FACILITY FORM 602

(ACCESSION NUMBER)	(THRU)
(PAGES)	(CODE)
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

FACILITY FORM 602

N65 13554	
(ACCESSION NUMBER)	(THRU)
48	1
(PAGES)	(CODE)
CR-59657	12
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

GPO PRICE $ ______

OTS PRICE(S) $ ______

Hard copy (HC) 2.00

Microfiche (MF) .50

TR AE 6407

ON NON-STEADY LIQUID DISCHARGE FROM RESERVOIRS

by

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TROY, NEW YORK

August, 1964

ACKNOWLEDGMENT

The authors wish to express their appreciation to Professors J. V. Foa, H. Hagerup and L. Segel of the RPI faculty for stimulating conversations and suggestions during the course of this work. The patience of our machinists, Messrs. Walter Wells and Donald Uline, in machining and remachining the series of nozzles is gratefully acknowledged. It is also a pleasure to acknowledge a receipt of a Rensselaer Research Grant with which the equipment was built.

One of us (JLW) acknowledges a traineeship supported by National Aeronautics and Space Administration.

SUMMARY

A group of problems involving time-dependent discharge of inviscid liquid from a reservoir is examined. Two geometries are treated in detail: the slender conical reservoir, and the cylindrical reservoir.

The differential equation governing free-surface motion during discharge is non-linear and cannot be integrated exactly except for the special geometry of a cylindrical reservoir. However it is shown that it is possible, in the two specific cases treated, to neglect the free-surface acceleration. This is called the approximate unsteady solution. This is compared to a numerical solution of the exact equation, the quasi-steady solution, and experiment. It is highly probable that free-surface acceleration plays no role in the flow regardless of the reservoir geometry.

For the cylindrical reservoir it turns out that only with an exit nozzle is it possible to make meaningful comparison with experiment, and even then special precautions must be taken to avoid free-surface collapse. The correction to the discharge time even for short nozzle lengths may be a significant fraction of the total. Experiments indicate that when the length/diameter ratio of the nozzle is large, friction in the nozzle is important. When the ratio is small the inlet shape must be carefully designed in order

to avoid flow separation of the jet from the nozzle walls.

Other theoretical work is discussed briefly. In particular the work of Stary is considered who employs momentum and kinetic energy correction factors to account for the lack of uniformity of the flow across any cross-section normal to the axis. It is shown for a particular geometry that the introduction of these correction factors is superfluous and it is concluded that their significance even for steady flows should be reexamined.

CONTENTS

LIST OF FIGURES

SYMBOLS

A	cross-sectional area
C	inverse of velocity coefficient
d	diameter
F, G	defined in eq. (19)
g	gravitational constant
$k_{1,2,3}$	Stary's correction factors
k	constant
m,n	indices or exponents
p	pressure
t	time, dimensional
w	vertical velocity component
x	dummy variable
y	free-surface coordinate, dimensionless
z	vertical coordinate, dimensional
α	A_2/A_1
β	defined by eq. (15)
ϵ	z_2/z_1 for cylindrical reservoir
ζ	free-surface coordinate, dimensional
λ	z_2/z_1 for conical reservoir
ρ	fluid density
τ	dimensionless time

Subscripts

1,2	denote exit and initial free-surface stations, respectively
d	denotes discharge time
s	denotes starting time
ϵ	denotes discharge time corrected for nozzle height

INTRODUCTION

The subject of unsteady liquid flow through ducts has not been greatly developed. For example, in the recent "Handbook of Fluid Dynamics", reference 1, no mention is made of the subject. It is possible to deal with flows with negligible viscous effects by standard methods of potential theory involving superposition of singularities. Roudebush and Pinkel, reference 2, treat the unsteady flow out of a cylindrical tank by this method but it is too complicated to have much promise except in special cases.

One reads the standard works in vain for an adequate treatment of the subject. In their famous work Prandtl and Tietjens, reference 3, approach the subject by means of the unsteady Bernoulli equation. They point out (correctly) for one situation that the governing differential equation is of second order but no details are given.

Kozeny, reference 4, in a work available only in German, develops the theme outlined by Prandtl. In a short note Stary, reference 5, applies the development of Kozeny to the problem of a cylindrical tank with a horizontal bottom which is provided with a discharge hole. Neglecting viscosity he obtains a solution for the discharge time in terms of the area ratio, the original free-surface height, and certain momentum and kinetic energy correction factors which have to be determined empirically. Bird, Stewart and

Lightfoot, reference 6, page 239, pose as an exercise, the same problem considered by Stary; their expression (taking into account what appears to be a misprint) for the discharge time agrees with that of Stary when his correction factors are put equal to unity. We shall also give the solution to this problem by an alternative approach.

Bird et al. also consider, page 226, the unsteady discharge from a conical tank, which is a special case of a class of flows considered in the present work. By neglecting the kinetic energy they obtain a result which is equivalent to a quasi-steady flow whose exit velocity is given by the Torricelli value based on the instantaneous free-surface height.

Figure 1 illustrates the conical reservoir. Assuming a fluid of negligible viscosity under action of gravity we ask: If at time $t = 0$ the valve at the exit is opened, what is the discharge time t_d required to empty the reservoir?

The quasi-steady solution. According to Torricelli's law the exit velocity at station (1) is

$$w_1 = -\sqrt{2g(\zeta - z_1)} \approx -\sqrt{2g\zeta}, \tag{1}$$

if $z_1 \ll \zeta$. From the continuity equation for an incompressible fluid the velocity of the free-surface $\zeta = \zeta(t)$ is $w(\zeta) = d\zeta/dt = w_1 z_1^2/\zeta^2$, and thus the discharge time is

$$t_d = \int_{z_2}^{z_1} \frac{d\zeta}{w(\zeta)} \approx \frac{1}{5}\left(\frac{z_2}{z_1}\right)^2 \left(\frac{2z_2}{g}\right)^{\frac{1}{2}}. \tag{2}$$

It is convenient to work with dimensionless quantities. Putting $\tau \equiv (g/z_2)^{\frac{1}{2}} t$, $\lambda \equiv z_2/z_1$, then

$$\tau_d = \lambda^2 \sqrt{2}/5. \tag{3}$$

It is noted that according to quasi-steady theory the discharge speed depends only on the free-surface height. Thus the discharge velocity theoretically becomes zero as the free surface reaches the exit. On the other hand in a non-steady incompressible flow the entire body of fluid must be accelerated from an initial velocity of zero. In a non-dissipative flow the free-surface velocity ought to increase in time and be a maximum at the exit. Therefore it is not obvious in advance whether the quasi-steady or non-steady theories ought to predict the lesser discharge times.

Comparison of the solution of the unsteady flow equations with the quasi-steady solution will provide a convenient measure of the importance of non-steady phenomena. It will be shown that in some cases use of the quasi-steady result can lead to significant error.

THE UNSTEADY EQUATIONS

We assume that the flow can be treated as one-dimensional. There is no particular difficulty, initially, in generalizing to include flow geometries involving ducts with area distributions following the law $A = kz^n$. For axially-symmetric ducts $n = 2$ designates a conical boundary, whereas $n \gtrless 2$ correspond to convex and concave interiors respectively (see Figure 2). Furthermore, it is simple to include in the initial analysis the effect of a pressurized reservoir such that the free-surface pressure remains constant.

For unsteady incompressible flow the continuity equation is

$$w(z,t)A(z) = w(z_1,t)A(z_1) = w(\zeta)A(\zeta). \tag{4}$$

Thus, specification of the velocity at any station automatically specifies the entire flow field.

The dynamical equation is

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{5}$$

Specifying a constant discharge pressure p_1, and constant free-surface pressure p_2, the boundary conditions are

$$p(z_1,t) = p_1 \quad , \quad p[\zeta(t)] = p_2 . \tag{6}$$

Initial conditions are

$$\zeta(0) = z_2 \quad , \quad w(z_2,0) = w[\zeta(0)] = 0. \tag{7}$$

Equation (5) can be rearranged and integrated from the discharge station to the free surface at an arbitrary time

$$\int_{z=z_1}^{\zeta} \frac{\partial w}{\partial t}\partial z + \int_{z=z_1}^{\zeta} \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \frac{w^2}{2} + gz\right)\partial z = 0,$$

or

$$\int_{z_1}^{\zeta} \frac{\partial w}{\partial t}\partial z + \frac{p_2 - p_1}{\rho} + w^2(\zeta) - w_1^2 + g(\zeta - z_1) = 0. \quad (8)$$

From (4)

$$w(z,t) = w(\zeta)\zeta^n/z^n = \dot{\zeta}\zeta^n/z^n, \quad (9)$$

where $\dot{\zeta} \equiv d\zeta/dt$. Therefore

$$\frac{\partial w}{\partial t} = (\ddot{\zeta}\zeta^n + n\zeta^{n-1}\dot{\zeta}^2)/z^n, \quad (10)$$

and

$$\int_{z_1}^{\zeta} \frac{\partial w}{\partial t}\partial z = (\ddot{\zeta}\zeta^n + n\zeta^{n-1}\dot{\zeta}^2)\left[z_1^{-(n-1)} - \zeta^{-(n-1)}\right]/(n-1), \quad (11)$$

for $n \neq 1$.

Combining (8) and (11), we obtain the following differential equation for the free-surface motion:

$$(n-1)^{-1}(\zeta\ddot{\zeta} + n\dot{\zeta}^2)\left[\left(\frac{\zeta}{z_1}\right)^{n-1} - 1\right] + \frac{1}{2}\left[1 - \left(\frac{\zeta}{z_1}\right)^{2n}\right]\dot{\zeta}^2 + \frac{p_2 - p_1}{\rho} + g(\zeta - z_1) = 0. \quad (12)$$

Now we define

$$\lambda \equiv z_2/z_1 , \quad y \equiv \zeta/z_2 , \quad \tau \equiv (g/z_2)^{1/2} t , \tag{13}$$

and equation (12) becomes

$$\frac{y\ddot{y}}{n-1}\left[(\lambda y)^{n-1}-1\right] + \left\{\frac{n}{n-1}\left[(\lambda y)^{n-1}-1\right] - \frac{1}{2}\left[(\lambda y)^{2n}-1\right]\right\}\dot{y}^2 + \beta + \frac{\lambda y-1}{\lambda} = 0, \tag{14}$$

with

$$\beta \equiv (p_2 - p_1)/\rho g z_2 . \tag{15}$$

For the conical duct of Figure 1, n = 2, and equation (14) reduces to

$$y\ddot{y} + \frac{1}{2}\left[3-(\lambda y)^3-(\lambda y)^2-(\lambda y)\right]\dot{y}^2 + \lambda^{-1} + \beta/(\lambda y - 1) = 0. \tag{16}$$

For n = 1 the integration of (10) introduces a logarithmic term. The final differential equation for this case is

$$y \ln(\lambda y)\ddot{y} + \frac{1}{2}\left[1-(\lambda y)^2 + 2\ln(\lambda y)\right]\dot{y}^2 + \beta + (\lambda y - 1)/\lambda = 0. \tag{17}$$

Note that (16) and (17) are independent of the constant <u>k</u>; i.e. in the case n = 2, the flow does not depend on the actual cone angle, and similarly for other shapes.

The governing initial conditions in all cases are

$$y(0) = 1 , \quad \dot{y}(0) = 0. \tag{18}$$

The solution terminates at $y = \lambda^{-1}$, when the free surface reaches the exit station.

Equations (14) through (17) are surprisingly formidable for what is apparently a simple flow situation, i.e., a flow which is one-dimensional, incompressible and inviscid. A general solution to (14) has not been obtained. Therefore we focus attention on the conical duct of Figure 1, for which $n = 2$, which is governed by equation (16).

INTEGRATION OF (16)

A variety of attempts to integrate (16) were not entirely successful. It is useful to mention some of these briefly because they point up the mathematical difficulties. Furthermore, it was in the attempt of trying to evaluate the utility of one of these approaches that the clue to a successful approximate solution was obtained.

Quadrature. Equation (16) can be integrated by quadrature. Putting

$$\left.\begin{aligned} F(y) &\equiv y^{-1}[3-(\lambda y)^3-(\lambda y)^2-\lambda y] \\ G(y) &\equiv -(\lambda y)^{-1} - \beta/y(\lambda y-1), \end{aligned}\right\} \qquad (19)$$

and noting that $\ddot{y} = d(\tfrac{1}{2}\dot{y}^2)/dy$, then (16) becomes

$$d(\tfrac{1}{2}\dot{y}^2)/dy + F(y)(\tfrac{1}{2}\dot{y}^2) = G(y). \qquad (20)$$

Multiplying by the integrating factor exp $\int F(y)dy$, then a first integral gives

$$\tfrac{1}{2}\dot{y}^2 = \exp\left[-\int F(\xi)d\xi\right]\int_1^y G(\xi)\exp\left[\int F(\xi)\right]d\xi, \tag{21}$$

and finally

$$\tau_d = \int_1^{\lambda^{-1}} \frac{dy}{\left\{\exp\left[-\int F(\xi)d\xi\right]\int_1^y G(\xi)\exp\left[\int F(\xi)d\xi\right]d\xi\right.} . \tag{22}$$

Evaluation of (22) depends on the integration indicated in (21) which has not been achieved. A numerical integration looks possible using a digital computer although great care must be employed because, in practical cases where values of $2 < \lambda < 100$ might be encountered, extreme variations of the integrands can be expected. For this reason it was decided to investigate other possibilities first.

Series solutions. Frobenius' method in which $y(\tau)$ is represented as an infinite series in τ fails because the number of terms required for adequate accuracy (values of τ_d as large as 2000 are encountered) would be very large. And then there is no guarantee that the series would be convergent.

Asymptotic expansion in powers of a small parameter. An attempt was made, for $\beta = 0$, $n = 2$, to put

$$y(\tau) = y_0(\tau) + \lambda^{-1}y_1(\tau) + \lambda^{-2}y_2(\tau) + \cdots + \lambda^{-m}y_m(\tau) + \cdots,$$

substitute in (16), and solve for the resulting sequence of differential equations in y_m. Unfortunately these differential were incompatible with each other and the method failed. It seems likely that some variation of this technique ought to work but it remains to be discovered.

THE METHOD OF SUCCESSIVE APPROXIMATIONS.

Equation (20) is particularly suitable in form for employing a well known approximate method for solving non-linear differential equations. One of the lower derivatives is replaced initially by an expression which allows the resulting equation to be integrated for the term which was originally replaced. This expression is then put into the original differential equation and, hopefully, the succeeding differential equation can be integrated. The process, if convergent, can be repeated until the actual solution is approached to any accuracy desired.

Thus equation (20) can be rewritten and approximated as follows:

$$\frac{1}{2}\dot{y}_m^2 = \int_1^y \left[G(\xi) - \frac{1}{2}\dot{y}_{m-1}^2 F(\xi) \right] d\xi, \tag{23}$$

where the subscript m denotes the order of the approximation. For n = 2 and $\beta = 0$ this becomes

$$\frac{1}{2}\dot{y}_n^2 = -\lambda \ln y - \int_1^y \frac{1}{2}\dot{y}_{n-1}^2 F(\xi)\,d\xi . \tag{24}$$

In principle we are indifferent to the choice of $\dot{y}_0$ as long as the process converges. Expecting that the exit velocity rapidly builds up it seems reasonable to try the Torricelli value, based on the initial free-surface height, as the zeroth approximation. However, substitution of this value, which is $\dot{y}_0 = -\sqrt{2}\lambda^{-2}$, in (24) leads to imaginary values for $\dot{y}_1$. This indicates that the procedure is very sensitive to the choice of the zeroth approximation.

On the other hand, choosing $\dot{y}_0 = 0$ leads to

$$\frac{1}{2}\dot{y}_1^2 = -\lambda^{-1}\ln y . \tag{25}$$

Substitution of (25) to obtain the next approximation yields

$$\frac{1}{2}\dot{y}_2^2 = -\frac{\lambda^2}{9}\left[3y^3\ln y + 1 - y^3\right] - \frac{\lambda}{4}\left[2y^2\ln y + 1 - y^2\right] - \left[y\ln y + 1 - y\right] + \frac{1}{\lambda}\left[\frac{3}{2}(\ln y)^2 - \ln y\right]. \tag{26}$$

Unfortunately equation (26) also predicts imaginary values for $\dot{y}_2$ and the method again fails. Furthermore the complexity of (26) indicates, even if a satisfactory zeroth approximation could be found, that if convergence were slow, requiring several iterations, the method might still be unwieldy. Furthermore, if an expression for the velocity could be obtained with adequate accuracy, evaluation of the discharge time would still require an integration in which the last approximation appears as a radical in the denominator of the integrand. Integration, except by a numerical procedure, of such a result appears unlikely.

This infelicitous behavior is not without some redeeming consequences, however. It is seen from (16), for $\beta = 0$, that initially, since $\dot{y} = 0$, the second order term is dominant. However, as the free-surface speed builds up, the magnitude of $\ddot{y}$ must decrease rapidly from its initial value of λ^{-1} because the coefficient of $\dot{y}^2$ is initially of order λ^3. Since $\ddot{y}$ cannot change sign this suggests that the second order term may in fact be negligible throughout most of the discharge. Before investigating this point a criterion for the time to "start" an unsteady flow is propounded.

THE STARTING TIME

There is no unambiguous way to define a starting time for an unsteady flow. At the exit the velocity is initially zero. Intuitively we expect that the velocity rapidly builds up until it reaches or surpasses the Torricelli value. When this condition is reached the free-surface velocity is $\dot{y} = -\sqrt{2}\,\lambda^{-2}$, for $n = 2$ and $\beta = 0$. We define the starting time τ_s as the time to reach this value.

An approximate lower bound on τ_s can be obtained as follows. Expecting that the free-surface build-up is attained rapidly we put $y \approx 1$, $\dot{y} \approx 0$ in (16), and obtain $\ddot{y} \approx -\lambda^{-1}$, which is integrated to give $\dot{y} \approx -\lambda^{-1}\tau$. This is equated to the Torricelli value for $y = 1$ and solved for the starting time which is

$$\tau_s \approx \sqrt{2}\,\lambda^{-1}. \tag{27}$$

Values of τ_s, according to (27), turn out to be of order λ^{-3} times the discharge time for the same configuration, which is an insignificant fraction of the total. Similar expressions can be derived for arbitrary n and β.

AN APPROXIMATE RESULT FOR THE DISCHARGE TIME

For $n = 2$ and $\beta = 0$ the free-surface acceleration is initially $\ddot{y} = -\lambda^{-1}$. As $\dot{y}^2$ increases, $\ddot{y}$ decreases rapidly in magnitude and remains small for most of the discharge run. As the free surface approaches the exit, however, the volume of liquid decreases at a rapidly increasing rate until for $y = \lambda^{-1}$, $\ddot{y} = -1$, which corresponds to the acceleration of gravity.

This suggests that in (16) it is possible to ignore the acceleration of the free surface, reducing the problem to the integration of the following first-order equation:

$$\tfrac{1}{2}\left[3 - (\lambda y)^3 - (\lambda y)^2 - \lambda y\right]\dot{y}^2 + \lambda^{-1} = 0. \tag{28}$$

It was first pointed out by Prandtl, whose discussion is reproduced by Schlichting, ref. 9, page 63, that it is not generally possible to neglect the highest order derivative in a differential equation because this means that one of the initial conditions cannot be satisfied. In the present case, however, the approximation leads to reasonably accurate discharge times. It is conjectured that from (28) although $\dot{y}(0) = -\sqrt{2}\lambda^{-2}$, which is the Torricelli value, it is nevertheless sufficiently close to the exact requirement $\dot{y}(0) = 0$ that the resulting error for the discharge time is negligible. Equivalently, use of (28) is tantamount to neglecting the time to start the flow as given by (27).

On the other hand (28) breaks down when the free surface reaches the exit, yielding an infinite velocity at that point. Since the time for the free surface to cover the last small fraction of its total displacement is small in any case, the singular behavior at the exit has no great effect on the discharge time.

Solving (28) for the negative root of $\dot{y}$ and integrating gives

$$\tau_d = \sqrt{\lambda\sqrt{2}} \int_{\lambda^{-1}}^{1} \left[(\lambda y)^3 + (\lambda y)^2 + \lambda y - 3 \right]^{1/2} dy, \qquad (29)$$

or, equivalently,

$$\tau_d = (\lambda/2)^{1/2} \int_{\lambda^{-1}}^{1} \left\{ (\lambda y) \left[(\lambda y + 1/2)^2 + 3(1/4 - 1/\lambda y) \right] \right\}^{1/2} dy. \qquad (30)$$

Since the second term in brackets is small compared to the first, except near the exit, we ignore it to obtain the explicit result

$$\tau_d = \lambda^2 \sqrt{2}/5 + \lambda/3\sqrt{2}, \qquad (31)$$

keeping only terms in λ to the first power or greater.

The first term of (31) is the same as the quasi-steady expression of equation (3). Consequently the solution for the full unsteady flow problem is a second-order correction to the quasi-steady solution. This is borne out by Figure 3 in which are plotted equation (3), equation* (31), a numerical integration of (29), and experimental values -- all for $n = 2$ -- and for $n = 1$, a numerical integration of equation (17) for $\beta = 0$, neglecting the term in $\ddot{y}$.

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* During the preparation of this paper equation (16) was programmed for the Philco "Transac" digital computer and several runs were made. For the case $\lambda = 40$, $\beta = 0$ the result differed only by one part in 900 from that of equation (31). For $\lambda = 10$, $\beta = 10$ the digital computation yielded $\tau_d = 7.24$, which is about 1% greater than that of a numerical integration of (16) with $\ddot{y} = 0$. It is amusing to note that in the digital computation the function $\ddot{y}$ oscillated irregularly in sign all the way to the exit although its magnitude averaged only about 10^{-4}. This behavior caused only a few wiggles in the velocity term y, and can be taken as another indication that $\ddot{y}$ can be safely neglected.

There is no significant difference between the values given by (29) and (30) except for $\lambda < 5$. For $\lambda = 10$ the value given by (31) is about 8% higher than the quasi-steady value, for $\lambda = 40$ this drops to 2% and to 0.8% at $\lambda = 100$.

Experimental data were obtained on equipment, described in the Appendix, which permitted use of any one of three different orifices. By varying also the liquid height, the range of values $6 \leq \lambda \leq 50$ was obtained. Experimental values for τ_d were slightly higher than predicted by (31) varying from 3% in most cases to 10% for a few. It is reasonable to expect that a part of the difference between theory and experiment is due to the neglected action of viscosity.

THE CYLINDRICAL RESERVOIR

Theory. The flow from a cylindrical reservoir with a small nozzle at the exit has been posed as an exercise by Bird, Stewart and Lightfoot to be treated by one-dimensional theory. Their result for the discharge time is given in the form of an unevaluated integral. Stary in reference 4 tackles the same problem using the theory developed by Kozeny, in which the unsteady Bernoulli equation for a streamline in a general potential flow is integrated first transversely and then streamwise, yielding a differential equation for the free-surface motion which is integrated to give an expression for the discharge time. His expression involves certain empirical factors intended to correct for the non-uniform distribution of kinetic energy and momentum flux when integrating over a cross-section. Surprisingly, no experimental results appear to be extant for this flow, so that it is not possible to evaluate either treatment.

It is desirable to compare the one-dimensional treatment with that of Stary to evaluate the importance of his correction factors. A schematic is shown in Figure 4a. We commence with a concave interior connecting the fixed cross-section area A_2, located at $z = z_2$, with an outlet area A_1 at $z = 0$. For the initial area distribution we choose

$$A = A_1\left[1 + (\alpha - 1)(z/z_2)^n\right], \tag{32}$$

where $\alpha \equiv A_2/A_1$. Omitting consideration of a pressurized reservoir, we repeat essentially the same procedure by which (14) was derived and are led to the following equation for arbitrary n:

$$\left\{\left[1 + (\alpha-1)y^n\right]\ddot{y} + n(\alpha-1)y^{n-1}\dot{y}^2\right\}\int_0^y \frac{dx}{1+(\alpha-1)x^n}$$

$$+\frac{1}{2}\left\{1 - \left[1+(\alpha-1)y^n\right]^2\right\}\dot{y}^2 + y = 0. \tag{33}$$

Taking the limit as $n \to 0$ produces the desired cylindrical contour, for which the governing differential equation is

$$y\ddot{y} + \tfrac{1}{2}(1-\alpha^2)\dot{y}^2 + y = 0, \tag{34}$$

with the same initial conditions given by (18). Equation (34) can be integrated as it stands in terms of gamma functions. We define $s \equiv \alpha^2 - 1$ and then, employing the transformation $y = u^2$, we obtain a form whose integral is given by Dwight, reference 9, formula No. 857.1:

$$\tau_d = [2(s-1)]^{1/2} \int_0^1 \frac{du}{[1-u^{2s-2}]^{1/2}}$$

$$= [\pi/2(s-1)]^{1/2} \frac{\Gamma\left[\frac{1}{2(s-1)}\right]}{\Gamma\left[\frac{s}{2(s-1)}\right]} \tag{35}$$

for $s > 1$; for $s = 1$

$$\tau_d = \sqrt{\pi} \tag{36}$$

An expression similar to (35) is obtainable for $s < 1$, but this case is of little practical interest except to note that in a free fall which is the limit as $\alpha \to 1$, $\tau_d \to \sqrt{2}$. We have verified that the expression of Bird, Stewart and Lightfoot, taking into account a misprint, produces these same results assuming a negligible nozzle length.

The approximate unsteady solution,* neglecting the second-order derivative in (34), is

$$\tau_d = [2(\alpha^2-1)]^{1/2}, \tag{37}$$

which for $\alpha \gg 1$ reduces to the approximation

$$\tau_d = \sqrt{2}\,\alpha. \tag{38}$$

Equation (38) is also the result obtained from quasi-steady theory. Obviously, the approximate theory of (37) breaks down for area ratios near unity since it predicts $\tau_d \to 0$ when $\alpha \to 1$. For area ratios $\alpha \gtrsim 10$, equations (35), (37) and (38) are indistinguishable for practical purposes. For example when $\alpha = 10$ exact theory gives $\tau_d = 14.10$, while the approximate and quasi-steady values are 14.07 and 14.14 respectively. We conclude that for all area ratios of practical interest, unsteady effects are negligible.

The quasi-steady theory starts to diverge somewhat from the exact theory below $\alpha = 5$ but curiously, for free fall, when $\alpha = 1$, they both yield $\tau_d = \sqrt{2}$. This is all the more remarkable when we consider that the free-surface velocity according to quasi-steady

* After completion of this analysis it was brought to our attention that Kaufman, in a book reference 8, recently translated from the German, has also treated the problem of reservoir discharge. For a general area distribution he obtains a quadrature similar to equation (22), and then for a cylindrical container he puts $\ddot{y} \approx 0$, and obtains eq. (38) modified by a velocity-coefficient correction factor.

theory is wrong at every instant except one, and that it varies in the wrong sense with respect to time. The reason it yields the same discharge time is that, if in the free-surface velocity function, according to quasi-steady theory, we substitute $\tau_d - \tau$ for τ we obtain the correct history in a free fall. Integration of either function of course must yield the same discharge time.

Discussion of Stary's solution. By neglecting the free-surface acceleration Stary obtains [eq. (4) of his note rewritten in the present notation] $\tau_d = \left[2(k_1\alpha^2 - k_2)\right]^{\frac{1}{2}} \approx k_3\sqrt{2}\,\alpha$ where k_1, k_2, k_3 are correction factors, and where $1.04 \leq k_3 \leq 1.08$. He does not explain how to choose the appropriate value for k_3. Stary terms this the quasi-steady solution. If we put $k_3 = 1$ then his result reduces to (38), which we have called the approximate unsteady solution. Of course for the cylindrical reservoir there is little distinction between the two. However for the conical reservoir the quasi-steady solution underestimates the actual discharge time which, on the other hand, is given quite accurately by the approximate theory. In other words neglecting the free-surface acceleration is not equivalent to the quasi-steady approximation.

Stary also obtains a solution of the complete equation which gives the ratio of the exact discharge time to the approximate discharge time. The ratio is less than 1.02 for $\alpha \gtrsim 10$ and approaches unity as the area ratio becomes large. If the correction factors are set equal to unity then his solution reduces to equation (35). Thus except for the correction factors the unsteady Bernoulli equation and the one-dimensional theory are equivalent.

The sharp-edged orifice. So far we have said nothing regarding flow conditions at the exit. For the conical reservoir, as long as the cone is slender, the contraction effect of the jet is small as our experiments show. For the cylindrical reservoir however, with a sharp orifice at the exit -- the problem considered by Stary -- the jet leaves with a significant radial component of velocity inward which causes a radical contraction of the jet area downstream of the exit. The effect on the discharge time is enormous as shown by Figure 6 in which Stary's approximate solution, with $k_3 = 1.08$, is compared with experiment for an initial free-surface height of $z_2 = 11.25$ in. The discrepancies are too great to be attributable to lack of precision in the correction factor k_3. Instead, following the usual procedure, it might be better to put $\tau_d = C\sqrt{2\alpha}$ where C is the inverse of a velocity coefficient, and is evaluated from the experimental data. Whether or not it is valid to employ a non-time-dependent velocity coefficient in an unsteady flow is not known.

Whether or not to treat an unsteady flow situation by quasi-steady theory depends on factors which are not easy to pinpoint.

Sabersky and Acosta, in a recently published text book reference 10, treat the flow out of a cylindrical reservoir by quasi-steady theory and conclude rightly that unsteady effects are negligible for area ratios $\alpha \gtrsim 10$. On the other hand their analysis is equally applicable without alteration to a reservoir of arbitrary geometry, a conical reservoir for example. As we have shown, even for area ratios up to 40, quasi-steady theory for the conical reservoir predicts discharge times from two to eight percent less than the approximate theory which itself is less than experiment. Thus it is our opinion that their method for determining a criterion for neglecting unsteady effects is faulty and should be used only with caution.

Theoretical correction for the nozzle length. Rather than introduce the additional complexity of a velocity coefficient we have elected to deal with the configuration of Figure 4b where a short transition section and straight nozzle enable the jet to be emitted without any radial component of velocity at the exit. The velocity coefficient of a well-shaped, smooth nozzle can be expected to be near unity except when the length/diameter ratio becomes large. In this event viscous effects may become non-negligible as can be visualized in the limit of decreasing diameter as the nozzle becomes a capillary tube.

For a short nozzle-plus-transition-section length the associated volume of liquid is negligible with respect to the whole. Therefore for analytical purposes the flow terminates effectively when $z = z_1$. By subtracting the dimensional discharge times for two flows of original heights z_2 and z_1, and then non-dimensionalizing using z_2 as the reference length, we obtain the discharge time, denoted τ_ϵ, for the nozzle configuration

$$\tau_\epsilon = \sqrt{2}\,\alpha\,(1 - \epsilon^{1/2}), \tag{39}$$

where $\epsilon \equiv z_1/z_2$. Equation (39) is based on quasi-steady theory but it is easy to show that it is also valid for exact theory to a close approximation, for values of ϵ not too near unity.

EXPERIMENTAL RESULTS FOR THE CYLINDRICAL RESERVOIR

Although measurement of the discharge time for a cylindrical reservoir appears almost trivial to carry out there are several surprises for the unsuspecting. It is possible that failure to cope with these unforeseen effects accounts for the dearth of published data for this flow.

Effect of nozzle shape. Early measurements with a straight exit preceded by a transition section which was simply a 90° circular arc section of 3/8 in. radius, resulted in discharge times substantially greater than predicted by theory, presumably due to flow separation from the nozzle wall. We then went to the nozzle shown in Figure 4b in which the inlet was shaped to contour A of Figure 5. This contour, which is not based on a theoretical design, was employed for all of the runs with the exception of a few at d = 3 in. We shall return to this point shortly. Details of the apparatus are given in the Appendix.

Free-surface collapse. Another phenomenon occurs for the smaller area ratios (for d = 1 in. and above). As the free-surface height decreases, a point is reached where the center is visually lower than that of the liquid adjacent to the walls. This is a consequence of the pressure gradient normal to the streamlines, near the transition section, as the fluid makes a turn more or less parallel to the nozzle boundary. When conditions are favorable the free surface, which remains at atmospheric pressure, pops through the nozzle forming a long finite cavity inside the jet. This destroys the almost one-dimensional nature of the flow, decreases the effective jet cross-sectional area, and consequently increases the discharge time drastically.

To eliminate this interesting but unwanted effect we employ a thin balsa wood disc, a little smaller in diameter than the reservoir, and which floats on the free surface. Surface tension "seals" the free surface and eliminates the popping through entirely.

Although this phenomenon does not occur for the smaller holes (d_1 = 0.25 and 0.5 in.), use of the disc is helpful there also.

As the free surface approaches the terminal level z_1, surface tension and friction become more important. The result is that, without the disc, instead of the flow terminating in a sharp cutoff it tails off gradually into a dripping process. With the disc however, there is a distinct transition from a continuous jet to the discrete droplet stage, which was taken to signal the end of the flow. We have not heard of this device being exploited elsewhere but if there are industrial processes where cavity formation is a problem (such as in the draining of a large tank of molasses) it may have some potential usefulness.

<u>Turbulence in the jet.</u> Even with the improved nozzle the flow in each case is initially turbulent on the jet boundary. This is understandable since the Reynolds' number, based on the initial quasi-steady velocity, and for water at $75^{\circ}F$, is 24,000 for the one-quarter inch hole, which value exceeds the minimum* critical Reynolds number 2300 and is great enough almost to guarantee the appearance of turbulence. To the eye the scale of turbulence increases as the exit diameter is increased. As the free-surface level falls the outflow Reynolds number decreases. Eventually a point is reached where the jet suddenly becomes laminar. The smaller the hole the sooner this occurs.

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* Ref. 7, page 35.

Experimental results. Experiment and theory are compared in Figure 7. Runs were made for an overall nozzle-plus-transition length of $z_1 = 1.25$ in. and a height of $z_2 = 25.15$ in., which correspond to $\epsilon = 0.0497$ and $1-\epsilon^{\frac{1}{2}} = 0.778$. Each experimental point is the average of two runs. In no case does the deviation of any run exceed one percent of its corresponding average. Because a log-log plot tends to obscure the magnitude of error, a summary of our results appears as Table I.

TABLE I
Experiment and Theory for the Cylindrical Reservoir

α	z_2 (in.)	ϵ	d_1 (in.)	τ_ϵ Eq. (39)	t_ϵ (sec.)	τ_ϵ (exp.)	Dev. (%)	Contour
3080	25.15	0.0497	0.25	3384	939.1	3679	8.7	A
770.1			0.50	846.1	227.8	892.5	5.5	
192.5			1.00	211.5	54.6	213.9	1.1	
85.56			1.50	94.00	24.6	96.38	2.5	
48.13			2.00	52.87	13.63	53.40	1.1	
30.80			2.50	33.84	9.25	36.24	7.1	
21.39			3.00	23.50	7.90	30.95	31.7	A
21.39			3.00	23.50	6.60	25.86	10.0	B
21.39	25.15	0.0497	3.00	23.50	6.60	25.86	10.0	C
3080	13.25	0.0943	0.25	3018	511.3	3309	9.3	A
192.5	13.25	0.0943	1.00	188.6	35.1	189.5	0.6	A

$z_1 = 1.25$ in. $d_2 = 13.88$ in.

There is some disagreement between theory and experiment on both ends of the curve. For large values of the area ratio (small diameter holes) it was foreseen that viscous effects would be greatest. A crude idea of the magnitude of the viscous contribution can be obtained as follows: for an initial Reynolds number of 24,000, and a nozzle length of 0.75 in., theory of turbulent flow through a smooth pipe predicts a head loss of 1.9 in., this from an initial head of about 25 in. For the three-inch hole the corresponding Reynolds number is 290,000 and the head loss 0.09 in. A theory for variable Reynolds number is not developed, but Kaufman, reference 8 page 123, gives an approximate scheme to handle it.

As the initial Reynolds number becomes larger the deviation of experiment from theory becomes almost negligible and continues so over most of the central region of the curve. Thus we conclude that viscous effects are of importance only for the smaller nozzles, more specifically when the length/diameter ratio exceeds unity. We also conclude that there is no need to incorporate the kinetic energy and momentum correction factors of Kozeny and Stary in the analysis. Only by putting $k_3 = 1$ would Stary's solution agree closely with experiment for values of the area ratio near $\alpha = 200$. Furthermore, as we have indicated, for large values of α the deviation is due to viscous effects and not the failure of the one-dimensional theory to account for variations of the kinetic energy and momentum flux in the transverse direction.

For $d_1 = 3$ in., the largest diameter nozzle tested, uncorking the hole results in a rather startling gush of water. The theoretical running time is only about six seconds. For contour A the actual discharge time was 30% above theory. For this configuration the jet was initially turbulent but turned laminar much sooner than for any other diameter. It is probable that the flow separated from the transition section wall although no attempt was made to verify this.

Instead, another run was made using contour B, which has a slightly more gradual transition terminating in a one-half inch long straight nozzle. This brought the experimental time to only 10% above theory. A third run, using contour C, resulted in no additional reduction in the discharge time. Since the theory must be right in the limiting case of free fall, and since the experimental deviation from theory is apparently increasing with decreasing α -- the trend can also be detected for $d_1 = 2.5$ in. -- we conjecture that separation still occurs for contours B and C although these runs remained turbulent much longer than for contour A. No elementary theory can account for this as it involves separation of either or both turbulent and laminar unsteady boundary layer on a curved wall. It might be possible by a better nozzle design, e.g., adapting the method of Thwaites, reference 11, to maintain attached flow throughout the run.

Several runs were made for the nozzle of contour A with z_2 = 13.15 in., ϵ = 0.0943. The agreement with theory was essentially the same as for ϵ = 0.0497, and the data are omitted from the graph. It is important to note, however, that even for a relatively short nozzle the factor $1 - \epsilon^{\frac{1}{2}}$ may be significantly less than unity.

CONCLUSIONS

From the two configurations investigated there is strong evidence that unsteady, inviscid, liquid flow from a reservoir, with a free surface, should be treated by quasi-steady theory only when variations in the cross-sectional area of the reservoir are restricted essentially to the neighborhood of the exit station.

On the other hand it appears that the discharge time of such a flow can be determined quite accurately by one-dimensional theory if the acceleration of the free surface is neglected. Integration of the resulting equation is called the approximate unsteady solution. The approximate solution is found not to differ significantly from a numerical solution of the exact equation for the conical reservoir, whereas for certain ranges of the height ratio it may deviate considerably from the quasi-steady solution. Agreement between experiment and theory for a slender conical reservoir is good.

Theoretical discharge times for the cylindrical reservoir configuration are about the same for all three theories as long as the area ratio is ten or above. For a cylindrical reservoir certain precautions are essential if meaningful measurements are to be obtained. In the first place to avoid collapse of the free surface it is necessary to use a thin, balsa wood disc which rides on the free surface. Furthermore to eliminate the jet contraction at the exit a short transition-section-plus-nozzle must be employed. If the length/diameter ratio of the nozzle is large compared to unity, and if the Reynolds number based on the instantaneous outflow velocity is not too high, frictional effects may be important. If the ratio is small, there appears to be a tendency of the jet to separate from the nozzle wall which results in larger experimental discharge times than predictable from theory. It ought to be possible by suitable design to avoid this separation. Introduction of the nozzle requires a correction to the theoretical discharge time which may be large even for short nozzles.

On the question of one-dimensional theory versus the non-steady Bernoulli equation, with weighting factors to account for non-uniformities, we conclude that the former is the more accurate. In one extreme, where experiments deviate from the one-dimensional theory, the error is due to neglected fluid viscosity which neither theory pretends to account for. The other indications are that the discrepancy is due to inadequate design of the nozzle contraction. Thus we do not believe that the use of the Kozeny-Stary correction factors can be justified for unsteady flows. In fact we suggest that their validity even in steady flows should be re-examined.

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APPENDIX

THE CONICAL RESERVOIR

The conical reservoir, Figure 1, was made in two sections. The lower part, Figure 8, which consists of three separate pieces bolted together, was machined from aluminum round stock, and provided with O-ring seals at the horizontal joints. To the upper end of this section is attached an extension (not shown) fabricated from 0.060 inch sheet metal, rolled to the conical shape. Spot checks of the extension show that the maximum deviation is less than one percent across any diameter at the upper station. The total height of this reservoir from the apex of the cone, extended is 30 inches.

To restrain the fluid and to start the flow we use a simple hinged arm supporting a rubber pad which is pressed against the orifice lip. The device is held in place by a trigger prior to starting. This arrangement works well enough since it does not seem to contribute any transients of its own.

THE CYLINDRICAL RESERVOIR

For the cylindrical reservoir, Figures 4a and 4b, an empty 120 lb. commerical grease drum is employed. The sharp-edged orifice is obtained by machining a hole in the drum bottom.

The nozzle configuration uses the same drum with the nozzle of

Figure 9 bolted to the bottom. The nozzle is enlarged by machining as we proceed from the one-quarter-inch to the three-inch hole. Rubber stoppers are used to cork the supply.

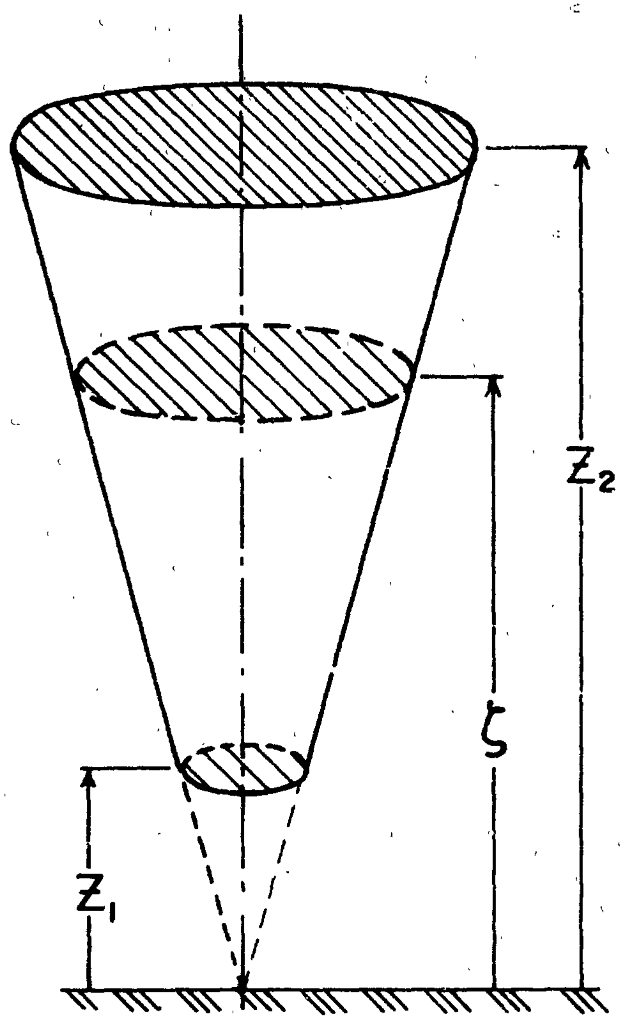

Fig. 1 The conical reservoir

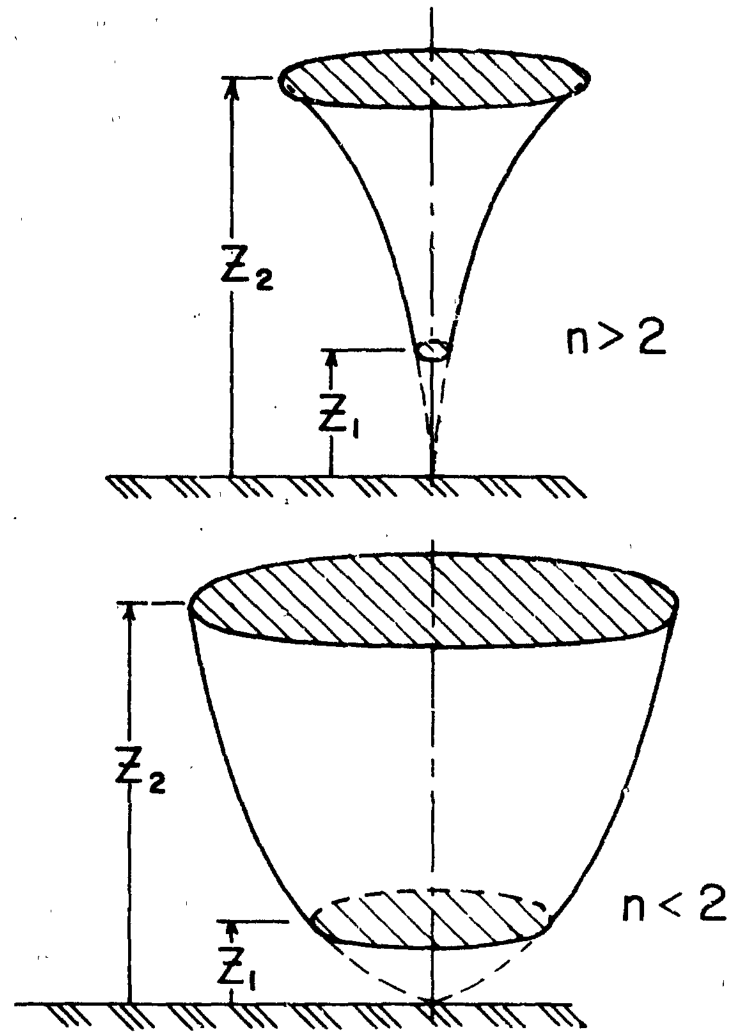

Fig. 2 Reservoirs with $A = kz^n$

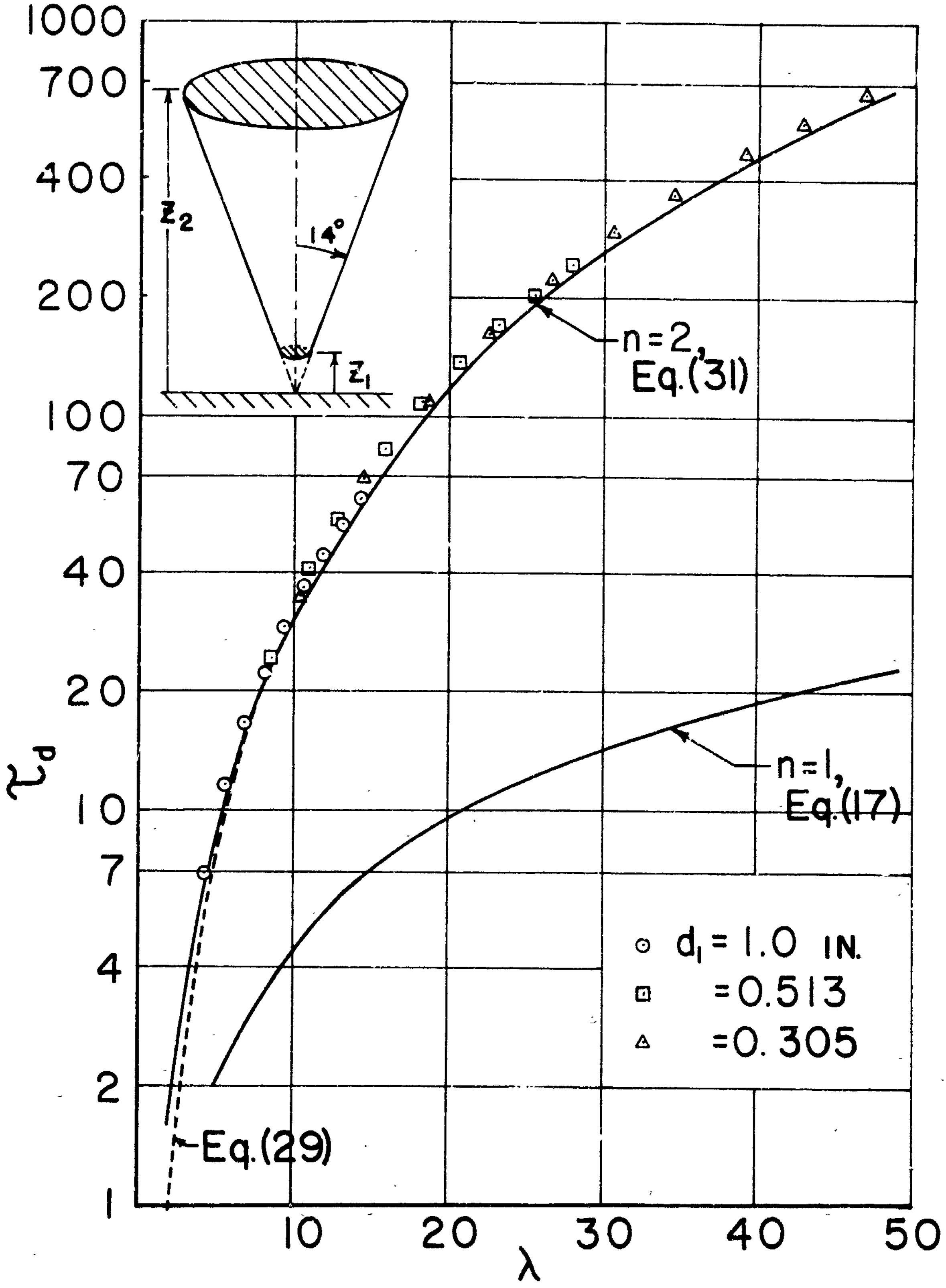

Fig. 3 Discharge times for n = 1, 2

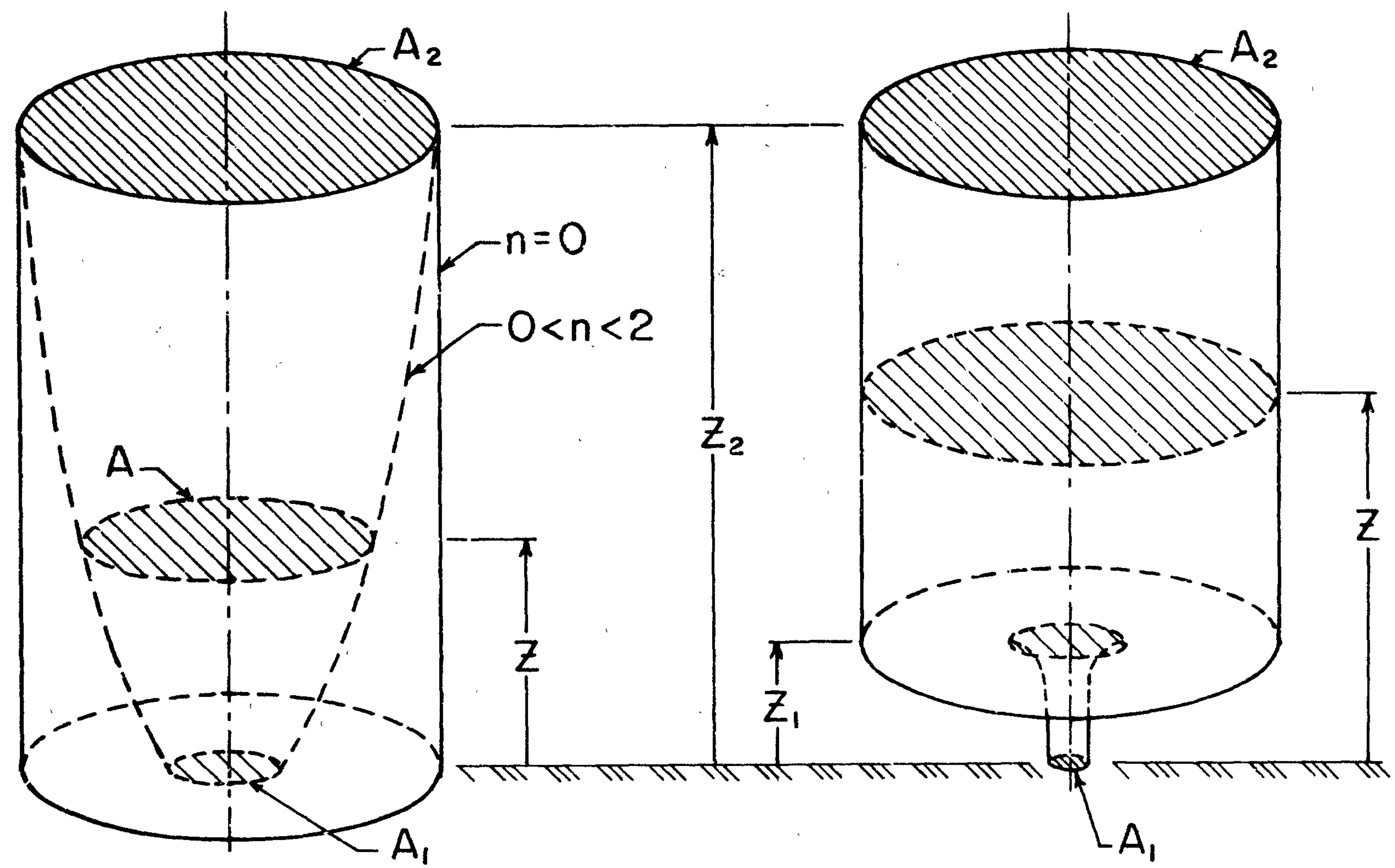

Fig. 4a Cylindrical reservoir as a limiting case of $n \rightarrow 0$

Fig. 4b Cylindrical reservoir with nozzle outlet

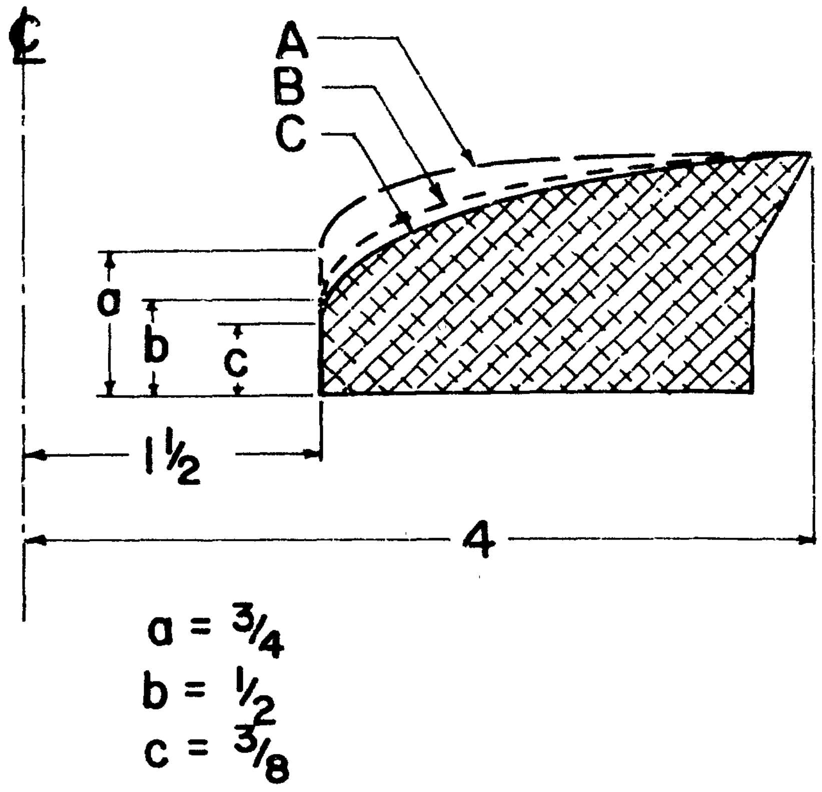

Fig. 5 Nozzle contours

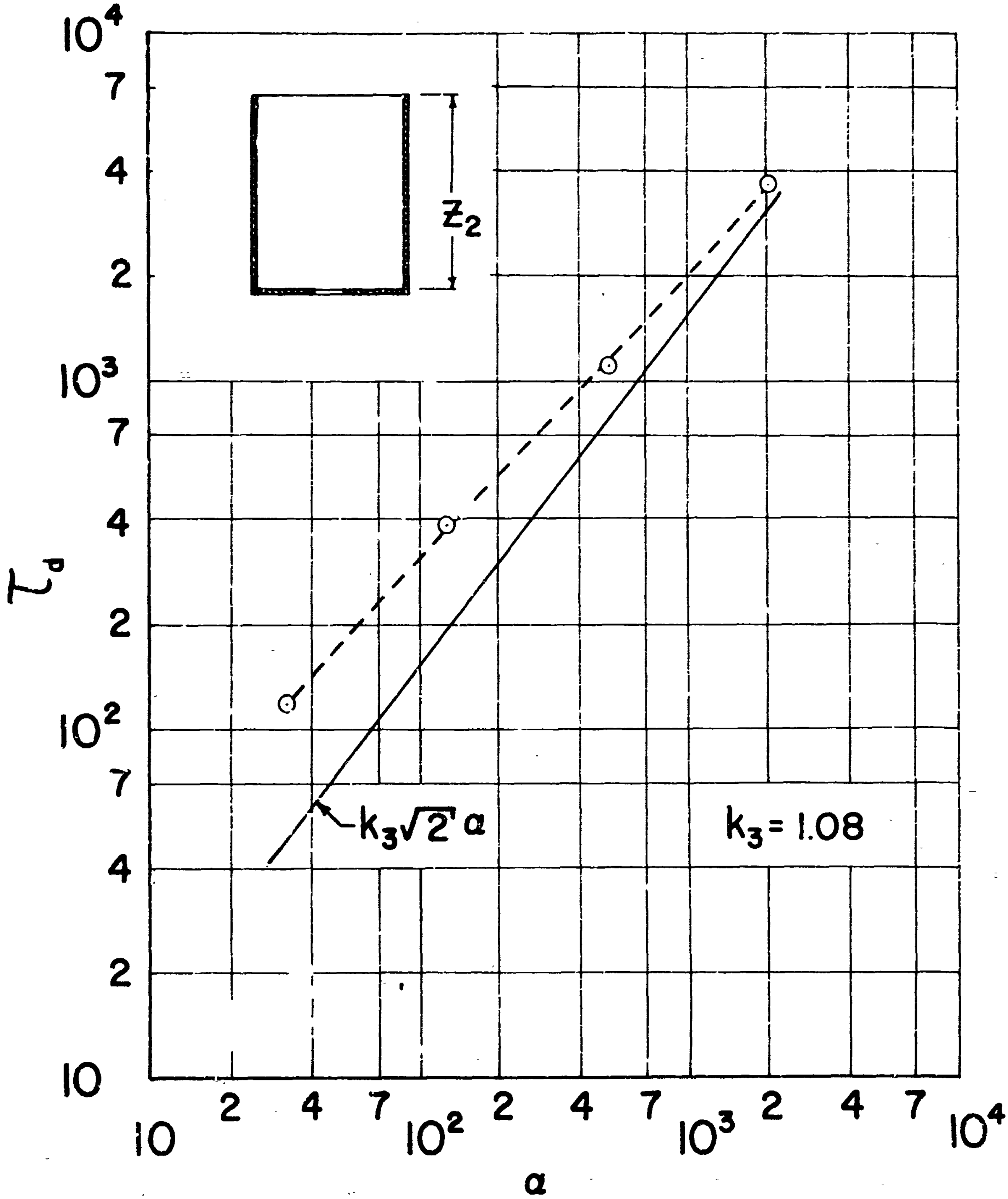

Fig. 6 Discharge times for sharp-edged orifice

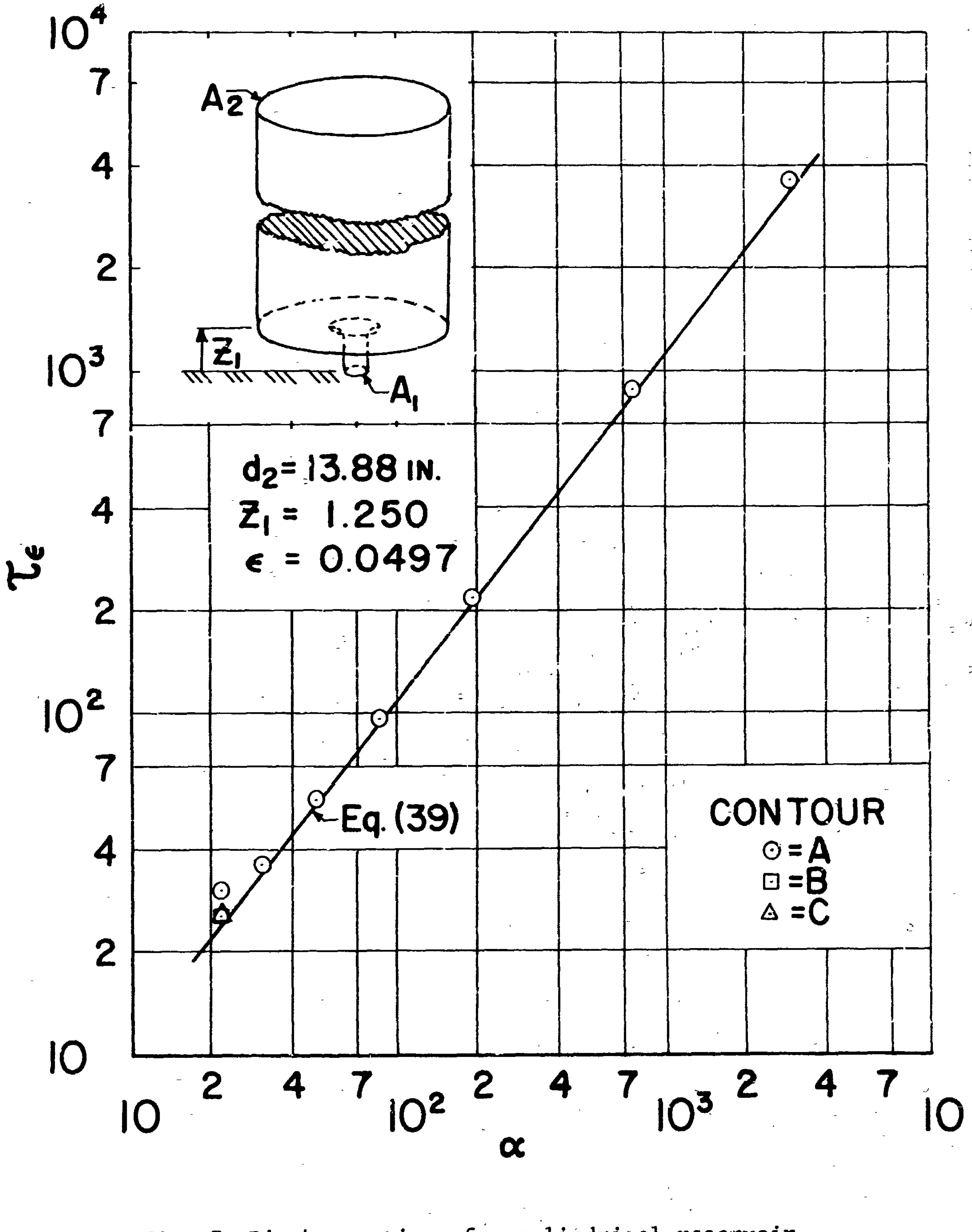

Fig. 7 Discharge times for cylindrical reservoir

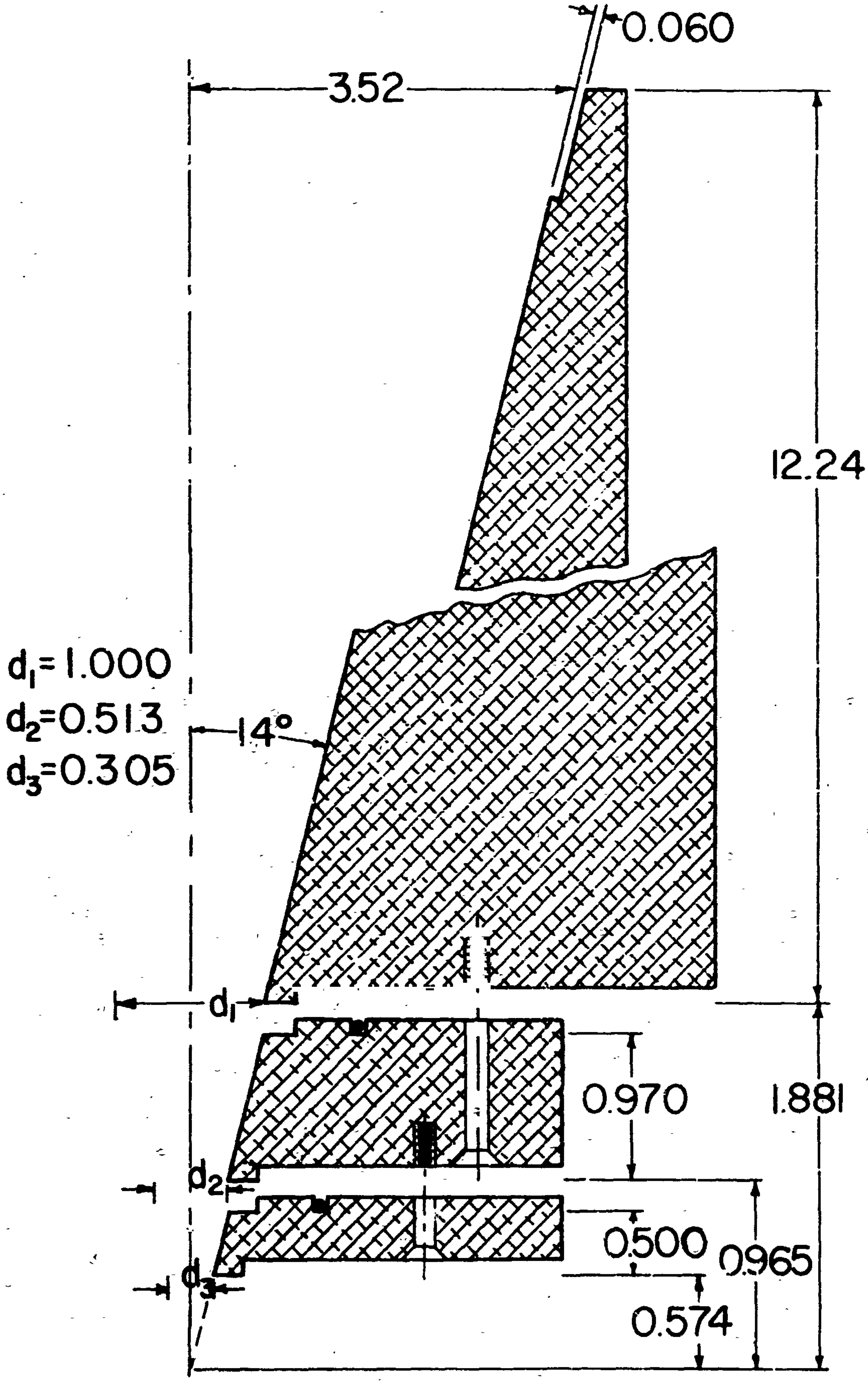

Fig. 8 Dimensions of conical reservoir

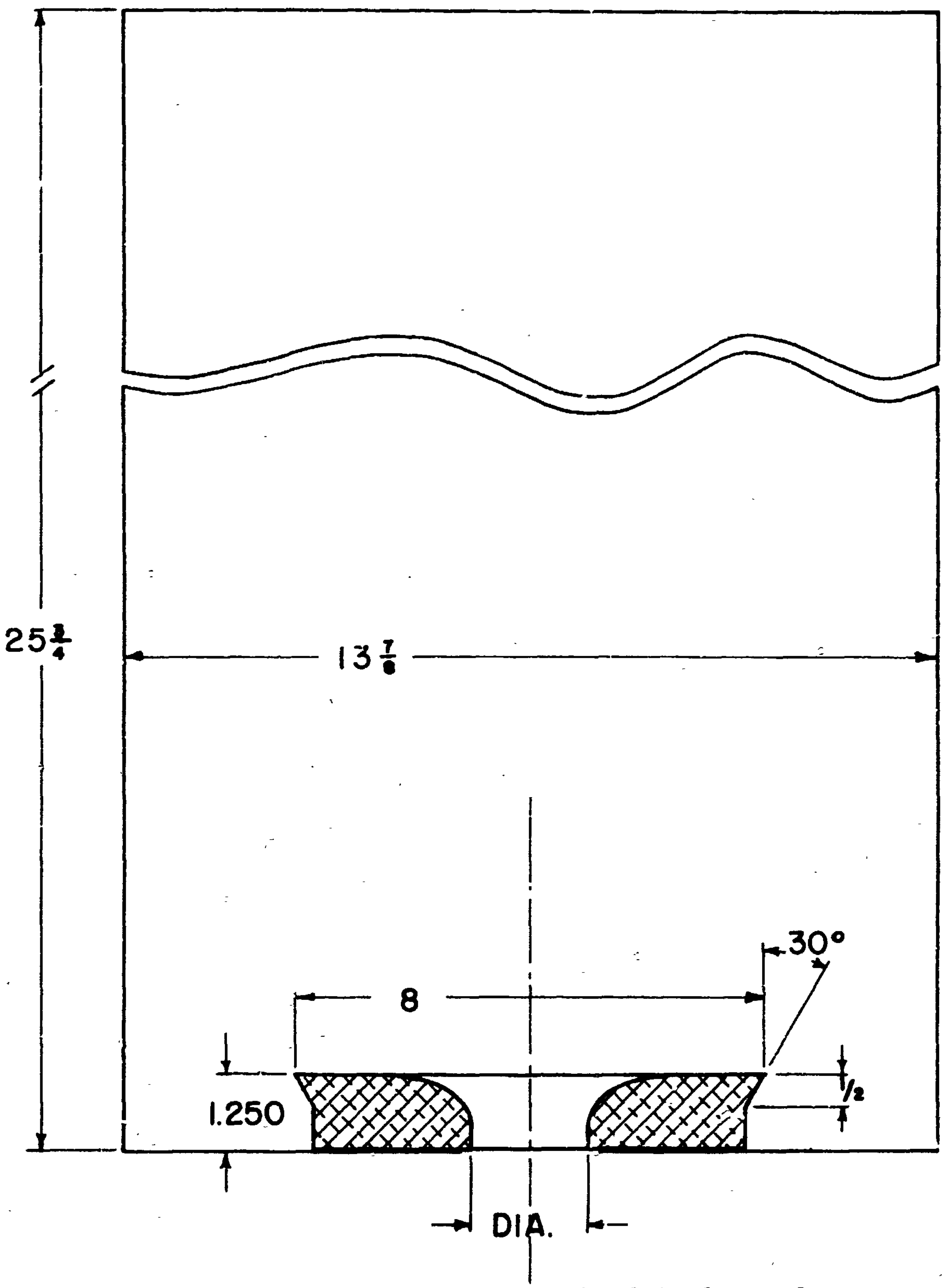

Fig. 9 Dimensions of cylindrical nozzle