For publication in the Journal of Mathematical Physics

Relativistic Rotation and the Disk Problem

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ABSTRACT

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It is shown that the Ehrenfest paradox arising from the rotating disk problem may be resolved without resorting to a non-Euclidean geometry on the rotating disk, and without postulating a radial contraction to compensate for the circumferential Lorentz contraction.

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INTRODUCTION

Disk rotation has been sporadically studied ever since the Ehrenfest paradox was proposed in 1909.[1] Using a qualitative argument in 1911, Einstein[2] concluded that the geometry of a rotating disk must be non-Euclidean as a result of circumferential shortening with no radial contraction. Lorentz[3] took issue with this view, and in 1921 stated that he

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had worked out the case of a thin, infinitely rigid disk and found that if v were the velocity of the rim, then the radius would be shortened "in the ratio of 1 to $1 - v^2/8c^2$," and that the disk surface would, therefore, remain Euclidean. Eddington,[4] using a different method, confirmed this result and, hence, the Euclidean geometry of a rotating disk, in 1924. Some 14 years later Einstein and Infeld[5] reiterated the belief that disk geometry was non-Euclidean, and in the next year, Levy[6] repeated this view. Berenda[7] supported Einstein's conjecture in 1942 by an analytical study of disk rotation in which he claimed that the hypercurvature of the disk surface was given by

$$-3(\Omega^2/c^2)[1 - (\Omega^2 r^2/c^2)]^{-2}$$

In developing this relation it was, however, necessary to ignore the vanishing of the Riemann-Christoffel tensor and to independently construct a so-called intrinsic geometry. Hill,[8] in 1942, assumed that relativistic rotation of itself could not limit the size of a rotating disk and employed a kinematic argument to propose that the Ehrenfest paradox might best be resolved by a nonlinear speed-distance law; that is, that the classical definition of uniform rotational motion must be abandoned. Four years later, Rosen[9] redefined the conditions for rigid body rotation and affirmed that "no rigid body would have a radius equal to, or exceeding, $1/\Omega$, but that there seems to be no reason why an idealized body with a smaller radius could not rotate according to the linear law." In defining spatial distance in a rotating system, he introduced a coordinate system similar to Berenda's, which required that the surface of a rotating disk be non-Euclidean.

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A partial reconciliation of the viewpoints represented by Einstein and Eddington was presented in 1952 by Møller,[10] who noted that regardless of whether the disk surface was Euclidean or not, the relation

$$p = 2\pi a$$

where p and a represent the stationary measurements of the disk circumference and radius, respectively, is not a rotational invariant. Instead, he suggested that the relation should be

$$p = \frac{2\pi a}{[1 - (\Omega^2 a^2/c^2)]^{1/2}} \tag{1}$$

His ensuing discussion of the geometry of the disk surface depended upon a metric tensor of the form used by Berenda, which Møller also specifically interpreted as implying a non-Euclidean geometry on the disk surface. In 1961 Weber[11] repeated this line of reasoning and wrote that the surface of a rotating disk was non-Euclidean.

All of the above investigators either implicitly or explicitly used an N-transformation[12]

$$\left.\begin{aligned} z^1 &= x^1 \cos \Omega x^4 + x^2 \sin \Omega x^4 \\ z^2 &= -x^1 \sin \Omega x^4 + x^2 \cos \Omega x^4 \\ z^3 &= x^3 \\ z^4 &= x^4 \end{aligned}\right\} \tag{2}$$

to relate the uniformly rotating coordinates z^α to the stationary coordinates x^i. Claims for nonorthogonality, such as those of Berenda and Rosen, are dependent upon a misinterpretation of this nonorthogonal transformation in the four-dimensional relativity space. In the subsequent development it is shown that the conclusions, but not necessarily the analyses, of Eddington and Lorentz were substantially correct; that is, that the geometry of the rotating disk is Euclidean.

Space-Time O-Transformations

Transformation (2) demands that

$$(ds)^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + 2\Omega(z^1 - z^2) - c^2 \left\{1 - \frac{\Omega^2}{c^2} [(z^1)^2 + (z^2)^2]\right\} \tag{3}$$

in the rotating space, while

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dx^4)^2 \tag{4}$$

represents arc length in the stationary frame.[13] Since it may be easily shown that the Riemann-Christoffel tensor vanishes in the z^α frame attached to the disk, it follows that the rotating space is still Euclidean. This observation is unchanged if $dx^3 = dz^3 = 0$, as in Berenda's analysis. Consequently, the off-diagonal terms in the rotating metric merely indicate use of a nonorthogonal coordinate system for the description of a Euclidean space. It has been shown[15] that Eq. (2) is, in fact, nothing more than a classical Newtonian N-transformation[16] from a stationary orthogonal system to a rotating nonorthogonal system. Transformation to an orthogonal cylindrical coordinate system y^i on the rotating disk may, however, be accomplished with the O-transformation

$$\left.\begin{aligned} y^1 &= \xi^1 \\ y^2 &= \gamma(\xi^2 - \Omega\xi^4) \\ y^3 &= \xi^3 \\ y^4 &= \gamma(\xi^4 - \alpha\xi^2) \end{aligned}\right\} \tag{5}$$

in which ξ^i represents a stationary circular cylindrical coordinate system. Its inverse is

$$\left.\begin{aligned} \xi^1 &= y^1 \\ \xi^2 &= \gamma(y^2 + \Omega y^4) \\ \xi^3 &= y^3 \\ \xi^4 &= \gamma(y^4 + \alpha y^2) \end{aligned}\right\} \quad (6)$$

in which the transformation parameters γ and α are defined in terms of the angular velocity Ω and the radial distance $r =; \xi^1 =; y^1$ according to

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad (7)$$

$$\alpha = \frac{v^2}{\Omega c^2} \quad (8)$$

and

$$v = \Omega r \quad (9)$$

where the notation $r =; y^1 =; \xi^1$ indicates that r is a transformation parameter. It takes the magnitude of y^1, or ξ^1, but does not play the same role as either y^1 or ξ^1.

Calculation of the metric of the space seen by a rotating observer depends upon the relations[17]

$$b_{ij} = C_{mn}\xi_i^m\xi_j^m$$

where

$$\xi_i^m \equiv \partial\xi^m/\partial y^i \text{ and } C_{11} = C_{33} = 1,\ C_{22} = r^2,\ C_{44} = -c^2$$

and where

$$\begin{array}{ll} \xi_1^1 = 1 & \xi_3^3 = 1 \\ \xi_2^2 = \gamma & \xi_2^4 = \gamma\alpha \\ \xi_4^2 = \gamma\Omega & \xi_4^4 = \gamma \end{array}$$

Thus, we find that

$$b_{11} = b_{33} = 1$$

$$b_{22} = \gamma^2(r^2 - c^2\alpha^2) = r^2$$

$$b_{44} = -\gamma^2(c^2 - v^2) = -c^2$$

which proves that the space seen by the rotating observer is indeed Euclidean. Moreover,

$$b_{ij} = C'_{ij}.$$

Although the transformation parameters contain factors proportional to y^1 and y^4, or ξ^1 and ξ^4, they appear as constants in

$$\partial\xi^n/\partial y^i \text{ and } \partial y^i/\partial\xi^n \qquad (10)$$

because they have no geometrical significance, as was shown.[15] Such parameters frequently appear whenever a transformation includes a change of units which is position dependent. For example, consider the two circular cylindrical coordinate systems X^n and Z^α, both stationary in E^3, such that X^2 is the usual radian measure of position and Z^2 is a measure of length, say meters, along a circular arc, according to

$$\left.\begin{aligned} Z^1 &= X^1 \\ Z^2 &= rX^2 \\ Z^3 &= X^3 \end{aligned}\right\} \qquad (11)$$

where $r =; X^1 =; Z^1$. Certainly the <u>geometry</u> of space is not affected by the choice units; that is, neither the orthogonality of the coordinate lines nor the lines themselves, as shown in Fig. 1, is altered by the choice of units for measuring distance along these lines. Regardless of the measuring units chosen, the arc length ds is given by $(ds)^2 = G_{mn}\, dX^n\, dX^m = H_{\alpha\beta}\, dZ^\alpha\, dZ^\beta$, with

$$G_{11} = G_{33} = 1,\ G_{22} = (X^1)^2 \qquad (12)$$

and

$$H_{11} = H_{22} = H_{33} = 1$$

Failure to recognize that r is a transformation parameter in Eq. (11) (i.e., the erroneous use of $r = X^1 = Z^1$) will lead to an erroneous metric; namely

$$H_{11} = 1 + \left(\frac{Z^2}{Z^1}\right)^2$$

$$H_{12} = H_{21} = -\frac{Z^2}{Z^1}$$

$$H_{22} = H_{33} = 1,$$

which implies that the system has become nonorthogonal just as a consequence of the change of measuring units. This contradiction demonstrates that r must indeed be taken as a transformation parameter.

Thus the unraveling of the Ehrenfest paradox and the resolution of the Einstein-Infeld versus Lorentz-Eddington controversy is immediate. The surface of a rotating disk remains flat, as implied by the vanishing of the Riemann-Christoffel tensor under either Eq. (2) or (5), and may be simply described by Eq. (5) with $x^3 = Z^3 = 0$. The circumferential shortening with rotation does not imply a non-Euclidean geometry, but rather that the quantity 2π should be replaced by $2\pi\gamma$, as noted earlier.

Footnotes

[1]P. Ehrenfest, Physik. Z., 10 (1909).

[2]A. Einstein, The Principle of Relativity (Dover Pub. Inc., New York, 1952), Sec. 3, and Ann. Physik., 48, 769-822 (1916), Sec. 3.

[3]H. A. Lorentz, Nature, 106, 793-795 (1921) and Collected Papers (Martinus Nijhoff, The Hague, 1934), vol. 7.

[4]A. S. Eddington, The Mathematical Theory of Relativity (Cambridge Univ. Press, 1924), 2nd ed., Sec 50, 112-113.

[5]A. Einstein and L. Infeld, The Evolution of Physics (Simon and Schuster, New York, 1938).

[6]H. Levy, Modern Science (A. A. Knopf, New York, 1939).

[7]C. W. Berenda, Phys. Rev. (2), 62, 280-290 (1942).

[8]E. L. Hill, Phys. Rev. (2), 69, 488-491 (1946).

[9]N. Rosen, Phys. Rev. (2), 71, 54-58 (1947).

[10]C. Møller, The Theory of Relativity (Clarendon Press, Oxford, 1952), p. 221.

[11]J. Weber, General Relativity and Gravitational Waves (Interscience Pub. Inc., New York, 1961).

[12]N stands for nonorthogonal, and later on O will stand for orthogonal.

[13]The choice of g_{ij} or $-g_{ij}$ is of no consequence. See reference 14.

[14]J. L. Synge, Relativity: The Special Theory (North-Holland Pub. Co., Amsterdam, 1958), p. 16.

[15]W. C. Orthwein, "Rotational invariance of Maxwell's equations", Tensor (to be published).

[16]The corresponding Newtonian O-transformation is

$$z^1 = x^1 \cos\varphi + x^2 \sin\varphi$$

$$z^2 = -x^1 \sin\varphi + x^2 \cos\varphi$$

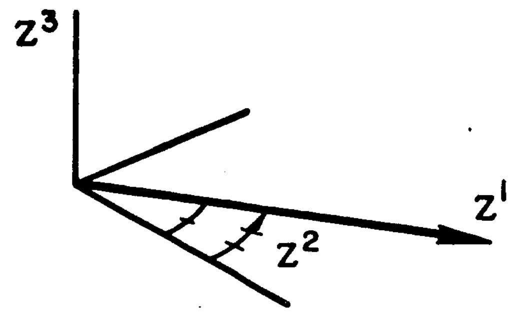
$$z^3 = x^3$$

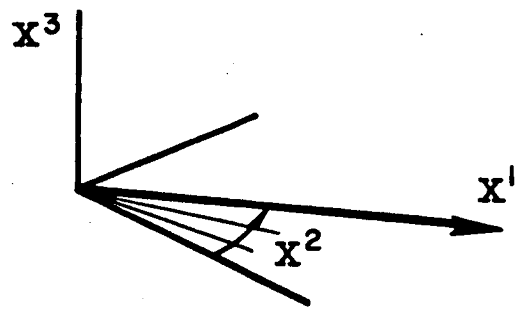
$$z^4 = x^4$$

in which $\varphi = \Omega t$ is a transformation parameter.[15] Note that in R^4 Newtonian O-transformations may correspond to translation, rotation, or a combination thereof, in a hyperplane normal to the time axis, while relativistic O-transformations correspond to rotation in a hyperplane containing the time axis.

[17]The Einstein summation convection is implied.

Fig. 1.- Coordinates used in Eq. (11).


Z3
Z1
Z2


X3
X1
X2