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TECHNIQUES FOR EXAMINING STATISTICAL AND POWER-SPECTRAL PROPERTIES OF RANDOM TIME HISTORIES

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TECHNIQUES FOR EXAMINING STATISTICAL AND POWER-SPECTRAL PROPERTIES OF RANDOM TIME HISTORIES*

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SUMMARY

A technique is presented for digitally generating random time histories having any desired shaped power spectra. Four random time histories having different statistical and power-spectral properties have been generated and analyzed to determine their instantaneous mean and amplitude distributions. In each, the distribution of instantaneous means could be approximated by a normal or Gaussian distribution and the distribution of instantaneous amplitudes could be approximated by the sum of a Rayleigh distribution and a normal distribution. An attempt was made to relate the coefficients of the equations used to represent the distributions of means and amplitudes to the power-spectral properties of the generated time histories. Two of the coefficients could be related to the power-spectral properties of the time histories. The remaining two coefficients were empirically determined since no apparent relationship was found between these coefficients and the power-spectral properties of the generated random time histories.

INTRODUCTION

Many of the loads encountered by aircraft and missiles are random in nature and, consequently, are usually described statistically. In order to reduce the mathematical complexity in utilizing such a description in analyzing the response of structures to loads, most investigators have made simplifying assumptions about the statistics of the random-load history (ref. 1). As an example, in fatigue studies the statistics of the load peaks are usually used. These statistics are obtainable either by actually counting the peak loads at various levels or by a relationship developed by Rice (ref. 2) which relates the peak load distribution to the power spectrum of the random load-time history. When programing fatigue tests, all peak loads are usually applied about a common mean load. In general, this mean load is representative of the overall mean

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of the random load-time history from which the peak load distribution was derived. A variation in mean load can have an effect on fatigue life (ref. 3). Thus, it appears that a statistical description of both the distribution of instantaneous mean loads (i.e., average of two successive peak loads) and associated instantaneous amplitude distributions (i.e., difference between peak load and instantaneous mean load) would be more useful than the peak load distribution alone for studying fatigue under random loading. The instantaneous mean load distributions and associated instantaneous amplitude distributions are hereafter referred to as simply the mean and amplitude distributions. These distributions can be obtained by actually counting the instantaneous means and instantaneous amplitudes, but there is no known relationship between these distributions and the power spectrum as was the case for the peak load distribution.

In the present investigation, an attempt is made to develop an empirical relationship between the power-spectral properties of a given random time history and the mean and amplitude distributions of this time history. This was done by digitally generating four random time histories with different power-spectral properties and counting the means and amplitudes in order to determine their distributions for each of the time histories generated. Equations describing the mean and amplitude distributions are developed and an attempt is made to relate the coefficients of these equations to the power-spectral properties of the generated time histories.

SYMBOLS

A amplitude of periodic function of time

a_K filter factors or Fourier coefficients

$F\left(\frac{f}{f_F}\right)$ frequency-response function

f frequency, cps

f_F Nyquist or folding frequency, cps

f_a number of amplitudes counted in interval $y_i \leqq y \leqq y_{i+1}$

f_b number of amplitudes counted which exceed $y = y_i$

f_c computed frequency of occurrence of amplitudes in interval $y_i \leqq y \leqq y_{i+1}$

f_o number of times per second zero axis is crossed with positive slope

f_p number of positive peaks per second

f_y	number of times per second that value y is exceeded
f_{y_i}	computed number of amplitudes which exceed $y = y_i$
K_1, K_2	constants of peak probability distributions
L_h	raw spectral density estimate
N_N	number of amplitudes normally distributed about specified mean value
N_R	number of amplitudes distributed according to Rayleigh distribution about specified mean value
N_T	number of positive amplitudes about specified mean value
P_N	normal probability
P_R	Rayleigh probability
$P_p(y)$	probability that peak will exceed given value of y
p	modified random number
$R(\tau)$	covariance function or autocorrelation function of continuous variable
R_N	generated random number
R_p	covariance or autocorrelation of discrete set of values
t	time, sec
Δt	uniform interval of time, sec
X	normal deviate
$Y(t)$	filtered time history
Y_i	discrete set of values obtained by sampling filtered time history $Y(t)$ at uniform intervals of time Δt
$y(t)$	original time history
y_i	discrete set of values obtained by sampling original time history $y(t)$ at uniform intervals of time Δt
z	standard variable, y/σ_y
α, β	dummy variables

σ_N	coefficient of normal probability
σ_R	coefficient of Rayleigh probability
σ_y	standard deviation or root-mean-square (rms) value of $y(t)$
σ_y^2	mean square value of $y(t)$
$\sigma_{\dot{y}}^2$	mean square value of first derivative of $y(t)$
$\sigma_{\ddot{y}}^2$	mean square value of second derivative of $y(t)$
$\phi(f)$	power spectral density of a continuous variable
ϕ_h	smoothed spectral density estimate

Matrix notations:

$[\ ]$	square matrix
$\{\ \}$	column matrix

A dot over a variable indicates differentiation with respect to time.

A bar over a term indicates the mean value of the term.

GENERATION OF RANDOM TIME HISTORIES

A digital random time history having the properties of band-limited white noise was generated and used as the input to several linear systems, each having significantly different frequency-response characteristics. The output responses obtained were used to calculate power-spectral properties and also to obtain the distributions of means and amplitudes.

A brief outline of the procedures used to simulate digitally random time histories having different shaped power spectra is as follows. A more detailed discussion of each of the following steps will appear in subsequent paragraphs.

1. Random numbers having a uniform probability distribution were generated.

2. The uniformly distributed random numbers were then transformed into a normal or Gaussian distribution having a mean of zero and a variance of one. The numbers obtained were assumed to be samples taken at 1/2-sec intervals from a continuous record. A power spectrum was calculated by using these numbers and was found to be essentially flat. The normally distributed numbers will be used as the input to a linear system.

3. In order to determine the frequency response of the linear system the following equation was used:

$$\phi_{in}\left|F\left(\frac{f}{f_F}\right)\right|^2 = \phi_{out}$$

where ϕ_{in} is the power spectrum of the normally distributed numbers obtained in step 2 above and ϕ_{out} is the desired shaped power spectrum. Knowing both the input and the desired output power spectrums, the magnitude of the frequency response $\left|F\left(\frac{f}{f_F}\right)\right|$ can be determined from the above relationship.

4. The frequency response was then used to filter the input (i.e., the normally distributed numbers) to the linear system. The filtering was done by utilizing the following equation:

$$Y_i = \sum_{K=-M}^{M} a_K y_{i+K}$$

where y_{i+K} is the input to the system, a_K represents the filter factors obtained by transforming the frequency response into the time domain, and Y_i is the output which represents a random time history having the desired shaped power spectrum.

5. Four time histories were generated in this manner. Power spectrums were calculated for each in order to insure that the proper filter factors had been obtained.

6. Once it was determined that the calculated power spectrums were essentially the same as the desired shaped power spectrums, the digital random time histories of step 5 were analyzed to determine their instantaneous mean and amplitude distributions.

The procedure described can be used equally well for digitally simulating other random time histories having arbitrarily shaped power spectra.

RANDOM NUMBER GENERATOR

Random numbers were obtained with a fixed-point pseudo random number generator developed by the National Bureau of Standards (ref. 4). Each generated random number R_N was obtained from the previous random number R_{N-1} by taking the last 11 digits of the product $R_0 R_{N-1}$ where $R_0 = 5^{15}$ and $N = 1, 2, 3, \ldots$. Numbers were then selected at random from the generated R_N. Only the first 6 digits p of the randomly selected 11-digit number were used in this investigation. Approximately 160 000 random numbers were selected in this manner each having an equally likely chance of occurring (i.e., uniform probability distribution). The set of numbers obtained were all greater than or equal to zero but less than or equal to 999 999.

TRANSFORMATION TO NORMAL DISTRIBUTION

The random numbers were transformed into a normal distribution with mean equal to zero and unit variance by an approximate equation developed by Tukey. (See ref. 5.) This transformation was made in order to simulate a stationary, Gaussian random process. The transformation requires that the random numbers be between zero and one. Therefore, all numbers p were divided by 10^6 and designated q. Tukey's transformation is

$$X' = 4.91\left[q^{0.14} - (1 - q)^{0.14}\right] \tag{1}$$

where q is the modified random number and X' is the normal deviate. It was found in this investigation that when X' became greater than 2.4, there were significant departures from the normal distribution. Hence, it was necessary to use a corrected normal deviate X, as follows:

$$\left.\begin{aligned} X &= X' && \left(|X'| \leqq 2.4\right) \\ X &= X' + \frac{X'}{|X'|}(0.13)(X' - 2.4)^2 && \left(|X'| > 2.4\right) \end{aligned}\right\} \tag{2}$$

It should be noted that these equations restrict the normal deviates to the range $-5.73 \leqq X \leqq 5.73$ which is no great handicap.

A power spectrum was calculated by using equations (B1) to (B3) and the first 40 000 normally distributed random numbers X. Due to storage limitations in the computer used, only 5000 numbers could be handled at one time. Therefore, 8 power spectra were calculated for each of the first 8 groups of 5000 numbers generated. The 8 power spectra were found to be essentially flat and varied only slightly from each other, indicating that the sample size of 5000 numbers was sufficiently large. An average power spectrum was obtained from the 8 groups of numbers (white noise) and the remaining properties were calculated based on this average spectrum.

FILTERING OF RANDOM NUMBERS

The generated random numbers X, which when taken at discrete uniform intervals of time define a time history having a flat power spectrum, can be modified by numerical filtering techniques in order to change their amplitude response characteristics and thus change the power spectrum of the time history. The amplitude response characteristics can be changed by utilizing the input-output relation of power-spectral analysis, which states that the product of the input power spectrum $\phi_{in}(f)$ and the square of the amplitude response $\left|F\left(\frac{f}{f_F}\right)\right|^2$ (sometimes called a transfer function) is equal to the output power spectrum $\phi_{out}(f)$. Thus,

$$\phi_{out}(f) = \left|F\left(\frac{f}{f_F}\right)\right|^2 \phi_{in}(f) \tag{3}$$

The amplitude response $\left|F\left(\frac{f}{f_F}\right)\right|$ can be determined from this equation since $\phi_{in}(f)$ is the flat power spectrum obtained by calculation from the generated random numbers and $\phi_{out}(f)$ is the specified or desired power spectrum. The amplitude response defines the changes that have to be made in the frequency domain in order to obtain the desired shaped power spectrum. These changes can be reflected in the time domain by taking the Fourier transform of the amplitude response. A time history comprised of discrete values Y_i and having the desired shaped power spectrum can be calculated with the use of the following equation:

$$Y_i = \sum_{K=-M}^{M} a_K y_{i+K} \tag{4}$$

where $y_{i+K} = 0$ when $i < M$. The Fourier coefficients a_K result from the Fourier transform of the amplitude response and the generated random numbers X are represented by $y_{i\pm K}$. Details for determining the coefficients of the Fourier cosine series representation of the amplitude response are given in appendix A. The four amplitude response functions used in this investigation are shown in figure 1. The symbols show the shape of the response actually used to filter the random numbers whereas the solid curve shows the desired response. Twenty points were used to represent this response. These four amplitude response functions represent the concepts of bandwidth-limited white noise, atmospheric turbulence phenomena, single-degree-of-freedom system, and a modified single-degree-of-freedom system (band pass), respectively. For brevity the filtered time histories obtained by using these response functions are referred to as time histories A to D, respectively. Statistical samples showing the characteristically different features of the four time histories obtained in this manner are shown in figure 2. For clarity, the values of Y_i have not been plotted but rather the curves faired through these values. The increment of time is $\Delta t = 1/2$ sec between values of Y_i. As a reference, $10\,\Delta t$ is shown in figure 2.

For the normally distributed numbers a power spectrum was calculated for each set of filtered random numbers using equations (B1) to (B3) and the first 40 000 numbers in each set. The power spectra were obtained by averaging the power spectra of 8 groups of 5000 numbers. Power-spectral properties were calculated based on the average power spectrum. This procedure was followed in order to determine whether the filtered time histories had power spectra equivalent to the specified or desired power spectra. The calculated power spectra were equivalent, within small tolerances, to the specified power spectra.

In calculating these power spectra, the assumption was made that the filtered random numbers represented a sampling from a continuous time history $y(t)$ at discrete uniform intervals of time $\Delta t = 1/2$ sec which resulted in a discrete set of values Y_i when $t = i\,\Delta t$. There is no loss of information from this sampling if the time history $y(t)$ contains no frequencies greater than the Nyquist or folding frequency f_F where

$$f_F = \frac{1}{2\,\Delta t} \tag{5}$$

The frequencies f, $(2f_F \pm f)$, $(4f_F \pm f)$, . . . cannot be distinguished in any frequency representation of $y(t)$ which is determined from the values of y_i. Thus, frequencies greater than f_F will appear to be in the range $0 \leqq f \leqq f_F$. It is said then that all the frequencies in $y(t)$ have been folded into the range $0 \leqq f \leqq f_F$. This folding property follows from the relations

$$\sin 2\pi(2mf_F \pm f)t = \sin(2m\pi i \pm 2\pi ft) = \pm\sin 2\pi ft$$

where

$$i = 0, 1, 2, 3, \ldots$$

$$m = 1, 2, 3, 4, \ldots$$

If frequencies higher than f_F are present in the sampling, they will appear to contribute power or energy to the lower frequencies which will result in errors in the power spectrum at the lower frequencies. This situation was automatically eliminated by properly selecting the frequency range of the shaped output power spectrum.

POWER-SPECTRAL-DENSITY CHARACTERISTICS

OF RANDOM TIME HISTORIES

In using the power-spectral-density approach for analyzing the fluctuations of a random process it will be assumed that the process is stationary (i.e., statistical properties are invariant with time) and also Gaussian in nature. The power spectrum or power spectral density $\phi(f)$ is a frequency distribution function which describes the frequency content of the time variation of a random disturbance $y(t)$. For stationary processes, the power spectrum $\phi(f)$ may be defined by the relationship which exists between $\phi(f)$ and the covariance or autocorrelation function $R(\tau)$. This relationship is expressed as a Fourier cosine transform pair as follows (ref. 6):

$$\left.\begin{aligned} \phi(f) &= 4\int_0^\infty R(\tau)\cos 2\pi f\tau \, d\tau \\ R(\tau) &= \int_0^\infty \phi(f)\cos 2\pi f\tau \, df \end{aligned}\right\} \tag{6}$$

The covariance function, which is the mean value of the product $y(t)y(t+\tau)$, gives a measure of the correlation between values of $y(t)$ separated by a time interval τ. Hence

$$R(\tau) = \overline{y(t)y(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t)y(t+\tau)\, dt \tag{7}$$

For the special case when $\tau = 0$

$$R(0) = \overline{y(t)^2} = \int_0^\infty \phi(f)\, df = \sigma_y^2 \tag{8}$$

The function $\phi(f)$ may be regarded as the contribution of any frequency f to the mean square of $y(t)$. The square root of the mean square value is known as the root mean square (rms) or standard deviation σ_y of $y(t)$.

The derivatives of a Gaussian random process are required in order to determine the statistics of such quantities as the number of times per unit time the disturbance crosses the axis $y(t) = 0$, the number of maxima of $y(t)$ per unit time, or the number of times per unit time that the disturbance exceeds a value of $y(t) = y_i$ where $i = 1, 2, 3, \ldots$. The following relationships are the ones developed by Rice (ref. 2) between the derivatives of $y(t)$ and the power spectral density $\phi(f)$:

$$\overline{\dot{y}(t)^2} = \sigma_{\dot{y}}^2 = \int_0^\infty (2\pi f)^2 \phi(f)\ df \tag{9}$$

and

$$\overline{\ddot{y}(t)^2} = \sigma_{\ddot{y}}^2 = \int_0^\infty (2\pi f)^4 \phi(f)\ df \tag{10}$$

The relationships involving these derivatives in obtaining zero crossings, peaks, and level crossings are as follows.

The number of times per second that the zero axis is crossed with a positive slope is

$$f_o = \frac{1}{2\pi} \frac{\sigma_{\dot{y}}}{\sigma_y} \tag{11}$$

The number of positive peaks per second is

$$f_p = \frac{1}{2\pi} \frac{\sigma_{\ddot{y}}}{\sigma_{\dot{y}}} \tag{12}$$

The number of times per second that a value of $y(t) = y_i$ is exceeded is

$$f_y = f_o \exp \frac{-y_i^2}{2\sigma_y^2} \tag{13}$$

(See ref. 6 for limits on y_i.)

Another relationship involving the derivatives of $y(t)$ can be used to obtain the probability $P_p(y)$ that a peak will exceed a given value of $y(t) = y_i$. The probability is expressed in terms of a standard variable z where $z = y_i/\sigma_y$.

The expression for the probability of obtaining a peak greater than y_i is (ref. 7)

$$P_p(y) = P_N\left(\frac{z}{K_1}\right) + \frac{f_o}{f_p} e^{-\frac{z^2}{2}}\left[1 - P_N\left(\frac{z}{K_2}\right)\right] \tag{14}$$

where $P_N\left(\frac{z}{K_1}\right)$ and $P_N\left(\frac{z}{K_2}\right)$ are the normal probabilities that $\frac{z}{K_1}$ and $\frac{z}{K_2}$ will be exceeded; that is

$$P_N\left(\frac{z}{K}\right) = \frac{1}{\sqrt{2\pi}} \int_{z/K}^{\infty} \exp \frac{-z^2}{2}\, dz \tag{15}$$

where

$$K = K_1 = \sqrt{1 - \left(\frac{f_o}{f_p}\right)^2} \tag{16}$$

or

$$K = K_2 = \frac{K_1}{f_o/f_p} \tag{17}$$

The previous relations are valid only for a stationary random process which is Gaussian in nature. A more detailed discussion of this subject may be found in references 6 and 8. For data-processing purposes, the operations representing these expressions are more conveniently expressed in other forms. Appendix B gives the expressions which are in a form amenable to digital computing (eqs. (B1) to (B6)).

ANALYSIS OF RANDOM TIME HISTORIES

An attempt is made in the present investigation to describe analytically the distributions of instantaneous means and amplitudes of several random time histories having different power-spectral properties and to develop relationships between these analytical expressions and the power-spectral properties of the random time histories used. This was done by analyzing four random time histories having different statistical and power-spectral properties which were generated with the aid of a digital computer. Some of the power-spectral properties of these time histories have been calculated and are given in table I. The distributions of instantaneous means and amplitudes were obtained by actually counting each mean and amplitude in the time histories. The number of occurrences of each of these values is listed in tables II to V. A discussion is presented of the distributions obtained by counting, the manner in which these distributions were described analytically, and the relationship between these distributions and the power spectra of the various time histories.

The frequency distributions of the means for the four time histories investigated are plotted in figure 3. All four distributions appear to be normally distributed. This normality was checked by plotting the probability of exceeding a given mean value on normal probability paper for each of the time histories (fig. 4). As a first approximation the means can be considered to be normally distributed.

The frequency distributions of the amplitudes f_a about specified means are tabulated in tables VI to IX. Only those distributions which were considered to have a sufficient sample size to be representative have been plotted in figures 5 to 8. The positive and negative amplitudes are approximately symmetrical. Amplitudes were considered to be either positive or negative depending on whether the slope of the line between successive peaks was positive or negative. The amplitude distributions are approximately symmetrical about the zero mean.

An equation, similar to the one developed by Rice (ref. 2) for determining the peak distribution, which gave the best fit to the data was found to represent the distributions of the amplitudes about any specified mean. The equation represents the sum of a normal distribution and a Rayleigh distribution and is expressed mathematically as follows:

$$f_c = N_N P_N + N_R P_R \tag{18}$$

where

f_c computed number of occurrences of amplitudes in range $y_i \leqq y \leqq y_{i+1}$

N_N number of amplitudes normally distributed

N_R number of amplitudes distributed according to Rayleigh distribution

P_N normal probability which may be expressed by the following expression:

$$P_N = \frac{1}{\sigma_N\sqrt{2\pi}} \int_{y_i}^{y_{i+1}} \exp \frac{-\alpha^2}{2\sigma_N^2}\, d\alpha \tag{19}$$

P_R Rayleigh probability which may be expressed by the following expression:

$$P_R = \frac{1}{\sigma_R^2} \int_{y_i}^{y_{i+1}} \exp \frac{-\beta^2}{2\sigma_R^2}\, d\beta \tag{20}$$

The general form for equation (18) is a modification of the peak probability distribution equation which is the sum of a normal and a modified Rayleigh distribution. The coefficient σ_R in the Rayleigh portion of the equation is the slope of the straight-line portion of the curve obtained from a plot of log of the cumulative frequency of amplitudes against the square of the amplitude for any specified mean. It was found that the coefficient σ_R remained approximately constant, regardless of the mean value. The following relationship was developed, by using a trial and error procedure, in order to relate the coefficient σ_R to some of the power-spectral characteristics of a random time history:

$$\sigma_R = \frac{f_o}{f_p} + \frac{\left(\sqrt{K_2} - K_1\right)^2}{\frac{f_o}{f_p} + f_p} \tag{21}$$

where f_o, f_p, K_1, and K_2 are derivable from the power spectrum of a random time history.

The coefficient σ_N in the normal portion of the equation which gave the best fit to the data was found to be related to the coefficient σ_R as follows:

$$\sigma_N = 1 - \sigma_R \tag{22}$$

The remaining coefficients in equation (18), namely N_N and N_R, were adjusted by a least-squares technique to give the best fit to the data (eqs. (C3) and (C4)). These coefficients might have some physical significance but for the purpose of the present paper they will be treated as being independent. The technique used is summarized in appendix C. Since the distributions of amplitudes are approximately symmetrical, only the positive amplitudes were used to determine the coefficients. The number of positive amplitudes at a specified mean N_T was found to be related to the coefficients N_N and N_R by the following expression:

$$N_T = N_R + \frac{1}{2} N_N \tag{23a}$$

which may be rewritten as follows:

$$1 = \frac{N_R}{N_T} + \frac{1}{2} \frac{N_N}{N_T} \tag{23b}$$

The ratio of N_R/N_T was found to be a nonlinear function of the mean, being fairly constant for means close to zero and becoming progressively smaller for larger means. An attempt was made to predict the quantity N_T by determining the probability of occurrence of the means and multiplying it by the total number of occurrences in the time history. Since it has already been established that the probability distribution of the means is approximately normal, all that is required is the standard deviation of the means. A relationship between the standard deviation of the means and the power-spectral characteristics of a random time history was found but the relationship was not sufficiently accurate to predict small standard deviations - that is, time histories C and D - and therefore not accurate enough to predict N_T. No apparent relationship was found between the power-spectral characteristics of a random time history and the coefficients N_N and N_R.

The coefficients derived to give the best fit to the observed frequencies for each of the time histories investigated are presented in table X. The computed frequencies based on these coefficients are given in tables VI to IX, and the observed frequencies are also given for comparison. In addition, the observed frequencies (open symbols) and the computed frequencies (solid symbols) are plotted in figures 5 to 8.

In the present paper, a strictly empirical approach was taken. An equation was fitted to the data using a least-squares technique and two variables, namely, N_N and N_R. Possibly a better fit could be achieved by adjusting the four coefficients σ_N, σ_R, N_N, and N_R simultaneously with a least-squares procedure. However, a strictly analytical approach would be more desirable. It is most probable that an expression could be derived analytically since the expression for amplitudes developed in this paper is quite similar to the expression for peaks developed by Rice (ref. 2). In addition, a definite relationship exists between the peaks and the means and amplitudes.

CONCLUSIONS

Four random time histories with significantly different statistical and power-spectral properties have been generated with the aid of a digital computer. The statistics of the means and amplitudes as well as the power-spectral characteristics have been obtained for each time history. The following conclusions have been drawn from an analysis of the data obtained:

1. The frequency distributions of the means are, in first approximation, normally distributed and symmetrical about a mean of zero.

2. The frequency distributions of the positive (or negative) amplitudes for a specified mean can be described by the sum of a Rayleigh and a normal distribution. The positive and negative distributions are approximately symmetrical. These distributions are also approximately symmetrical about a mean of zero.

3. The standard deviations of both the normal and Rayleigh distributions representing the frequency distributions of the amplitudes are essentially constant over the entire range of mean values and can be approximated from the power-spectral characteristics of the time histories.

4. The coefficients N_N and N_R in the general equation defining the distribution of amplitudes have been obtained empirically but no apparent relationship between these coefficients and the power-spectral properties of the time histories has been found.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., December 3, 1964.

APPENDIX A

DETERMINATION OF COEFFICIENTS OF A FOURIER COSINE SERIES

REPRESENTATION OF FREQUENCY-RESPONSE FUNCTION

Consider the continuous periodic function of time, amplitude A, and frequency f, such that

$$y(t) = A \cos 2\pi ft$$

If this function of time is sampled at discrete time intervals Δt, the continuous time history is replaced by a discrete set of values y_i equal to $y(t)$ when $t = i\ \Delta t$ and undefined in between. Thus,

$$y(t) = y_i = A \cos 2\pi fi\ \Delta t \qquad (t = i\ \Delta t)$$

When $\Delta t = \dfrac{1}{2f_F}$,

$$y_i = A \cos i\pi \frac{f}{f_F} \tag{A1}$$

The above time history can be modified to change its frequency characteristics (i.e., numerically filtered) as follows:

$$Y_i = \sum_{K=-M}^{M} a_K y_{i+K} \tag{A2}$$

where

Y_i	filtered time history
y_{i+K}	original time history
a_K	filter factors or coefficients
M	number of points used to approximate the amplitude response

Equation (A2) represents, in numerical form, the passage of an input signal $y(t)$ through some linear system which results in an output signal $Y(t)$. Upon substituting y_i from equation (A1) for y_{i+K} in equation (A2),

$$Y_i = A\left\{a_0 \cos \pi \frac{f}{f_F} i + \sum_{K=1}^{M}\left[a_{-K} \cos \pi \frac{f}{f_F}(i - K) + a_K \cos \pi \frac{f}{f_F}(i + K)\right]\right\}$$

Where $a_K = a_{-K}$

$$Y_i = A \cos \pi \frac{f}{f_F} i \left(a_0 + 2 \sum_{K=1}^{M} a_K \cos \pi \frac{f}{f_F} K \right)$$

$$Y_i = y_i \left(a_0 + 2 \sum_{K=1}^{M} a_K \cos \pi \frac{f}{f_F} K \right) \tag{A3}$$

The term in parentheses is in the form of a Fourier cosine series. It is necessary therefore to represent the amplitude response function $\left| F\left(\frac{f}{f_F}\right) \right|$ in the form of a Fourier cosine series in order to filter the generated time history. Therefore let

$$\frac{f}{f_F} = \frac{h}{H}$$

where $h = 0, 1, 2, \ldots H$, then

$$\left| F\left(\frac{f}{f_F}\right) \right| \rightarrow F\left(\frac{h}{H}\right) = F_h = a_0 + 2 \sum_{h=1}^{H} a_K \cos \pi \frac{Kh}{H}$$

where

$$a_K = \frac{1}{H} \int_0^H F_h \cos \pi \frac{Kh}{H} \, dh$$

Using the trapezoidal rule of numerical integration

$$\left\{a_K\right\} = \frac{1}{H} \left[\cos \pi \frac{Kh}{H}\right] \left[I_T\right] \left\{F_h\right\}$$

where

$$\left[I_T\right] = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & & & & \\ 0 & & 1 & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \frac{1}{2} \end{bmatrix}$$

APPENDIX B

EXPRESSIONS USED FOR DIGITAL COMPUTING

Given a time history $y(t)$, digitized at discrete uniform time intervals Δt, and assuming that the origin of time occurs at one of these intervals, then

$$t = i\,\Delta t$$

where $i = 0, 1, 2, \ldots N$ and

$$y(t) \rightarrow y(i\,\Delta t) = y_i$$

The covariance or autocorrelation function shall be defined as a quantity R_p where

$$R_p = \frac{1}{N + 1 - p} \sum_{i=0}^{N-P} y_i y_{i+p} \qquad (p = 0, 1, 2, \ldots M = 60) \tag{B1}$$

Equation (B1) is the numerical integration of equation (7).

The power spectral density is the Fourier cosine transform of the covariance function R_p. For convenience it is obtained in two steps. The preliminary step gives estimates of the raw spectral density L_h, and the final step gives estimates of the smoothed spectral density ϕ_h. Estimates of spectral density are termed raw when they are obtained from the covariance function R_p by Fourier cosine series transformation and smoothed when hanned (operation of smoothing with weights 1/4, 1/2, 1/4) from the raw estimates (ref. 9). The smoothing operation partially accounts for the fact that a finite sample rather than an infinite sample was used when taking the Fourier transform. There are as many estimates L_h as there are terms in R_p, that is, M. The raw spectral density estimates are given by the matrix equation:

$$\left\{L_h\right\} = 4\,\Delta t \left[\cos \frac{\pi p h}{M}\right] \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & 1 & \\ \cdot & & & & & \frac{1}{2} \end{bmatrix} \left\{R_p\right\} \tag{B2}$$

where h is used to represent a frequency

$$f_h = \frac{h}{M} f_F = \frac{h}{2M\,\Delta t}$$

and

$$p = h = 0,\ 1,\ 2,\ .\ .\ .\ M = 60$$

The smoothed spectral density estimates are given by the matrix equation

$$\{\phi_h\} = \frac{1}{4}\begin{bmatrix} 2 & 2 & 0 & . & . & . \\ 1 & 2 & 1 & & & \\ 0 & 1 & 2 & 1 & & \\ . & & & . & & \\ . & & & 1 & 2 & 1 \\ . & & & 0 & 2 & 2 \end{bmatrix}\{L_h\} \tag{B3}$$

The mean square of $y(t)$ and its derivatives are defined as follows:

$$\sigma_y^2 = \Delta f_h\left(\frac{1}{2}\,\phi_0 + \phi_1 + \phi_2 + .\ .\ .\ \phi_{M-1} + \frac{1}{2}\,\phi_M\right) \tag{B4}$$

where $\Delta f_h = \frac{h}{2M\,\Delta t} = \frac{1}{2M\,\Delta t}$

$$\sigma_{\dot{y}}^2 = \frac{(2\pi)^2}{(2M\,\Delta t)^3}\left(\sum_{h=0}^{M-1} h^2\phi_h + \frac{1}{2}\,M^2\phi_h\right) \tag{B5}$$

$$\sigma_{\ddot{y}}^2 = \frac{(2\pi)^4}{(2M\,\Delta t)^5}\left(\sum_{h=0}^{M-1} h^4\phi_h + \frac{1}{2}\,M^4\phi_h\right) \tag{B6}$$

APPENDIX C

LEAST-SQUARES TECHNIQUE FOR DETERMINING THE COEFFICIENTS N_R AND N_N

The equation chosen to represent the frequency distribution of amplitudes about a specified mean is

$$f_c = N_N P_N + N_R P_R \tag{C1}$$

where f_c is the computed number of occurrences of amplitudes in the range $y_i \leqq y \leqq y_{i+1}$. In order to facilitate computation of the number of occurrences of amplitudes in the range $y > y_i$, f_{y_i} is computed first. The desired value f_c can then be computed from the following relation:

$$f_c = f_{y_i} - f_{y_{i+1}}$$

Thus,

$$f_{y_i} = N_N P_N(y_i) + N_R P_R(y_i) \tag{C2}$$

where

$$P_N(y_i) = \frac{1}{\sigma_N \sqrt{2\pi}} \int_{y_i}^{\infty} \exp \frac{-\alpha^2}{2\sigma_N{}^2}\, d\alpha = 1 - \frac{1}{\sigma_N \sqrt{2\pi}} \int_{0}^{y_i} \exp \frac{-\alpha^2}{2\sigma_N{}^2}\, d\alpha = \frac{1}{2}\left[1 - \phi(x)\right]$$

where $\phi(x)$ = Error function

$$P_R(y_i) = \frac{1}{\sigma_R{}^2} \int_{y_i}^{\infty} \beta \exp \frac{-\beta^2}{2\sigma_R{}^2}\, d\beta = \exp \frac{-y_i^2}{2\sigma_R{}^2}$$

The following approximation, obtained from reference 10, was used to facilitate computation of the error function $\phi(x)$ in the computer:

$$\phi(x) = 1 - \frac{1}{\left(1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right)^4}$$

where

$$x = \frac{y_i}{2\sigma_N}$$

$$a_1 = 0.278393$$

$$a_2 = 0.230389$$

$$a_3 = 0.000972$$

$$a_4 = 0.078108$$

The least-squares technique involves minimizing with respect to each of the undetermined coefficients the sum of the differences squared between the actual and calculated number of times a value $y = y_i$ is exceeded - that is, $\sum \left(f_b - f_{y_i}\right)^2$. Since the coefficients σ_N and σ_R have been previously determined to be constant it is only necessary to minimize with respect to N_R and N_N. Thus,

$$\frac{\partial}{\partial N_R}\left\{\sum_{i=0}^{\infty}\left[f_a - N_N P_N\left(y_i\right) - N_R P_R\left(y_i\right)\right]^2\right\} = 0$$

$$\frac{\partial}{\partial N_N}\left\{\sum_{i=0}^{\infty}\left[f_a - N_N P_N\left(y_i\right) - N_R P_R\left(y_i\right)\right]^2\right\} = 0$$

Solving these two simultaneous linear equations for N_N and N_R gives

$$N_R = \frac{\sum_{i=0}^{\infty}\left[P_N\left(y_i\right)\right]^2 \sum_{i=0}^{\infty} f_a P_R\left(y_i\right) - \sum_{i=0}^{\infty} P_R\left(y_i\right) P_N\left(y_i\right) \sum_{i=0}^{\infty} f_a P_N\left(y_i\right)}{\sum_{i=0}^{\infty}\left[P_R\left(y_i\right)\right]^2 \sum_{i=0}^{\infty}\left[P_N\left(y_i\right)\right]^2 - \left[\sum_{i=0}^{\infty} P_R\left(y_i\right) P_N\left(y_i\right)\right]^2} \tag{C3}$$

$$N_N = \frac{\sum_{i=0}^{\infty} f_a P_R\left(y_i\right) \sum_{i=0}^{\infty} P_R\left(y_i\right) P_N\left(y_i\right) - \sum_{i=0}^{\infty} f_a P_N\left(y_i\right) \sum_{i=0}^{\infty}\left[P_R\left(y_i\right)\right]^2}{\left[\sum_{i=0}^{\infty} P_R\left(y_i\right) P_N\left(y_i\right)\right]^2 - \sum_{i=0}^{\infty}\left[P_N\left(y_i\right)\right]^2 \sum_{i=0}^{\infty}\left[P_R\left(y_i\right)\right]^2} \tag{C4}$$

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TABLE I

POWER-SPECTRAL-DENSITY CHARACTERISTICS OF THE

FOUR RANDOM TIME HISTORIES

	Time history			
	A	B	C	D
σ_y	0.9978	1.0017	0.9924	0.9915
$\sigma_{\dot{y}}^2$	2.5147	.6489	2.3571	2.6216
$\sigma_{\ddot{y}}^2$	11.5483	1.4698	7.7025	8.9284
f_o	.2529	.1280	.2462	.2599
f_p	.3411	.2395	.2877	.2937
f_o/f_p	.7414	.5344	.8558	.8849
K_1	.6711	.8452	.5173	.4658
K_2	.9052	1.5816	.6045	.5264

TABLE II

FREQUENCY OF OCCURRENCE OF INSTANTANEOUS MEANS AND INSTANTANEOUS AMPLITUDES FOR TIME HISTORY A

Mean / Amp.	0.0	-0.2	+0.2	-0.4	+0.4	-0.6	+0.6	-0.8	+0.8	-1.0	+1.0	-1.2	+1.2	-1.4	+1.4	-1.6	+1.6	-1.8	+1.8	-2.0	+2.0	-2.2	+2.2	-2.4	+2.4	-2.6	+2.6	Total
-0.1	323	306	297	251	257	202	217	141	155	87	87	53	42	28	27	15	20	4	5			2						2,519
+.1	328	301	313	263	230	192	176	136	152	74	75	43	44	28	29	16	15	3	8	2	1	1		2				2,431
-.3	339	360	376	271	298	198	206	146	162	93	102	44	56	22	25	22	15	3	8		1	1	1					2,749
+.3	325	322	379	291	312	216	235	146	155	97	123	50	56	35	25	15	9	5	7	1	2		2					2,808
-.5	430	430	401	355	342	279	257	190	181	111	102	60	69	29	16	13	13	7	4	4	3	2	1					3,299
+.5	441	440	391	330	380	256	253	182	183	107	117	56	62	32	26	19	18	4	4	3	2		1					3,307
-.7	486	439	469	409	383	277	318	173	195	118	121	69	57	26	26	14	15	7	5	3	2		1					3,613
+.7	483	450	463	405	356	309	286	193	207	129	110	60	78	24	42	14	16	7	4	4	2							3,642
-.9	451	472	470	404	404	311	284	196	205	112	114	61	76	36	32	12	17	8	7	1	1							3,674
+.9	492	443	444	371	387	296	326	195	197	116	113	59	59	26	35	22	12	5	5	2	4				1			3,610
-1.1	378	390	388	349	367	234	256	159	161	115	125	65	70	26	24	8	20	4	4	1	1	1	1					3,147
+1.1	411	399	413	381	367	262	269	174	162	87	99	68	66	34	26	11	7	3	2	1	1		1					3,244
-1.3	371	300	341	314	274	185	221	136	135	86	79	47	53	23	22	9	7	4	2		1					1		2,611
+1.3	337	312	318	259	275	188	212	132	145	71	104	59	43	22	23	13	13	6	3				1					2,536
-1.5	239	238	250	203	202	152	147	98	103	55	46	41	40	16	15	7	9	3	2	1	2							1,869
+1.5	235	278	251	207	202	142	139	110	110	64	53	41	42	15	15	2	6	4	2		3		1					1,922
-1.7	181	170	157	124	143	97	114	46	70	45	43	18	28	13	14	2	3	1	3		1							1,273
+1.7	166	177	164	125	135	91	121	74	74	40	49	24	30	8	11	3	5		1		2							1,300
-1.9	113	106	102	80	87	65	54	49	45	20	24	14	14	6	8	4	2	1	1			1						796
+1.9	109	104	99	89	76	65	70	30	35	35	17	13	16	8	7	1	5	2	1	1								783
-2.1	55	64	48	48	42	35	36	32	34	19	20	13	11	1	2		3	2										465
+2.1	60	51	63	46	41	34	31	22	25	19	15	11	7	2	4	1												432
-2.3	38	37	27	31	25	16	20	20	10	5	8	2	4	3	2	1												249
+2.3	36	21	24	24	24	18	22	15	12	8	8	4	2			2	1	1										222
-2.5	10	16	11	8	7	8	5	4	4	6	4	1	3		1			1										89
+2.5	5	14	17	13	7	8	8	5	5	5	3	1	2		1	1	1		1									97
-2.7	6	9	3	4	3	5	3	2	3					1			1		1									41
+2.7	8	9	8	2	5	5	2	1	4	1	2	2	1	1														51
-2.9	3	2	5	2	1		1	1	1	1	1	1																19
+2.9	4	2	3	2	1		2	2			1																	17
-3.1			1		1				1		1																	4
+3.1	1	1	1	1							1						1											6
-3.3			1																									1
+3.3	1																											1
Total	6,865	6,663	6,698	5,662	5,634	4,146	4,291	2,810	2,931	1,726	1,767	980	1,031	465	458	227	234	85	80	24	28	8	10	2	1	1		52,827

TABLE III

FREQUENCY OF OCCURRENCE OF INSTANTANEOUS MEANS AND INSTANTANEOUS AMPLITUDES FOR TIME HISTORY B

Mean / Amp.	0.0	-0.2	+0.2	-0.4	+0.4	-0.6	+0.6	-0.8	+0.8	-1.0	+1.0	-1.2	+1.2	-1.4	+1.4	-1.6	+1.6	-1.8	+1.8	-2.0	+2.0	-2.2	+2.2	-2.4	+2.4	-2.6	+2.6	-2.8	+2.8	-3.0	+3.0	-3.2	+3.2	Total
-0.1	366	376	369	344	359	297	305	218	222	158	177	132	126	76	95	36	41	33	40	16	18	10	12	4	2	2	4							3,838
+.1	386	380	364	368	338	286	290	211	234	183	182	132	139	74	100	54	49	34	39	16	24	9	9	4	3	1	3	1						3,913
-.3	302	310	318	255	273	223	209	187	175	133	142	85	86	66	52	26	43	15	24	15	17	6	7	3	3								1	2,976
+.3	313	330	295	251	294	212	213	166	176	128	144	86	92	63	43	37	37	10	15	10	14	5	7	4	5				1				1	2,952
-.5	250	250	301	259	267	209	205	169	155	109	108	86	81	30	51	37	27	10	21	6	8	5	4	2	1	2								2,653
+.5	249	264	291	223	244	211	219	163	189	107	115	77	91	45	35	30	23	12	25	10	11	5	1		1	1			1					2,643
-.7	239	236	232	221	239	183	177	174	147	78	91	76	62	39	33	23	25	15	23	4	5	4	2	1	1						1			2,331
+.7	256	243	245	213	204	186	182	147	120	96	75	71	73	43	37	19	27	15	14	4	11	6	4	1			1	1	1					2,295
-.9	203	205	232	184	198	140	145	105	108	84	89	50	59	30	39	17	22	14	6	10	3	3	4					1	1					1,952
+.9	192	223	189	178	191	165	152	94	100	77	92	57	52	32	25	15	19	11	10	4	5		1											1,884
-1.1	171	153	153	140	139	109	119	92	82	68	53	42	39	19	29	14	11	9	8	4	2	2												1,458
+1.1	187	154	159	122	172	116	125	76	73	52	57	39	33	24	19	16	18	5	6	2	1		1				1							1,458
-1.3	154	123	124	107	117	103	94	47	72	40	42	29	30	7	11	4	11	4	1		1		4											1,125
+1.3	130	118	146	109	90	84	89	73	61	45	57	27	41	13	20	10	9	7	2	1			1		1									1,134
-1.5	104	82	89	49	74	72	57	36	45	22	54	21	22	5	11	6	8	2	3		2					1								765
+1.5	84	71	100	88	84	60	70	47	44	37	29	17	17	10	5	5	7	3	3		1													782
-1.7	60	64	48	58	44	48	34	36	27	16	22	14	16	12	7	4	2			1														513
+1.7	69	60	62	59	48	50	39	28	36	20	27	5	4	5	9	1	5	1	1	1														530
-1.9	50	36	39	28	29	28	24	20	22	9	11	7	7	5	5	1	1	1	2	1														326
+1.9	39	36	42	37	35	36	43	13	16	17	9	10	5	3	5	5	3	1						1										356
-2.1	23	21	29	21	19	21	12	11	5	6	4	2	4	3	2	2						1												186
+2.1	15	25	24	13	21	8	24	6	8	2	10	6	1	2	3		1			1														170
-2.3	6	13	9	5	11	6	9	6	5	2	4		2																					78
+2.3	8	12	11	10	8	8	7	5	6	3	2	1	2		1																			84
-2.5	11	12	6	4	2	1	6		5	3	1		1																					52
+2.5	4	8	6	4	5	7	6	1	2	1		1	1																					46
-2.7	4		3	3	1	2	3		1		1																							18
+2.7	2	3		7	5	1	1		1	1	1			1																				23
-2.9	1	3	1			1	1	1																										8
+2.9	2							1	1																									4
-3.1			1			1						1																						3
+3.1	1		1	1																														3
-3.3																																		
+3.3			1	1																														2
Total	3,881	3,811	3,890	3,362	3,511	2,874	2,860	2,133	2,138	1,497	1,599	1,074	1,086	607	637	362	389	202	242	107	122	57	57	20	17	7	9	3	4		1		2	36,561

TABLE IV

FREQUENCY OF OCCURRENCE OF INSTANTANEOUS MEANS AND INSTANTANEOUS AMPLITUDES FOR TIME HISTORY C

Amp. \ Mean	0.0	-0.2	+0.2	-0.4	+0.4	-0.6	+0.6	-0.8	+0.8	-1.0	+1.0	-1.2	+1.2	-1.4	+1.4	-1.6	+1.6	-1.8	+1.8	-2.0	+2.0	-2.2	+2.2	-2.4	+2.4	Total
-0.1	401	324	364	232	224	144	132	49	46	17	22	6	2	1	2	0	1									1,967
+.1	375	344	355	249	219	133	141	56	55	18	13	6	5	1	0	0	1									1,971
-.3	356	332	308	192	195	89	91	31	28	6	9	2												1		1,640
+.3	334	321	310	173	199	73	89	33	25	8	6	2	1		1											1,575
-.5	467	374	425	218	229	80	90	30	28	3	6	3	1													1,954
+.5	443	426	437	240	221	86	117	35	36	5	7	1	1		1											2,058
-.7	608	462	492	286	291	115	129	26	30	11	10		1													2,461
+.7	585	453	501	264	294	124	124	31	32	3	2	1														2,414
-.9	690	509	403	278	290	142	116	38	34	11	9	1	1	1												2,623
+.9	642	526	496	318	285	124	122	40	31	15	12												1			2,612
-1.1	598	516	522	298	322	116	138	37	37	5	6			1	1											2,597
+1.1	599	527	540	291	317	121	121	33	43	10	8		1													2,611
-1.3	576	444	468	262	278	115	126	37	38	5	8	1														2,358
+1.3	541	429	448	270	290	129	130	40	41	6	7	2	2													2,335
-1.5	438	340	380	244	226	108	115	35	38	7	4	2	1													1,938
+1.5	445	365	391	214	232	102	112	28	39	7	8	2	3	1												1,949
-1.7	312	254	295	165	186	78	77	18	22	3	6	1		1												1,418
+1.7	325	261	295	163	172	71	85	34	18	5	3						1									1,433
-1.9	224	227	176	118	142	53	60	18	8	3	6	1														1,036
+1.9	245	213	185	121	126	62	65	14	27	3	5	1						1								1,068
-2.1	170	162	139	97	87	39	36	10	10	4	3		1	1												759
+2.1	175	179	127	94	91	35	26	16	9	2	4															758
-2.3	104	98	94	57	60	16	19	7	9	1	2															467
+2.3	104	85	110	36	49	22	24	12	12		1		1													456
-2.5	73	50	69	36	39	23	21	6	7		1															325
+2.5	64	59	57	33	32	23	18	5	9		2															302
-2.7	42	36	40	23	22	6	13	5	5	1																193
+2.7	46	33	35	21	25	6	7	2	3	1																179
-2.9	20	16	17	12	8	6	9	2			1															91
+2.9	23	25	19	14	12	11	7	2	2																	115
-3.1	13	15	18	7	3	2																				58
+3.1	10	7	11	8	4	2	5				1															48
-3.3	4	3	10	1	3																					21
+3.3	10	4	8	1	3		2																			28
-3.5	2	1	6		1		1																			11
+3.5	3	1	3	1																						8
-3.7	1	1	1	1					1																	5
+3.7			2	1																						3
-3.9	1				1																					2
+3.9							1																			1
-4.1				1																						1
+4.1																										
-4.3																										
+4.3																										
-4.5																										
+4.5		1																								1
Total	10,069	8,425	8,657	5,040	5,178	2,256	2,369	730	723	160	172	32	21	7	5		3	1					1	1		43,850

TABLE V

FREQUENCY OF OCCURRENCE OF INSTANTANEOUS MEANS AND INSTANTANEOUS AMPLITUDES FOR TIME HISTORY D

Mean / Amp.	0.0	-0.2	+0.2	-0.4	+0.4	-0.6	+0.6	-0.8	+0.8	-1.0	+1.0	-1.2	+1.2	-1.4	+1.4	-1.6	+1.6	-1.8	+1.8	-2.0	+2.0	-2.2	+2.2	Total
-0.1	484	388	367	228	214	68	79	24	22	4	9	1		1										1,889
+.1	486	395	355	231	194	73	94	27	23	5	4	2	3											1,892
-.3	492	356	349	168	137	34	30	7	6	2	2													1,583
+.3	491	340	341	149	154	48	46	2	4	2	3		1											1,581
-.5	649	419	429	153	163	34	35	7	9		2		1											1,901
+.5	650	406	444	176	163	29	40	6	4		1													1,919
-.7	824	573	553	183	199	31	43	6	6	1	1													2,420
+.7	768	573	556	179	207	44	37	8	3	1	2												1	2,379
-.9	889	645	609	218	244	42	38	2	6															2,693
+.9	903	647	649	227	236	43	44	3	2															2,754
-1.1	872	587	664	238	216	43	39	8	8															2,675
+1.1	840	619	635	209	214	43	40	6	8															2,614
-1.3	764	542	550	205	214	42	30	5	3		1													2,356
+1.3	738	525	538	215	237	50	43	7	5															2,358
-1.5	603	415	462	201	202	48	51	4	4															1,990
+1.5	650	443	437	190	193	40	45	8	4	1														2,011
-1.7	449	358	378	144	131	36	37	2	1	1														1,537
+1.7	460	355	351	146	169	38	30	5	7										1					1,562
-1.9	322	248	229	120	107	22	26	5	3															1,082
+1.9	316	263	251	105	107	24	29	7	5															1,107
-2.1	255	198	191	77	81	15	18	4	4															843
+2.1	242	200	190	74	62	17	17	3	1															806
-2.3	155	144	119	50	74	18	15			1														576
+2.3	176	122	122	48	43	20	15	1	1									1						549
-2.5	110	71	81	36	32	3	5																	338
+2.5	110	74	86	36	40	7	17		2															372
-2.7	62	44	63	29	20	7	8		1															234
+2.7	56	43	61	18	24	5	7																	214
-2.9	43	33	24	11	16	4	5		1															137
+2.9	37	42	20	13	12	3	1																	128
-3.1	22	13	18	11	1		1																	66
+3.1	24	19	18	5	5	2																		73
-3.3	14	9	7	3	2																			35
+3.3	11	10	12	1	2		1		1															38
-3.5	3	1	5	1	4																			14
+3.5	6	2	2		4		1																	15
-3.7	1	1	3				1																	6
+3.7	3	1	3	1	1																			9
-3.9	2		2																					4
+3.9	1				1																			2
-4.1					1																			1
+4.1																								
-4.3																								
+4.3																								
-4.5						1																		1
+4.5																								
-4.7																								
+4.7				1																				1
Total	13,983	10,124	10,174	4,100	4,126	934	968	157	144	18	25	3	5	1				1	1				1	44,765

TABLE VI

PREDICTED AND OBSERVED FREQUENCIES OF OCCURRENCE OF INSTANTANEOUS AMPLITUDES

ABOUT A SPECIFIED INSTANTANEOUS MEAN FOR TIME HISTORY A

Mean	-1.4		-1.2		-1.0		-0.8		-0.6		-0.4		-0.2		0.0		+0.2		+0.4		+0.6		+0.8		+1.0		+1.2		+1.4	
Amp. class mark	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c
0.1	28	28	43	46	74	87	136	153	192	183	263	243	301	325	328	321	313	325	230	215	176	183	152	153	75	87	44	46	29	28
.3	35	26	50	51	97	91	146	152	216	213	291	280	322	342	325	348	379	342	312	293	235	213	155	152	123	91	56	51	25	26
.5	32	29	56	60	107	104	182	171	256	257	330	335	440	395	441	407	391	395	380	347	253	257	183	171	117	104	62	60	26	29
.7	24	33	60	69	129	119	193	195	309	294	405	384	450	451	483	466	463	451	356	386	286	294	207	195	110	119	78	69	42	33
.9	26	33	59	69	116	120	195	197	296	297	371	388	443	456	492	470	444	456	387	388	326	297	197	197	113	120	59	69	35	33
1.1	34	30	68	63	87	109	174	178	262	269	380	351	399	412	411	426	413	412	367	351	269	269	162	178	99	109	66	63	26	30
1.3	22	25	59	52	71	90	132	147	188	222	260	289	312	340	337	351	318	340	275	289	212	221	145	147	104	90	43	52	23	25
1.5	15	19	41	39	64	68	110	111	142	168	207	219	278	258	235	266	251	258	202	219	139	168	110	111	53	68	42	39	15	19
1.7	8	13	24	28	40	48	74	78	91	117	125	154	177	181	166	186	164	181	135	154	121	118	74	78	49	48	30	28	11	13
1.9	8	8	13	18	35	31	30	51	65	77	89	100	104	118	109	121	99	118	76	100	70	77	35	51	17	31	16	18	7	8
2.1	2	5	11	11	19	19	22	31	34	46	46	61	51	71	60	74	63	71	41	61	31	46	25	31	15	19	7	11	4	5
2.3	0	3	4	6	8	11	15	17	18	26	24	34	21	40	36	42	24	40	24	34	22	26	12	17	8	11	2	6	0	3
2.5	0	2	1	3	5	6	5	9	8	14	13	18	14	21	5	22	17	21	7	18	8	14	5	9	3	6	2	3	1	2
2.7	1	1	2	2	1	3	1	5	5	7	2	9	9	11	8	11	8	11	5	9	2	7	4	5	2	3	1	2		
2.9							2	2			2	4	2	5	4	2	3	5	1	4	2	3			1	1				
3.1											1	2	1	2	1	1	1	2							1	1				

f_a - observed frequency of occurrence (i.e., number of amplitudes counted in an interval)
f_c - predicted frequency of occurrence (eq. (C1))

TABLE VII

PREDICTED AND OBSERVED FREQUENCIES OF OCCURRENCE OF INSTANTANEOUS AMPLITUDES

ABOUT A SPECIFIED INSTANTANEOUS MEAN FOR TIME HISTORY B

Mean	-1.4		-1.2		-1.0		-0.8		-0.6		-0.4		-0.2		0.0		+0.2		+0.4		+0.6		+0.8		+1.0		+1.2		+1.4	
Amp. class mark	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c
0.1	74	89	132	136	183	184	211	236	286	282	268	344	380	363	386	382	364	363	338	344	290	282	234	236	182	184	139	136	100	89
.3	63	62	86	104	128	143	166	187	212	237	251	288	330	310	313	320	295	310	294	288	213	237	176	187	144	143	92	104	43	62
.5	45	36	77	72	107	102	163	137	211	187	223	225	264	250	249	250	291	250	244	225	219	187	189	137	115	102	91	72	35	36
.7	43	27	71	61	96	88	147	120	186	171	213	205	243	231	256	228	245	231	204	205	182	171	120	120	75	88	73	61	37	27
.9	32	24	57	56	77	81	94	111	165	159	178	191	223	215	192	212	189	215	191	191	152	159	100	111	92	81	52	56	25	24
1.1	24	20	39	48	52	69	76	94	116	136	122	162	154	183	187	180	159	183	172	162	125	136	73	94	57	69	33	48	19	20
1.3	13	16	27	37	45	53	73	73	84	104	109	125	118	141	130	139	146	141	90	125	89	104	61	73	57	53	41	37	20	16
1.5	10	11	17	26	37	37	47	51	60	73	88	87	71	99	84	97	100	99	84	87	70	73	44	51	29	37	17	26	5	11
1.7	5	7	5	16	20	24	28	32	50	47	59	56	60	63	69	62	62	63	48	56	39	47	36	32	27	24	4	16	9	7
1.9	3	4	10	10	17	14	13	19	36	27	37	33	36	37	39	37	42	37	35	33	43	27	16	19	9	14	5	10	5	4
2.1	2	2	6	5	2	8	6	10	8	15	13	18	25	20	15	20	24	20	21	18	24	15	8	10	10	8	1	5	3	2
2.3	0	1	1	3	3	4	5	5	8	7	10	9	12	10	8	10	11	10	8	9	7	7	6	5	2	4	2	3	1	1
2.5	0	1	1	1	1	2	1	2	7	3	4	4	8	5	4	5	6	5	5	4	6	3	2	2	0	2	1	1		
2.7	1	0			1	1	0	1	1	1	7	2	3	2	2	2	0	2	5	2	1	1	1	1	1	1				
2.9							1	0			0	1			2	1	0	1					1	0						
3.1											1	0			1	0	1	0												

f_a - observed frequency of occurrence (i.e., number of amplitudes counted in an interval)
f_c - predicted frequency of occurrence (eq. (C1))

TABLE VIII

PREDICTED AND OBSERVED FREQUENCIES OF OCCURRENCE OF INSTANTANEOUS AMPLITUDES ABOUT A SPECIFIED INSTANTANEOUS MEAN FOR TIME HISTORY C

Mean	-0.8		-0.6		-0.4		-0.2		0.0		+0.2		+0.4		+0.6		+0.8	
Amp. class mark	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c
0.1	56	55	133	141	249	219	344	355	375	375	355	355	219	219	141	141	55	55
.3	33	22	73	74	173	165	321	279	334	323	310	279	199	165	89	74	25	22
.5	35	33	86	108	240	245	428	415	443	481	437	415	221	245	117	108	36	33
.7	31	40	124	131	264	297	453	504	585	584	501	504	294	297	124	131	32	40
.9	40	42	124	140	318	316	526	535	642	621	496	535	285	316	122	140	31	42
1.1	33	41	121	135	291	304	527	516	599	598	540	516	317	304	121	135	43	41
1.3	40	36	129	120	270	270	429	458	541	532	448	458	290	270	130	120	41	36
1.5	28	30	102	99	214	223	365	379	445	440	391	379	232	223	112	99	39	30
1.7	34	23	71	77	163	173	261	293	325	340	295	293	172	173	85	77	18	23
1.9	14	17	62	56	121	126	213	214	245	248	185	214	126	126	65	56	27	17
2.1	16	12	35	38	94	87	179	147	175	170	127	147	91	87	26	38	9	12
2.3	12	7	22	25	36	56	85	95	104	110	110	95	49	46	24	25	12	7
2.5	5	5	23	15	33	34	59	58	64	68	57	58	32	34	18	15	9	5
2.7	2	3	6	9	21	20	33	34	46	39	35	34	25	20	7	9	3	3
2.9	2	1	11	5	14	11	25	19	23	22	19	19	12	11	7	5	2	1
3.1			2	3	8	6	7	10	10	11	11	10	4	6	5	3		
3.3					1	3	4	5	10	6	8	5	3	3	2	1		
3.5					1	1	1	2	3	3	3	2			0	1		
3.7					1	1	0	1			2				0	0		
3.9							0	0							1	0		
4.1							0	0										
4.3							0	0										
4.5							1	0										

f_a - observed frequency of occurrence (i.e., number of amplitudes counted in an interval)

f_c - predicted frequency of occurrence (eq. (C1))

TABLE IX

PREDICTED AND OBSERVED FREQUENCIES OF OCCURRENCE OF

INSTANTANEOUS AMPLITUDES ABOUT A SPECIFIED

INSTANTANEOUS MEAN FOR TIME HISTORY D

Mean	-0.6		-0.4		-0.2		0.0		+0.2		+0.4		+0.6	
Amp. class mark	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c	f_a	f_c
0.1	73	94	231	194	395	355	486	485	355	355	194	194	94	94
.3	48	25	149	121	340	309	491	427	341	309	154	121	46	25
.5	29	38	176	184	406	470	650	650	444	470	163	184	40	38
.7	44	46	179	225	573	576	768	795	556	576	207	225	37	46
.9	43	50	227	242	647	618	903	854	649	618	236	242	44	50
1.1	43	49	209	236	619	604	840	834	635	604	214	236	40	49
1.3	50	44	215	213	525	545	738	753	538	545	237	213	43	44
1.5	40	37	190	179	443	459	650	635	437	459	193	179	45	37
1.7	38	29	146	142	355	363	460	502	351	363	169	142	30	29
1.9	24	22	105	106	263	271	316	375	251	271	107	106	29	22
2.1	17	15	74	75	200	191	242	264	190	191	62	75	17	15
2.3	20	10	48	50	122	128	176	177	122	128	43	50	15	10
2.5	7	7	36	32	74	81	110	112	86	81	40	32	17	7
2.7	5	4	18	19	43	49	56	67	61	49	24	19	7	4
2.9	3	2	13	11	42	28	37	39	20	28	12	11	1	2
3.1	2	1	5	6	19	15	24	21	18	15	5	6	0	1
3.3			1	3	10	8	11	11	12	8	2	3	1	1
3.5			0	2	2	4	6	5	2	4	4	2	1	0
3.7			1	1	1	2	3	3	3	2	1	1		
3.9			0	0			1	1			1	0		
4.1			0	0										
4.3			0	0										
4.5			0	0										
4.7			1	0										

f_a - observed frequency of occurrence (i.e., number of amplitudes counted in an interval)

f_c - predicted frequency of occurrence (eq. (C1))

TABLE X

COEFFICIENTS DETERMINED TO GIVE BEST FIT TO OBSERVED FREQUENCIES

	Time history A $\sigma_R = 0.814$ $\sigma_N = 0.186$		Time history B $\sigma_R = 0.746$ $\sigma_N = 0.254$		Time history C $\sigma_R = 0.915$ $\sigma_N = 0.085$		Time history D $\sigma_R = 0.942$ $\sigma_N = 0.058$	
Mean	N_R	N_N	N_R	N_N	N_R	N_N	N_R	N_N
-1.4	224	60	153	295				
-1.2	473	88	358	433				
-1.0	819	174	515	584				
-.8	1,340	316	709	743	319	96		
-.6	2,026	343	1,019	865	1,058	237	390	170
-.4	2,644	458	1,221	1,059	2,392	332	1,888	304
-.2	3,104	650	1,378	1,104	4,056	529	4,830	495
.0	3,204	630	1,356	1,177	4,705	538	6,675	674
+.2	3,104	650	1,378	1,104	4,056	529	4,830	495
+.4	2,644	458	1,221	1,059	2,392	332	1,888	304
+.6	2,026	343	1,019	865	1,058	237	390	170
+.8	1,340	316	709	743	319	96		
+1.0	819	174	515	584				
+1.2	473	88	358	433				
+1.4	224	60	153	295				

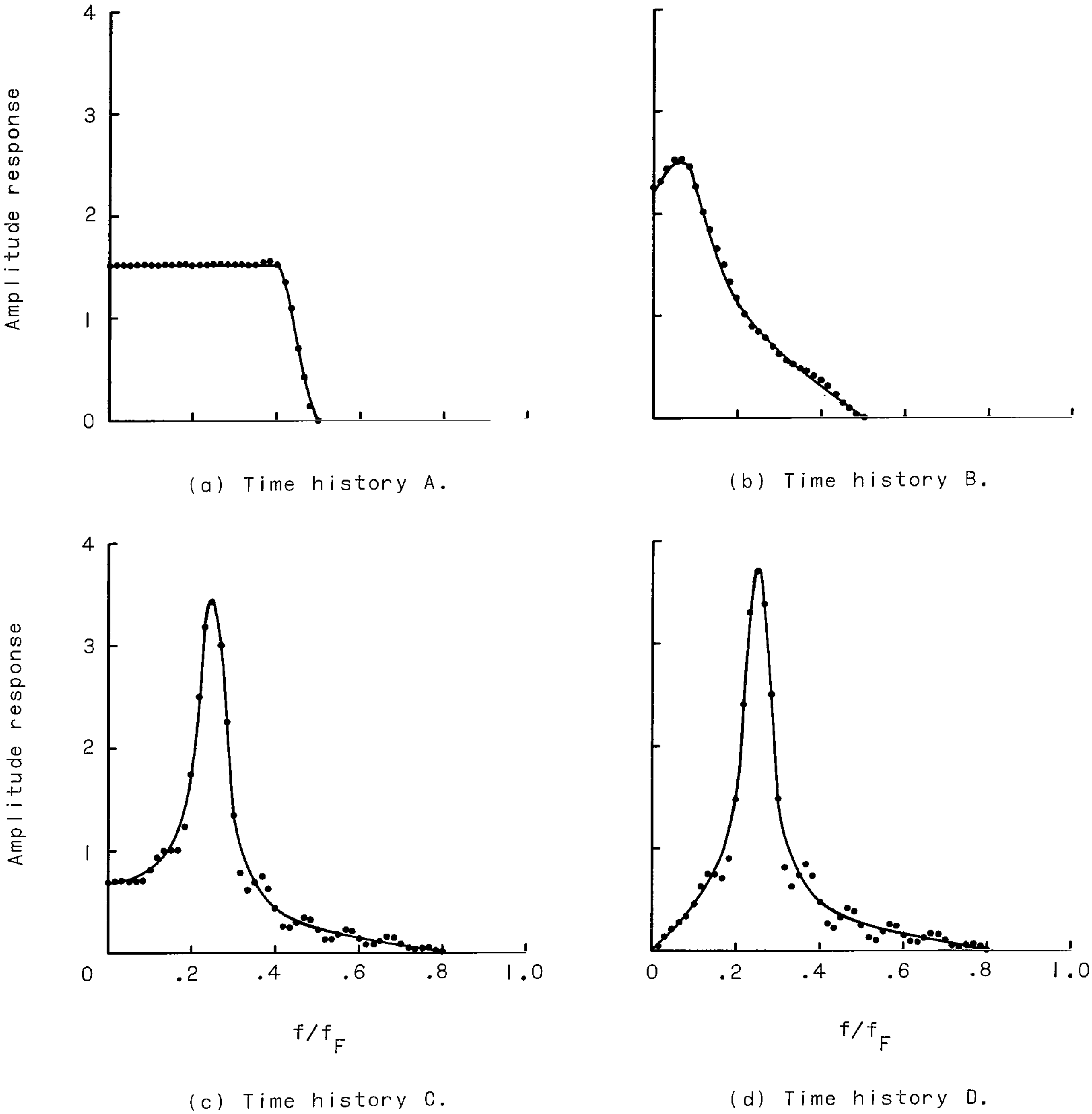
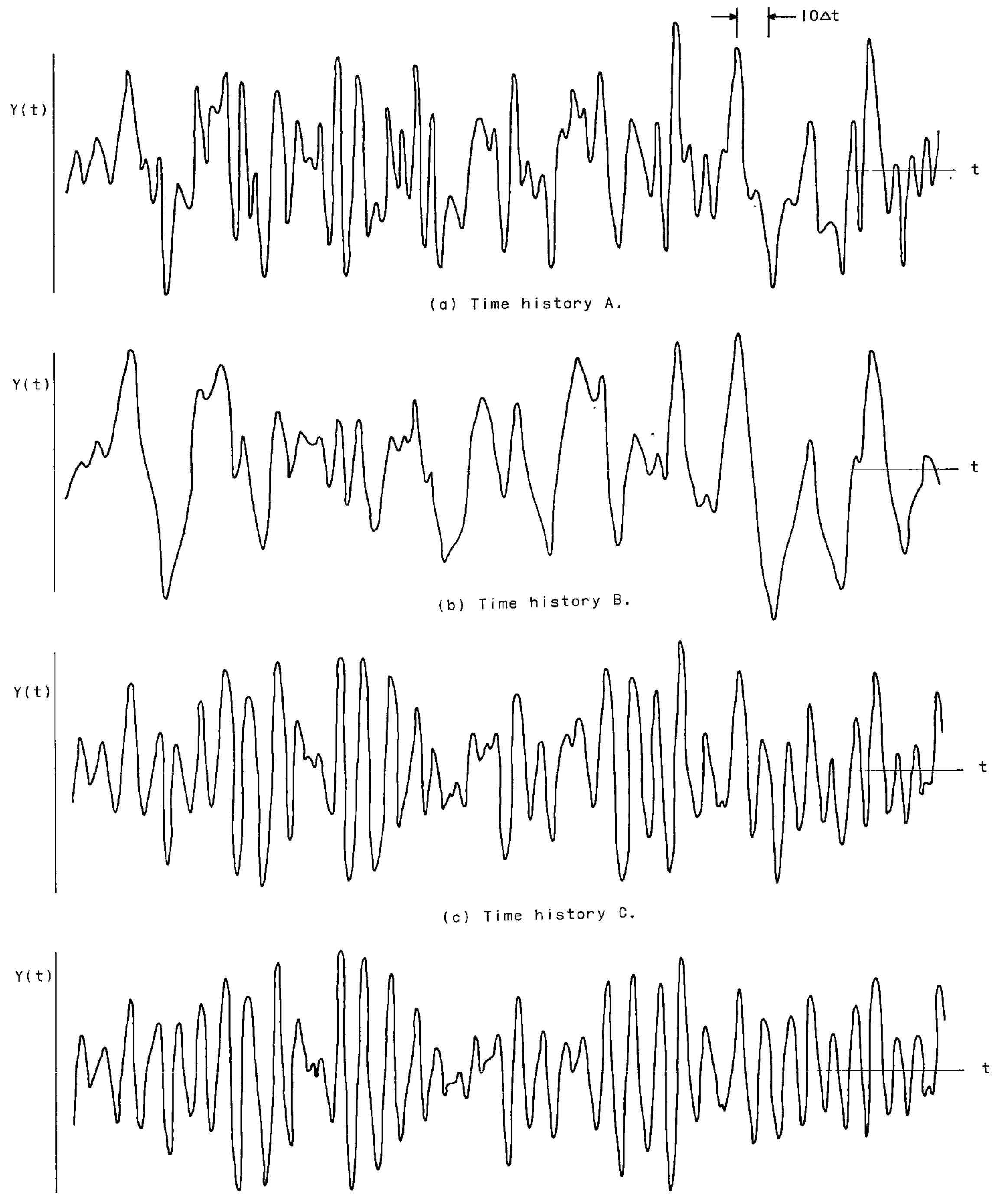


Figure 1.- Amplitude responses employed in filtering. Curves represent the desired response; symbols represent the response obtained using Fourier coefficients.



(a) Time history A.

(b) Time history B.

(c) Time history C.

(d) Time history D.

Figure 2.- Samples of the four time histories obtained by filtering.

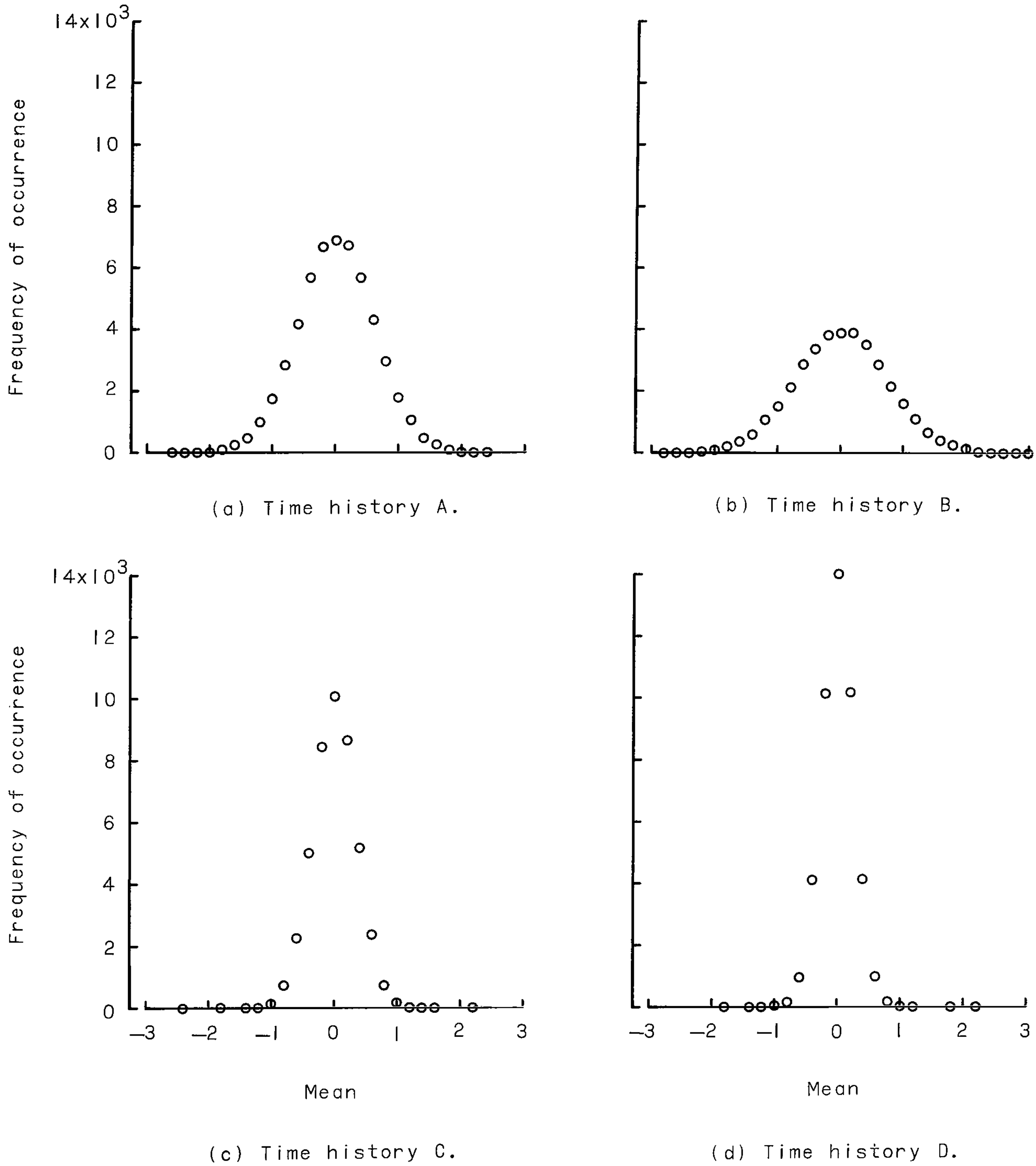


Figure 3.- Statistical distributions of instantaneous mean values for the four time histories.

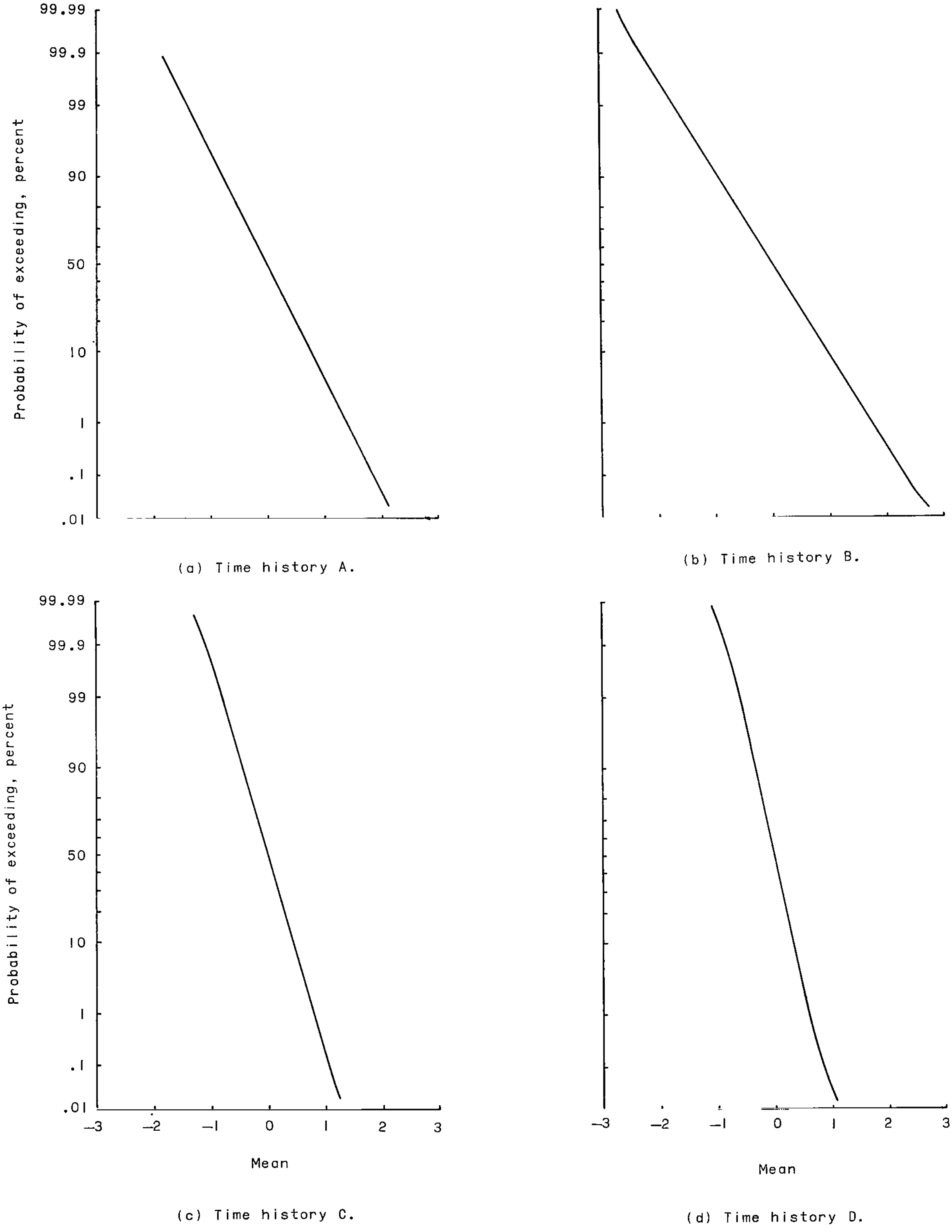


Figure 4.- Probability distributions of instantaneous mean values for the four time histories.

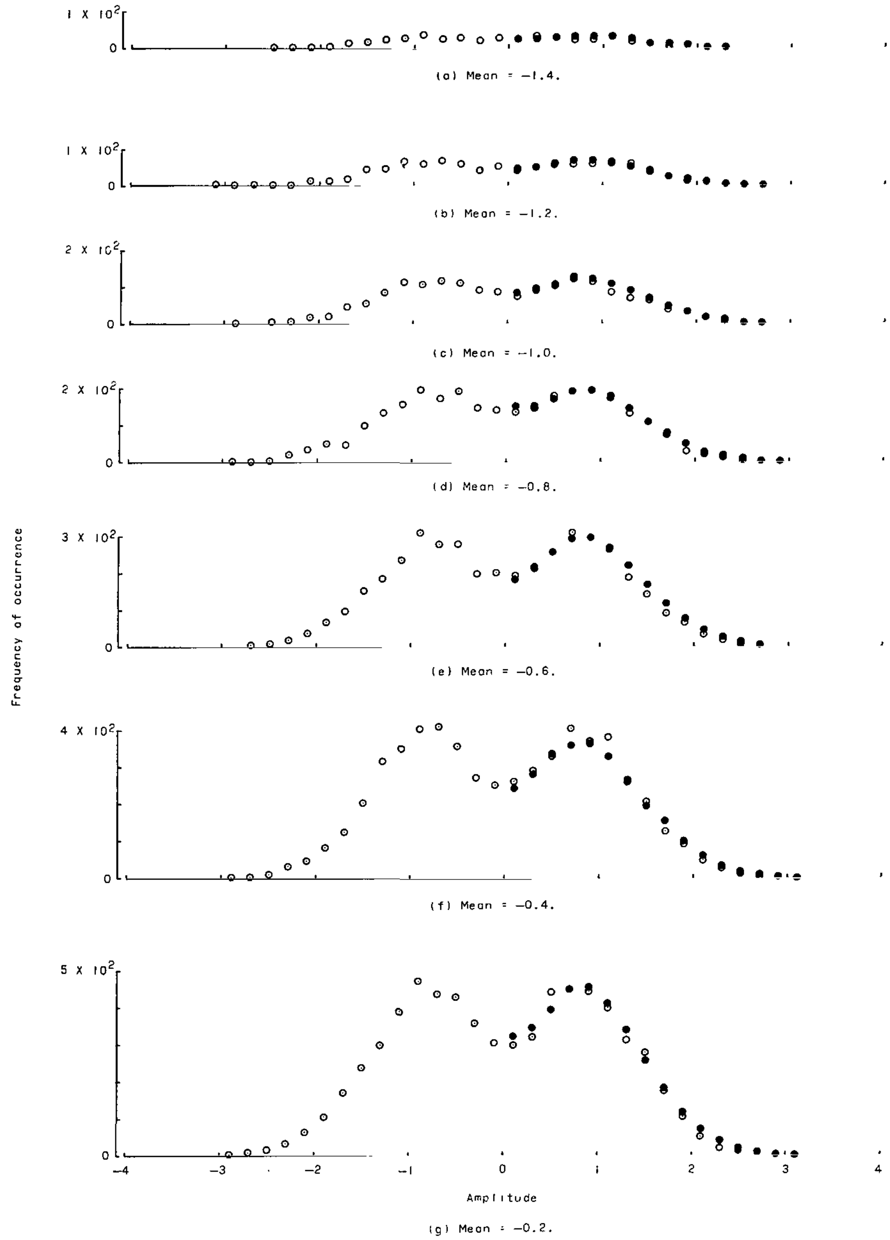
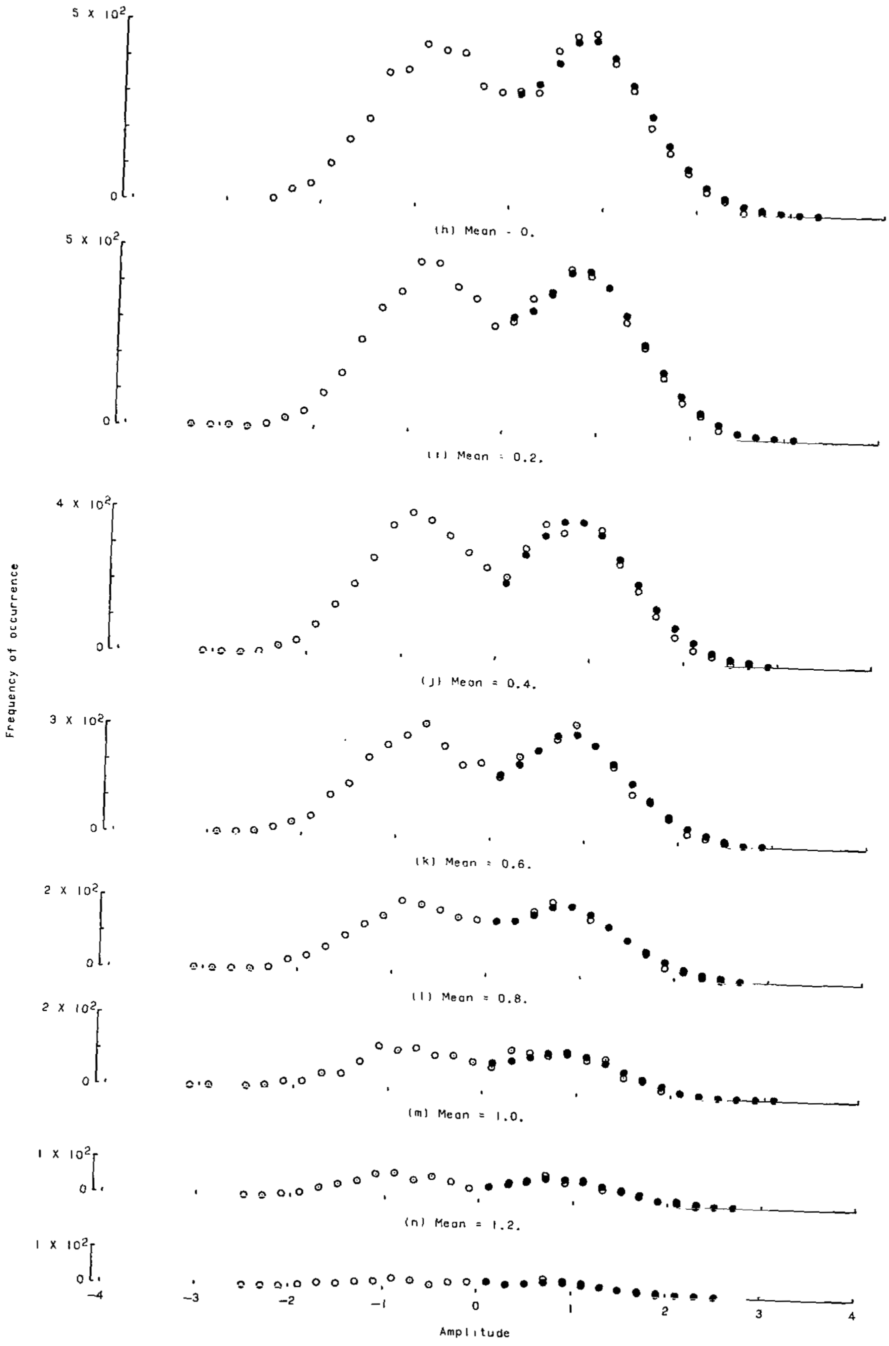


Figure 5.- Statistical distributions of instantaneous amplitudes with respect to a specified instantaneous mean value for time history A. Open symbols represent actual values; solid symbols represent computed values.



(o) Mean = 1.4.

Figure 5.- Concluded.

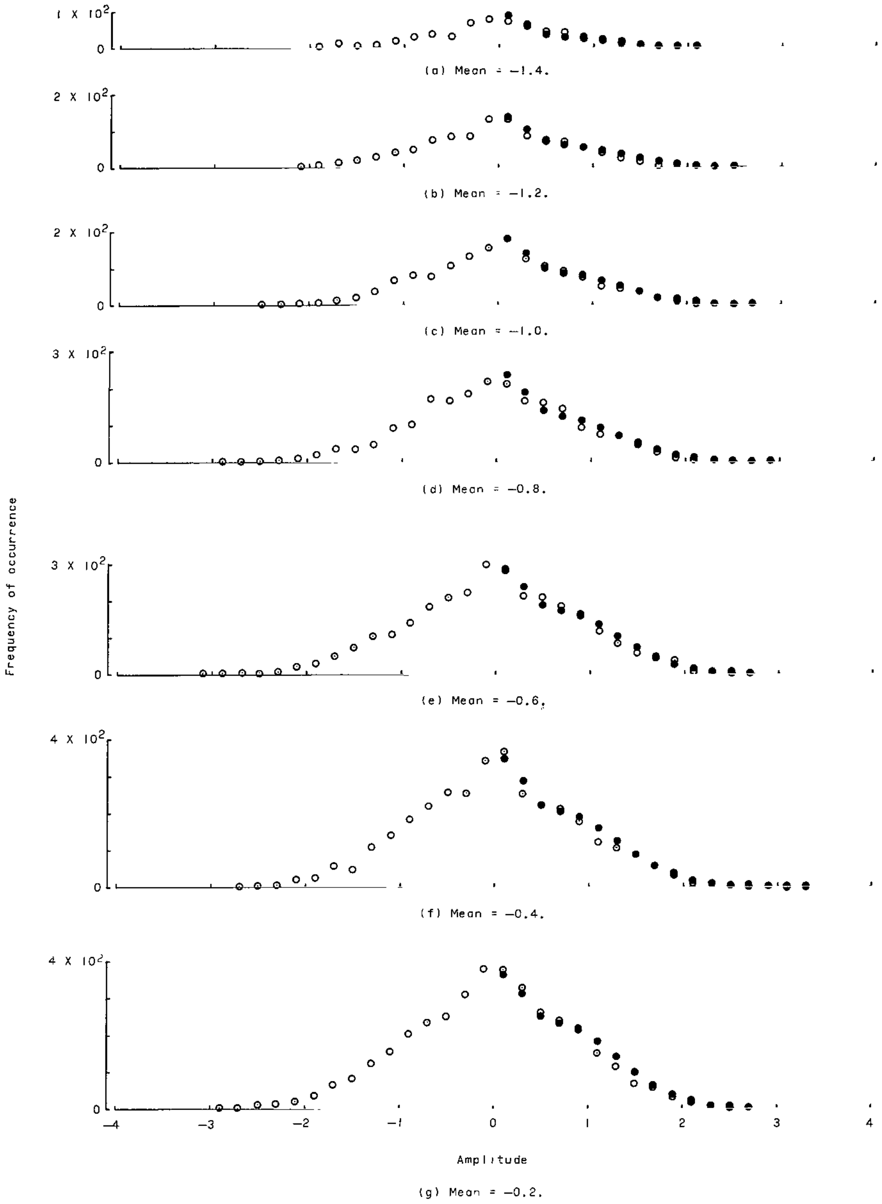


Figure 6.- Statistical distributions of instantaneous amplitudes with respect to a specified instantaneous mean value for time history B. Open symbols represent actual values; solid symbols represent computed values.

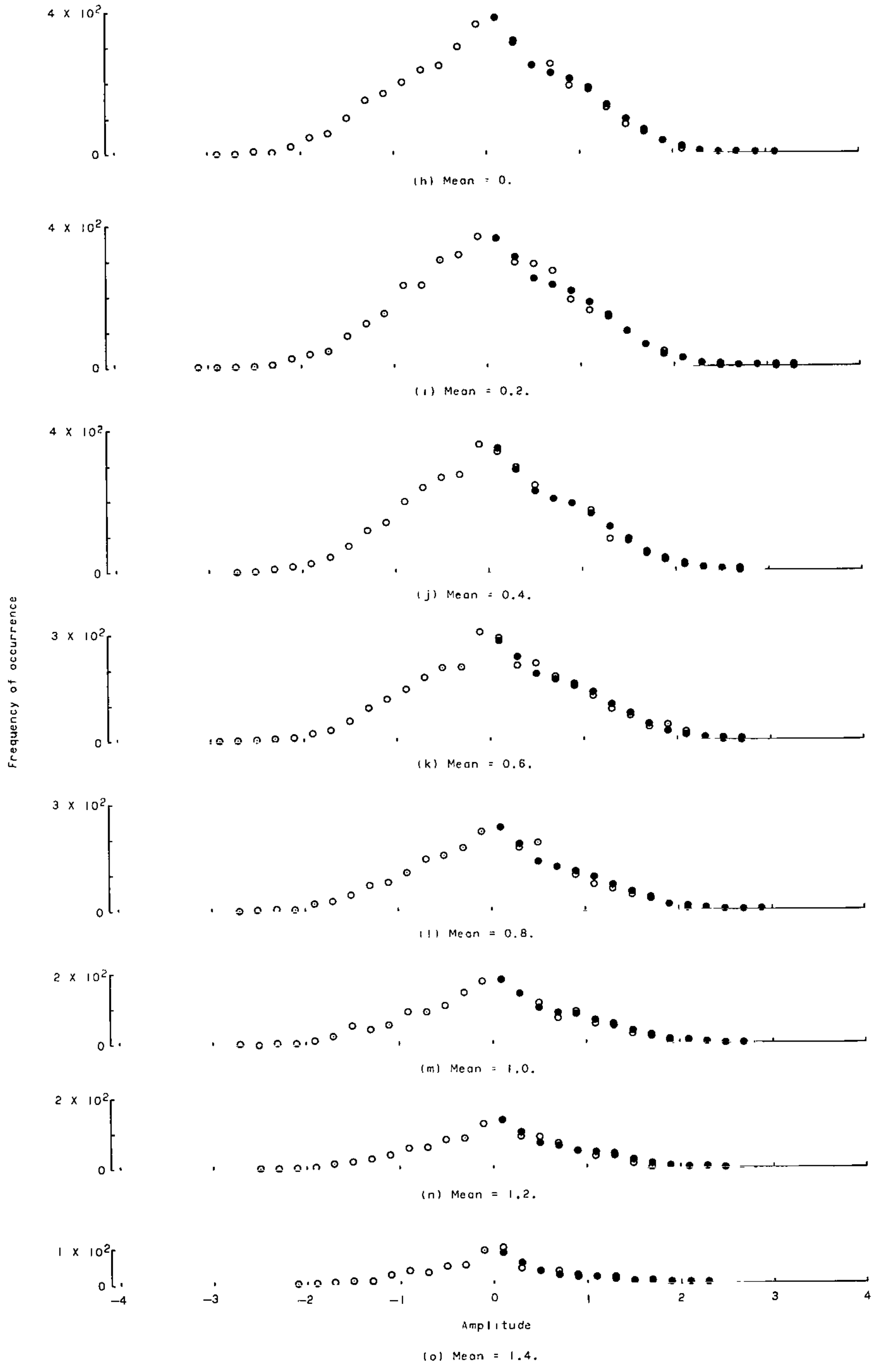


Figure 6.- Concluded.

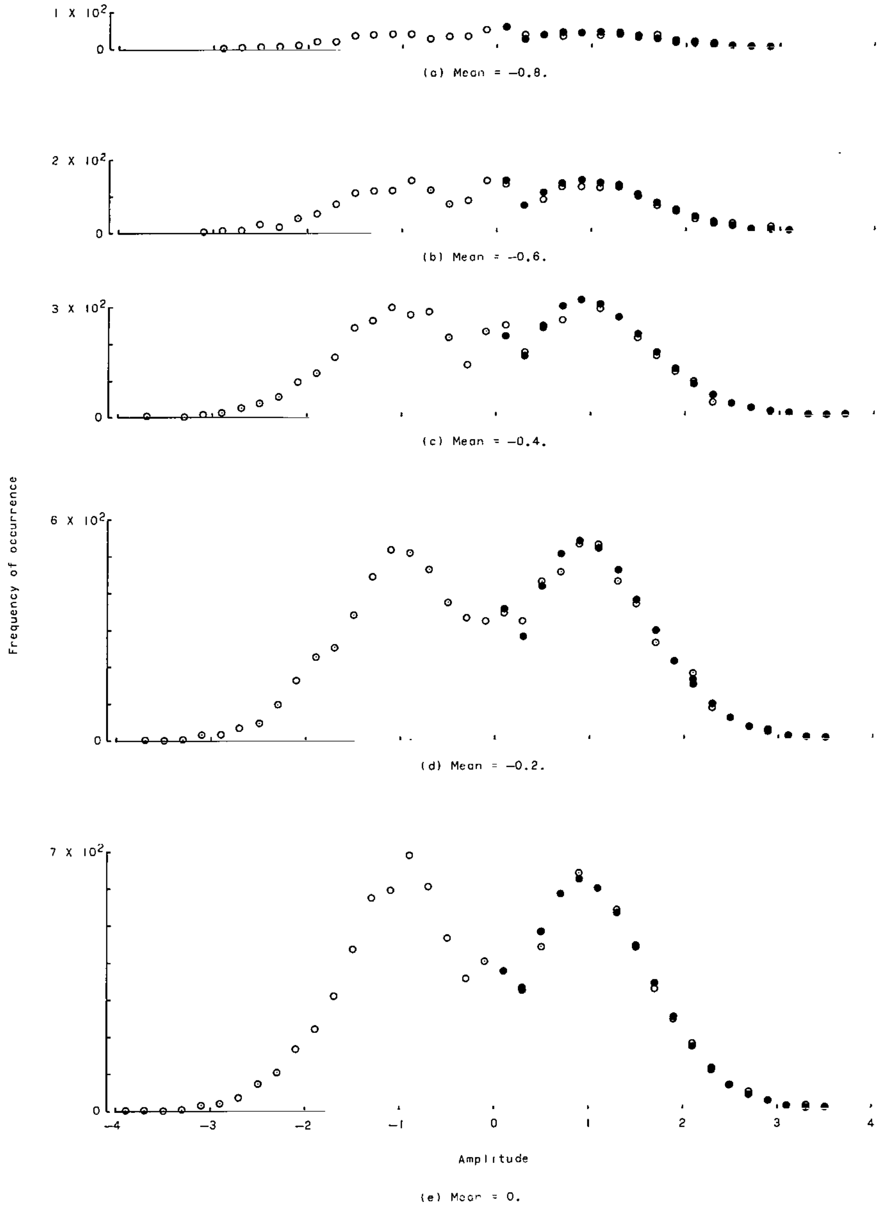


Figure 7.- Statistical distributions of instantaneous amplitudes with respect to a specified instantaneous mean value for time history C. Open symbols represent actual values; solid symbols represent computed values.

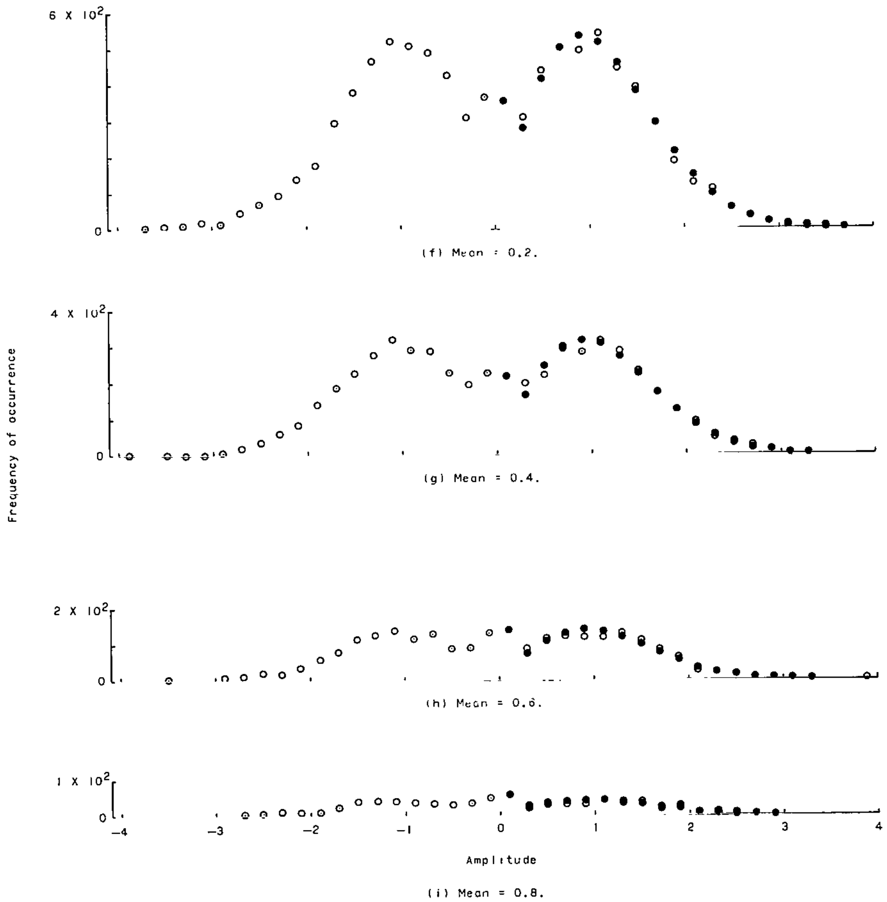


Figure 7.- Concluded.

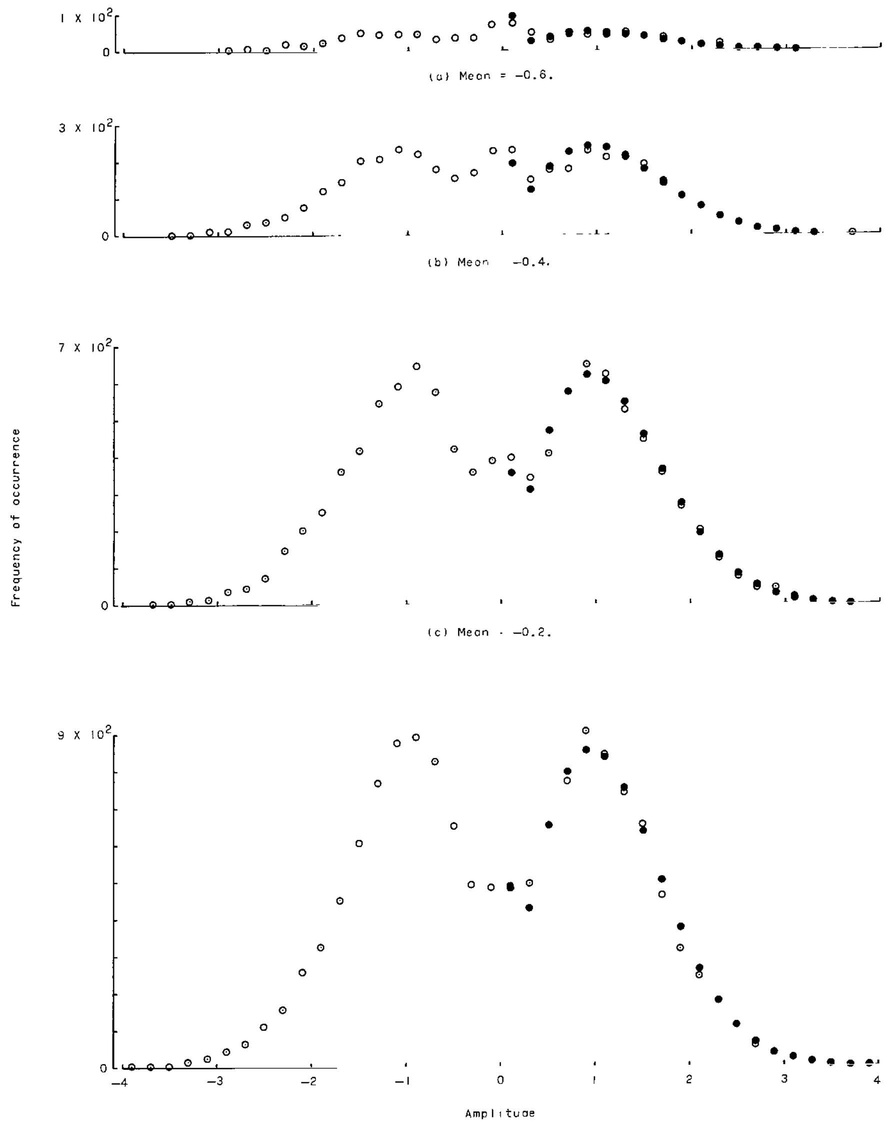


Figure 8.- Statistical distributions of instantaneous amplitudes with respect to a specified instantaneous mean value for time history D. Open symbols represent actual values; solid symbols represent computed values.

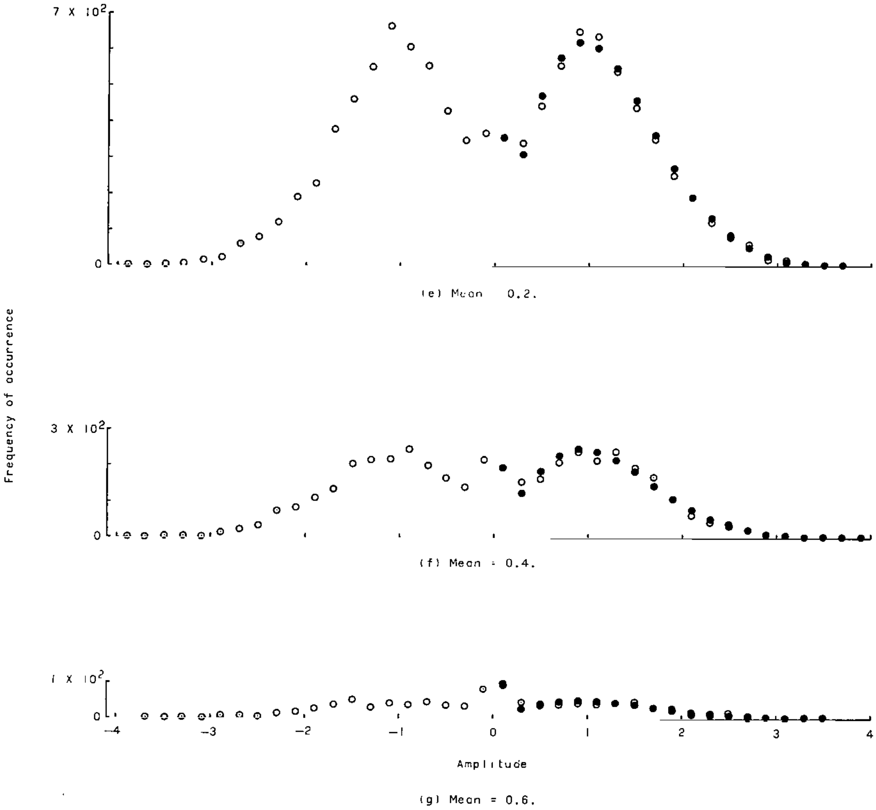


Figure 8.- Concluded.

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