{\alpha \to \pi/2} \frac{f'(\alpha)}{g'(\alpha)} = 0
\]

and l'Hôpital's Rule may be used again.

\[
f''(\alpha) = 2PQ'' \xrightarrow[\alpha \to \pi/2]{} \frac{2P}{2P} = 1
\]

\[
g''(\alpha) = -2Q' \cos \alpha \sin \alpha + Q'' \cos^2 \alpha - 2Q (\cos^2 \alpha - \sin^2 \alpha) -
\]

\[-2Q' \cos \alpha \sin \alpha \xrightarrow[\alpha \to \pi/2]{} 0 + 0 - 4P(0 - 1) - 0 = 4P
\]

so that

\[
\frac{dv}{d\alpha} = \frac{1}{4P}
\]

\[
\alpha = \frac{\pi}{2}
\]

and

\[
N_\alpha = \frac{N_0}{8P} \quad \text{for} \quad \alpha = \pi/2.
\]

(41)
SECTION IV. DISTRIBUTION OF MOLECULES ACROSS THE Y-AXIS UPON EXIT FROM ARRAY

A. Circular Array

1. General Expression

From Section II, page 4, the number of molecules per radian reflected from a point \( P_\alpha \) at an angle \( \phi \) with the normal is:

\[
N_{\alpha, \phi} = \frac{1}{2} N_{\alpha} \cos \phi.
\]

Now the total number of molecules per radian from \( P_\alpha \) that passes through a section of the opening from the origin \((0, 0)\) to point \((0, z)\) is (see Fig. 6):

\[
N_{z,\alpha} = \int_{0}^{\delta} N_{\alpha, \phi} \, d\phi = \int_{0}^{\delta} \frac{1}{2} N_{\alpha} \cos \phi \, d\phi = \frac{1}{2} N_{\alpha} \sin \delta.
\]
and the total number that passes through this section from all points on the surface is:

\[ N_z = \int_0^\pi N_{z,\alpha} \, d\alpha = \int_0^\pi \frac{1}{2} N_\alpha \sin \delta \, d\alpha. \]  

(42)

Thus, to calculate \( N_z \), the initial distribution over the surface, \( N_\alpha \), and \( \sin \delta \) must be expressed in terms of \( \alpha \).

In the case of the circle of unit radius, the calculation of \( \sin \delta \) is especially simple. From Figure 6:

\[ \frac{\sin \delta}{z} = \frac{\sin (\pi - \alpha)}{b} \]

or

\[ \sin \delta = \frac{z \sin \alpha}{b}. \]

Also

\[ b = \left(1 + z^2 - 2z \cos (\pi - \alpha)\right)^\frac{1}{2}, \]

and

\[ \cos (\pi - \alpha) = -\cos \alpha \]

so that

\[ b = \left(1 + z^2 + 2z \cos \alpha\right)^\frac{1}{2}. \]

Then

\[ \sin \delta = \frac{z \sin \alpha}{\left(1 + z^2 + 2z \cos \alpha\right)^\frac{1}{2}}, \]
so that the expression for the distribution, equation (42), becomes:

\[ N_z = \frac{z}{2} \int_0^\pi \frac{N_\alpha \sin \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{3}{2}}} \]  

(43)

2. Incident Distribution Uniform over the Surface

In this case, \( N_\alpha = \frac{N_\alpha}{\pi} = \text{constant} \), so that:

\[ N_z = \frac{N_0 z}{2\pi} \int_0^\pi \frac{\sin \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{3}{2}}} \]

\[ = \frac{N_0 z}{2\pi} \left[ -\frac{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}}{z} \right]_0^\pi \]

\[ = -\frac{N_0}{2\pi} \left[ (1 + z^2 - 2z)^{\frac{1}{2}} - (1 + z^2 + 2z)^{\frac{1}{2}} \right] \]

\[ = -\frac{N_0}{2\pi} \left[ \pm (1 - z) - \pm (1 + z) \right] \]

But \((1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}} \equiv b\), which is a distance, and therefore positive. Since \(0 \leq z \leq 1\), only the positive square roots are meaningful, so that:

\[ N_z = \frac{N_0 z}{\pi}. \]

Thus the exit distribution is uniform across the y-axis, for molecules exiting the array after the first collision.
3. Incident Distribution Uniform across the Y-Axis

In this case, from equation (21), page 54:

$$N_{\alpha} = \frac{1}{2} N_o \sin \alpha$$

so that, from equation (43):

$$N_z = \frac{1}{2} \int_0^\pi \frac{N_{\alpha} \sin \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}} = \frac{z}{2} \int_0^\pi \frac{\frac{1}{2} N_o \sin^2 \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}}$$

$$N_z = \frac{1}{4} N_o z \int_0^\pi \frac{\sin^2 \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}}.$$ (44)

This integral cannot be evaluated in closed form, but has been numerically integrated by a General Electric 225 Computer. The results of this calculation are given in Section VI of this report, for $N_o = 10,000$ and several values of $z$.

B. Elliptical Array

Consider now a beam of molecules incident from the right onto a semi-elliptical surface as shown in Figure 7. The calculation of the exit distribution of those molecules which exit the array after one collision will depend upon the relations of the variables $y$, $y_o$, and $z$, as follows.

Consider first the case shown in Figure 7, where $y_o \leq 0 \leq z \leq y$. In this case the number of molecules per radian from $P_\alpha$ that passes through a section of the opening from $(0, 0)$ to $(0, z)$ is given by:

$$N_{z,\alpha} = \frac{1}{2} \int_\delta^\epsilon N_{\alpha} \cos \varphi \, d\varphi = \frac{1}{2} N_{\alpha} (\sin \epsilon - \sin \delta)$$ (45)
where $N_{\alpha}$ is the incident distribution function defined in Section III, page 43. The total number of molecules that pass through this section from all points on the semi-elliptical surface would then be:

$$N_z = \frac{1}{2} \int_0^\pi N_{\alpha} (\sin \epsilon - \sin \delta) \, d\alpha,$$

(46)

however, care must be taken to consider several different cases, and to define the angles $\delta$ and $\epsilon$ more precisely. For example, in Figure 8, $\epsilon$ is less than $\delta$ so that equation (45) would yield a negative result for $N_{z,\alpha}$, the number of molecules passing through the section from $(0, 0)$ to $(0, z)$. This of course must be avoided.

Another ambiguity is introduced when $\sin \delta$ and $\sin \epsilon$ are expressed in terms of $\alpha$. From Figure 7:

$$\frac{\sin \epsilon}{z - y_0} = \frac{\sin \theta}{b}$$

or

$$\sin \epsilon = \left(\frac{z - y_0}{b}\right) \sin \theta.$$
But, in the case shown in Figure 9:

\[
\frac{\sin \varepsilon}{z - y_0} = \sin (\pi - (-\theta)) \frac{b}{b}
\]

or

\[
\sin \varepsilon = \frac{(z - y_0) \sin (-\theta)}{b}
\]

which is the negative of the first case.

Now, a careful examination of Figures 7, 8, and 9 shows that these three cases are the only ones that need be considered, as follows.

In the calculation of \(\sin \varepsilon\) and \(\sin \delta\), only the triangles containing these angles are important; and since the variable \(y\) is not a dimension of any of these triangles, its relationship to \(y_0\), \(z\) and the origin \((0, 0)\) need not be considered. This reduces the number of
possible cases to six, the number of permutations of three items 
(y₀, z, and 0). Furthermore, half of the remaining cases are for 
negative z. Since all the arrays considered in this study are sym-
metric with respect to the x-axis, all the results for negative z are 
identical to those for positive z, i.e., N₋z = N_z. It therefore remains 
to consider only the three cases shown in Figures 7, 8, and 9.

Case 1.

\[ y_0 \leq 0 \leq z \quad \text{(See Figure 7.)} \]

\[ N_{z,\alpha} = \frac{1}{2} N_{\alpha} \sin \epsilon \quad \text{and} \quad \sin \epsilon = \frac{(z - y_0) \sin \theta}{b} \]

and

\[ \sin \delta = -\frac{y_0 \sin \theta}{s} \]

so that

\[ N_{z,\alpha} = \frac{1}{2} N_{\alpha} \sin \theta \left[ \frac{(z - y_0) + y_0}{b} \right]. \quad (47) \]

Case 2.

\[ 0 \leq z \leq y_0 \quad \text{(See Figure 8.)} \]

\[ N_{z,\alpha} = \frac{1}{2} N_{\alpha} (\sin \delta - \sin \epsilon) \]

where now

\[ \sin \delta = \frac{y_0 \sin (-\theta)}{s} = -\frac{y_0 \sin \theta}{s} \]
and

\[
\sin \epsilon = \frac{(y_0 - z) \sin (-\theta)}{b} = \frac{(z - y_0) \sin \theta}{b}
\]

so that

\[
N_{z, \alpha} = \frac{1}{2} N_{\alpha} \sin \theta \left[ -\frac{y_0}{s} - \frac{(z - y_0)}{b} \right]
\]

which is the negative of the result for Case 1 above.

**Case 3.**

\(0 \leq y_0 \leq z\) (See Figure 9.)

\[
N_{z, \alpha} = \frac{1}{2} N_{\alpha} (\sin \delta + \sin \epsilon)
\]

where now

\[
\sin \delta = \frac{y_0 \sin (-\theta)}{s} = -\frac{y_0 \sin \theta}{s}
\]

and

\[
\sin \epsilon = \frac{(z - y_0) \sin (\pi - (-\theta))}{b} = \frac{(y_0 - z) \sin \theta}{b}
\]

so that

\[
N_{z, \alpha} = \frac{1}{2} N_{\alpha} \sin \theta \left[ -\frac{y_0}{s} + \frac{(y_0 - z)}{b} \right]
\]

which is the same as the result for Case 2.

To take care of the difference in sign between Cases 1, 2, and 3, it is necessary only to insert absolute value signs. Thus, the desired expression for the exit distribution is:
\[
N_Z = \frac{1}{2} \int_0^\pi N_{\alpha} \left| \left( \frac{y_0 - z}{b} - \frac{y_0}{s} \right) \sin \theta \right| \, d\alpha
\]  

(50)

where the quantities \(N_{\alpha}, y_0, b, s\) and \(\theta\) must now be expressed in terms of \(\alpha\) (\(z\) is the independent variable). The incident distribution, \(N_{\alpha}\), for an ellipse, has been calculated in Section III-A, page 43, for an incident flux uniform across the \(y\)-axis. It therefore remains only to calculate \(y_0, b, s, \) and \(\theta\):

From Figure 7:

\[\tan \theta = -\frac{x}{y - y_0}\]

and from Figure 8 or 9:

\[\tan (-\theta) = -\frac{x}{y_0 - y},\]

so that, in all cases:

\[y_0 = y + x \cot \theta.\]  

(51)

Now the equation for the ellipse is:

\[\frac{x^2}{A^2} + y^2 = 1\]

so that

\[\frac{2x}{A^2} \, dx + 2y \, dy = 0,\]

and

\[\frac{dy}{dx} = -\frac{x}{yA^2};\]

but

\[\frac{dy}{dx} = \tan \theta,\]
so that

\[ \tan \theta = -\frac{x}{yA^2} \]  \hspace{1cm} (52)

Also from Figure 7:

\[ \tan (\alpha - \frac{\pi}{2}) = \frac{y}{-x}, \]

but

\[ \tan (\alpha - \frac{\pi}{2}) = \frac{\sin (\alpha - \frac{\pi}{2})}{\cos (\alpha - \frac{\pi}{2})} = \frac{-\cos \alpha}{\sin \alpha} = -\cot \alpha, \]

so that

\[ \cot \alpha = \frac{y}{x}. \]

From Figure 8 or 9:

\[ \tan \alpha = -\frac{x}{-y}, \]

so that in all cases:

\[ \begin{align*}
  x &= y \tan \alpha \\
  y &= x \cot \alpha
\end{align*} \]  \hspace{1cm} (53)

Then from equation (52):

\[ \tan \theta = -\tan \alpha/A^2 \]

and

\[ \theta = \tan^{-1} \left( -\tan \alpha/A^2 \right) \]  \hspace{1cm} (54)
so that, from equations (53) and (51):

\[ y_o = y + y \tan \alpha \cot \theta \]

\[ = y \left\{ 1 + \tan \alpha \cdot \cot \left[ \tan^{-2} (-\tan \alpha/A^2) \right] \right\} \]

\[ = y \left\{ 1 + \tan \alpha \left( \frac{A^2}{-\tan \alpha} \right) \right\} \]

\[ y_o = y (1 - A^2). \quad (55) \]

The expression for \( y \) is given in Section III-A, page 45:

\[ y = -A \cot \alpha (1 + A^2 \cot^2 \alpha)^{-\frac{1}{2}}. \quad (56) \]

Now, from Figure 7:

\[ b = + \sqrt{(y - z)^2 + x^2} \]

and from Figure 8 or 9:

\[ b = + \sqrt{(z - y)^2 + x^2} \quad (57) \]

Also, from Figure 7, 8, or 9:

\[ s = + \sqrt{x^2 + y^2} \quad (58) \]

so that, inserting equations (55), (57) and (58) into equation (50):

\[ N_z = \frac{1}{2} \int_{0}^{\pi} N_\alpha \left\| \frac{y(1 - A^2) - z - y(1 - A^2)}{\sqrt{(y - z)^2 + x^2}} \sin \theta \right\| d\alpha \quad (59) \]
where, from page 43:

\[ N_\alpha = \frac{1}{2} N_0 \sin \alpha \left( \sin^2 \alpha + A^2 \cos^2 \alpha \right)^{-3/2}; \]

from equation (56):

\[ y = -A \cot \alpha \left( 1 + A^2 \cot^2 \alpha \right)^{-\frac{1}{2}}; \]

from equation (53):

\[ x = y \tan \alpha; \]

and, from equation (54):

\[ \theta = \tan^{-1} \left( -\tan \alpha/A^2 \right). \]

\[ z \] is of course the independent variable; \( A \) is the constant of the ellipse; \( N_0 \) is the number of incident molecules.

Now, at \( \alpha = 0 \) and \( \alpha = \pi \), the expression for \( y \) (and therefore \( x \)) becomes indeterminate, since \( \cot 0 = \cot \pi = \infty \). However, from the way in which the ellipses are defined, at \( \alpha = 0 \), \( y = -1 \) and \( x = 0 \), while at \( \alpha = \pi \), \( y = +1 \) and \( x = 0 \). Also, at \( \alpha = \pi/2 \), \( \tan \alpha = \infty \), but this simply means that \( \theta = \pi/2 \) also (which can be seen from Figure 7, 8, or 9).

Note that for \( A = 1 \), equation (59) reduces to the equation for the circle, as it should:

\[ N_\alpha = \frac{1}{2} \sin \alpha \left( \sin^2 \alpha + \cos^2 \alpha \right)^{-3/2} = \frac{1}{2} N_0 \sin \alpha \]

\[ y = -\cot \alpha \left( 1 + \cot^2 \alpha \right)^{-\frac{1}{2}} \]

\[ = -\cot \alpha \left( 1 + \frac{\cos^2 \alpha}{\sin^2 \alpha} \right)^{-\frac{1}{2}} \]
\[
\begin{align*}
    &\cot \alpha \left( \frac{\sin^2 \alpha + \cos^2 \alpha}{\sin^2 \alpha} \right)^{-\frac{1}{2}} \\
    &\cot \alpha (\sin \alpha)^{-1} \\
y &= -\cos \alpha \\
x &= y \tan \alpha = -\cos \alpha \tan \alpha \\
x &= -\sin \alpha \\
\theta &= \tan^{-1} (-\tan \alpha) \\
\tan \theta &= -\tan \alpha \\
\sin \theta &= \tan \theta \sqrt{\frac{1}{1 + \tan^2 \theta}} = -\tan \alpha \left(1 + \tan^2 \alpha\right)^{-\frac{1}{2}} \\
&= -\tan \alpha \left(1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}\right)^{-\frac{1}{2}} = -\tan \alpha (\pm \cos \alpha) \\
\sin \theta &= \pm \sin \alpha, \\
\text{but since only the absolute value enters into the calculation:} \\
\sin \theta &= \sin \alpha
\end{align*}
\]

so that:

\[
N_z = \frac{1}{2} \int_{0}^{\pi} \frac{1}{2} N_0 \sin \alpha \left| \frac{-z \sin \alpha}{\sqrt{(-\cos \alpha - z)^2 + \sin^2 \alpha}} \right| \, d\alpha
\]
\[ \frac{1}{4} N_0 z \int_0^\pi \frac{\sin^2 \alpha \, d\alpha}{\sqrt{\cos^2 \alpha + 2z \cos \alpha + z^2 + \sin^2 \alpha}} \]

\[ \frac{1}{4} N_0 z \int_0^\pi \frac{\sin^2 \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{1/2}} \]

which is identical to equation (44), page 54.

Equation (59) for the distribution of molecules exiting a semi-elliptical array after one collision is not integrable in closed form, but has been numerically integrated by the GE 225 Computer, for an incident flux uniform across the y-axis. The results are given in Section VI of this report for \( N_0 = 10,000 \) and several values of \( A \) and \( z \).

C. Parabolic Array

By referring to Figure 7 of the previous section, it can be seen that the expression for the exit distribution from a parabolic section will be very similar to that for the semi-elliptical array, except that the expressions for \( x, y, \theta, y_0 \) and \( N_\alpha \) will be different, as follows.

The equation for the parabola is:

\[ y^2 = 4Px + 1, \]

and from Section III-B, page 46:

\[ y = x \cot \alpha \quad \text{(60)} \]

so that

\[ x^2 \cot^2 \alpha - 4Px - 1 = 0. \]
and

\[ x = \frac{4P \pm (16P^2 + 4 \cot^2 \alpha)^{\frac{1}{2}}}{2 \cot^2 \alpha} \]

\[ x = \frac{2P - (4P^2 + \cot^2 \alpha)^{\frac{1}{2}}}{\cot^2 \alpha} \] (61)

the negative sign being chosen because the domain of interest is for \( x \leq 0 \) and \( P \geq 0 \). Again using the equation for the parabola:

\[ y^2 = 4Px + 1 \]

\[ 2ydy = 4Pdx \]

\[ \tan \theta = \frac{dy}{dx} = \frac{2P}{y} \]

\[ \theta = \tan^{-1} \left( \frac{2P}{y} \right) \] (62)

Now,

\[ y_0 = y + x \cot \theta \] (63)

as before, and the expression for \( N_\chi \) (for incident beam uniform across the y-axis) is given by equation (40) page 48. The expression for the exit distribution is then given by

\[ N_\chi = \frac{1}{2} \int_0^\pi \frac{N_0 P}{\cos^2 \alpha} \left[ 1 - \frac{2P}{(4P^2 + \cot^2 \alpha)^{\frac{1}{2}}} \right] \left\{ \frac{y_0 - z}{\sqrt{(y - z)^2 + x^2}} - \frac{y_0}{\sqrt{x^2 + y^2}} \right\} |\sin \theta| d\alpha \] (64)
where \( x, y, y_0 \) and \( \theta \) are given by equations (61), (60), (63) and (62) above. \( z \), of course, is the independent variable; \( P \) is the constant of the parabola; \( N_0 \) is the number of incident molecules. As in the case of the ellipse, care must be taken at \( \alpha = 0, \pi/2, \pi \). At \( \alpha = 0, \pi \): \( \cot \alpha = \infty \), but again \( x = 0 \) and \( y = \pm 1 \) at these points. At \( \alpha = \pi/2 \), it has already been shown in Section III-B, page 46, that \( N_\alpha = N_0/8P \).

Again the expression for \( N_z \) is not integrable in closed form, but results of the numerical integration are given in Section VI of this report.

SECTION V. DISTRIBUTION OF MOLECULES ALONG THE CENTER LINE (X-AXIS) OF AN ELLIPSE AND A PARABOLA

A. General Expressions

The distribution of molecules along the center line of an ellipse (or parabola or hyperbola) can be found by calculating the number of molecules which passes through a section of the x-axis between \( x_1 \) and \( x_2 \) (Fig. 10).

[Diagram of an ellipse with labeled points and angles]
Using the same argument as in Section IV (page 51), this number would be given by:

\[
N_X = \int_{\pi/2}^{\pi} N_\alpha (\sin \epsilon - \sin \delta) \, d\alpha
\]  

(65)

if now the angles \( \epsilon \) and \( \delta \) are defined in Figure 10. Since all the arrays considered are symmetric with respect to the \( x \)-axis, the integration extends from \( \pi/2 \) to \( \pi \) and is doubled to obtain the total contribution.

The problem now is to express \( \sin \epsilon \) and \( \sin \delta \) in terms of the independent variables \( \alpha \) and \( x_1 \). For purposes of numerical evaluation of the integral, \( x_2 \) has been taken to be \( x_1 + A/8 \) for the ellipse and \( x_1 + 1/48p \) for the parabola. The initial distribution function, \( N_\alpha \), has been calculated for the ellipse and parabola in Section III above.

Now, from Figure 10, it is clear that there will again be a number of cases to consider, depending upon the relative positions of \( x, x_0, x_1 \) and \( x_2 \). (The position of the origin does not matter, since none of the pertinent triangles contain the origin.) If all permutations of these variables were to be considered, there would be \( 4! = 24 \) possible cases. Fortunately, this number can be reduced, as follows: (a) as mentioned above, \( x \) is always greater than \( x_1 \). This eliminates half of the 24 cases. (b) An examination of ellipses and parabolas shows that \( x < x_0 \) in all cases. Since this condition is independent of (a), 6 of the remaining 12 cases are eliminated. There are, therefore, six cases to consider:

1. \( x \leq x_1 \leq x_2 \leq x_0 \)
2. \( x \leq x_1 \leq x_0 \leq x_2 \)
3. \( x \leq x_0 \leq x_1 \leq x_2 \)
4. \( x_1 \leq x_2 \geq x \leq x_0 \)
5. \( x_1 \leq x \leq x_2 \leq x_0 \)
6. \( x_1 \leq x \leq x_0 \leq x_2 \).
It will now be convenient to introduce the following definitions:

\[
\begin{align*}
    a &\equiv x_0 - x_1 \\
    b &\equiv + \sqrt{y^2 + (x_0 - x)^2} \\
    c &\equiv x_0 - x_2 \\
    d &\equiv + \sqrt{y^2 + (x_1 - x)^2} \\
    r &\equiv + \sqrt{y^2 + (x_2 - x)^2}
\end{align*}
\]

(66)

Case 1: \( x \leq x_1 \leq x_2 \leq x_0 \) (Fig. 11)

\[
\sin \epsilon = \frac{a}{d} \sin \left( \frac{\pi}{2} - \theta \right)
\]

or

\[
\sin \epsilon = \frac{a \cos \theta}{d}.
\]
Also

\[ \sin \delta = \sin \left( \frac{\pi}{2} - \theta \right) \]

The desired quantity is

\[ (\sin \epsilon - \sin \delta) = \left( \frac{a}{d} - \frac{c}{r} \right) \cos \theta. \]

Expressions must now be obtained for \( a, c, d, r \) and \( \theta \) in terms of the independent variables \( \alpha, x_1 \) and \( x_2 \). From the definitions of \( a, c, d \) and \( r \) (see above), this means that \( x_0, x \) and \( y \) (and \( \theta \)) must be expressed in terms of \( \alpha, x_1 \) and \( x_2 \). But this has already been done for \( x, y \) and \( \theta \) in Section IV for the ellipses and parabolas. The remaining quantity, \( x_0 \), will be calculated later (see page 72).

Case 2. \( x \leq x_1 \leq x_0 \leq x_2 \) (See Figure 12).

\[ \text{FIGURE 12} \]
From Figure 12:

\[
\frac{\sin \epsilon}{a} = \frac{\sin \left( \frac{\pi}{2} - \theta \right)}{d}
\]

or

\[
\sin \epsilon = \frac{a \cos \theta}{d}.
\]

Also

\[
\frac{\sin \delta}{-c} = \frac{\sin \left( \frac{\pi}{2} - \theta \right)}{r}
\]

or

\[
\sin \delta = -\frac{c \cos \theta}{r}.
\]

Now, from Figure 12, it is clear that the desired quantity in this case is not \((\sin \epsilon - \sin \delta)\), but is \((\sin \epsilon + \sin \delta)\). This may be obtained by using the same expression as in Case 1 above for \((\sin \epsilon - \sin \delta)\), namely:

\[
\left( \frac{a}{d} - \frac{c}{r} \right) \cos \theta,
\]

the negative sign being taken care of, since \(c\) is negative in this case. Note that all angles are considered positive, regardless of their direction.

Case 3. \(x \leq x_0 \leq x_1 \leq x_2\) (See Figure 13).

\[
\frac{\sin \epsilon}{a} = \frac{\sin \left( \frac{\pi}{2} + \theta \right)}{d}
\]

or

\[
\sin \epsilon = -\frac{a \cos \theta}{d}
\]
\[
\frac{\sin \delta}{-c} = \frac{\sin \left( \frac{\pi}{2} + \theta \right)}{r}
\]

or

\[
\sin \delta = -\frac{c \cos \theta}{r}
\]

FIGURE 13

In this case, the desired quantity is \((\sin \delta - \sin \epsilon)\). Again, if the original (Case 1) expression is used, the correct answer is obtained, i.e.,

\[
\left( \frac{a}{d} - \frac{c}{r} \right) \cos \theta = (\sin \delta - \sin \epsilon)
\]

Case 4. \(x_1 \leq x \leq x_2 \leq x_0\).

At this point it is clear that the relative position of \(x\) does not enter into the calculation of \((\sin \epsilon - \sin \delta)\). Case 4 is
therefore the same as Case 1. Similarly, Cases 5 and 6 are the same as Cases 1 and 2, respectively. Thus, the desired expression is, for all cases:

\[(\sin \epsilon - \sin \delta) = \left(\frac{a}{d} - \frac{c}{r}\right) \cos \theta.\]  

(67)

The only remaining task is to calculate \(x_0\) in terms of \(\alpha\); referring back to Figure 10:

\[\frac{\sin \eta}{x_0} = \frac{\sin \xi}{s},\]

or

\[x_0 = -\frac{s \sin \eta}{\sin \xi},\]

but

\[\theta + \eta + \frac{\pi}{2} + \left(\alpha - \frac{\pi}{2}\right) = \pi\]

so that

\[\eta = \pi - (\alpha + \theta)\]

Then

\[\sin \eta = \sin (\alpha + \theta)\]

Also

\[\sin \xi = \sin \left(\frac{\pi}{2} + \theta\right) = \cos \theta\]
and

\[ s = (x^2 + y^2)^{1/2}, \]

so that

\[ x_0 = - \frac{(x^2 + y^2)^{1/2} \sin (\alpha + \theta)}{\cos \theta} \] (68)

Equation (65), with the appropriate substitutions from equations (66), (67) and (68) and the expressions for \( N_\alpha, \theta, x \) and \( y \) from Section III, yields the distribution \( N_x \) along the \( x \)-axis as a function of \( x_1 \). Calculations for specific cases follow below.

B. Ellipse

From page 44, equations (32), (33) and (34):

\[ x = - A (1 + A^2 \cot^2 \alpha)^{-1/2}, \]

and

\[ y = x \cot \alpha. \]

From page 60, equation (54):

\[ \theta = \tan^{-1} (- \tan \alpha/A^2). \]

From page 46, equation (35):

\[ N_\alpha = \frac{1}{2} N_0 A \sin \alpha (\sin^2 \alpha + A^2 \cos^2 \alpha)^{-3/2}. \]
where \( N_0 \) is an arbitrary number of incident molecules and \( A \) is the "shape parameter" of the ellipse:

\[
\frac{x^2}{A^2} + y^2 = 1.
\]

Summarizing, then: \((x_1, x_2, A, \alpha, N_0 \) are independent variables).

\[
\theta = \tan^{-1} \left( -\frac{\tan \alpha}{A} \right)
\]

\[
x = -A \left( 1 + A^2 \cot^2 \alpha \right)^{-1/2}
\]

\[
y = x \cot \alpha
\]

\[
N_\alpha = \frac{1}{2} N_0 A \sin \alpha \left( \sin^2 \alpha + A^2 \cos^2 \alpha \right)^{-3/2}
\]

\[
x_0 = -(x^2 + y^2)^{1/2} \sin (\alpha + \theta)/\cos \theta
\]

\[
a = x_0 - x_1
\]

\[
b = [y^2 + (x_0 - x)^2]^{1/2}
\]

\[
c = x_0 - x_2
\]

\[
d = [y^2 + (x_1 - x)^2]^{1/2}
\]

\[
r = [y^2 + (x_2 - x)^2]^{1/2}
\]

\[
N_x = \int_{\pi/2}^{\pi} N_\alpha \left[ \frac{a}{d} - \frac{c}{r} \right] \cos \theta \, d\alpha.
\]
C. Parabola

From page 47, equations (38) and (37):

\[ x = \tan^2 \alpha [2P - (4P^2 + \cot^2 \alpha)^{1/2}] \]

\[ y = x \cot \alpha. \]

From page 48, equation (40):

\[ N\alpha = \frac{N P}{\cos^2 \alpha} [1 - 2P (4P^2 + \cot^2 \alpha)^{-1/2}] \]

where \( P \) is the shape parameter of the parabola

\[ y^2 = 4Px + 1. \]

Summarizing:

\[ x = \tan^2 \alpha [2P - (4P^2 + \cot^2 \alpha)^{1/2}] \]

\[ y = x \cot \alpha \]

\[ \theta = \tan^{-1} \left( \frac{2P}{y} \right) \]

\[ N\alpha = \frac{N P}{\cos^2 \alpha} [1 - 2P (4P^2 + \cot^2 \alpha)^{-1/2}]. \]

The remaining steps are identical to those for the ellipse.
SECTION VI. NUMERICAL RESULTS

A. Fraction of Molecules Making nth Collision with a Semi-Circular Surface.

From the expressions derived for distributions over the semi-circular surface (page 39), the total number of molecules making the nth collision (with the surface) can be found by integrating $N_\alpha^{(n)}$ from $\alpha = 0$ to $\pi$. It is easier, however, to calculate $2 \times N_\beta^{(n)}$ at $\beta = 0$, which is the same quantity (see Figure 1, page 2). The results follow:

1. Incident Distribution Uniform over the Surface

$$2 \times N_\beta^{(1)} = 2 \times \frac{N_0}{\pi} \left(\frac{\pi}{2}\right) = N_0$$

$$2 \times N_\beta^{(2)} = 2 \times \frac{N_0}{\pi} \left(\frac{\pi}{2} - 1\right) = N_0 \left(\frac{\pi}{\pi} - \frac{2}{\pi}\right) \approx 0.3634 N_0$$

$$2 \times N_\beta^{(3)} = 2 \times \frac{N_0}{4\pi} \left[2\pi + \left(-7 + \frac{\pi}{2}\right)\right] = N_0 \left(\frac{5\pi}{4\pi} - \frac{14}{4\pi}\right) \approx 0.1359 N_0$$

$$2 \times N_\beta^{(4)} = 2 \times \frac{N_0}{16\pi} \left[8\pi + \left(-38 + \frac{9\pi}{2}\right)\right] = N_0 \left(\frac{25\pi}{16\pi} - \frac{76}{16\pi}\right) \approx 0.0505 N_0$$

$$2 \times N_\beta^{(5)} = 2 \times \frac{N_0}{64\pi} \left[32\pi + \left(-187 + \frac{57\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24}\right)\right] \approx 0.0188 N_0$$

2. Incident Distribution Uniform across the Y-Axis

$$2 \times N_\beta^{(1)} = 2 \times \frac{N_0}{2} = N_0$$

$$2 \times N_\beta^{(2)} = 2 \times \frac{N_0}{3} \left(1 - \frac{1}{2}\right) = \frac{N_0}{3} \approx 0.3333 N_0$$
\[ 2 \times N^{(3)}_{\beta=0} = 2 \times \frac{N_0}{12} \left( \frac{5}{3} \cdot \frac{\pi}{2} + \frac{2}{3} \right) = N_0 \left( \frac{14 - 3\pi}{36} \right) \approx 0.1271 \ N_0 \]

\[ 2 \times N^{(4)}_{\beta=0} = 2 \times \frac{N_0}{48} \left( \frac{77}{9} - \frac{11\pi}{6} - \frac{8}{9} \right) = \frac{N_0}{24} \left( \frac{8}{9} - \frac{11\pi}{6} \right) \approx 0.0471 \ N_0 \]

\[ 2 \times N^{(5)}_{\beta=0} = 2 \times \frac{N_0}{48} \left( \frac{617}{108} - \frac{145\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{8}{27} \right) \approx 0.0175 \ N_0 \]

From these results, the fraction of molecules exiting the array after \( n \) collisions can be calculated:

\[
\alpha_n = \frac{\text{(\# making } n^{\text{th}} \text{ collision}) - \text{(\# making } (n+1)^{\text{th}} \text{ collision})}{\text{(\# making } n^{\text{th}} \text{ collision})}
\]

The results of the trivial calculation are given in Table I.

B. Distributions across Openings and Center-Lines of Arrays after One Collision, with Incident Distribution Uniform Across Y-Axis.

![Figure 14](image-url)
In the analysis of the preceding sections, expressions were derived for the following:

\[ N_z = \text{number of molecules that pass through a section of the opening between (0, 0) and (0, z), after one collision with the surface.} \]

\[ N_x = \text{number of molecules that pass through a section of the x-axis between (x_1, 0) and (x_1 + 0.1, 0), after one collision with the surface.} \]

Programs were written for a General Electric 225 Computer to evaluate the integrals for these expressions, for several different geometries. The results of these calculations are given in Tables II through V, and are shown graphically in Figures 15 through 30.

In Tables II and III (for exit distributions), the independent variable \( z \) is given in column (A), and \( N_z \) is given in column (B). But the quantity of greater interest is the number between \( z \) and \( z - 0.1 \). This number, \( N_{\Delta z}(225) \), is given in column (C). The numbers in column (D), \( N_{\Delta z}(7090) \), are the corresponding values obtained by the Monte Carlo method, using the IBM 7090 Computer\(^1\). The difference between these two, \( \Delta \), is given in column (E) and the percentage difference in column (G). The probable error, P. E., given in column (F), is calculated as follows.

In general, if \( N \) measurements of a quantity are taken and \( p \) is the probability of obtaining a given measurement, then the standard deviation of a measurement is given by:

\[ \sigma = \sqrt{Np(1-p)}, \]

and the probable error, P. E., is given by \( \text{P. E.} = 0.6745\sigma \). The probability, \( p \), is found from the 225 results (see next paragraph) and \( N = 10,000 \) is the number of "measurements." For example, the first value of \( N_{\Delta z} \) (for \( z = 0.1 \)) in Table II is 957. This means that the probability, \( p \), of a molecule falling into the section between \( z = 0 \) and \( z = 0.1 \) is 957/10,000 = 0.0957 (since the incident number of molecules is 10,000). Then \( (1-p) = 0.9042 \) and

\[ \sigma = \sqrt{(10,000)(0.0957)(0.9042)} = \sqrt{865.3194} = 29.42, \]

\(^1\)This method and the results have been reported in Marshall Technical Paper MTP-AERO-62-53 by J. O. Ballance, W. K. Roberts, and D. W. Tarbell, and were presented at the Cryogenic Engineering Conference, August 14-16, 1962.
so that

\[ P. E. = (0.6745) \times (29.42) = 19.8 \approx 20. \]

In Tables IV and V (for center-line distributions), the independent variable, \( x \), is given in column (A) and \( N_x(225) \) in column (B). Since \( N_x \) is already for a fixed interval, no subtraction is necessary. \( N_x(7090) \) is given for the same interval in column (C); the difference, \( \Delta \), in column (D); the % difference in (F), and the probable error, P. E., in column (E).

In all cases, the GE 225 Program employed the Simpson Rule to evaluate the integral. Since the accuracy of these evaluations depends on the size of the increment, \( \Delta \alpha \), used in the Simpson Rule, this increment was taken to be just small enough so that the GE 225 result contained as many significant figures as the 7090 result. The value used for \( \Delta \alpha \) ranged from 1° down to 0.005°, depending on the particular case; in each case, the increment used was made smaller until two results were identical (to the same number of significant figures as there were in the 7090 result). The numbers given in column (C) for \( N_{\alpha x}(225) \) can therefore be considered exact, with the error in the Monte Carlo results only.

SECTION VII. DISCUSSIONS AND CONCLUSIONS

A. Surface Distributions

The distribution of molecules on the inner surface of a semicircle (or a long cylinder in three dimensions) has been derived as a function of the number of collisions made by the molecules. Initial distributions considered were for completely uniform incidence upon the surface and for a directed, uniform flow on the entrance to the surface. Figure 31 graphically presents the percentage of particles colliding per radian for a semicircle and the two initial distributions considered. It is seen that, for either initial distribution, after two collisions the distribution on the surface becomes approximately constant. Of course, the percentage of particles making the third, fourth, and fifth collisions becomes quite small (i.e., 1.75% make 5 collisions). Chahine [5] has derived the total number flux incident at any point on a semicircular surface per unit time and unit area with a directed uniform flow. By summing the distributions for the five collisions this same flux may be found. This summation is within 2% of the Chahine result. Since approximately 99.4% of the incident particles have exited after 5 collisions, this difference is not significant and shows that, for this case, less than 2% of the flux results from particles making more than 5 collisions.
B. Exit and Center Line Distributions

The distributions of particles exiting the surface array and along the center line of symmetry of the array have been derived. Due to the complexity of these derivations, analytical solutions were obtained only for single collisions with the surface; however, computer programs can consider any number of collisions. Early in the study it was thought that concentration of particles (focusing) would occur at the exit of the array. No such focusing was observed, but the distributions at the exit and along the center line does yield very interesting results. For example, for an elliptical surface with the parameter $A$ equal to two, it is seen from the Table II and Figure 18 that 50% of the incident particles cross the inner 50% of the center line after the first collision. A practical application of the result is shown in Figure 22. Here is a cryogenic pumping array which could be used for highly directed flow fields such as in a low density wind tunnel, a rocket exhaust test chamber, or a rocket sounding probe for atmospheric sampling. Radiation shields reduce the radiant heat losses of the 20°K surface and also provides some impedance to molecules from leaving the array. Collection efficiency (that is, the percentage of particles condensed by the 20°K surface compared to the total number of particles entering the array) should be very high (hopefully, 80-90 percent).

C. Monte Carlo Computer Techniques

The applicability and accuracy of Monte Carlo computer techniques for the study of free molecular flow has been shown. These techniques are necessary for this type of study since the analytical solutions become quite complex. As future plans in the study are for extension of the program into transition flow where intermolecular collisions will be considered, it is felt that only through the use of Monte Carlo methods can meaningful studies be made.
APPENDIX A

Useful Integrals

\[
\int \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \, d\alpha = \sin^2 \frac{\alpha}{2} \quad (1)
\]

\[
\int \sin^2 \frac{\alpha}{2} \, d\alpha = \frac{1}{2} (\alpha - \sin \alpha) \quad (2)
\]

\[
\int \cos^2 \frac{\alpha}{2} \, d\alpha = \frac{1}{2} (\alpha + \sin \alpha) \quad (3)
\]

\[
\int \alpha \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \, d\alpha = \frac{1}{2} (\sin \alpha - \alpha \cos \alpha) \quad (4)
\]

\[
\int \alpha^2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \, d\alpha = \alpha \sin \alpha + \cos \alpha - \frac{1}{2} \alpha^2 \cos \alpha \quad (5)
\]

\[
\int \sin \alpha \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \, d\alpha = \frac{1}{4} (\alpha - \frac{1}{2} \sin 2\alpha) \quad (6)
\]

\[
\int \cos \alpha \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \, d\alpha = \frac{1}{4} \sin^2 \alpha \quad (7)
\]

\[
\int \alpha \cos^2 \frac{\alpha}{2} \, d\alpha = \frac{1}{2} \left( \frac{1}{2} \alpha^2 + \alpha \sin \alpha + \cos \alpha \right) \quad (8)
\]
\[ \int \alpha^2 \cos^2 \frac{\alpha}{2} \, d\alpha = \frac{\alpha^3}{6} + \left( \frac{\alpha^2}{2} - 1 \right) \sin \alpha + \alpha \cos \alpha \]  
(9)

\[ \int \sin \alpha \cos^2 \frac{\alpha}{2} \, d\alpha = -\cos^4 \frac{\alpha}{2} \]  
(10)

\[ \int \cos \alpha \cos^2 \frac{\alpha}{2} \, d\alpha = \frac{1}{4} \left( \alpha + \frac{3}{2} \sin \alpha + \sin \alpha \cos^2 \frac{\alpha}{2} + \frac{1}{4} \sin 2\alpha \right) \]  
(11)

\[ \int \alpha \sin^2 \frac{\alpha}{2} \, d\alpha = \frac{1}{2} \left( \frac{\alpha^2}{2} - \alpha \sin \alpha - \cos \alpha \right) \]  
(12)

\[ \int \alpha^2 \sin^2 \frac{\alpha}{2} \, d\alpha = \frac{\alpha^3}{6} - \left( \frac{\alpha^2}{2} - 1 \right) \sin \alpha - \alpha \cos \alpha \]  
(13)

\[ \int \sin \alpha \sin^2 \frac{\alpha}{2} \, d\alpha = \sin^4 \frac{\alpha}{2} \]  
(14)

\[ \int \cos \alpha \sin^2 \frac{\alpha}{2} \, d\alpha = \frac{1}{4} \left( -\alpha + \frac{3}{2} \sin \alpha + \sin \alpha \cos^2 \frac{\alpha}{2} - \frac{3}{4} \sin 2\alpha \right) \]  
(15)

\[ \int \cos \frac{\alpha}{2} \, d\alpha = 2 \sin \frac{\alpha}{2} \]  
(16)

\[ \int \sin \frac{\alpha}{2} \, d\alpha = -2 \cos \frac{\alpha}{2} \]  
(17)
APPENDIX A (Cont'd)

\[ \int \alpha \sin \frac{\alpha}{2} \, d\alpha = 4 \sin \frac{\alpha}{2} - 2\alpha \cos \frac{\alpha}{2} \]  

(18)

\[ \int \alpha \cos \frac{\alpha}{2} \, d\alpha = 4 \cos \frac{\alpha}{2} + 2\alpha \sin \frac{\alpha}{2} \]  

(19)

\[ \int \frac{\alpha^2}{2} \cos \frac{\alpha}{2} \, d\alpha = 4\alpha \cos \frac{\alpha}{2} + \left( \alpha^2 - 8 \right) \sin \frac{\alpha}{2} \]  

(20)

\[ \int \frac{\alpha^2}{2} \sin \frac{\alpha}{2} \, d\alpha = 4\alpha \sin \frac{\alpha}{2} - \left( \alpha^2 - 8 \right) \cos \frac{\alpha}{2} \]  

(21)

\[ \int \alpha^3 \cos \frac{\alpha}{2} \, d\alpha = (12\alpha^2 - 96) \cos \frac{\alpha}{2} + (2\alpha^3 - 48\alpha) \sin \frac{\alpha}{2} \]  

(22)

\[ \int \alpha^3 \sin \frac{\alpha}{2} \, d\alpha = (12\alpha^2 - 96) \sin \frac{\alpha}{2} - (2\alpha^3 - 48\alpha) \cos \frac{\alpha}{2} \]  

(23)
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**TABLE I**

FRACTION OF MOLECULES THAT EXIT SEMICIRCULAR ARRAY AFTER THE \( n^{th} \) COLLISION
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Figure 16 Distribution of Particles Exiting After Single Collision
(Incident Distribution Uniform Across Opening)
Figure 17: Distribution of Particles Exiting After Single Collision
(Uniform Distribution Across Opening)

Ellipses: $x = \frac{A^2}{2}, y = \frac{A^2}{2}$

N Exit
N Incident
A = 1.800

Percent of Total Particles Exiting
Figure 18  Distribution of Particles Exiting After Single Collision
(Incident Distribution Uniform Across Opening)
Figure 19: Distribution of Particles Exiting After Single Collision
(Incident Distribution Uniform Across Opening)
Figure 22  Distribution of Particles Exiting After Single Collision
(Incident Distribution Uniform Across Opening)
Theoretical Model

Equation: \( X = \frac{2}{A} \)

Figure 2: Distribution of Particles Intersecting Gitter Line After Single Collision (Uniform Distribution Across Opening)
Figure 25  Distribution of Particles Intercepting Center Line After Single Collision
(Incident Distribution Uniform Across Opening)

Ellipse \[ X = A(1 - y^2)^{1/2} \]

\[ A = 1.500 \]
Figure 26  Distribution of Particles Intercepting Center Line After Single Collision

(Incident Distribution Uniform Across Opening)
Figure 28: Distribution of Particles Intercepting Center Line After Single Collision
(Incident Distribution Uniform Across Opening)
Figure 29  Distribution of Particles Intercepting Center Line After Single Collision

(Incident Distribution Uniform Across Opening)

Parabola $y^2 = 4px + 1$

$P = 0.125$
Figure 30  Distribution of Particles Intercepting Center Line After Single Collision
(Incident Distribution Uniform Across Opening)
Fig. 31 Distribution of Particles Over a Semi-Circular Surface at First Five Collisions
Semi-Elliptic Surface (77°K)

Radiation Shield (77°K)

Condensing Surface (26°K)

Note: 50% of incoming molecules should hit Condensing Surface after one collision with surface.

Fig. 32 Sketch of Possible Cryopanel for Low Density Wind Tunnel
REFERENCES


A STUDY OF THE DISTRIBUTION OF MOLECULES
UNDER FREE MOLECULAR FLOW CONDITIONS
AFTER COLLISIONS WITH SIMPLE GEOMETRIES

DAVID W. TARBERLL and JAMES O. BALLANCE

The information in this report has been reviewed for security
classification. Review of any information concerning Department of
Defense or Atomic Energy Commission programs has been made by the
MSFC Security Classification Officer. This report, in its entirety,
has been determined to be unclassified.

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M-DEP-DIR
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M-ASTR
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Mr. Boehm
Mr. Feather
Mr. Allen

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Mr. Hueter

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Mr. Schwinghamer

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Mr. Riehl
Dr. Gayle
Dr. Egger
Mr. Deuel

M-QUAL
Mr. Grau

M-RP
Dr. Stuhlinger
Mr. Heller
Dr. Shelton
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DISTRIBUTION

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Mr. Grafton
Mr. Connor

M-AERO
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Mr. Tarbell (20)
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EXTERNAL

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