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AN APPROXIMATE NONLINEAR ANALYSIS OF THE STABILITY OF SLOSHING MODES UNDER TRANSLATIONAL AND ROTATIONAL EXCITATION

by T. R. Rogge and H. J. Weiss

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ABSTRACT

This paper considers the irrotational motion of an incompressible, inviscid fluid contained in a partially filled tank. The tank is subjected to both transverse and rotational vibrations whose frequencies are near the first natural frequency of small free-surface oscillations. Following a method suggested by Hutton[3], the analysis is performed retaining higher order terms in the free-surface dynamic and kinematic boundary conditions. The theoretical investigation predicts the forcing frequency ranges, for various combinations of rotational and translational motion, over which there are stable, steady-state, harmonic, planar and nonplanar motions. The least stable case occurs when a combination of motions occurs in mutually orthogonal planes.

SECTION 1

INTRODUCTION

The linear theory of the small oscillations of a free surface in a gravitational field appears in essentially complete form in Lamb's Hydrodynamics[1]. The advent of the missile age has stimulated renewed interest in this problem, and in recent years numerous papers have been written on specific problems in this area.

The dynamic response of the liquid propellant in the tanks of a space vehicle can affect the stability of the vehicle; this influence can be alleviated in many ways, among which are the proper choice of tank form, of tank location, or the introduction of baffles into the tank.

These fluid oscillations, resulting from such sources as perturbation of the trajectory, have been shown experimentally to be most critical when the excitation frequency is in the region of a natural frequency of lower mode fluid oscillations.

Eulitz and Glaser[2] have compared experimental results with the previously obtained theoretical solutions, which are obtained from a linear boundary value problem. Within the framework of linear theory, the free surface of the fluid in a container undergoing transverse harmonic vibrations should exhibit a steady-state, planar, harmonic motion at all frequencies except resonance. Eulitz and Glaser claimed thorough agreement between the experimental results and the linearized theory.

Hutton[3] notes that the free surface of a fluid in a container undergoing transverse harmonic vibrations does not necessarily exhibit a steady-state harmonic motion. In fact, if the container is excited at a frequency well below the lowest natural frequency, p_{11}, of small, free-surface oscillations, the steady-state fluid motion is harmonic with a constant peak wave height and a single nodal diameter perpendicular to the direction of excitation. The wave height increases with an increase of the excitation frequency. When the excitation frequency is close to but smaller than p_{11}, the smoothly oscillating free surface changes to a violently splashing condition. As the frequency increases, this motion continues until a frequency greater than p_{11} is attained. Additional increases in the excitation frequency reduce the wave height up to the point where the cycle begins again as the next resonant frequency is approached.

Hutton shows that the sloshing motion can be accurately predicted in an inviscid liquid if the analysis includes appropriate nonlinear effects.

The present paper is an extension and generalization of Hutton's work to include not only transverse harmonic oscillations of the container but also rotational harmonic

oscillations. Again, the appropriate nonlinear effects are included. A comparison is made between the two solutions and, in particular, the stability of the nonlinear motion is studied.

SECTION 2

DEFINITION OF THE BOUNDARY VALUE PROBLEM

The problem under consideration is that of a tank, partially filled with a nonviscous, incompressible liquid, which is mounted in a system which is moving along a prescribed path. Perturbations of the path of the system cause the liquid to oscillate. There exist two possible types of motion that should be considered. The first is that of surface waves of large amplitudes, possibly of low frequency, which could actually damage the tank structure. For the most part this type can be controlled by suitable baffles in the tank. The second type, which will be considered here, is that of surface waves of small amplitudes with a frequency near the natural frequency of the control system on the tank, i.e., the natural frequency of the liquid-tank configuration.

Since the tank is in motion along some path, it seems reasonable to refer its motion to an inertial coordinate system, for example the earth. However, if any type of measuring device is attached to the tank, then it measures quantities in terms of a tank-fixed reference frame which is moving relative to the inertial system. Thus, it is necessary to be able to express the tank-fixed system in terms of the inertial system and vice versa.

Let Y_i be an inertial Cartesian coordinate system with origin O' and coordinates y_i; and let X_i be a Cartesian coordinate system moving relative to Y_i, with origin O and coordinates x_i. Then, instantaneously, the position of a particle moving with the X_i system can be described in the Y_i system by

$$y_i = \bar{Z}_i + a_{ji} x_j \tag{2.1}$$

where the summation convention is being used and Latin subscripts take on the values 1, 2, and 3. In (2.1), $\bar{Z}_i(t)$, with components measured in Y_i, give the instantaneous displacement of O relative to O'; and

$$a_{ij}(t) = \cos\left(x_i, y_j\right) \tag{2.2}$$

measures the instantaneous rotation of x_i with respect to y_j. Subsequently the following notation will be used: a barred vector has components measured in Y_i and an unbarred vector has components measured in X_i.

Since a_{ij} are a set of direction cosines, they satisfy, for any t,

$$a_{ik} a_{jk} = \delta_{ij} \tag{2.3}$$

where δ_{ij} is the Kronecker delta. Denote df/dt by $\dot{f}$ and take the derivative of (2.3); then

$$a_{ik}\dot{a}_{jk} + \dot{a}_{ik}a_{jk} = 0 \tag{2.4}$$

Define

$$\omega_{ij} = a_{ik}\dot{a}_{jk} \tag{2.5}$$

Thus, using (2.4) and (2.5), one has

$$\omega_{ji} = a_{jk}\dot{a}_{ik} = -\dot{a}_{jk}a_{ik} = -a_{ik}\dot{a}_{jk} = -\omega_{ij} \tag{2.6}$$

That is, ω_{ij} is a skew-symmetric second order quantity which can be shown to be a second-order tensor.

A dual vector ω_i can then be defined such that

$$\omega_{ij} = -\epsilon_{ijk}\omega_k$$

where ϵ_{ijk} is the third-order alternating tensor. Thus, from (2.6),

$$a_{ik}\dot{a}_{jk} = -\epsilon_{ijk}\omega_k \tag{2.7}$$

where ω_k is the angular velocity of X_i with respect to Y_i measured in X_i.

The absolute velocity of a particle whose position is described by (2.1) can be found by differentiating (2.1) with respect to time. This operation gives

$$\dot{y}_i = \dot{\bar{Z}}_i + a_{ji}\dot{x}_j + \dot{a}_{ji}x_j = \bar{q}_i \tag{2.8}$$

where $\bar{q}_i$ is the velocity measured in the Y_i system. The measuring device fixed on the tank measures q_i, where q_i is the velocity measured in the X_i system and

$$q_i = a_{ij}\bar{q}_j \tag{2.9}$$

Using (2.8), q_i can be written as

$$q_i = a_{ij}\left(\dot{\bar{Z}}_j + a_{kj}\dot{x}_k + \dot{a}_{kj}x_k\right) \tag{2.10}$$

With (2.3), (2.7), and the fact that $\dot{Z}_i = a_{ij}\dot{\bar{Z}}_j$, (2.10) becomes

$$q_i = \dot{Z}_i + \dot{x}_i - \epsilon_{ikj}\omega_j x_k \tag{2.11}$$

or

$$q_i = \dot{Z}_i + \dot{x}_i + \epsilon_{ijk}\omega_j x_k \tag{2.12}$$

where the skew-symmetric property of the alternating tensor ϵ_{ijk} has been used.

The next quantity of interest is the absolute acceleration

$$\bar{a}_i = \dot{\bar{q}}_i \tag{2.13}$$

From (2.9), (2.3) can be written as

$$\bar{a}_i = \frac{d}{dt}\left(a_{ji}q_j\right) = a_{ji}\frac{dq_j}{dt} + \dot{a}_{ji}q_j \tag{2.14}$$

The quantity of interest is $a_i = a_{ij}\bar{a}_j$; thus from (2.14),

$$a_k = a_{ki}a_{ji}\frac{dq_j}{dt} + a_{ki}\dot{a}_{ji}q_j \tag{2.15}$$

or

$$a_k = \frac{dq_k}{dt} + \epsilon_{ijk}\omega_i q_j \tag{2.16}$$

Here it is noted that the velocity q_i is a function of not only the time, but also the coordinates x_i, which are also functions of time; thus

$$\frac{dq_k}{dt} = \frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial x_i}\frac{dx_i}{dt} \tag{2.17}$$

From (2.12),

$$\frac{dx_i}{dt} = \dot{x}_i = q_i - \dot{Z}_i - \epsilon_{ijk}\omega_j x_k \tag{2.18}$$

Then (2.18), (2.17), and (2.16) give, finally,

$$a_k = \frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial x_i}\left[q_i - \dot{Z}_i - \epsilon_{ijs}\omega_j x_s\right] + \epsilon_{ijk}\omega_i q_j \tag{2.19}$$

In vector form, (2.11) and (2.19) appear as

$$\vec{q} = \vec{q}_o + \vec{\omega}\times\vec{r} + \dot{\vec{r}} \tag{2.20}$$

and

$$\vec{a} = \frac{\partial\vec{q}}{\partial t} + \vec{\omega}\times\vec{q} + \left[\left(\vec{q} - \vec{q}_o - \vec{\omega}\times\vec{r}\right)\cdot\nabla\right]\vec{q} \tag{2.21}$$

where $\vec{q}_o$ is the velocity of O relative to O′

The Eulerian equations of motion for an incompressible, inviscid fluid are, in the inertial system,

$$\bar{a}_i = \bar{F}_i - \frac{1}{\rho}\frac{\partial p}{\partial y_i} \tag{2.22}$$

where $\bar{F}_i$ is the specific body force, ρ is the density, and p is the pressure. Since

$$y_i = y_i\left(x_1, x_2, x_3\right),$$

$$\frac{\partial p}{\partial y_i} = \frac{\partial p}{\partial x_k}\frac{\partial x_k}{\partial y_i} = a_{ki}\frac{\partial p}{\partial x_k} \tag{2.23}$$

Transform (2.22) to the tank fixed system using (2.23); then

$$a_j = F_j - \frac{1}{\rho}a_{ji}a_{ki}\frac{\partial p}{\partial x_k} \tag{2.24}$$

or

$$a_j = F_j - \frac{1}{\rho}\frac{\partial p}{\partial x_j} \tag{2.25}$$

Assuming that the motion is irrotational, there exists a potential, ϕ, such that

$$\bar{q}_i = -\frac{\partial\phi}{\partial y_i} \tag{2.26}$$

The incompressibility assumption implies that

$$\frac{\partial \bar{q}_i}{\partial y_i} = 0 \tag{2.27}$$

Equations (2.26) and (2.27) then lead to

$$\nabla^2 \phi = \frac{\partial}{\partial y_i}\left(\frac{\partial \phi}{\partial y_i}\right) = 0 \tag{2.28}$$

Transform the above to the tank fixed system, noting that

$$\frac{\partial \phi}{\partial y_i} = \frac{\partial \phi}{\partial x_k}\frac{\partial x_k}{\partial y_i} = a_{ki}\frac{\partial \phi}{\partial x_k}$$

Then

$$q_j = -\frac{\partial \phi}{\partial x_j} \tag{2.29}$$

$$\frac{\partial q_i}{\partial x_j} = 0 \tag{2.30}$$

$$\nabla^2 \phi = \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_j}\right) = 0 \tag{2.31}$$

Thus, the solution of Laplace's equation furnishes a possible potential function for an incompressible, irrotational flow. In order to determine exactly which potential function is the solution, certain boundary conditions need to be prescribed.

Consider a tank of arbitrary shape partially filled with fluid. Assume a constant acceleration is acting along the x_3 axis. The surface of the liquid then assumes a planar surface normal to this axis, this surface being called the free surface or quiescent free surface. The origin of the X_i system is taken at the center of gravity of the accelerating fluid system. The motion of the tank-fixed system X_i relative to Y_i, characterized by $\dot{Z}_i$ and ω_i, are oscillatory motions superimposed on the constant-acceleration motion. These motions will induce perturbations or disturbances of the free surface. The measuring device traveling with the tank sees only the forcing motions or perturbations.

In this analysis it will be assumed that the tank is rigid. With this in mind, the boundary condition on the wetted surface of the tank must be that the velocity of the liquid normal to the tank wall must equal the normal component of velocity of the tank itself. Thus, if υ_i is the unit exterior normal to the tank and $q_i = -\frac{\partial \phi}{\partial x_i}$,

$$-\nu_i \frac{\partial \phi}{\partial x_i} = \nu_i \left[\dot{Z}_i + \epsilon_{ijk} \omega_j x_k \right] \tag{2.32}$$

where $\dot{x}_i = 0$ for a rigid tank.

There are two conditions at the free surface. Denoting the disturbed free surface by $\eta(x_1, x_2, t)$ and the unit normal to the quiescent free surface by n_i, the kinematic condition that a particle of fluid which travels with the free surface as it moves must have the same velocity as the free surface itself is given as

$$\frac{d}{dt}\left(x_3 - \eta\right)\Big|_{x_3 = \eta} = 0 \tag{2.33}$$

where x_3 is the displacement of a particle in the x_3 direction, and, as in (2.17),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial x_k}{\partial t}\frac{\partial}{\partial x_k} \tag{2.34}$$

Expanding (2.33) and using (2.34) and (2.12)

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_1}\dot{x}_1 + \frac{\partial \eta}{\partial x_2}\dot{x}_2 = q_3 - \dot{Z}_3 - \epsilon_{3jk}\omega_j x_k, \text{ on } x_3 = \eta \tag{2.35}$$

But since $n_1 = n_2 = 0$ and $n_3 = 1$, the right hand side of (2.35) can be written as

$$\left(q_i - \dot{Z}_i - \epsilon_{ijk}\omega_j x_k\right) n_i$$

Thus (2.35) becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_i}\dot{x}_i = \left(q_i - \dot{Z}_i - \epsilon_{ijk}\omega_j x_k\right) n_i, \text{ on } x_3 = \eta \tag{2.36}$$

The second condition at the free surface is a dynamic one which states that the pressure at the free surface of the fluid must equal the ambient pressure. To find the form of

this boundary condition it is necessary to integrate the equations of motion. Substitute (2.19) into (2.25); the equations of motion become

$$\frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial x_i}\left[q_i - \dot{Z}_i - \epsilon_{its}\omega_t x_s\right] + \epsilon_{ijk}\omega_i q_j = F_k - \frac{1}{\rho}\frac{\partial p}{\partial x_k} \tag{2.37}$$

If the only specific body force is that due to the gravitational field in which the tank system is moving, (2.37) can be integrated directly:

$$\frac{p - p_o}{\rho} = -\left[\alpha x_3 + \frac{1}{2}\left(q_i - \dot{Z}_i\right)^2 - \left(\epsilon_{ijk}\omega_j x_k\right)q_i - \frac{\partial \phi}{\partial t}\right] \tag{2.38}$$

where p_o is the ambient pressure and α is the magnitude of the acceleration of the tank system. It is assumed here that p_o is a constant. Thus, the second boundary condition at the free surface is

$$\frac{\partial \phi}{\partial t} = \alpha\eta + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i\right)^2 + \epsilon_{ijk}\,\omega_j x_k \frac{\partial \phi}{\partial x_i}, \text{on } x_3 = \eta \tag{2.39}$$

In summary, the mathematical description of the motion of an incompressible, irrotational fluid confined in a moving, partially filled tank subject to translational and rotational perturbations is

$$\nabla^2 \phi = 0 \tag{2.40}$$

$$-\nu_i \frac{\partial \phi}{\partial x_i} = \nu_i\left[\dot{Z}_i + \epsilon_{ijk}\omega_j x_k\right] \tag{2.41}$$

on the wetted surface, where ν_i is the unit exterior normal to the tank;

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_i}\dot{x}_i = -\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i + \epsilon_{ijk}\omega_j x_k\right)n_i, \text{ on } x_3 = \eta \tag{2.42}$$

where n_i is the normal to the free surface; and

$$\frac{\partial \phi}{\partial t} = \alpha\eta + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i\right)^2 + \epsilon_{ijk}\omega_j x_k \frac{\partial \phi}{\partial x_i}, \text{ on } x_3 = \eta \tag{2.43}$$

If the free surface oscillations are sufficiently small, terms of second order in the velocities can be neglected. Then (2.40) to (2.43) become

$$\nabla^2 \phi = 0 \tag{2.1a}$$

$$-\nu_i \frac{\partial \phi}{\partial x_i} = \nu_i \left[\dot{Z}_i + \epsilon_{ijk} \omega_j x_k \right], \text{ on } S_\omega \tag{2.2a}$$

where S_ω is the wetted surface.

$$\frac{\partial \eta}{\partial t} = -\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i + \epsilon_{ijk} \omega_j x_k \right) n_i, \text{ on } x_3 = \eta \tag{2.3a}$$

$$\frac{\partial \phi}{\partial t} = \alpha \eta \text{ , on } x_3 = \eta \tag{2.4a}$$

The last two equations can be combined into a single condition:

$$\frac{1}{\alpha} \frac{\partial^2 \phi}{\partial t^2} = -\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i + \epsilon_{ijk} \omega_j x_k \right) n_i, \text{ on } x_3 = \eta \tag{2.5a}$$

If is is assumed that $\dot{Z}_i$ and ω_j can furthermore be represented as harmonic oscillations,

$$\left. \begin{aligned} \dot{Z}_i &= Z_i^{(0)} e^{i\beta t} \\ \omega_i &= \omega_i^{(0)} e^{i\beta t} \end{aligned} \right\} \tag{2.6a}$$

then it may be assumed that

$$\phi \left(x_i, t \right) = \psi \left(x_i \right) e^{i\beta t} \tag{2.7a}$$

The problem then reduces to

$$\nabla^2 \psi = 0 \tag{2.8a}$$

$$-\nu_i \frac{\partial \psi}{\partial x_i} = \nu_i \left[Z_i^{(0)} + \epsilon_{ijk} \omega_j^{(0)} x_k \right], \text{ on } S_\omega \tag{2.9a}$$

$$-\frac{\beta^2}{\alpha} \psi = -\left(\frac{\partial \psi}{\partial x_i} + Z_i^{(0)} + \epsilon_{ijk} \omega_j^{(0)} x_k \right) n_i, \text{ on } x_3 = \eta \tag{2.10a}$$

The solution to (2.8a) through (2.10a) can always be obtained if the tank is a prismatic cylinder with x_3 parallel to a generator and with cross section such that $\nabla^2 \psi$ is separable in the appropriate three-dimensional coordinate system.

SECTION 3

NONLINEAR SLOSHING IN A CIRCULAR CYLINDRICAL TANK

The natural course here is to express the problem in terms of cylindrical polar coordinates (r, θ, z). Before doing this, consider (2.41); there exist two segments of wetted surface: the side of the tank and the bottom of the tank. Considering the side first, the normal ν_i has the following components:

$$\left.\begin{aligned} \nu_1 &= \cos\theta \\ \nu_2 &= \sin\theta \\ \nu_3 &= 0 \end{aligned}\right\} \tag{3.1}$$

Thus, (2.41) becomes

$$-\nu_1 \frac{\partial \phi}{\partial x_1} - \nu_2 \frac{\partial \phi}{\partial x_2} = \nu_1 \dot{Z}_1 + \nu_2 \dot{Z}_2 + \nu_1 \epsilon_{1jk} \omega_j x_k + \nu_2 \epsilon_{2jk} \omega_j x_k \tag{3.2}$$

On the bottom of the tank the normal ν_i has the components

$$\left.\begin{aligned} \nu_1 &= 0 \\ \nu_2 &= 0 \\ \nu_3 &= -1 \end{aligned}\right\} \tag{3.3}$$

Thus, (2.41) becomes

$$\frac{\partial \phi}{\partial x_3} = -\dot{Z}_3 - \epsilon_{3jk} \omega_j x_k \tag{3.4}$$

Note that $\dot{Z}_3$ is absent in (3.2) and that ω_3 is absent in (3.4). This condition will in fact be the case for any cylindrical tank whose generators are parallel to the x_3 axis and is not merely a peculiarity of the circular-cylindrical tank.

In the following, (2.40) through (2.43) will be transformed into cylindrical polar coordinates. So far all the quantities in these equations have been measured with respect

to the tank-fixed rectangular Cartesian coordinate system. These quantities may also be expressed in a tank-fixed, cylindrical, polar coordinate system. In doing so, it is also convenient to shift the origin from the center of gravity of the fluid to the geometric center of the quiescent free surface. Let

$$x_1 = x$$

$$x_2 = y$$

$$x_3 = z$$

and use the usual transformation equations to cylindrical polar coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \tag{3.5}$$

Then (2.40) becomes

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{3.6}$$

Let

$$\left.\begin{aligned} b_i &= \epsilon_{ijk}\omega_j x_k \\ b_1 &= \omega_2 z - \omega_3 r\sin\theta \\ b_2 &= \omega_3 r\cos\theta - \omega_1 z \\ b_3 &= \omega_1 r\sin\theta - \omega_2 r\cos\theta \end{aligned}\right\} \tag{3.7}$$

Denote the quantity $\dot{Z}_i$ by u_i; (3.2) and (3.4) become, respectively,

$$\frac{\partial\phi}{\partial r} = -u_1\cos\theta - u_2\sin\theta - z\omega_2\cos\theta + z\omega_1\sin\theta \tag{3.8}$$

$$\frac{\partial\phi}{\partial z} = -u_3 - \omega_1 r\sin\theta + \omega_2 r\cos\theta \tag{3.9}$$

It may be noted in (3.8) that ω_3 is absent. This is only true for a circular cylinder. To see this, consider the last term, $\nu_i \epsilon_{ijk} \omega_j x_k$ in (2.41). In vector form this is

$$\vec{\nu} \cdot (\vec{\omega} \times \vec{\rho}) = \vec{\omega} \cdot (\vec{\rho} \times \vec{\nu}) \tag{3.10}$$

where

$$\vec{\rho} = (r \cos\theta,\ r \sin\theta,\ z) .$$

On the side of the circular-cylindrical tank, ν_i has components given by (3.1). Thus, the third component of $\vec{\rho} \times \vec{\nu}$ is

$$(\vec{\rho} \times \vec{\nu})_3 = r \cos\theta \sin\theta - r \cos\theta \sin\theta \equiv 0 \tag{3.11}$$

Thus, from (3.10), ω_3 is absent in the expression $\nu_i \epsilon_{ijk} \omega_j x_k$ for a circular cylindrical tank. Since this term in (2.9a) is the only place ω_j enters, ω_3 is absent from this boundary condition for this special tank configuration.

On $x_3 = \eta(r, \theta, t)$, the unit exterior normal has the following components:

$$\left.\begin{aligned} n_1 &= n_2 = 0 \\ n_3 &= 1 \end{aligned}\right\} \tag{3.12}$$

Therefore, from (2.11), (2.42), (3.7), and (3.12), one of the boundary conditions at $z = \eta$ becomes

$$\begin{aligned} -\frac{\partial\phi}{\partial z} - u_3 - \omega_1 r \sin\theta + \omega_2 r \cos\theta &= \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial r}\frac{\partial\phi}{\partial r} \\ &- \frac{1}{r^2}\frac{\partial\eta}{\partial\theta}\frac{\partial\phi}{\partial\theta} - u_1\left(\cos\theta\frac{\partial\eta}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\eta}{\partial\theta}\right) - u_2\left(\sin\theta\frac{\partial\eta}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\eta}{\partial\theta}\right) \\ &+ \omega_1\eta\left(\sin\theta\frac{\partial\eta}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\eta}{\partial\theta}\right) - \omega_2\eta\left(\cos\theta\frac{\partial\eta}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\eta}{\partial\theta}\right) - \omega_3\frac{\partial\eta}{\partial\theta} \end{aligned} \tag{3.13}$$

The other boundary condition at $z = \eta$ can be obtained from (2.43) using (3.7) and (3.12):

$$\begin{aligned} \frac{\partial\phi}{\partial t} = \alpha\eta + \frac{1}{2}\Bigg[&\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{2\partial\phi}{\partial r}\left(u_1\cos\theta + u_2\sin\theta\right) \\ &+ \frac{2}{r}\frac{\partial\phi}{\partial\theta}\left(u_2\cos\theta - u_1\sin\theta\right) + 2\frac{\partial\phi}{\partial z}u_3 + u_1^2 + u_2^2 + u_3^2\Bigg] \end{aligned}$$

$$+ \omega_2 \frac{\partial \phi}{\partial r} \eta \cos\theta - \frac{\omega_2}{r} \eta \frac{\partial \phi}{\partial \theta} \sin\theta + \omega_3 \frac{\partial \phi}{\partial \theta} + r \frac{\partial \phi}{\partial z} (\omega_1 \sin\theta - \omega_2 \cos\theta)$$

$$- \omega_1 \eta \left(\frac{\partial \phi}{\partial r} \sin\theta + \frac{\cos\theta}{r} \frac{\partial \phi}{\partial \theta} \right) \tag{3.14}$$

Thus, if the cylindrical tank has radius a, and the original depth of the fluid is given by $z = -h$, the motion of the contained fluid in a circular cylindrical tank is governed by the following partial differential equation, subject to the given boundary conditions:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{3.15}$$

for

$$0 \le r < a$$

$$0 \le \theta \le 2\pi$$

and

$$-h < z < \eta$$

On $r = a$

$$\frac{\partial \phi}{\partial r} = -u_1 \cos\theta - u_2 \sin\theta - z\omega_2 \cos\theta + z\omega_1 \sin\theta \tag{3.16}$$

On $z = -h$

$$\frac{\partial \phi}{\partial z} = -u_3 - \omega_1 r \sin\theta + \omega_2 r \cos\theta \tag{3.17}$$

On $z = \eta$

$$-\frac{\partial \phi}{\partial z} - u_3 - \omega_1 r \sin\theta + \omega_2 r \cos\theta = \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial \eta}{\partial \theta} \frac{\partial \phi}{\partial \theta}$$

$$- u_1 \left[\cos\theta \frac{\partial \eta}{\partial r} - \frac{\sin\theta}{r} \frac{\partial \eta}{\partial \theta} \right] + \omega_1 \eta \left[\sin\theta \frac{\partial \eta}{\partial r} + \frac{\cos\theta}{r} \frac{\partial \eta}{\partial \theta} \right]$$

$$- u_2 \left[\sin\theta \frac{\partial \eta}{\partial r} + \frac{\cos\theta}{r} \frac{\partial \eta}{\partial \theta} \right] - \omega_2 \eta \left[\cos\theta \frac{\partial \eta}{\partial r} - \frac{\sin\theta}{r} \frac{\partial \eta}{\partial \theta} \right] - \omega_3 \frac{\partial \eta}{\partial \theta} \tag{3.18}$$

$$\frac{\partial \phi}{\partial t} = \alpha\eta + \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right] + \frac{\partial \phi}{\partial r} \left[u_1 \cos\theta + u_2 \sin\theta \right]$$

$$+ \frac{1}{r} \frac{\partial \phi}{\partial \theta} \left[u_2 \cos\theta - u_1 \sin\theta \right] + \frac{\partial \phi}{\partial z} u_3 + \frac{1}{2} \left[u_1^2 + u_2^2 + u_3^2 \right]$$

$$+ \omega_2 \eta \left[\frac{\partial \phi}{\partial r} \cos\theta - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \sin\theta \right] + \omega_3 \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial z} r \left[\omega_1 \sin\theta - \omega_2 \cos\theta \right]$$

$$- \omega_1 \eta \left[\frac{\partial \phi}{\partial r} \sin\theta + \frac{\cos\theta}{r} \frac{\partial \phi}{\partial \theta} \right] \tag{3.19}$$

To simplify the problem described, make the following transformation:

$$\psi(r, \theta, z, t) = \phi(r, \theta, z, t) + u_1 r \cos\theta + u_2 r \sin\theta - z r \omega_1 \sin\theta$$

$$+ z \omega_2 r \cos\theta \tag{3.20}$$

Equations (3.15) through (3.19) become:

$$\nabla^2 \psi = 0 \tag{3.21}$$

On $r = a$

$$\frac{\partial \psi}{\partial r} = 0 \tag{3.22}$$

On $z = -h$

$$\frac{\partial \psi}{\partial z} = -u_3 - 2 r \omega_1 \sin\theta + 2 r \omega_2 \cos\theta \tag{3.23}$$

On $z = \eta$

$$\frac{\partial \psi}{\partial t} - \dot{u}_1 r \cos\theta - \dot{u}_2 r \sin\theta + \eta r \dot{\omega}_1 \sin\theta - \eta \dot{\omega}_2 r \cos\theta$$

$$= \alpha\eta + \frac{1}{2}\left[\left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 \right] - \frac{\eta^2}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{u_3^2}{2}$$

$$+ \frac{3}{2} r^2 \left(\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta\right) + 2 \frac{\partial \psi}{\partial z}\left(r\omega_1 \sin\theta - r\omega_2 \cos\theta + \frac{u_3}{2}\right)$$

$$- 3 r^2 \omega_1 \omega_2 \sin\theta \cos\theta + u_3 \omega_1 r \sin\theta - u_3 \omega_2 r \cos\theta - \eta \omega_2 u_1 - u_2 \omega_3 r \cos\theta$$

$$+ \eta \omega_1 u_2 + \eta \omega_3 \omega_2 r \sin\theta + u_1 \omega_3 r \sin\theta + \omega_3 \frac{\partial \psi}{\partial \theta} + \eta \omega_3 \omega_1 r \cos\theta \tag{3.24}$$

and

$$-\frac{\partial \psi}{\partial z} - u_3 + 2 r \omega_2 \cos\theta - 2 r \omega_1 \sin\theta = \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\frac{\partial \eta}{\partial \theta}\frac{\partial \psi}{\partial \theta} - \omega_3 \frac{\partial \eta}{\partial \theta} \tag{3.25}$$

The free surface η is one of the unknowns, but it may be eliminated between (3.24) and (3.25) as shown in Appendix A. Equations (3.24) and 3.25) may thus be replaced by (A.12):

$$
\begin{aligned}
& a_{00} + a_{01} - \frac{a_{11}b_{00}}{\alpha} + a_{02} - \frac{a_{11}b_{01}}{\alpha} + \frac{a_{12}b_{00}}{\alpha} + \frac{a_{11}b_{11}b_{00}}{\alpha^2} + \frac{a_{22}b_{00}^2}{2\alpha^2} \\
& + B_{11} + B_{12} + B_{13} - a_{11}\Big(\alpha^{-1}\dot{u}_2 r \sin\theta + \alpha^{-2}\Big[r\dot{\sigma}_1 \dot{u}_2 r \sin\theta \\
& - b_{11}\dot{u}_2 r \sin\theta + r\dot{\sigma}_1 b_{00}\Big] + \alpha^{-1}\Big[\frac{3}{2}r^2\omega_1^2 \sin^2\theta + \frac{3}{2}r^2\omega_2^2\cos^2\theta \\
& + 2\xi_r r\left(u_3 - \sigma_1\right) - 3r^2\omega_1\omega_2 \sin\theta\cos\theta + 2u_3 r\sigma_1 + u_1\omega_3 r\sin\theta \\
& + \omega_3\xi_\theta\Big]\Big) - B_{22}\Big(\alpha^{-1}\left[b_{00} + \dot{u}_2 r\sin\theta\right] - \alpha^{-2}\left[b_{11} - r\dot{\sigma}_1\right]\left[b_{00} + \dot{u}_2 r\sin\theta\right] \\
& + \alpha^{-1}\Big[b_{01} + \frac{3}{2}r^2\omega_1^2\sin^2\theta + \frac{3}{2}r^2\omega_2^2\cos^2\theta + \xi_z r\left(u_3 - \sigma_1\right) \\
& - 3r^2\omega_1\omega_2\sin\theta\cos\theta + 2u_3 r\sigma_1 + u_1\omega_3 r\sin\theta + \omega_3\xi_\theta\Big]\Big) \\
& + a_{12}\alpha^{-1}\dot{u}_2 r\sin\theta + B_{23}\alpha^{-1}\left[b_{00} + \dot{u}_2 r\sin\theta\right] + \frac{a_{22}}{2}\alpha^{-2}\Big(2b_{00}\dot{u}_2 r\sin\theta \\
& + \dot{u}_2^2 r^2\sin^2\theta\Big) + \frac{B_{33}}{2}\alpha^{-2}\left[b_{00} + \dot{u}_2 r\sin\theta\right]^2 + 0\left(\eta^4\right) = 0 \qquad \text{(A.12)}
\end{aligned}
$$

where

$$\xi_\theta = \frac{\partial\psi}{\partial\theta}(r, \theta, 0, t)$$

$$\xi_z = \frac{\partial\psi}{\partial z}(r, \theta, 0, t)$$

$$\xi_r = \frac{\partial\psi}{\partial r}(r, \theta, 0, t)$$

a_{ij}, b_{ij}, and B_{ij} are functions of the potential ψ and its partial derivatives, all evaluated at $z = 0$ (see Appendix A); and

$$\sigma_1 = \omega_2 \cos\theta - \omega_1 \sin\theta$$

This replacement leads to the boundary value problem consisting of (3.21), (3.22), (3.23), and (A.12), which involves only the potential function ψ and the prescribed tank displacements. The tank displacements are assumed to be

$$\left.\begin{aligned} x_i(t) &= \epsilon_0^i \sin\omega t \\ \theta_i(t) &= \theta_0^i \sin\omega t \\ i &= 1, 2, 3 \end{aligned}\right\} \tag{3.26}$$

with ϵ_0^i and θ_0^i "small" and ω close to or equal to the lowest natural frequency p_{11} given by

$$p_{11} = \sqrt{\alpha\lambda_{11} \tanh \lambda_{11} h} \tag{3.27}$$

where λ_{11} is the first non-zero root of

$$J_1'\left(\lambda_{1n} a\right) = 0$$

Here the $x_i(t)$ correspond to translational motion, and the $\theta_i(t)$ correspond to rotational motions. Since ϵ_0^i and θ_0^i are small, it can be effectively assumed each set is the same for all i, say ϵ_0 and θ_0, respectively; and furthermore it can be assumed that

$$\theta_0 = \frac{\epsilon_0}{h} \tag{3.28}$$

The tank velocities $\dot{x}_i(t)$ and $\dot{\theta}_i(t)$ are

$$\begin{aligned} \dot{x}_i(t) &= \epsilon \cos\omega t \\ \dot{\theta}_i(t) &= \frac{\epsilon}{h} \cos\omega t \end{aligned} \tag{3.29}$$

where

$$\epsilon = \omega\epsilon_0$$

Following Hutton[3], a steady-state harmonic solution to this boundary value problem is posed in a perturbation form, in analogy with the Duffing problem[4, 5, 6] in terms of the parameter ϵ

$$\psi = \epsilon^{\frac{1}{3}}\left[\psi_1(\vec{r}, t)\cos\omega t + X_1(\vec{r}, t)\sin\omega t\right]$$
$$+ \epsilon^{\frac{2}{3}}\left[\psi_0(\vec{r}) + \psi_2(\vec{r})\cos 2\omega t + X_2(\vec{r})\sin 2\omega t\right]$$
$$+ \epsilon\left[\psi_3(\vec{r})\cos 3\omega t + X_3(\vec{r})\sin 3\omega t\right] \tag{3.30}$$

where the functions ψ_n and X_n for each value of n, each satisfy

$$\left.\begin{aligned} &\nabla^2\Phi = 0 \\ &\frac{\partial\Phi}{\partial r} = 0, \text{ on } r = a \\ &\frac{\partial\Phi}{\partial z} = 0, \text{ on } z = -h \end{aligned}\right\} \tag{3.31}$$

Here $\vec{r}$ means dependence upon r, θ, and z. A set of normal modes of vibration which satisfies (3.31) identically is

$$\left[A_{mn}(t)\cos m\theta + B_{mn}(t)\sin m\theta\right] J_m\left(\lambda_{mn} r\right)\frac{\cosh\lambda_{mn}(z + h)}{\cosh\lambda_{mn} h} \tag{3.32}$$

where the J_m are Bessel functions of the first kind of order m, for m a positive integer or zero; and λ_{mn} are an infinite set of numbers for each m obtained from the equation

$$J_m'\left(\lambda_{mn} a\right) = 0 \tag{3.33}$$

The functions $A_{mn}(t)$ and $B_{mn}(t)$ will be called the generalized coordinates of the mn'th mode; they depend only on the time, t. The natural frequency of small, free-surface oscillations in the mn'th mode is denoted by p_{mn}. When the tank displacements are harmonic motions at a frequency close to or at the lowest natural frequency, p_{11}, associated with the J_1 mode, the generalized coordinates A_{11} and B_{11} dominate all other generalized coordinates. Thus, it is assumed that the first order terms, ψ_1 and X_1, in (3.9) contain only the J_1 mode; thus ψ_1 and X_1 are chosen as

$$\psi_1 = \left[f_1(\tau)\cos\theta + f_3(\tau)\sin\theta\right] J_1\left(\lambda_{11} r\right) \frac{\cosh\left[\lambda_{11}(z+h)\right]}{\cosh\lambda_{11} h} \tag{3.34}$$

$$X_1 = \left[f_2(\tau)\cos\theta + f_4(\tau)\sin\theta\right] J_1\left(\lambda_{11} r\right) \frac{\cosh\left[\lambda_{11}(z+h)\right]}{\cosh\lambda_{11} h} \tag{3.35}$$

where the transformations

$$\left.\begin{aligned} \tau &= \frac{1}{2}\epsilon^{\frac{2}{3}}\omega t \\ p_{11}^2 &= \omega^2\left[1 - \nu\epsilon^{\frac{2}{3}}\right] \end{aligned}\right\} \tag{3.36}$$

have been used. In (3.36), ν is a dimensionless measure of frequency and τ is a dimensionless time parameter. A derivation of the above transformation is given in the appendix.

As shown in the appendix, (3.30) is then substituted into (A.12). Equate the coefficient of $\epsilon^{\frac{1}{3}}$ to zero:

$$\left(\alpha\psi_{1z} - p_{11}^2\psi_1\right)\cos\omega t + \left(\alpha X_{1z} - p_{11}^2 X_1\right)\sin\omega t = 0, \text{ on } z = 0 \tag{A.22}$$

Thus (A.22) is satisfied identically for all time if ψ_1 and X_1 are chosen as in (3.34) and (3.35). As can be seen, the coefficient of $\epsilon^{\frac{1}{3}}$ involves only the J_1 mode.

The vanishing of the coefficient of $\epsilon^{\frac{2}{3}}$ gives

$$\psi_{0z} = 0 \tag{A.27a}$$

$$\alpha\psi_{2z} - r p_{11}^2\psi_2 = 2p_{11}\left(X_{1r}\psi_{1r} + \frac{X_{1\theta}\psi_{1\theta}}{r^2} + \frac{3\zeta_{11}^2 - 1}{2}\lambda_{11}^2\psi_1 X_1\right) \tag{A.27b}$$

$$\begin{aligned} \alpha X_{2z} - r p_{11}^2 X_2 = p_{11}\Bigg(&X_{1r}^2 - \psi_{1r}^2 + \frac{X_{1\theta}^2}{r^2} - \frac{\psi_{1\theta}^2}{r^2} \\ &+ \frac{3\zeta_{11}^2 - 1}{2}\lambda_{11}^2\left(X_1^2 - \psi_1^2\right)\Bigg) \end{aligned} \tag{A.27c}$$

The functions ψ_0, ψ_2, and X_2 are chosen to satisfy (A.27a), (A.27b), and (A.27c). If ψ_0 is taken to be constant, (A.27a) will be satisfied identically. Choose ψ_2 and X_2 to be

$$\psi_2 = \sum_{n=1}^{\infty} \hat{A}_{0n} J_0\left(\lambda_{0n} r\right) \frac{\cosh\left[\lambda_{0n}(z+h)\right]}{\cosh \lambda_{0n} h}$$

$$+ \sum_{n=1}^{\infty} \left(\hat{A}_{2n} \cos 2\theta + \hat{B}_{2n} \sin 2\theta\right) J_2\left(\lambda_{2n} r\right) \frac{\cosh\left[\lambda_{2n}(z+h)\right]}{\cosh \lambda_{2n} h} \tag{3.37}$$

and

$$X_2 = \sum_{n=1}^{\infty} \hat{C}_{0n} J_0\left(\lambda_{0n} r\right) \frac{\cosh\left[\lambda_{0n}(z+h)\right]}{\cosh \lambda_{0n} h}$$

$$+ \sum_{n=1}^{\infty} \left(\hat{C}_{2n} \cos 2\theta + \hat{D}_{2n} \sin 2\theta\right) J_2\left(\lambda_{2n} r\right) \frac{\cosh\left[\lambda_{2n}(z+h)\right]}{\cosh \lambda_{2n} h} \tag{3.38}$$

where

$$J_0'\left(\lambda_{0n} a\right) = J_2'\left(\lambda_{2n} a\right) = 0.$$

By finding the appropriate generalized coordinates in ψ_2 and X_2, (A.27b) and (A.27c) can be satisfied. These generalized coordinates can be found by introducing (3.34), (3.35), (3.37), and (3.38) into (A.27b) and (A.27c) and applying a Fourier-Bessel technique using the following orthogonality conditions:

$$\int_0^a r J_0\left(\lambda_{0m} r\right) J_0\left(\lambda_{0n} r\right) dr = \begin{cases} 0, & m \neq n \\ \frac{a^2}{2} J_0^2\left(\lambda_{0n} a\right), & m = n \end{cases} \tag{3.39}$$

$$\int_0^a r J_2\left(\lambda_{2m} r\right) J_2\left(\lambda_{2n} r\right) dr = \begin{cases} 0, & m \neq n \\ \frac{\lambda_{2n}^2 a^2 - 4}{2\lambda_{2n}^2} J_2^2\left(\lambda_{2n} a\right), & m = n \end{cases} \tag{3.40}$$

These conditions give the generalized coordinates of the J_0 and J_2 modes as:

$$
\left.
\begin{aligned}
\hat{A}_{0n} &= \Omega_{0n}\left(f_1 f_2 + f_3 f_4\right) \\
\hat{A}_{2n} &= \Omega_{2n}\left(f_1 f_2 - f_3 f_4\right) \\
\hat{B}_{2n} &= \Omega_{2n}\left(f_1 f_4 + f_2 f_3\right) \\
\hat{C}_{0n} &= \frac{\Omega_{0n}}{2}\left(f_2^2 + f_4^2 - f_1^2 - f_3^2\right) \\
\hat{C}_{2n} &= \frac{\Omega_{2n}}{2}\left(f_2^2 + f_3^2 - f_1^2 - f_4^2\right) \\
\hat{D}_{2n} &= \Omega_{2n}\left(f_2 f_4 - f_1 f_3\right)
\end{aligned}
\right\} \tag{3.41}
$$

where Ω_{0n} and Ω_{2n} are constants defined in the appendix.

The terms a_{00}, $a_{01} - \frac{a_{11} b_{00}}{\alpha}$, $a_{02} - \frac{a_{11} b_{01}}{\alpha} + \frac{a_{12} b_{00}}{\alpha} + \frac{a_{11} b_{00} b_{11}}{\alpha^2} - \frac{a_{22} b_{00}^2}{2\alpha^2}$, and B_{11}, each contribute to the coefficient of ϵ. The coefficient of ϵ contains $\sin \omega t$, $\sin 2\omega t$, $\sin 3\omega t$, $\cos \omega t$, $\cos 2\omega t$, and $\cos 3\omega t$. With this type of approximation it is assumed that only the first harmonic terms need vanish. The first harmonic terms from a_{00} and B_{11} are

$$
p_{11}^2 \left\{ \left[\frac{dX_1}{d\tau} - \nu \psi_1 - r \cos\theta - r \sin\theta \right] \cos \omega t - \left(\frac{d\psi_1}{d\tau} + \nu X_1 \right) \sin \omega t \right\}
$$

$$
- \left(\alpha - \frac{2r}{h} \cos \theta + \frac{2r}{h} \sin \theta \right) \cos \omega t \tag{3.42}
$$

where $p_{11}^2 r \cos\theta \cos\omega t$ corresponds to the translational motion u_1, $p_{11}^2 r \sin\theta \cos\omega t$ corresponds to the translational motion u_2, $\alpha \cos \omega t$ corresponds to the translational motion u_3, $\frac{2r}{h} \cos \theta \cos \omega t$ corresponds to the rotational motion ω_2, and $\frac{2r}{h} \sin \theta \cos \omega t$ corresponds to the rotational motion ω_1.

The first harmonic terms from $a_{01} - \frac{a_{11} b_{00}}{\alpha}$ are

$$
p_{11} \Big[\psi_{1r} X_{2r} - X_{1r} \psi_{2r} + \frac{1}{r^2} \left(\psi_{1\theta} X_{2\theta} - X_{1\theta} \psi_{2\theta} \right) - \lambda_{11}^2 \left(\zeta_{11}^2 - 1 \right) \left(X_1 \psi_2 - \psi_1 X_2 \right)
$$

$$
- \lambda_{11} \zeta_{11} \left(\psi_1 X_{2z} - X_1 \psi_{2z} \right) + \frac{1}{2} \left(\psi_1 X_{2zz} - X_1 \psi_{2zz} \right) \Big] \cos \omega t
$$

$$- p_{11}\Big[X_{1r}X_{2r} + \psi_{1r}\psi_{2r} + \frac{1}{r^2}\left(X_{1\theta}X_{2\theta} + \psi_{1\theta}\psi_{2\theta}\right)$$

$$+ \lambda_{11}^2\left(\zeta_{11}^2 - 1\right)\left(X_1X_2 + \psi_1\psi_2\right) - \lambda_{11}\zeta_{11}\left(X_1X_{2z} + \psi_1\psi_{2z}\right)$$

$$+ \frac{1}{2}\left(X_1X_{2zz} + \psi_1\psi_{2zz}\right)\Big]\sin\omega t \tag{3.43}$$

Similarly, there are first harmonic terms from $a_{02} - \dfrac{a_{11}b_{01} - a_{12}b_{00}}{\alpha} + \dfrac{2a_{11}b_{00}b_{11} + a_{22}b_{00}^2}{2\alpha^2}$ which because of their length are not written out here.

The equation obtained by setting the first harmonic terms of the coefficient of ϵ equal to zero is now satisfied in a Rayleigh-Ritz, or averaged, sense by multiplying the equation by

$$J_1\left(\lambda_{11}r\right)\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix} r\,dr\,d\theta \tag{3.44}$$

integrating over the free surface, $0 \le r < a$, $0 \le \theta \le 2\pi$, and using the known results

$$\int_0^a\int_0^{2\pi} r\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix} J_1\left(\lambda_{11}r\right)dr\,d\theta = 0 \tag{3.45}$$

$$\int_0^a\int_0^{2\pi} r^2\sin\theta\cos\theta\,J_1\left(\lambda_{11}r\right)dr\,d\theta = 0 \tag{3.46}$$

$$\int_0^a\int_0^{2\pi} r^2\begin{Bmatrix}\sin^2\theta\\ \cos^2\theta\end{Bmatrix} J_1\left(\lambda_{11}r\right)dr\,d\theta = \frac{\pi a}{\lambda_{11}^2}J_1\left(\lambda_{11}a\right) \tag{3.47}$$

The contributions from $a_{00} + B_1$ are

$$\int_0^a\int_0^{2\pi}\left(a_{00} + B_{11}\right)\cos\theta\,J_1\left(\lambda_{11}r\right)r\,dr\,d\theta =$$

$$\pi p_{11}^2 \left\{ \frac{\lambda_{11}^2 a^2 - 1}{2\lambda_{11}^2} J_1^2\left(\lambda_{11} a\right)\left(\frac{df_2}{d\tau} - \nu f_1\right) - \left[\frac{a}{\lambda_{11}^2} J_1\left(\lambda_{11} a\right)\right]^{(u_1)} \right\} \cos \omega t$$

$$- \pi p_{11}^2 \left\{ \frac{\lambda_{11}^2 a^2 - 1}{2\lambda_{11}^2} J_1^2\left(\lambda_{11} a\right)\left(\frac{df_1}{d\tau} + \nu f_2\right) \right\} \sin \omega t$$

$$+ \left[\frac{2\pi a}{\lambda_{11}^2 h} J_1\left(\lambda_{11} a\right)\right]^{(\omega_2)} \cos \omega t \tag{3.48}$$

$$\int_0^a \int_0^{2\pi} \left(a_{00} + B_{11}\right) \sin \theta J_1\left(\lambda_{11} r\right) r\, dr\, d\theta = \pi p_{11}^2 \left\{ \frac{\lambda_{11}^2 a_{11}^2 - 1}{2\lambda_{11}^2} J_1^2\left(\lambda_{11} a\right) \right.$$

$$\left. \left(\frac{df_4}{d\tau} - \nu f_3\right) - \left[\frac{\pi a}{\lambda_{11}^2} J_1\left(\lambda_{11} a\right)\right]^{(u_2)} \right\} \cos \omega t - \pi p_{11}^2 \left\{ \frac{\lambda_{11}^2 a^2 - 1}{2\lambda_{11}^2} J_1^2\left(\lambda_{11} a\right) \right.$$

$$\left. \left(\frac{df_3}{d\tau} + \nu f_4\right) \right\} \sin \omega t - \left[\frac{2\pi a}{\lambda_{11}^2 h} J_1\left(\lambda_{11} a\right)\right]^{(\omega_1)} \cos \omega t \tag{3.49}$$

The contributions from $a_{01} - \frac{a_{11} b_{00}}{\alpha} = B_2$ are

$$\int_0^a \int_0^{2\pi} B_2 \cos \theta\, J_1\left(\lambda_{11} r\right) r\, dr\, d\theta = \pi p_{11} \left[f_1\left(f_j f_j\right)\hat{G}_1 + f_4\left(f_2 f_3 \right.\right.$$

$$\left.\left. - f_1 f_4\right)\hat{G}_2 \right] \cos \omega t + \pi p_{11} \left[f_2\left(f_j f_j\right)\hat{G}_1 - f_3\left(f_2 f_3 - f_1 f_4\right)\hat{G}_2 \right] \sin \omega t \tag{3.50}$$

$$\int_0^a \int_0^{2\pi} B_2 \sin \theta\, J_1\left(\lambda_{11} r\right) r\, dr\, d\theta = \pi p_{11} \left[f_3\left(f_j f_j\right)\hat{G}_1 - f_2\left(f_2 f_3 \right.\right.$$

$$\left.\left. - f_1 f_4\right)\hat{G}_2 \right] \cos \omega t + \pi p_{11} \left[f_4\left(f_j f_j\right)\hat{G}_1 + f_1\left(f_2 f_3 - f_1 f_4\right)\hat{G}_2 \right] \sin \omega t \tag{3.51}$$

where $f_j f_j = f_1^2 + f_2^2 + f_3^2 + f_4^2$, $\hat{G}_1$, and $\hat{G}_2$ are constants defined in the appendix.

The contributions from

$$B_3 = a_{02} - \frac{a_{11} b_{01} - a_{12} b_{00}}{\alpha} + \frac{2 a_{11} b_{00} b_{11} + a_{22} b_{00}^2}{2\alpha^2}$$

are

$$\int_0^a \int_0^{2\pi} B_3 \cos\theta \, J_1(\lambda_{11} r) r \, dr \, d\theta = -\frac{p_{11}^2}{4}\Big[f_1(f_j f_j)\hat{H}_1 + f_4(f_2 f_3 - f_1 f_4)\hat{H}_2\Big]\cos\omega t - \frac{p_{11}^2}{4}\Big[f_2(f_j f_j)\hat{H}_1 - f_3(f_2 f_3 - f_1 f_4)\hat{H}_2\Big]\sin\omega t \tag{3.52}$$

$$\int_0^a \int_0^{2\pi} B_3 \sin\theta \, J_1(\lambda_{11} r) r \, dr \, d\theta = -\frac{p_{11}^2}{4}\Big[f_3(f_j f_j)\hat{H}_1 - f_2(f_2 f_3 - f_1 f_4)\hat{H}_2\Big]\cos\omega t - \frac{p_{11}^2}{4}\Big[f_4(f_j f_j)\hat{H}_1 + f_1(f_2 f_3 - f_1 f_4)\hat{H}_2\Big]\sin\omega t \tag{3.53}$$

where $\hat{H}_1$ and $\hat{H}_2$ are constants defined in the appendix.

From the Rayleigh-Ritz process two ordinary differential equations are obtained. Setting the coefficients of sin ωt and cos ωt equal to zero in each of these equations results in four, first-order, nonlinear, ordinary differential equations.

This system is

$$\frac{df_i}{d\tau} = G_i(f_1, f_2, f_3, f_4), \quad i = 1, 2, 3, 4 \tag{3.54}$$

where

$$\left.\begin{aligned} G_1 &= -H_{,2} \\ G_2 &= H_{,1} \\ G_3 &= -H_{,4} \\ G_4 &= H_{,3} \end{aligned}\right\} \tag{3.55}$$

and

$$H = \left(F_1 + A_2\right)f_1 + \left(U_2 + A_2\right)f_4 + \frac{1}{2}\nu f_j f_j$$

$$+ \frac{1}{4}K_1\left(f_j f_j\right)^2 - \frac{1}{2}K_2\left(f_2 f_3 - f_1 f_4\right)^2 \tag{3.56}$$

where

$$H,i = \frac{\partial H}{\partial f_i} \tag{3.57}$$

The constants F_1, K_1, K_2, A_1, A_2, U_2 are defined in the appendix by (A.18), (A.19), (A.20), and (A.21).

A steady-state harmonic solution to the boundary value problem is given by the roots of the four equations

$$G_i\left(f_1, f_2, f_4, f_4\right) = 0, \quad i = 1, 2, 3, 4 \tag{3.58}$$

The roots of (3.58) are functions of f_i where the f_i are independent of the time τ. The form of (3.54) is similar to the equations derived by Miles[7] for the undamped spherical pendulum.

There are two solutions to (3.58). The first, called planar motion, is

$$\left.\begin{aligned} f_1 &= \gamma \\ f_3 &= \gamma Q \\ f_2 &= f_4 = 0 \end{aligned}\right\} \tag{3.59}$$

where γ is a parameter independent of time. The transformed frequency is

$$\nu = P\gamma^{-1} - K_1 R\gamma^2 \tag{3.60}$$

where

$$\left.\begin{aligned} P &= -\left(F_1 + A_2\right) \\ Q &= \frac{A_1 + U_2}{F_1 + A_2} \\ R &= 1 + Q^2 \end{aligned}\right\} \tag{3.61}$$

The second solution to (3.58), called nonplanar motion, is

$$\left.\begin{aligned} f_1 &= \gamma \\ f_2 &= -\left(\gamma^2 - \frac{P\gamma^{-1}}{K_2 R}\right)^{\frac{1}{2}} Q = -LQ \\ f_3 &= \gamma Q \\ f_4^2 &= \gamma^2 - \frac{P\gamma^{-1}}{K_2 R} = L^2 \end{aligned}\right\} \tag{3.62}$$

with

$$\nu = -\gamma^{-1}\left(K_3 + \frac{A_2 K_1}{K_2}\right) + R K_4 \gamma^2 \tag{3.63}$$

$$\left.\begin{aligned} K_3 &= \frac{K_1}{K_2} F_1 \\ K_4 &= K_2 - 2K_1 \end{aligned}\right\} \tag{3.64}$$

It is seen that the nonplanar solution is real and, hence, exists for $\gamma > 0$ when $\gamma^3 - \frac{P}{K_2 R} > 0$, and for $\gamma < 0$ when $\gamma^3 - \frac{P}{K_2 R} < 0$.

The names planar and nonplanar motion are used in analogy with Miles' terminology for the spherical pendulum. It is not to be implied that the motion of the free surface is necessarily described by the names given to the two solutions of (3.58).

SECTION 4

STABILITY OF THE STEADY-STATE HARMONIC SOLUTION

To determine the stability of the motion corresponding to a given steady-state solution, consider the perturbed solution

$$\left.\begin{aligned} & f_i(\tau) = f_i^{(0)} + c_i e^{\lambda\tau} \\ & |c_i| \ll 1 \\ & i = 1, 2, 3, 4 \end{aligned}\right\} \tag{4.1}$$

The $f_i^{(0)}$ are constants corresponding to the steady-state amplitudes of the harmonic solutions of (3.54). The corresponding steady-state solution will be stable if $\mathrm{Re}(\lambda) \leq 0$ and unstable if $\mathrm{Re}(\lambda) > 0$.

Substitute (4.1) into (3.54), neglect products of the c_i's, and use the fact that the $f_i^{(0)}$ are solutions of (3.58); the following set of homogeneous algebraic equations are obtained:

$$\begin{bmatrix} d_{11}+\lambda & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22}-\lambda & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33}+\lambda & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44}-\lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.2}$$

where

$$d_{11} = 2K_1 f_1^{(0)} f_2^{(0)} + K_2 f_3^{(0)} f_4^{(0)}$$

$$d_{12} = \nu + K_1\left(f_j^{(0)} f_j^{(0)}\right) + 2K_1 f_2^{(0)^2} - K_2 f_3^{(0)^2}$$

$$d_{13} = 2K_1 f_2^{(0)} f_3^{(0)} + K_2\left[f_1^{(0)} f_4^{(0)} - 2f_2^{(0)} f_3^{(0)}\right]$$

$$d_{14} = 2K_1 f_2^{(0)} f_4^{(0)} + K_2 f_1^{(0)} f_3^{(0)}$$

$$d_{21} = \nu + K_1\left(f_j^{(0)} f_j^{(0)}\right) + 2K_1 f_1^{(0)^2} - K_2 f_4^{(0)^2}$$

$$d_{22} = d_{11}$$

$$d_{23} = 2K_1 f_1^{(0)} f_3^{(0)} + K_2 f_2^{(0)} f_4^{(0)}$$

$$d_{24} = 2K_1 f_1^{(0)} f_4^{(0)} + K_2\left[f_2^{(0)} f_3^{(0)} - 2f_1^{(0)} f_4^{(0)}\right]$$

$$d_{31} = d_{24}$$

$$d_{32} = d_{14}$$

$$d_{33} = 2K_1 f_3^{(0)} f_4^{(0)} + K_2 f_1^{(0)} f_2^{(0)}$$

$$d_{34} = \nu + K_1\left[f_j^{(0)} f_j^{(0)}\right] + 2K_1 f_4^{(0)^2} - K_2 f_1^{(0)^2}$$

$$d_{41} = d_{23}$$

$$d_{42} = d_{13}$$

$$d_{43} = \nu + K_1\left(f_j^{(0)} f_j^{(0)}\right) + 2K_1 f_3^{(0)^2} - K_2 f_2^{(0)^2}$$

$$d_{44} = d_{33}$$

The solutions of (4.2) will be nontrivial only if the determinant of the coefficient matrix is zero. This condition gives an equation for the allowable values of λ.

4.1 STABILITY OF PLANAR MOTION. Substituting (3.59) into the expressions for the d_{ij}'s and expanding the determinant of the coefficient matrix in (4.2), one obtains

$$\lambda^4 + \lambda^2\left(M_1' + M_2' + 2M_3\right) + M_1'M_2' + 4K_1^2\gamma^4Q^2M_4 - K_2^2Q^2\gamma^4M_5 + M_3^2 = 0 \quad (4.3)$$

where

$$M_1' = \begin{vmatrix} \nu + \gamma^2\left(K_1R - K_2Q\right) & 0 \\ 0 & \nu + K_1\gamma^2\left(3 + Q^2\right) \end{vmatrix}$$

$$M_2' = \begin{vmatrix} \nu + K_1\gamma^2\left(1 + 3Q^2\right) & 0 \\ 0 & \nu + \gamma^2\left(K_1R - K_2\right) \end{vmatrix}$$

$$M_3 = \begin{vmatrix} K_2\gamma^2Q & 0 \\ 0 & 2K_1\gamma^2Q \end{vmatrix}$$

$$M_4 = \begin{vmatrix} 0 & \nu + \gamma^2\left(K_1R - K_2\right) \\ \nu + \gamma^2\left(K_1R - K_2Q\right) & 0 \end{vmatrix}$$

$$M_5 = \begin{vmatrix} \nu + K_1\gamma^2\left(3 + Q^2\right) & 0 \\ 0 & \nu + K_1\gamma^2\left(1 + 3Q^2\right) \end{vmatrix}$$

The boundary between stable and unstable planar motion corresponds to $\lambda = 0$. Set $\lambda = 0$ in (4.3) and substitute for ν from (3.60):

$$\gamma^{-4}\left[N_4\gamma^{12} - N_3P\gamma^9 + N_2P^2\gamma^6 + P^3N_1\gamma^3 + P^4\right] = 0 \quad (4.4)$$

where

$$\left.\begin{aligned}
N_1 &= -K_4 + 2K_1Q^2 - K_2Q \\
N_2 &= K_2K_4Q - K_2Q^2\left(K_2 + 2K_1\right) - 2K_1K_2\left(1 + Q^3\right) \\
N_3 &= 2K_1K_2^2\left(Q^4 - Q^3 + Q^2 - Q\right) \\
N_4 &= -rK_2^2K_1^2Q^2\left(Q^2 + 1\right)
\end{aligned}\right\} \tag{4.5}$$

One possible solution to (4.4) is $\gamma = \pm\infty$. Since γ is actually an amplitude this would correspond to unstable motion. Let $\sigma = \gamma^3$; then the other possible solution to (4.4) is

$$N_4\sigma^4 - N_3P\sigma^3 + N_2P^2\sigma^2 + P^3N_1\sigma + P^4 = 0 \tag{4.5}$$

This equation can then be solved for $\sigma = \gamma^3$. The solution of (4.5) for various perturbations is given in Section 5.

4.2 STABILITY OF NONPLANAR MOTION. Substitute (3.62) in the expressions for the d_{ij}'s:

$$d_{11} = \gamma QK_4L$$

$$d_{12} = \nu + \gamma^2\left(2K_1R - Q^2K_4\right) - \frac{K_1P\gamma^{-1}}{K_2R}\left(1 + 3Q^2\right)$$

$$d_{13} = L\gamma\left(K_4Q^2 + K_2R\right)$$

$$d_{14} = Q\left(K_4\gamma^2 + \frac{2K_1P\gamma^{-1}}{K_2R}\right)$$

$$d_{21} = \nu + \gamma^2\left(2K_1R - K_4\right) + P\gamma^{-1}\left(\frac{1}{R} - \frac{K_1}{K_2}\right)$$

$$d_{22} = d_{11}$$

$$d_{23} = -Q\left(K_4\gamma^2 - \frac{P\gamma^{-1}}{R}\right)$$

$$d_{24} = -\gamma L\left(K_4 + K_2 R\right)$$

$$d_{31} = d_{24}$$

$$d_{32} = d_{14}$$

$$d_{33} = -d_{11}$$

$$d_{34} = \nu - \gamma^2\left(2K_1 R - K_4\right) - \frac{K_1 P \gamma^{-1}}{K_2 R}\left(3 + Q^2\right)$$

$$d_{41} = d_{23}$$

$$d_{42} = d_{13}$$

$$d_{43} = \nu + \gamma^2\left(2K_1 R - K_4 Q^2\right) - P\gamma^{-1}\left(\frac{K_1}{K_2} + \frac{Q^2}{R}\right)$$

$$d_{44} = d_{33}$$

Substitute the value of ν given by (3.63) into the expressions for d_{12}, d_{21}, d_{34}, d_{43}:

$$d_{12} = \gamma^2\left(2K_1 R + K_4\right) - \frac{2K_1 P Q^2 \gamma^{-1}}{K_2 R}$$

$$d_{21} = \gamma^2\left(2K_1 R + K_4 Q^2\right) + \frac{P\gamma^{-1}}{R}$$

$$d_{34} = \gamma^2\left(2K_1 R + K_4 Q^2\right) - \frac{2K_1 P \gamma^{-1}}{K_2 R}$$

$$d_{43} = \gamma^2\left(2K_1 R + K_4\right) + \frac{P Q^2 \gamma^{-1}}{R}$$

Substitute the above values of the d_{ij}'s into (4.2) and set the determinant of the coefficient matrix equal to zero to give a fourth-degree polynomial for the determination of the parameter λ, such that (4.2) has nontrivial solutions. Stability of the

steady-state nonplanar solution is determined by examining the roots of this quartic equation. The regions of stable and unstable motion are given in Section 5 for various perturbations.

SECTION 5

NUMERICAL EXAMPLE

To compare results obtained here with those obtained by Hutton, use the following values of the parameters:

$$\left.\begin{aligned} a &= \text{tank radius} = 5.938 \text{ inches} \\ h &= \text{water depth} = 8.907 \text{ inches} \\ \lambda_{11} a &= 1.84119 \\ J_1\left(\lambda_{11} a\right) &= 0.581865 \end{aligned}\right\} \tag{5.1}$$

Then,

$$\left.\begin{aligned} F_1 &= 8.53992 \\ \zeta &= 0.99205 \\ p_{11} &= 10.987 \text{ rad/sec} = 1.734 \text{ cps} \\ A_2 &= -0.6389 \\ A_1 &= -A_2 \\ U_2 &= F_1 \end{aligned}\right\} \tag{5.2}$$

To evaluate K_1 and K_2, use only the first five terms in the infinite series defined by $\hat{G}_1$ and $\hat{G}_2$ to approximate the series. In the calculation of $\hat{G}_1$ and $\hat{G}_2$ the last three terms are about one percent of the first two terms. Thus,

$$\left.\begin{aligned} K_1 &= 0.4853 \times 10^{-5} \\ K_2 &= 0.13707 \times 10^{-4} \\ K_3 &= -3.0235 \\ K_4 &= 4.0010 \times 10^{-6} \end{aligned}\right\} \tag{5.3}$$

5.1 PLANAR MOTION. Equation (3.60) gives the transformed frequency, ν, in terms of the parameter γ. The coefficients in this equation depend on the perturbations given to the liquid-tank system. Thus,

$$\nu = -\left(F_1 + A_2\right)\gamma^{-1} - K_1\left[1 + \left(\frac{A_1 + U_2}{F_1 + A_2}\right)^2\right]\gamma^2 \tag{5.4}$$

Equation (4.4) gives the values of γ which separate stable and unstable regions. The coefficients in this equation also depend on the perturbations given to the liquid-tank system.

5.1.1 Case 1. Consider $\omega_1 = \omega_2 = u_2 = 0$, $u_1 = \epsilon \cos \omega t$, which is the case considered by Hutton. Then,

$$\nu = -8.5399\gamma^{-1} - 0.4853 \times 10^{-5}\gamma^2 \tag{5.5}$$

The motion is unstable for

$$-0.1337 < \nu < 0.06459 \tag{5.6}$$

where

$$\gamma = 95.82 \text{ when } \nu = -0.1337$$

and

$$\gamma = -85.41 \text{ when } \nu = 0.06439$$

This case agrees with the result obtained by Hutton[3].

5.1.2 Case 2

$$\left.\begin{aligned} \omega_1 &= u_1 = u_2 = 0 \\ \omega_2 &= \frac{\epsilon}{h} \cos \omega t \\ \nu &= 0.6389\gamma^{-1} - 0.4853 \times 10^{-5}\gamma^2 \end{aligned}\right\} \tag{5.7}$$

The motion is unstable for

$$-0.0237 < \nu < 0.0115 \tag{5.8}$$

where

$$\gamma = -40.377 \text{ when } \nu = -0.0237$$

and

$$\gamma = 35.988 \text{ when } \nu = 0.0115$$

5.1.3 Case 3

$$\left.\begin{aligned} u_1 &= u_2 = 0 \\ \omega_1 &= \omega_2 = \frac{\epsilon}{h} \cos \omega t \\ \nu &= 0.6389\,\gamma^{-1} - 0.9706 \times 10^{-6}\,\gamma^2 \end{aligned}\right\} \tag{5.9}$$

The motion is unstable for

$$-0.1863 < \nu < 0.1934 \tag{5.10}$$

where

$$\gamma = -3.43 \text{ when } \nu = -0.1863$$

and

$$\gamma = 3.30 \text{ when } \nu = 0.1934$$

5.1.4 Case 4

$$\left.\begin{aligned} \omega_1 &= \frac{\epsilon}{h} \cos \omega t \\ \omega_2 &= u_2 = 0 \\ u_1 &= \epsilon \cos \omega t \\ \nu &= -8.5399\,\gamma^{-1} - 0.488 \times 10^{-5}\,\gamma^2 \end{aligned}\right\} \tag{5.11}$$

The motion is unstable for

$$-5.735 < \nu < 5.662 \tag{5.12}$$

where

$$\gamma = -1.508 \text{ when } \nu = -5.735$$

and

$$\gamma = 1.489 \text{ when } \nu = 5.662$$

5.1.5 Case 5

$$\left.\begin{aligned} \omega_1 &= u_2 = 0 \\ \omega_2 &= \frac{\epsilon}{h} \cos \omega t \\ u_1 &= \epsilon \cos \omega t \\ \nu &= -7.901\,\gamma^{-1} - 0.4853 \times 10^{-5}\,\gamma^2 \end{aligned}\right\} \tag{5.13}$$

The motion is unstable for

$$-0.1269 < \nu < 0.06133 \tag{5.14}$$

where

$$\gamma = 93.37 \text{ when } \nu = -0.1269$$

and

$$\gamma = -83.23 \text{ when } \nu = 0.06133$$

5.1.6 Case 6

$$\left.\begin{aligned} u_1 &= u_2 = \epsilon \cos \omega t \\ \omega_1 &= \omega_2 = \frac{\epsilon}{h} \cos \omega t \\ \nu &= -7.9010\,\gamma^{-1} - 1.144 \times 10^{-5}\,\gamma^2 \end{aligned}\right\} \tag{5.15}$$

The motion is unstable for

$$-4.772 < \nu < 4.834 \tag{5.16}$$

where

$$\gamma = 1.655 \text{ when } \nu = -4.773$$

and

$$\gamma = -1.634 \text{ when } \nu = 4.834$$

5.1.7 Case 7

$$\omega_2 = u_1 = 0$$

$$\omega_1 = \frac{\epsilon}{h} \cos \omega t$$

$$u_2 = \epsilon \cos \omega t$$

The results are the same as in Case 5 (by symmetry).

5.1.8 Case 8

$$\omega_1 = u_1 = 0$$

$$\omega_2 = \frac{\epsilon}{h} \cos \omega t$$

$$u_2 = \epsilon \cos \omega t$$

The results are the same as Case 4 (by symmetry).

Cases 1 and 2, according to (2), should not differ. However, it is seen that the unstable region for Case 2 is much smaller than the unstable region for Case 1, indicating that, at least for stability considerations, rotational oscillations about x_2 are not equivalent to translational oscillations in the x_1 direction. Also note from Case 5 that the unstable region is slightly smaller but nearly equivalent to the unstable region in Case 1. Here the rotational and translational motions are taking place in the same plane. The rotational motion thus has a much smaller effect on the free-surface motion than does the translational motion, even to the extent that the combination is essentially not different than the situation for translation alone.

Cases 4 and 8 consider the combination of rotational motion and translational motion in planes perpendicular to one another. From (5.12) the region of unstable motion in these cases is much greater than any of the other cases considered.

5.2 NONPLANAR MOTION

5.2.1 Case 1

$$\omega_1 = \omega_2 = u_2 = 0$$

$$u_1 = \epsilon \cos \omega t$$

The quartic equation for this case is

$$\lambda^4 + \left(M_3 + M_4\right)\lambda^2 + M_5 M_6 = 0 \tag{5.17}$$

where

$$M_3 = \begin{vmatrix} d_{34} & d_{13} \\ d_{24} & d_{12} \end{vmatrix}, \qquad M_4 = \begin{vmatrix} d_{12} & d_{13} \\ d_{24} & d_{21} \end{vmatrix} \tag{5.18}$$

$$M_5 = \begin{vmatrix} d_{13} & d_{12} \\ d_{12} & d_{13} \end{vmatrix}, \qquad M_6 = \begin{vmatrix} d_{24} & d_{21} \\ d_{34} & d_{24} \end{vmatrix} \tag{5.19}$$

and the d_{ij}'s are defined in (4.6) and (4.7).

The transformed frequency, υ, is

$$\nu = -\left(F_1 + A_2\right)\frac{K_1}{K_2}\gamma^{-1} + RK_4\gamma^2 \tag{5.20}$$

For this case,

$$\left.\begin{aligned} A_2 &= 0 \\ R &= 1 \\ F_1 &= 8.53992 \\ \nu &= -3.0235\,\gamma^{-1} + 4.001 \times 10^{-6}\,\gamma^2 \end{aligned}\right\} \tag{5.21}$$

The generalized coordinates f_i for this case are

$$\left.\begin{aligned} f_1 &= \gamma \\ f_2 &= f_3 = 0 \\ f_4^2 &= \gamma^2 + \frac{F_1}{K_2}\gamma^{-1} \end{aligned}\right\} \tag{5.22}$$

Since f_4 must be real, $f_4^2 > 0$. Thus, γ cannot be in the range

$$(-85.41)^3 = -\frac{F_1}{K_2} < \gamma^3 < 0 \tag{5.23}$$

Evaluate $M_3 + M_4$ and $M_5 M_6$:

$$M_3 + M_4 = 7.5148 \times 10^{-10}\,\gamma\left(\gamma^3 + 3.5699 \times 10^5\right) \tag{5.24}$$

$$M_5 M_6 = 2.5677 \times 10^{-15}\,\gamma^{-1}\left(\gamma^3 + 6.2305 \times 10^5\right)\left(\gamma^3 + 3.7784 \times 10^5\right) \tag{5.25}$$

Consider (5.17) as a quadratic in λ^2:

$$\left(\lambda^2\right)^2 + \left(M_3 + M_4\right)\left(\lambda^2\right) + M_5 M_6 = 0 \tag{5.26}$$

If

$$\gamma^3 < -6.2305 \times 10^5$$

or

$$\gamma < -85.41$$

$$M_3 + M_4 > 0$$

$$M_5 M_6 < 0$$

Thus, by Descartes' rule of signs there is one positive real root of (5.26). This corresponds to a region of unstable motion. The case $-6.2305 \times 10^5 < \gamma^3 < 0$ is dealt with in (5.23).

To examine stability for $\gamma > 0$, compute the discriminant of (5.26):

$$\Delta = 52.99 \times 10^{-20} \gamma^{9} + 361.4 \times 10^{14} \gamma^{6} + 572.5 \times 10^{-10} \gamma^{3}$$

$$- 241.75 \times 10^{-5} \tag{5.27}$$

Set $\Delta = 0$, multiply by 10^{20}, and let $\sigma = \gamma^{3}$ to obtain

$$52.99\sigma^{3} + 36.14 \times 10^{5} \sigma^{2} + 572.5 \times 10^{10} \sigma - 241.75 \times 10^{15} = 0 \tag{5.28}$$

The only positive real root of (5.28) corresponds to $\gamma = 64.47$. Thus for $0 < \gamma < 64.47$, the roots of (5.26) are complex. If $\gamma > 0$, $M_3 + M_4$ and $M_5 M_6$ are both positive. Thus by Descartes' rule of signs there is no positive real root of Equation 5.17. Replace γ^2 by $-\alpha$ in (5.26) to obtain

$$\alpha^{2} - \left(M_3 + M_4\right)\alpha + M_5 M_6 = 0 \tag{5.29}$$

Thus by Descartes' rule of sign there exist either two positive real roots or no positive real roots of (5.29). If $\gamma > 64.47$, the roots of (5.29) are real. Hence for $\gamma > 64.47$, Re $(\gamma) \leq 0$, and this relation corresponds to stable motion.

In summary, the steady-state nonplanar motion is stable for

$$\left.\begin{array}{l} 64.47 < \gamma < \infty \\ -0.03027 < \nu < \infty \end{array}\right\} \tag{5.30}$$

and is unstable for

$$\left.\begin{array}{l} 0 < \gamma < 64.47 \\ -\infty < \nu < -0.03027 \end{array}\right\} \tag{5.31}$$

$$\left.\begin{array}{l} -\infty < \gamma < -85.41 \\ 0.06459 < \nu < \infty \end{array}\right\} \tag{5.32}$$

The solution does not exist when

$$-85.41 < \gamma < 0$$

$$-\infty < \nu < 0.06459$$

since then $f_4^2 < 0$.

This motion is stable for a small range of driving frequencies, ω, which includes the first natural frequency, p_{11}. This condition can be seen from the range of transformed frequencies, ν, given by (5.30).

The results have agreed with those obtained by Hutton[3].

5.2.2 Case 2

$$\omega_1 = u_1 = u_2 = 0$$

$$\omega_2 = \frac{\epsilon}{h} \cos \omega t$$

Proceeding as in Case 1, with $A_1 = U_2 = F_1 = 0$ and $A_2 = -0.6389$, the quartic equation to be examined is

$$\lambda^4 + \left(M_3 + M_4\right)\lambda^2 + M_5 M_6 = 0 \tag{5.33}$$

where M_3, M_4, M_5, and M_6 are defined by (5.18) and (5.19). Here the d_{ij}'s have different values than in Case 1.

The transformed frequency, ν, is

$$= -A_2 \frac{K_1}{K_2} \gamma^{-1} + K_4 \gamma^2 \tag{5.34}$$

or

$$= 0.2262\, \gamma^{-1} + 4.001 \times 10^{-6}\, \gamma^2 \tag{5.35}$$

The generalized coordinates for this case are

$$\left.\begin{aligned} f_1 &= \gamma \\ f_2 &= f_3 = 0 \\ f_4^2 &= \gamma^2 + \frac{A_2}{K_2} \gamma^{-1} \end{aligned}\right\} \tag{5.36}$$

Since f_4 must be real, $f_4^2 > 0$. Thus, γ cannot lie in the range

$$0 < \gamma^3 < -\frac{A_2}{K_2} \tag{5.37}$$

where $A_2 < 0$. This restriction gives

$$\left.\begin{array}{l} 0 < \gamma \ 35.8 \\ 0.01047 < \nu < \infty \end{array}\right\} \tag{5.38}$$

Regard (5.33) as a quadratic in λ^2 and examine the roots of

$$\left(\lambda^2\right)^2 + \left(M_3 + M_4\right)\lambda^2 + M_5 M_6 = 0 \tag{5.39}$$

This gives the following regions of unstable and stable motion, respectively:

$$-\infty < \gamma < 0 \text{ and } -\infty < \nu < \infty \tag{5.40}$$

$$35.8 < \gamma < \infty \text{ and } 0.01047 < \nu < \infty \tag{5.41}$$

This motion is not stable about the first natural frequency, as opposed to the situation in Case 1.

5.2.3 Case 3

$$\omega_1 = 0$$

$$\omega_2 = \frac{\epsilon}{h} \cos \omega t$$

$$u_1 = \epsilon \cos \omega t$$

$$u_2 = 0$$

The quartic equation for this case is (5.17) with M_3, M_4, M_5, and M_6 given by (5.18) and (5.19). The d_{ij}'s are evaluated from (4.6) and (4.7), with

$$A_1 = U_2 = 0$$

$$Q = 0$$

$$R = 1$$

The transformed frequency, ν, is

$$\nu = -2.797\,\gamma^{-1} + 4.001 \times 10^{-6}\,\gamma^2 \tag{5.42}$$

The generalized coordinates are

$$\left.\begin{aligned} f_1 &= \gamma \\ f_2 &= f_3 = 0 \\ f_4^2 &= \gamma^2 + \frac{\left(F_1 + A_2\right)}{K_2}\gamma^{-1} \end{aligned}\right\} \tag{5.43}$$

Since f_4 is real, $f_4^2 > 0$. Thus, γ cannot lie in the range

$$-\left(\frac{F_1 + A_2}{K_2}\right) < \gamma^3 < 0 \tag{5.44}$$

Thus, the solution does not exist for

$$\left.\begin{aligned} -75.6 < \gamma < 0 \\ -\infty < \nu < 0.0597 \end{aligned}\right\} \tag{5.45}$$

The regions of unstable motion are

$$-\infty < \gamma < -8.31 \text{ and } 0.0613 < \nu < \infty \tag{5.46}$$

$$0 < \gamma < 62.58 \text{ and } -\infty < \nu < -0.0399 \tag{5.47}$$

The region of stable motion is

$$62.58 < \gamma < \infty \text{ and } -0.0399 < \nu < \infty \tag{5.48}$$

Note the similarity between Cases 1 and 3 with regard to the stability of the nonplanar motion. The rotational motion has a much smaller effect on the stability of the free-surface motion than does the translational motion, even to the extent that the combination is essentially not different than the situation for translation alone.

In all cases considered for the stability of nonplanar motion, there is a region of υ for which it is possible to have both stable planar motion and stable nonplanar motion.

5.2.4 <u>Case 4</u>

$$u_2 = \omega_2 = 0$$

$$\omega_1 = \frac{\epsilon}{h} \cos \omega t$$

$$u_1 = \epsilon \cos \omega t$$

The quartic equation for the determination of the allowable values of λ is

$$\lambda^4 + M \lambda^2 + N = 0 \tag{5.49}$$

where

$$M = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} + \begin{vmatrix} d_{11} & d_{12} \\ d_{32} & d_{33} \end{vmatrix} + \begin{vmatrix} d_{11} & d_{12} \\ d_{43} & d_{44} \end{vmatrix}$$

$$+ \begin{vmatrix} d_{22} & d_{23} \\ d_{32} & d_{33} \end{vmatrix} + \begin{vmatrix} d_{22} & d_{23} \\ d_{43} & d_{44} \end{vmatrix} + \begin{vmatrix} d_{33} & d_{34} \\ d_{43} & d_{44} \end{vmatrix} \tag{5.50}$$

and N is the determinant of the coefficient matrix in (4.2) with $\lambda = 0$.

The transformed frequency, υ, is

$$\nu = -3.0235 \gamma^{-1} + 4.024 \times 10^{-6} \gamma^2 \tag{5.51}$$

and the generalized coordinates are

$$\left.\begin{aligned} f_1 &= \gamma \\ f_2 &= -0.0748 \left[\gamma^2 + 6.24 \times 10^5 \gamma^{-1}\right]^{\frac{1}{2}} \end{aligned}\right\} \tag{5.52}$$

$$\left.\begin{aligned} f_3 &= 0.0748 \gamma \\ f_4^2 &= \gamma^2 + 6.24 \times 10^5 \gamma^{-1} \end{aligned}\right\} \tag{5.53}$$

Since f_4 is real, $f_4^2 > 0$. Thus, γ cannot lie in the range

$$-85.41 < \gamma < 0 \text{ and } 0.06459 < \nu < \infty \tag{5.54}$$

The motion is unstable for

$$-\infty < \gamma < -85.41 \text{ and } 0.06459 < \nu < \infty \tag{5.55}$$

$$59.1 < \gamma < \infty \text{ and } -0.0371 < \nu < \infty \tag{5.56}$$

The motion is stable for

$$31.6 < \gamma < 59.1 \text{ and } -0.0915 < \nu < -0.0371 \tag{5.57}$$

This case, as in the planar motion, has the largest region of unstable motion.

SECTION 6

CONCLUSION

This paper considers the irrotational motion of an incompressible, inviscid fluid contained in a partially filled tank. The tank is subjected to both transverse and rotational vibrations whose frequencies are near the first natural frequency of small free-surface oscillations. The analysis was performed using a method originally suggested by Hutton[3], retaining higher-order terms in the free-surface dynamic and kinematic boundary conditions. The theoretical investigation predicts the forcing frequency ranges, for various combinations of rotational and translational motion, over which there are stable, steady-state, harmonic, planar and nonplanar motions. The least stable case occurs when a combination of motions occurs in planes perpendicular to one another. This condition substantiates the findings of Hutton in that the mechanism that apparently causes sloshing in the unstable regions is a nonlinear coupling of the fluid motions parallel with and perpendicular to the plane in which the translational motion is taking place.

SECTION 7

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APPENDIX A

In the sloshing problem being considered, the free-surface height, η, is an unknown which may be eliminated by replacing (3.24) and (3.25) by a single equation which does not contain η. Solving (3.24) for η, one obtains

$$\alpha\eta = \Gamma\left(r, \theta, \eta(r, \theta, t), t\right) \tag{A.1}$$

where

$$\begin{aligned}
\frac{1}{\alpha}\Gamma\left(r, \theta, \eta(r, \theta, t), t\right) = + \Big[&\alpha - r\dot{\omega}_1 \sin\theta + r\dot{\omega}_2 \cos\theta - \omega_2 u_1 \\
&+ \omega_1 u_2 + \omega_3\omega_2 r \sin\theta + \omega_3\omega_1 r \cos\theta\Big] \Big/ \left(\omega_1^2 + \omega_2^2\right) - \Bigg[\Big[\alpha - r\dot{\omega}_1 \sin\theta \\
&+ r\dot{\omega}_2 \cos\theta - \omega_2 u_1 + \omega_1 u_2 + \omega_3\omega_2 r \sin\theta + \omega_3\omega_1 r \cos\theta\Big]^2 \\
&+ 2\left(\omega_1^2 + \omega_2^2\right)\Bigg\{\frac{1}{2}\left(\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_z^2\right) - \psi_t + \dot{u}_1 r\cos\theta + \dot{u}_2 r\sin\theta + u_3\frac{\partial\psi}{\partial z} \\
&+ \frac{3}{2}r^2\left(\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta\right) + 2\left(\psi_z + \frac{u_3}{2}\right)\left(r\omega_1 \sin\theta - r\omega_2 \cos\theta\right) + \frac{u_3^2}{2} \\
&+ \omega_3 r\left(u_1 \sin\theta - u_2 \cos\theta\right) - 3r^2\omega_1\omega_2 \sin\theta\cos\theta + \omega_3\psi_\theta\Bigg\}\Bigg]^{\frac{1}{2}} \Big/ \left(\omega_1^2 + \omega_2^2\right)
\end{aligned}$$

and the negative sign in the quadratic formula is chosen so that $\alpha\eta$ remains finite if ω_1, ω_2, $\dot{\omega}_1$, $\dot{\omega}_2$ are all equal to zero. Equation (A.1) can now be used to obtain the partial derivatives, η_t, η_r, and η_θ. Thus,

$$\left.\begin{aligned}
\left(\alpha - \Gamma_\eta\right)\eta_t &= \Gamma_t \\
\left(\alpha - \Gamma_\eta\right)\eta_r &= \Gamma_r \\
\left(\alpha - \Gamma_\eta\right)\eta_\theta &= \Gamma_\theta
\end{aligned}\right\} \tag{A.2}$$

Multiply (3.25) by $(\alpha - \Gamma_\eta)$ and use (A.2):

$$\left(-\psi_z - u_3 + 2r\omega_2\cos\theta - 2r\omega_1\sin\theta\right)\left(\alpha - \Gamma_\eta\right)$$

$$= \Gamma_t - \Gamma_r\psi_r - \left(\frac{1}{r^2}\psi_\theta + \omega_3\right)\Gamma_\theta, \quad \text{on } z = \eta \tag{A.3}$$

The potential functions in (A.3) are evaluated on $z = \eta$, and, thus, (A.3) depends upon η implicitly. However, (3.24) depends upon η both explicitly and implicitly. The wave height, η, is eliminated between these two equations by first expanding the functions defined by these two equations in Taylor series about $z = 0$.

Introduce the notation

$$\xi_{r^m\theta^n z^p t^s} = \left.\frac{\partial k_\psi}{\partial r^m \partial\theta^n \partial z^p \partial t^s}\right|_{z = 0} \tag{A.4}$$

where

$$m + n + p + s = k$$

The Taylor series of the function defined by (A.3) and (3.24) can then be written in the form

$$a_0 + a_1\eta + \frac{a_2}{2!}\eta^2 + \frac{a_3}{3!}\eta^3 + \ldots = 0 \tag{A.5}$$

and

$$b_0 + b_1\eta + \frac{b_2}{2!}\eta^2 + \frac{b_3}{3!}\eta^3 + \ldots = 0 \tag{A.6}$$

respectively, where

$$b_0 = -b_{00} - b_{01} - \dot{u}_2 r\sin\theta - \frac{3r^2}{2}\left(\omega_1^2\cos^2\theta + \omega_2^2\sin^2\theta\right)$$

$$- 2\xi_z\left(r\omega_1\sin\theta - r\omega_2\cos\theta + \frac{u_3}{2}\right) + 3r^2\omega_1\omega_2\sin\theta\cos\theta + \frac{u_3^2}{2}$$

$$- \dot{u}_3 r\omega_1\sin\theta + u_3\omega_2 r\cos\theta - u_1\omega_3 r\sin\theta + \omega_3 u_2 r\cos\theta - \omega_3\xi_\theta$$

$$b_1 = -b_{11} - b_{12} - \alpha + r\dot{\omega}_1 \sin\theta - \dot{\omega}_2 r\cos\theta - 2\xi_{zz}\left(r\omega_1 \sin\theta - r\omega_2 \cos\theta + \frac{u_3}{2}\right) + \omega_2 u_1 - \omega_3\omega_2 r\sin\theta - \omega_3 \xi_{\theta z} - \omega_3\omega_1 r\cos\theta - \omega_1 u_2$$

$$b_2 = -b_{22} - b_{23} + \left(\omega_1^2 + \omega_2^2\right) - 2\xi_{zz}\left(r\omega_1 \sin\theta - r\omega_2\cos\theta + \frac{u_3}{2}\right) - \omega_3 \xi_{\theta zz}$$

$$b_{00} = -\xi_t + \dot{u}_1 r\cos\theta$$

$$b_{01} = \frac{1}{2}\left(\xi_r^2 + \frac{\xi_\theta^2}{r^2} + \xi_z^2\right)$$

$$b_{11} = \frac{\partial b_{00}}{\partial z}$$

$$b_{12} = \frac{\partial b_{01}}{\partial z}$$

$$\dot{b}_{00} = \frac{\partial b_{00}}{\partial t}$$

$$b_{m+1,n+1} = \frac{\partial b_{mn}}{\partial z} \quad \text{for} \quad m, n = 1, 2, 3 \ldots \tag{A.7}$$

It is evident that the potential functions are of the same order as the wave height, as can be seen by neglecting the products of η and ξ in (A.6); then, the first approximation becomes

$$-b_{00} - \alpha\eta = \dot{u}_2 r \sin\theta$$

or

$$\eta = \frac{1}{\alpha}\left[\xi_t - \dot{u}_1 r\cos\theta - \dot{u}_2 r\sin\theta\right] \tag{A.8}$$

With this fact in mind, one can expand the function defined by (A.1) in a binomial expansion and neglect terms of $0(\eta^4)$. Here it is assumed that ω_1, ω_2, ω_3, u_1, u_2, u_3, and their time derivatives are of the same order as the wave height. This assumption gives:

$$
\begin{aligned}
\Gamma(r,\theta,\eta,t) = -\Bigg[& -\psi_t + \dot{u}_1 r\cos\theta + \frac{1}{2}\left(\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_z^2\right) + \dot{u}_2 r\sin\theta + u_3\psi_z \\
& + \frac{3r^2}{2}\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right) + 2\left(\psi_z + \frac{u_3}{2}\right)\left(r\omega_1\sin\theta - r\omega_2\cos\theta\right) + \frac{u_3^2}{2} \\
& + \omega_3 r\left(u_1\sin\theta - u_2\cos\theta\right) - 3r^2\omega_1\omega_2\sin\theta\cos\theta + \omega_3\psi_\theta\Bigg] \\
& + \frac{1}{\alpha}\left[-r\dot{\omega}_1\sin\theta + r\dot{\omega}_2\cos\theta\right]\Bigg[-\psi_t + \dot{u}_1 r\cos\theta + \dot{u}_2 r\sin\theta \\
& + \frac{1}{2}\left(\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_z^2\right) + \frac{3r^2}{2}\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right) + u_3\psi_z + \frac{u_3^2}{2} \\
& + 2\left(\psi_z + \frac{u_3}{2}\right)\left(r\omega_1\sin\theta - r\omega_2\cos\theta\right) + \omega_3 r\left(u_1\sin\theta - u_2\cos\theta\right) \\
& - 3r^2\omega_1\omega_2\sin\theta\cos\theta + \omega_3\psi_\theta\Bigg] + \alpha^{-1}\Bigg[-\omega_2 u_1 + \omega_1 u_2 + \omega_3 r\Big(\omega_2\sin\theta \\
& + \omega_1\cos\theta\Big)\Bigg]\left[-\psi_t + \dot{u}_1 r\cos\theta + \dot{u}_2 r\sin\theta\right] - \alpha^{-2}\Big[r^2\dot{\omega}_1^2\sin^2\theta \\
& + r^2\dot{\omega}_2^2\cos^2\theta - 2r^2\dot{\omega}_1\dot{\omega}_2\sin\theta\cos\theta\Big]\left[-\psi_t + \dot{u}_1 r\cos\theta + \dot{u}_2 r\sin\theta\right] \\
& + 0(\eta^4)
\end{aligned}
\tag{A.9}
$$

Use (A.9) to find Γ_r, Γ_θ, and Γ_t. Note that on $z = \eta$, $\Gamma_\eta = \Gamma_z$; then $\alpha - \Gamma_\eta$ can be computed. Substitute these values into (A.3) and expand the function defined by the result in a Taylor series about $z = 0$. This operation leads to (A.5), where a_n is defined below:

$$
a_k = \left.\frac{\partial^k \Theta}{\partial z^k}\right|_{z=0} \quad \text{for} \quad k = 0, 1, 2, 3 \ldots
$$

where

$$\Theta = -\alpha\psi_z - \psi_{tt} + \ddot{u}_1 r\cos\theta + 2\psi_r\psi_{rt} + 2\frac{\psi_\theta\psi_{\theta t}}{r^2} + 2\psi_z\psi_{zt} + u_3\dot{u}_3$$

$$- \dot{u}_1\left(\psi_r\cos\theta - \psi_\theta\frac{\sin\theta}{r}\right) - \psi_r^2\psi_{rr} - \frac{\psi_\theta^2\psi_{\theta\theta}}{r^4} - \psi_z^2\psi_{zz} + \frac{\psi_r\psi_\theta^2}{r^3}$$

$$- 2\psi_r\psi_{rz}\psi_z - 2\frac{\psi_r\psi_\theta\psi_{r\theta}}{r^2} - 2\frac{\psi_z\psi_\theta\psi_{\theta z}}{r^2} + \Big(-u_3 + 2r\Big[\omega_2\cos\theta$$

$$- \omega_1\sin\theta\Big]\Big)\Big[\alpha + \psi_r\psi_{rz} + \frac{\psi_\theta\psi_{\theta z}}{r^2} + \psi_z\psi_{zz} - \psi_{tz} - 2r\psi_{zz}\Big(\omega_2\cos\theta$$

$$- \omega_1\sin\theta\Big) + u_3\psi_{zz} + \omega_3\psi_{\theta z} - \alpha^{-1} r\Big(\dot{\omega}_2\cos\theta - \dot{\omega}_1\sin\theta\Big)\Big(\psi_r\psi_{rz}$$

$$- \psi_{tz} + \frac{\psi_\theta\psi_{\theta z}}{r^2} + \psi_z\psi_{zz}\Big)\Big] - \psi_z\Big[-2\psi_{zz}\Big(\omega_2\cos\theta - \omega_1\sin\theta\Big) + \omega_3\psi_{\theta z}$$

$$+ u_3\psi_{zz} + \alpha^{-1} r\Big(\dot{\omega}_2\cos\theta - \dot{\omega}_1\sin\theta\Big)\psi_{tz}\Big] - \alpha^{-1} r\Big(\dot{\omega}_2\cos\theta - \dot{\omega}_1\sin\theta\Big)\Big[\psi_r\psi_{rt}$$

$$+ \frac{\psi_\theta\psi_{\theta t}}{r^2} + \psi_z\psi_{zt} - \psi_{tt} + \ddot{u}_1 r\cos\theta + \ddot{u}_2 r\sin\theta + 3r^2\Big(\omega_1\dot{\omega}_1\sin^2\theta + \omega_2\dot{\omega}_2\cos^2\theta\Big)$$

$$- 2\Big(\psi_{zt} + \frac{\dot{u}_3}{2}\Big)r\Big(\omega_2\cos\theta - \omega_1\sin\theta\Big) - 2r\Big(\psi_z + \frac{u_3}{2}\Big)\Big(\dot{\omega}_2\cos\theta - \omega_1\sin\theta\Big)$$

$$+ \dot{\omega}_3 r\Big(u_1\sin\theta - u_2\cos\theta\Big) + \omega_3 r\Big(\dot{u}_1\sin\theta - \dot{u}_2\cos\theta\Big) + u_3\dot{u}_3 + u_3\psi_{zt} + \dot{u}_3\psi_z$$

$$- 3r^2\sin\theta\cos\theta\Big(\dot{\omega}_1\omega_2 + \omega_1\dot{\omega}_2\Big) + \dot{\omega}_3\psi_\theta + \omega_3\psi_{\theta t}\Big] + \Big[\ddot{u}_2 r\sin\theta + \dot{u}_3\psi_z$$

$$+ 3r^2\Big(\omega_1\dot{\omega}_1\sin^2\theta + \omega_2\dot{\omega}_2\cos^2\theta\Big) - 2\Big(\psi_{zt} + \frac{\dot{u}_3}{2}\Big)r\Big(\omega_2\cos\theta - \omega_1\sin\theta\Big)$$

$$- 2r\Big(\psi_z + \frac{u_3}{2}\Big)\Big(\dot{\omega}_2\cos\theta - \dot{\omega}_1\sin\theta\Big) + \dot{\omega}_3 r\Big(u_1\sin\theta - u_2\cos\theta\Big) + u_3\psi_{zt}$$

$$+ \omega_3 r\left(\dot{u}_1 \sin\theta - \dot{u}_2 \cos\theta\right) - 3 r^2 \sin\theta \cos\theta\left(\dot{\omega}_1 \omega_2 + \omega_1 \dot{\omega}_2\right) + \dot{\omega}_3 \psi_\theta$$

$$+ \omega_3 \psi_{\theta t}\Big] - \alpha^{-1} r\left(\ddot{\omega}_2 \cos\theta - \ddot{\omega}_1 \sin\theta\right)\Big[-\psi_t + \dot{u}_1 r \cos\theta + \frac{1}{2}\left(\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_z^2\right)$$

$$+ \dot{u}_2 r \sin\theta + \frac{3}{2} r^2\left(\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta\right) - 2 r\left(\psi_z + \frac{u_3}{2}\right)\left(\omega_2 \cos\theta\right.$$

$$\left. - \omega_1 \sin\theta\right) + \omega_3 r\left(u_1 \sin\theta - u_2 \cos\theta\right) - 3 r^2 \omega_1 \omega_2 \sin\theta \cos\theta + u_3 \psi_z + \frac{u_3^2}{2}$$

$$+ \omega_3 \psi_\theta\Big] - \alpha^{-1}\Big[-\omega_2 u_1 + \omega_1 u_2 + \omega_3 r\left(\omega_2 \sin\theta + \omega_1 \cos\theta\right)\Big]\Big[-\psi_{tt}$$

$$+ \ddot{u}_1 r \cos\theta + \ddot{u}_2 r \sin\theta\Big] - \alpha^{-1}\Big[-\dot{\omega}_2 u_1 - \omega_2 \dot{u}_1 + \dot{\omega}_1 u_2 + \omega_1 \dot{u}_2$$

$$+ \dot{\omega}_3 r\left(\omega_2 \sin\theta + \omega_1 \cos\theta\right) + \omega_3 r\left(\dot{\omega}_2 \sin\theta + \dot{\omega}_1 \cos\theta\right)\Big]\Big[-\psi_t + \dot{u}_1 r \cos\theta$$

$$+ \dot{u}_2 r \sin\theta\Big] + \alpha^{-2}\Big[2 r^2 \dot{\omega}_1 \ddot{\omega}_1 \sin^2\theta + 2 r^2 \dot{\omega}_2 \ddot{\omega}_2 \cos^2\theta$$

$$- 2 r^2 \sin\theta \cos\theta\left(\ddot{\omega}_1 \dot{\omega}_2 + \dot{\omega}_1 \ddot{\omega}_2\right)\Big]\Big[-\psi_t + \dot{u}_1 r \cos\theta + \dot{u}_2 r \sin\theta\Big]$$

$$+ \alpha^{-2}\Big[r^2 \dot{\omega}_1^2 \sin^2\theta + r^2 \dot{\omega}_2^2 \cos^2\theta - 2 r^2 \dot{\omega}_1 \dot{\omega}_2 \cos\theta \sin\theta\Big]\Big[-\psi_{tt} + \ddot{u}_1 r \cos\theta$$

$$+ \ddot{u}_2 r \sin\theta\Big] + \psi_r\Big[-\dot{u}_2 \sin\theta - 3 r\left(\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta\right) + 2 \psi_{zr} r\left(\omega_2 \cos\theta\right.$$

$$\left. - \omega_1 \sin\theta\right) + 2\left(\psi_z + \frac{u_3}{2}\right)\left(\omega_2 \cos\theta - \omega_1 \sin\theta\right) - \omega_3\left(u_1 \sin\theta - u_2 \cos\theta\right) - u_3 \psi_{zr}$$

$$+ 6 r \omega_1 \omega_2 \sin\theta \cos\theta - \omega_3 \psi_{\theta r} + \alpha^{-1} r\left(\ddot{\omega}_2 \cos\theta - \ddot{\omega}_1 \sin\theta\right)\left(-\psi_{rt}\right.$$

$$\left. + \dot{u}_1 \cos\theta + \dot{u}_2 \sin\theta\right) + \alpha^{-1}\left(\dot{\omega}_2 \cos\theta - \dot{\omega}_1 \sin\theta\right)\left(-\psi_t + \dot{u}_1 r \cos\theta\right.$$

$$\left. + \dot{u}_2 r \sin\theta\right)\Big] + \omega_3\Big[\psi_{\theta t} + \dot{u}_1 r \sin\theta - \dot{u}_2 r \cos\theta - \psi_r \psi_{\theta r} - \frac{\psi_\theta \psi_{\theta\theta}}{r^2} - u_3 \psi_{\theta z}$$

$$- \psi_z \psi_{\theta z} - 3 r^2 \omega_1^2 \sin\theta \cos\theta + 3 r^2 \omega_2^2 \cos\theta \sin\theta + 2 \psi_{z\theta} r\left(\omega_2 \cos\theta\right.$$

$$\left. - \omega_1 \sin\theta\right) - 2 r\left(\psi_z + \frac{u_3}{2}\right)\left(\omega_2 \sin\theta + \omega_1 \cos\theta\right) - \omega_3 r\left(u_1 \cos\theta + u_2 \sin\theta\right)$$

$$+ 3 r^2 \omega_1 \omega_2\left(\cos^2\theta - \sin^2\theta\right) - \omega_3 \psi_{\theta\theta} + \alpha^{-1} r\left(\dot{\omega}_2 \cos\theta - \dot{\omega}_1 \sin\theta\right)\left(-\psi_{\theta t}\right.$$

$$\left. - \dot{u}_1 r \sin\theta + \dot{u}_2 r \cos\theta\right) - \alpha^{-1} r\left(\dot{\omega}_2 \sin\theta + \dot{\omega}_1 \cos\theta\right)\left(-\psi_t + \dot{u}_1 r \cos\theta\right.$$

$$\left.\left. + \dot{u}_2 r \sin\theta\right)\right] + \frac{\psi_\theta}{r^2}\left[- \dot{u}_2 r \cos\theta - 3 r^2 \omega_1^2 \sin\theta \cos\theta + 3 r^2 \omega_2^2 \cos\theta \sin\theta\right.$$

$$+ 2 \psi_{z\theta} r\left(\omega_2 \cos\theta - \omega_1 \sin\theta\right) - 2 r\left(\psi_z + \frac{u_3}{2}\right)\left(\omega_2 \sin\theta + \omega_1 \cos\theta\right)$$

$$- \omega_3 r\left(u_1 \cos\theta + u_2 \sin\theta\right) + 3 r^2 \omega_1 \omega_2\left(\cos^2\theta - \sin^2\theta\right) - \omega_3 \psi_{\theta\theta} - u_3 \psi_{z\theta}$$

$$+ \alpha^{-1} r\left(\dot{\omega}_2 \cos\theta - \dot{\omega}_1 \sin\theta\right)\left(-\psi_{\theta t} - \dot{u}_1 r \sin\theta + \dot{u}_2 r \cos\theta\right)$$

$$\left. - \alpha^{-1} r\left(\dot{\omega}_2 \sin\theta + \dot{\omega}_1 \cos\theta\right)\left(-\psi_t + \dot{u}_1 r \cos\theta + \dot{u}_2 r \sin\theta\right)\right] + 0\left(\eta^4\right)$$

Equation (A.6) is solved for η:

$$\eta = -\frac{b_0}{b_1} - \left(\frac{b_2}{2 b_1}\right)\eta^2 + \ldots = -\frac{b_0}{b_1} + 0\left(\eta^3\right) \tag{A.10}$$

since $b_2 = 0(\eta)$. Substitute (A.10) into (A.5):

$$a_0 + a_1\left[-\frac{b_0}{b_1} + 0\left(\eta^3\right)\right] + \frac{a_2}{2}\left[-\frac{b_0}{b_1} + 0\left(\eta^3\right)\right]^2 + 0\left(\eta^4\right) = 0$$

or

$$a_0 + a_1\left(-\frac{b_0}{b_1}\right) + \frac{a_2}{2}\left(\frac{b_0^2}{b_1^2}\right) + 0\left(\eta^4\right) \ldots = 0 \tag{A.11}$$

Compute the indicated multiplication; (A.11) is then

$$
\begin{aligned}
& a_{00} + a_{01} - \frac{a_{11} b_{00}}{\alpha} + a_{02} - \frac{a_{11} b_{01}}{\alpha} + \frac{a_{12} b_{00}}{\alpha} + \frac{a_{11} b_{11} b_{00}}{\alpha^2} \\
& + \frac{a_{22} b_{00}^2}{2!\,\alpha^2} + B_{11} + B_{12} + B_{13} - a_{11}\Big(\alpha^{-1} \dot{u}_2 r \sin\theta + \alpha^{-2} r \dot{\sigma}_1 \dot{u}_2 r \sin\theta \\
& - \alpha^{-2} b_{11} \dot{u}_2 r \sin\theta + \alpha^{-2} r \dot{\sigma}_1 b_{00} + \alpha^{-1}\Big[\frac{3}{2} r^2 \omega_1^2 \sin^2\theta + \frac{3}{2} r^2 \omega_2^2 \cos^2\theta \\
& + 2\xi_r\left(u_3 - \sigma_1\right) - 3 r^2 \omega_1 \omega_2 \sin\theta \cos\theta + 2 u_3 r \sigma_1 + u_1 \omega_3 r \sin\theta + \omega_3 \xi_\theta\Big]\Big) \\
& - B_{22}\Big(\alpha^{-1}\left[b_{00} + \dot{u}_2 r \sin\theta\right] - \alpha^{-2}\left[b_{11} - r\dot{\sigma}_1\right]\left[b_{00} + \dot{u}_2 r \sin\theta\right] \\
& + \alpha^{-1}\Big[b_{01} + \frac{3}{2} r^2 \omega_1^2 \sin^2\theta + \frac{3}{2} r^2 \omega_2^2 \cos^2\theta + 2\xi_z r\left(u_3 - \sigma_1\right) \\
& - 3 r^2 \omega_1 \omega_2 \sin\theta \cos\theta + 2 u_3 r \sigma_1 + u_1 \omega_3 r \sin\theta + \omega_3 \xi_\theta\Big]\Big) + a_{12} \alpha^{-1} \dot{u}_2 r \sin\theta \\
& + B_{23} \alpha^{-1}\left[b_{00} + \dot{u}_2 r \sin\theta\right] + \frac{a_{22}}{2!} \alpha^{-2}\left(2 b_{00} \dot{u}_2 r \sin\theta + \dot{u}_2^2 r^2 \sin^2\theta\right) \\
& + \frac{B_{33}}{2!} \alpha^{-2}\left[b_{00} + \dot{u}_2 r \sin\theta\right]^2 + 0\left(\eta^4\right) = 0
\end{aligned}
\tag{A.12}
$$

where

$$a_{00} = -\alpha \xi_z - \xi_{tt} + \ddot{u}_1 r \cos\theta$$

$$a_{01} = 2\xi_r \xi_{rt} + \frac{2\xi_\theta \xi_{\theta t}}{r^2} + 2\xi_z \xi_{tz} - \dot{u}_1\left(\xi_r \cos\theta - \xi_\theta \frac{\sin\theta}{r}\right)$$

$$
\begin{aligned}
a_{02} = & -\xi_r^2 \xi_{rr} - \frac{\xi_\theta^2 \xi_{\theta\theta}}{r^4} - \xi_z^2 \xi_{zz} + \frac{\xi_r \xi_\theta^2}{r^3} - 2\xi_r \xi_{rz} \xi_z \\
& - \frac{2\xi_r \xi_\theta \xi_{r\theta}}{r^2} - \frac{2\xi_z \xi_\theta \xi_{\theta z}}{r^2}
\end{aligned}
$$

$$a_{11} = -\alpha\xi_{zz} - \xi_{ttz}$$

$$a_{m+1,n+1} = \frac{\partial a_{mn}}{\partial z}$$

$$\sigma_1 = \omega_2 \cos\theta - \omega_1 \sin\theta$$

$$\sigma_{1,\theta} = \frac{\partial \sigma_1}{\partial \theta}$$

$$\dot{\sigma}_1 = \frac{\partial \sigma_1}{\partial t}$$

$$\beta_1 = u_1 \sin\theta - u_2 \cos\theta$$

$$\beta_{1,\theta} = \frac{\partial \beta_1}{\partial \theta}$$

$$\dot{\beta}_1 = \frac{\partial \beta_1}{\partial t}$$

$$B_{11} = \alpha\left(-u_3 + 2r\sigma_1\right) + \ddot{u}_2 r \sin\theta$$

$$\begin{aligned} B_{12} = {} & \left(-u_3 + 2r\sigma_1\right)b_{11} - \alpha^{-1} r\dot{\sigma}_1\left(\dot{b}_{00} + \ddot{u}_2 r\sin\theta\right) + 4r^2\left(\omega_1\dot{\omega}_1\sin^2\theta \right. \\ & \left. + \omega_2\dot{\omega}_2\cos^2\theta\right) - 2\left(\xi_{zt} + \frac{\dot{u}_3}{2}\right)r\sigma_1 - 2\left(\xi_z + \frac{\dot{u}_3}{2}\right)r\dot{\sigma}_1 + \dot{\omega}_3 r\beta_1 + \omega_3 r\dot{\beta}_1 \\ & - 3r^2\sin\theta\cos\theta\left(\dot{\omega}_1\omega_2 + \omega_1\dot{\omega}_2\right) + \dot{\omega}_3\xi_\theta + \omega_3\xi_{\theta t} + u_3\dot{u}_3 + u_3\xi_{zt} + \dot{u}_3\xi_z \\ & - \alpha^{-1} r\ddot{\sigma}_1\left(b_{00} + \dot{u}_2 r\sin\theta\right) + \xi_r\left(-\dot{u}_2\sin\theta\right) + \omega_3\left(\xi_{\theta t} + \dot{u}_1 r\sin\theta \right. \\ & \left. - \dot{u}_2 r\cos\theta\right) + \frac{\xi_\theta}{r^2}\left(-\dot{u}_2 r\cos\theta\right) \end{aligned}$$

$$B_{13} = \left(-u_3 + 2r\sigma_1\right)\left(b_{12} - 2r\xi_{zz}\sigma_1 + \omega_3\xi_{\theta z} - \alpha^{-1} r\dot{\sigma}_1 b_{11}\right)$$

$$- \xi_z\left(-2\xi_{zz}\dot{\sigma}_1 + \omega_3\xi_{\theta z} - \alpha^{-1} r\dot{\sigma}_1 b_{11}\right) - \alpha^{-1} r\dot{\sigma}_1\Big[\dot{b}_{01} + 4r^2\Big(\omega_1\dot{\omega}_2\sin^2\theta$$

$$+ \omega_2\dot{\omega}_2\cos^2\theta\Big) - 2\left(\xi_{zt} + \frac{\dot{u}_3}{2}\right)r\sigma_1 - 2\left(\xi_z + \frac{u_3}{2}\right)r\dot{\sigma}_1 + \omega_3 r\dot{\beta}_1 + \dot{\omega}_3 r\beta_1$$

$$- 3r^2\sin\theta\cos\theta\left(\dot{\omega}_1\omega_2 + \omega_1\dot{\omega}_2\right) + \omega_3\xi_\theta + \omega_3\xi_{\theta t}\Big] - \alpha^{-1} r\ddot{\sigma}_1\Big[b_{01} + \frac{u_3^2}{2}$$

$$+ \frac{3}{2}r^2\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right) - 2\left(\xi_z + \frac{u_3}{2}\right)r\sigma_1 + \omega_3 r\beta_1 + u_3\psi_z$$

$$- 3r^2\omega_1\omega_2\sin\theta\cos\theta + \omega_3\xi_\theta\Big] - \alpha^{-1}\Big[-\omega_2 u_1 + \omega_1 u_2 - \omega_3 r\sigma_{1,\theta}\Big]\Big[\dot{b}_{00}$$

$$+ \ddot{u}_2 r\sin\theta\Big] - \alpha^{-1}\Big[-\dot{\omega}_2 u_1 - \omega_2\dot{u}_1 + \dot{\omega}_1 u_2 + \omega_1\dot{u}_2 - \dot{\omega}_3 r\sigma_{1,\theta}$$

$$- \omega_3 r\dot{\sigma}_{1,\theta}\Big]\Big[b_{00} + \dot{u}_2 r\sin\theta\Big] + \alpha^{-2}\Big[2r^2\dot{\omega}_1\ddot{\omega}_1\sin^2\theta + 2r^2\dot{\omega}_2\ddot{\omega}_2\cos^2\theta$$

$$- 2r^2\sin\theta\cos\theta\left(\ddot{\omega}_1\dot{\omega}_2 + \dot{\omega}_1\ddot{\omega}_2\right)\Big]\Big[b_{00} + \dot{u}_2 r\sin\theta\Big] + \alpha^{-2}\Big[r^2\dot{\omega}_1^2\sin^2\theta$$

$$+ r^2\dot{\omega}_2^2\cos^2\theta - 2r^2\dot{\omega}_1\dot{\omega}_2\cos\theta\sin\theta\Big]\Big[\dot{b}_{00} + \ddot{u}_2 r\sin\theta\Big]$$

$$+ \xi_r\Big[-3r\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right) + 2\xi_{zr}r\sigma_1 - 2\left(\xi_z + \frac{u_3}{2}\right)\sigma_1 - \omega_3\beta_1$$

$$+ 6r\omega_1\omega_2\sin\theta\cos\theta - \omega_3\xi_{\theta r} + \alpha^{-1} r\dot{\sigma}_1\left(-\xi_{rt} + \dot{u}_1\cos\theta + \dot{u}_2\sin\theta\right) - u_3\xi_{rz}$$

$$+ \alpha^{-1}\dot{\sigma}_1\left(b_{00} + \dot{u}_2 r\sin\theta\right)\Big] + \omega_3\Big[-\xi_r\xi_{\theta r} - \frac{\xi_\theta\xi_{\theta\theta}}{r^2} - \xi_z\xi_{\theta z}$$

$$- 3r^2\omega_1^2\sin\theta\cos\theta + 3r^2\omega_2^2\cos\theta\sin\theta + 2\xi_z r\sigma_1 + 2\left(\xi_z + \frac{u_3}{2}\right)r\sigma_{1,\theta}$$

$$- \omega_3 r\beta_{1,\theta} + 3r^2\omega_1\omega_2\Big[\cos^2\theta - \sin^2\theta\Big] - \omega_3\xi_{\theta\theta} + \alpha^{-1} r\dot{\sigma}_1\Big(-\xi_{\theta t}$$

$$- \dot{u}_1 r\sin\theta + \dot{u}_2 r\cos\theta\Big) + \alpha^{-1}\dot{\sigma}_{1,\theta} r\left(b_{00} + \dot{u}_2 r\sin\theta\right)\Big]$$

$$+ \frac{\xi_\theta}{r^2}\Big[-3r^2\omega_1^2\sin\theta\cos\theta + 3r^2\omega_2^2\sin\theta\cos\theta + \xi_{z\theta} r\sigma_1$$

$$+ 2\left(\xi_z + \frac{u_3}{2}\right) r\sigma_{1,\theta} - \omega_3 r\beta_{1,\theta} + 3r^2\omega_1\omega_2\left[\cos^2\theta - \sin^2\theta\right] - \omega_3\xi_{\theta\theta} + u_3\xi_{z\theta}$$

$$+ \alpha^{-1} r\dot{\sigma}_1\left(-\xi_{\theta t} - \dot{u}_1 r\sin\theta + \dot{u}_2 r\cos\theta\right) + \alpha^{-1} r\dot{\sigma}_{1,\theta}\left(b_{00} + \dot{u}_2 r\sin\theta\right)\Bigg]$$

$$B_{m+1,n+1} = \frac{\partial B_{mn}}{\partial z} \tag{A.13}$$

In summary, the boundary value problem in terms of the potential function, with the higher order approximation for the free-surface condition, becomes

$$\nabla^2\psi(r,\theta,z,t) = 0 \quad \text{in} \quad \begin{cases} 0 \le r < a \\ 0 \le \theta \le 2\pi \\ -h < z < \eta \end{cases} \tag{A.14}$$

$$\psi_r = 0, \quad \text{on } r = a$$

$$\psi_z = -u_3 - 2r\omega_1\sin\theta + 2r\omega_2\cos\theta, \quad \text{on } z = -h$$

and (A.12).

The frequency of the forcing motion is close to or equal to the first natural frequency. The neighborhood of resonance considered is for

$$\left|\omega^2 - p_{11}^2\right| = 0\left(\epsilon^{\frac{2}{3}} p_{11}^2\right) \quad \text{as } \epsilon \to 0$$

Thus,

$$p_{11}^2\left(1 + \epsilon^{\frac{2}{3}}\nu\right) = \omega^2 \tag{A.15}$$

or

$$p_{11}^2 = \omega^2\left(1 - \epsilon^{\frac{2}{3}}\nu\right) \tag{A.16}$$

Here p_{11} is the lowest natural frequency of small, free-surface oscillations;

$$p_{11} = \sqrt{\alpha\lambda_{11}\tanh\lambda_{11}h} \tag{A.17}$$

where λ_{11} corresponds to the first nonzero root of

$$J_1'\left(\lambda_{1n} a\right) = 0$$

Constants are:

$$\Omega_{0n} = \frac{I_{01}^n + I_{02}^n + K_0 I_{03}^n}{\left(4p_{11}^2 - p_{0n}^2\right)\frac{a^2}{2p_{11}} J_0^2\left(\lambda_{0n} a\right)}$$

$$\Omega_{2n} = \frac{I_{21}^n - I_{22}^n + K_0 I_{23}^n}{\left(4p_{11}^2 - p_{2n}^2\right)\frac{\lambda_{2n}^2 a^2 - 4}{2\lambda_{2n}^2 p_{11}^2} J_2^2\left(\lambda_{2n} a\right)}$$

$$I_{q1}^n = \int_0^{\lambda_{11} a} u J_q\left(\frac{\lambda_{qn}}{\lambda_{11}} u\right)\left[\frac{d}{du} J_1(u)\right]^2 du, \quad q = 0,2$$

$$I_{q2}^n = \int_0^{\lambda_{11} a} \frac{1}{u} J_q\left(\frac{\lambda_{qn}}{\lambda_{11}} u\right) J_1^2(u)\, du, \quad q = 0,2$$

$$I_{q3}^n = \int_0^{\lambda_{11} a} u J_q\left(\frac{\lambda_{qn}}{\lambda_{11}} u\right) J_1^2(u)\, du, \quad q = 0,2$$

$$I_{q4}^n = \int_0^{\lambda_{11} a} u J_1(u) \frac{d}{du}\left[J_1(u)\right] \frac{d}{du}\left[J_q\left(\frac{\lambda_{qn}}{\lambda_{11}} u\right)\right] du, \quad q = 0,2$$

$$K_0 = \frac{3\zeta_{11}^2 - 1}{2}$$

$$G_1 = \frac{1}{2}\sum_{n=1}^{\infty}\left\{\left[\zeta_{0n}\zeta_{11}\lambda_{0n}\lambda_{11} - \frac{1}{2}\lambda_{0n}^2 + \lambda_{11}^2\left(1 - \zeta_{11}^2\right)\right]\Omega_{0n} I_{03}^n\right.$$

$$- \Omega_{0n} I_{04}^{n} - \Omega_{2n} I_{22}^{n} - \frac{1}{2} \Omega_{2n} I_{24}^{n} + \frac{1}{2} \left[\zeta_{11} \zeta_{2n} \lambda_{11} \lambda_{2n} - \frac{1}{2} \lambda_{2n}^{2} \right.$$

$$\left. + \lambda_{11}^{2} \left(1 - \zeta_{11}^{2} \right) \right] \Omega_{2n} I_{2n}^{n} \Bigg\}$$

$$G_2 = \sum_{h=1}^{\infty} \Bigg\{ \left[\zeta_{on} \zeta_{11} \lambda_{0n} \lambda_{11} - \frac{1}{2} \lambda_{0n}^{2} + \lambda_{11}^{2} \left(1 - \zeta_{11}^{2} \right) \right] \Omega_{0n} I_{03}^{n}$$

$$- \Omega_{0n} I_{04}^{n} + \Omega_{2n} I_{22}^{n} + \frac{1}{2} \Omega_{2n} I_{24}^{n} - \frac{1}{2} \left[\zeta_{11} \zeta_{2n} \lambda_{11} \lambda_{2n} \right.$$

$$\left. - \frac{1}{2} \lambda_{2n}^{2} + \lambda_{11}^{2} \left(1 - \zeta_{11}^{2} \right) \right] \Omega_{2n} I_{23}^{n} \Bigg\}$$

$$H_1 = 3 k_1 + k_2$$

$$H_2 = 2 \left(k_1 - k_3 + k_4 \right)$$

$$\zeta_{mn} = \tanh \left(\lambda_{mn} h \right)$$

$$p_{mn}^{2} = \alpha \lambda_{mn} \zeta_{mn}$$

$$k_1 = \frac{\pi}{2} \frac{\lambda_{11}^{2}}{K p_{11}^{2}} k_{10}$$

$$k_2 = \frac{\pi}{2} \frac{\lambda_{11}}{K \alpha \zeta_{11}} k_{20}$$

$$k_3 = \frac{\pi}{2} \frac{\lambda_{11}}{K \alpha \zeta_{11}} k_{30}$$

$$k_4 = \frac{\pi}{2} \frac{\lambda_{11}}{K \alpha \zeta_{11}} k_{40}$$

$$k_{10} = -\frac{1}{4}K\Big[6I_1 - I_3 - 2I_4 - 3I_5 + 4I_6 - 2I_7 + \zeta_{11}^2\left(3I_2 + 5I_3 + 15I_5\right)$$

$$+ 3\zeta_{11}^4 I_2\Big]$$

$$k_{20} = \frac{1}{2}K\zeta_{11}^2\left(9I_2 + 4I_3 + 12I_5 + 3\zeta_{11}^2 I_2\right)$$

$$k_{30} = \frac{1}{2}K\zeta_{11}^2\left(3I_2 - 4I_3 + 4I_5 + \zeta_{11}^2 I_2\right)$$

$$k_{40} = K\zeta_{11}^2\left(3I_2 + 4I_3 + 4I_5 + \zeta_{11}^2 I_2\right)$$

$$K = \frac{1}{\left(\lambda_{11}^2 a^2 - 1\right)J_1^2\left(\lambda_{11} a\right)}\left(\frac{\lambda_{11}^3}{\alpha\zeta_{11}}\right)$$

$$I_1 = \int_0^{\lambda_{11}a} uJ_1\left(\frac{dJ_1}{du}\right)^2\frac{d^2J_1}{du^2}\,du = \frac{1}{\lambda_{11}^2}\int_0^a rJ_1\left(\frac{dJ_1}{dr}\right)^2\frac{d^2J_1}{dr^2}\,dr$$

$$I_2 = \int_0^{\lambda_{11}a} uJ_1^4\,du = \lambda_{11}^2\int_0^a rJ_1^4\,dr$$

$$I_3 = \int_0^{\lambda_{11}a}\frac{1}{u}J_1^4\,du = \int_0^a\frac{1}{r}J_1^4\,dr$$

$$I_4 = \int_0^{\lambda_{11}a}\frac{1}{u^3}J_1^4\,du = \frac{1}{\lambda_{11}^2}\int_0^a\frac{1}{r^3}J_1^4\,dr$$

$$I_5 = \int_0^{\lambda_{11}a} uJ_1^2\left(\frac{dJ_1}{du}\right)^2 du = \int_0^a rJ_1^2\left(\frac{dJ_1}{dr}\right)^2 dr$$

$$I_6 = \int_0^{\lambda_{11} a} \frac{1}{u} J_1^2\left(\frac{dJ_1}{du}\right)^2 du = \frac{1}{\lambda_{11}^2} \int_0^a \frac{1}{r} J_1^2\left(\frac{dJ_1}{dr}\right)^2 dr$$

$$I_7 = \int_0^{\lambda_{11} a} \frac{1}{u^2} J_1^3\left(\frac{dJ_1}{du}\right) du = \frac{1}{\lambda_{11}^2} \int_0^a \frac{1}{r^2} J_1^3\left(\frac{dJ_1}{dr}\right) dr$$

$$F_1 = \frac{2a}{\left(\lambda_{11}^2 a^2 - 1\right) J_1\left(\lambda_{11} a\right)} \tag{A.18}$$

$$A_2 = -\frac{2F_1}{p_{11}^2 h} \tag{A.19}$$

$$A_1 = \frac{2F_1}{p_{11}^2 h} \tag{A.20}$$

$$U_2 = F_1 \tag{A.21}$$

$$K_1 = K_{10} + \Delta K_1$$

$$K_2 = K_{20} + \Delta K_2$$

$$K_{10} = \frac{K}{16}\Big[-18 I_1 + 3 I_3 + 6 I_4 + 9 I_5 - 12 I_6 + 6 I_7 + \zeta_{11}^2\left(9 I_2 - 7 I_3 - 21 I_5\right)$$
$$- 3\zeta_{11}^4 I_2\Big]$$

$$K_{20} = \frac{K}{8}\Big[-6 I_1 + I_3 + 2 I_4 + 3 I_5 - 4 I_6 + 2 I_7 + \zeta_{11}^2\left(3 I_2 + 19 I_3 - 7 I_5\right)$$
$$- \zeta_{11}^4 I_2\Big]$$

$$\Delta K_1 = \frac{2p_{11}}{\lambda_{11}^2} K \hat{G}_1$$

$$\Delta K_2 = -\frac{2p_{11}}{\lambda_{11}^2} K \hat{G}_2$$

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