COMPARISON OF GRYZINSKI'S AND BORN'S APPROXIMATIONS FOR INELASTIC SCATTERING IN ATOMIC HYDROGEN

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SUMMARY

Excitation cross sections for electron scattering in atomic hydrogen were determined by the Born approximation from the ground state to all excited states up to the 4f level. Gryzinski's semiclassical theory was used to obtain similar cross sections. Because of the small differences in energy between various energy levels of a principal quantum level, however, only the sum of cross sections for all transitions to the principal quantum level can be determined. Results of this method were compared with the sum of the individual sublevel cross sections determined by the Born approximation. Cross sections are presented for transitions from the ground state to \( n = 2, 3, 4, \) and ionization. The theoretical cross sections are compared with the experiment for the \( 1s - 2s, 1s - 2p, \) and \( 1s - \) ionization transitions. In addition, calculations of Gryzinski's electron exchange cross section were performed for the 2s, 3s, and 4s levels. Gryzinski's theory gave lower excitation cross sections than Born's approximation for all transitions up to \( n = 4 \) and for ionization. In the energy range considered (threshold to 360 eV), the cross sections resulting from the two theories are quite similar in shape and differ at various electron energies by a factor between 2 and 4. The exchange cross section of Gryzinski agrees very well with experiment for the \( 1s - 2s \) transition below 40 electron volts.

INTRODUCTION

One of the major energy loss mechanisms for a hot plasma is radiation. To predict the magnitude of this energy loss, a detailed knowledge of the cross sections involved is necessary. Hydrogen plasmas are of major importance for many research programs. For such plasmas, radiation losses are contributed by both the molecule and the atom constituents. The present investigation will concern itself only with atomic hydrogen cross sections.
Atomic hydrogen, the simplest atom, is attractive for theoretical studies of inelastic scattering by electron impact. Since atomic hydrogen readily recombines to form molecular hydrogen, its cross sections are difficult to obtain experimentally. Only the cross sections for the $1s - 2s$, $1s - 2p$, and $1s$ → ionization transitions have been measured (refs. 1 to 4). Most of the theoretical inelastic scattering calculations for atomic hydrogen cross sections have also been for these transitions (refs. 3 to 10). In the present investigation, cross sections for all hydrogen atom transitions from the ground state to the fourth quantum level and for ionization are presented for incident electron energies from threshold to 360 electron volts.

Two methods of calculating the pertinent cross sections are the Born approximation and the semiclassical theory of Gryzinski (refs. 11 and 12). The Born approximation is known to give valid results at high energies (ref. 13, p. 124). This fact permits normalization of experimental results to the Born approximation at energies above 200 electron volts (refs. 1 to 4). Gryzinski's semiclassical approach, on the other hand, agrees more closely with experiments in the low energy range (refs. 8 to 12 and 14). In this report, results obtained from the two theories are compared with each other and with some experimentally measured cross sections.

SYMBOLS

<table>
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<tr>
<td>$A$</td>
<td>defined by eq. (A21)</td>
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<tr>
<td>$a_0$</td>
<td>Bohr radius, $5.29 \times 10^{-9}$ cm</td>
</tr>
<tr>
<td>$b$</td>
<td>$(n + 1) - (n - 1)u$ \left/ \right. \quad 2(1 - u)$</td>
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<tr>
<td>$C_{n-\ell-1}^{\ell+2}(X)$</td>
<td>Gegenbauer polynomial defined by eq. (12)</td>
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<td>$E$</td>
<td>energy loss of incident electron</td>
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<td>kinetic energy of bound electron</td>
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<tr>
<td>$h$</td>
<td>Planck's constant, $6.6256 \times 10^{-27}$ (erg)(sec)</td>
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<tr>
<td>$I_{0n}(K)$</td>
<td>Born's differential cross section, eq. (6)</td>
</tr>
</tbody>
</table>
I_r \quad \text{defined by eq. (A11)}
I_\theta \quad \text{defined by eq. (A10)}
I_\varphi \quad \text{defined by eq. (A6)}
J_s(t) \quad \text{Bessel function of first kind of order } s \text{ and argument } t
K \quad \text{magnitude of momentum change}
K_{\text{max}} \quad \text{maximum momentum change, } k + k_{0n}
K_{\text{min}} \quad \text{minimum momentum change, } k - k_{0n}
\bar{K} \quad \text{momentum change}
k \quad \text{wave number of incident electron}
k_{0n} \quad \text{wave number of scattered electron}
\ell \quad \text{orbital angular momentum quantum number}
m \quad \text{magnetic quantum number}
N_{n\ell m} \quad \text{normalizing factor defined by eq. (A4)}
n \quad \text{principal quantum number}
P^m_{\ell}(\cos \theta) \quad \text{Legendre polynomial}
Q(U_{n'}) \quad \text{excitation cross section (Gryzinski theory)}
Q_e(U_{n'}) \quad \text{exchange cross section}
Q_{0n}(K) \quad \text{excitation cross section (Born approximation)}
R \quad \text{radius vector of incident electron}
r, \theta, \varphi \quad \text{spherical coordinates}
\vec{r} \quad \text{radius vector of bound electron}
s, p, d, f \quad \text{energy levels for } \ell = 0, 1, 2, 3, \text{ respectively}
U_i \quad \text{ionization potential}
U_n \quad \text{energy of level } n
u \quad \text{dummy variable}
V(r, R) \quad \text{Coulomb potential between incident electron and an atom}
v \quad \text{defined by eq. (A19)}
v_e \quad \text{speed of incident electron}
X \quad \text{defined by eq. (10)}
Cross sections for inelastic scattering of electrons by atomic hydrogen can be calculated in a number of ways. The methods investigated in this report are Gryzinski's semiclassical theory and Born's approximation.

Semiclassical (Gryzinski) Theory

Using a semiclassical approach, Gryzinski developed a theory giving excitation cross sections (refs. 11 and 12) and exchange cross sections (ref. 12). In this theory, s → s transitions are assumed to take place only by an electron exchange process. The cross section for this process is given by
According to Gryzinski's model, all transitions other than \( s \to s \) transitions take place by an excitation process. Excitation cross sections are given by

\[
Q(U_n) = \frac{\sigma_0}{U_n^2} g_j \left( \frac{E_2}{U_n}; \frac{E_1}{U_n} \right)
\]

where

\[
g_j \left( \frac{E_2}{U_n}; \frac{E_1}{U_n} \right) = \left( \frac{E_2}{E_1 + E_2} \right)^{3/2}
\]

\[
\begin{cases} 
2 \frac{E_1}{E_2} + \frac{U_n}{E_1} \left( 1 - \frac{E_1}{E_2} \right) - \left( \frac{U_n}{E_2} \right)^2 & \text{if } U_n + E_1 \leq E_2 \\
2 \left[ \frac{E_1}{E_2} + \frac{U_n}{E_1} \left( 1 - \frac{E_1}{E_2} \right) - \left( \frac{U_n}{E_2} \right)^2 \right] \left( \frac{U_n}{E_1} \right) \left( 1 - \frac{U_n}{E_2} \right) & \text{if } U_n + E_1 > E_2 
\end{cases}
\]

and

\[
\sigma_0 = 6.53 \times 10^{-14} \text{ cm}^2 \text{ eV}^2
\]
In equations (1) to (4) \( U_n \) and \( U_{n+1} \) are the energies of the \( n \) and \( n+1 \) levels. The ionization potential is \( U_i \), and \( E_1 \) and \( E_2 \) are the kinetic energies of the bound and the incident electrons, respectively.

The excitation cross section \( Q(U_n) \) is proportional to the probability that the incident electron will lose an amount of energy equal to or greater than the energy of the \( n \) level. Thus, the cross section for a given transition with loss of energy in the interval \( U_n \leq E < U_{n+1} \) is given by

\[
Q(U_n, U_{n+1}) = Q(U_n) - Q(U_{n+1})
\]

Since to excite the \( n+1 \) level an amount of energy greater than or equal to \( U_{n+1} \) must be lost, equation (5) is just the cross section for exciting the \( n \) level.

For atomic hydrogen the sublevels of a given principal quantum level are separated by a small amount of energy, and the Gryzinski approach does not give meaningful results for individual transitions. An example is the \( n = 2 \) level of atomic hydrogen, which has one 2s level and two 2p levels separated by less than half a wave number. The 2s transition from the ground state can take place only by an electron exchange process in Gryzinski's theory, while the two 2p transitions occur by an excitation process. Agreement with experiment and the Born approximation was obtained when all the energy levels, except the s level, of a principal quantum level were averaged to determine an energy for the principal quantum level. This energy was used to determine an excitation cross section from the ground state to the principal quantum level. Since the Gryzinski method gives only one cross section for excitation to any principal quantum level, it is necessary to add all the Born approximation cross sections of the sublevels for that principal quantum level for comparison. The Born approximation cross section to the s level is not included, since it is not considered in Gryzinski's excitation cross section.

**Born Approximation**

For fast electrons \( (v_e >> e^2/h) \), the first Born approximation is sufficient to determine the excitation cross section of an atom. The differential cross section for excitation from the ground state to some excited state \( n \) with momentum change between \( K \) and \( K + dK \) is given by reference 13 as

\[
I_{0n}(K)dK = \frac{8\pi^3 m^2}{h^4} \frac{K dK}{k^2} \left| \int \int V(R,r) e^{i\vec{K} \cdot \vec{R}} \psi_0(r) \psi^*_n(r) d\tau_r d\tau_R \right|^2
\]
where the magnitude of the momentum change is given by

\[ K = \left| k_{0n} \hat{\mathbf{r}}_1 - k_{0n} \hat{\mathbf{r}}_0 \right| \]  (7)

in which \( \frac{k\hbar}{2\pi} \hat{\mathbf{r}}_0 \) and \( \frac{k_{0n} \hbar}{2\pi} \hat{\mathbf{r}}_1 \) are the initial and final momentum vectors of the scattered electron, respectively. The ground state and excited state wave functions of the atom are \( \psi_0 \) and \( \psi_n^* \), respectively, and \( V(\mathbf{r}, \mathbf{R}) \) is the Coulomb potential between the incident electron and the atom. The differential volume elements in the coordinate space of the atomic and incident electrons are \( d\tau_{\mathbf{r}} \) and \( d\tau_{\mathbf{R}} \), respectively.

The indicated integration given in appendix 1 for atomic hydrogen gives

\[ I_{0n}(K)dK = \frac{128\pi^5 m^2 e^4}{k^2 \hbar^4} \frac{dK}{K^3} \left| \epsilon_{n\ell m}(K) \right|^2 \]  (8)

where

\[ \epsilon_{n\ell m}(K) = 2^{2\ell+3} n^{\ell+1}(2\ell + 1)^{1/2}(\ell + 1)! \left[ (n - \ell - 1)! \right]^{1/2} \left[ (n + \ell)! \right]^{-1/2} \]

\[ (K\alpha_0)^\ell \frac{[(n - 1)^2 + 4\xi^2]^{(n - \ell - 3)/2}}{[(n + 1)^2 + 4\xi^2]^{(n + \ell + 3)/2}} \left\{ (n + 1) [(n - 1)^2 + 4\xi^2] C_{n-\ell-1}^{\ell+2}(X) \right. \]

\[ - 2n [(n - 1)^2 + 4\xi^2]^{1/2} [(n + 1)^2 + 4\xi^2]^{1/2} C_{n-\ell-2}^{\ell+2}(X) \]

\[ + (n - 1) [(n + 1)^2 + 4\xi^2] C_{n-\ell-3}^{\ell+2}(X) \} \]  (9)

\[ X = (n^2 - 1 + 4\xi^2) \left\{ [(n + 1)^2 + 4\xi^2][(n - 1)^2 + 4\xi^2] \right\}^{-1/2} \]  (10)

\[^1\text{The correct evaluation of the integral in eq. (6) is given in ref. 13. Ref. 15 contains an additional factor, } [P_{\ell}^m(k - k_{0n} \cos \theta/K)]^2, \text{ which should not be there. The discrepancy is due to the orientation of coordinates; the proper orientation yields the answer given in ref. 13 and the appendix of this report.}\]
\[ \zeta = \frac{Kn_0}{2} \]  

\[ \sum_{n-\ell-1=0}^{\infty} C_{n-\ell-1}(x)u^{n-\ell-1} = (1 - 2ux + u^2)^{-\ell} \]  

To obtain the total cross section for a transition, integration over the limits of allowed momentum change must be performed; thus,

\[ Q_{0n}(K) = \int_{K_{\min}}^{K_{\max}} I_{0n}(K) dK \]  

where

\[ K_{\max} = k + k_{0n} \]

\[ K_{\min} = k - k_{0n} \]

RESULTS AND DISCUSSION

The calculated results for the \( 1s \rightarrow ns \) transitions are shown in figures 1(a) to (c). The experimental results of Stebbings, et al., (ref. 1) for the \( 1s \rightarrow 2s \) transition are also shown in figure 1(a). Below 40 electron volts, Gryzinski's approximation yields cross sections which fit the experimental results very well. Since for the \( 1s \rightarrow ns \) transitions the Born approximation involves an excitation process whereas the Gryzinski approximation assumes an exchange process, it is not surprising that the results from the two methods do not agree.

The Born approximation results for the \( 1s \rightarrow np, 1s \rightarrow nd, \) and \( 1s \rightarrow nf \) transitions are shown in figures 2, 3, and 4, respectively. In addition, a composite cross section for excitation from the ground state to all sublevels of a principal quantum level was calculated using the Gryzinski model. The results are compared in figures 2(a), 5, and 6 (p. 10 to 12) with composite values obtained from the Born approximation by summing the sublevel cross sections. Experimental results for the \( 1s \rightarrow 2p \) transition from reference 2 are also included in figure 2(a); these results were normalized to the Born approximation at 200 electron volts. Finally, in figure 7 (p. 12), the Born approximation and the
Figure 1. Atomic hydrogen cross section for $1s - ns$ transitions.
Figure 2. - Atomic hydrogen cross section for $1s \rightarrow np$ transitions.
Figure 3. - Born approximation of atomic hydrogen cross section for 1s–nd transitions.

(a) Transition, 1s – 3d.

(b) Transition, 1s – 4d.

Figure 4. - Born approximation of atomic hydrogen cross section for 1s – 4f transition.

Figure 5. - Atomic hydrogen cross section for 1s – (3p + 3d) transition.
absolute experimental results reported in reference 3 for the 1s → ionization cross section are compared with the results obtained by using Gryzinski's theory.

Semiclassical results for the 1s → 2p transition have been obtained by others (refs. 6 to 8, 12, and 14), but the calculations were not carried to as high an energy as those reported herein. The results reported herein for this transition agree with those of references 8, 12, and 14, but they are about a factor of 2 less than those in references 6 and 7. In reference 6, Saraph presents only the results of his calculations, and thus it is impossible to determine why his results differ. Kingston (ref. 7) did not use the exact Gryzinski model, and therefore, his results should not be expected to agree, but they agree better with experimental results than the results reported herein.

Comparison of the results of the Gryzinski method and the Born approximation with the experimental results of Fite and Brackmann (ref. 2) for the 1s → 2p transition (fig. 2(a), p. 10) shows the maximum value of the experimental cross section to be about half way between the maximum values of the two approximations. The experimental maximum occurs at about twice the energy predicted by the two approximations.

Examination of figures 2(a) and 5 to 7 shows that as the principal quantum number increases the percent difference between the two approximations decreases. As shown in figure 2(a), the 1s → 2p transition at an electron energy of 360 electron volts is found by the Gryzinski approximation to have a cross section about 24 percent of that obtained by the Born approximation. For the 1s → ionization transition (fig. 7), the Gryzinski cross section is about 75 percent of the Born cross section.
CONCLUDING REMARKS

Gryzinski's electron exchange model gives results in good agreement with experimental data for the $1s \rightarrow 2s$ transition at electron energies below 40 electron volts. At higher energies, the electron exchange process contributes very little to the transition; the calculated cross sections become much smaller than the experimental ones.

In calculating composite cross sections for the excitation of atomic hydrogen to various principal quantum levels, Gryzinski's approximation gives values lower than those of the Born approximation although the shapes of the two curves are nearly the same. The difference between the two approximations becomes less as the principal quantum number increases.

For the $1s \rightarrow$ ionization cross section, Gryzinski's method agrees well with the experimental absolute cross section. Except at low energies, where Gryzinski's theory predicts values higher than those determined by experiment, Gryzinski's theory gives results lower than the result of either the Born approximation or the experiment.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, May 3, 1965.
APPENDIX - INTEGRATION OF EQUATION (6)

The integral to be evaluated is

\[ I_0n(K)dK = \frac{8\pi^3 m^2}{h^4} \left| \int \int V(\vec{r}, \vec{R}) e^{i\vec{K} \cdot \vec{R}} \psi_0(\vec{r})\psi^*_n(\vec{r})d\tau \right|^2 \tag{6} \]

where the potential \( V(\vec{r}, \vec{R}) \) is the Coulomb interaction between the incident and the atomic electrons \( \frac{e^2}{|\vec{r} - \vec{R}|} \). The Coulomb interaction between the incident electron and the atomic nucleus \( \frac{e^2}{R} \) vanishes because of the orthogonality of the atomic wave functions.

Integration over the coordinate space of the incident electron by a method outlined in reference 16 yields

\[ \int \frac{e^2}{|\vec{r} - \vec{R}|} e^{i\vec{K} \cdot \vec{R}} d\tau = \frac{4\pi}{K^2} e^{2iK \cdot \vec{r}} \tag{A1} \]

Substitution of equation (A1) into equation (6) gives

\[ I_0n(K)dK = \frac{128\pi^5 m^2 e^4}{K^2 h^4} \left| \int e^{i\vec{K} \cdot \vec{r}} \psi_0(\vec{r})\psi^*_n(\vec{r})d\tau \right|^2 \tag{A2} \]

In order to evaluate equation (A2), the wave functions must be expressed in the coordinate system of the bound electron. With spherical coordinates, the wave function of atomic hydrogen becomes

\[ \psi_n(\vec{r}) = \psi_{n\ell m}(\vec{r}) = N_{n\ell m} r^{\ell} L^{2\ell+1}_{n+\ell} \left( \frac{2r}{na_0} \right) e^{-r/na_0} P^m_{\ell}(\cos \theta)e^{im\phi} \tag{A3} \]

where \( N_{n\ell m} \) is a normalizing factor given by

\[ N_{n\ell m}^2 = \frac{2^{2\ell}(2\ell + 1)(\ell - |m|)! (n - \ell - 1)!}{n\pi (na_0)^{2\ell+3} (\ell + |m|)! [n(\ell + 1)]^3} \tag{A4} \]

Thus, the integral in equation (A2) becomes
\[ \int e^{i\vec{K} \cdot \vec{r}} \psi_{100}^{(\ell)}(\vec{r}) \psi_{n \ell m}(\vec{r}) d\tau = N_{100} N_{n \ell m} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} e^{-r/a_0} \]

\[ r^{\ell}(2\ell+1) \left( \frac{2r}{a_0} \right) e^{-r/a_0} P_{n+\ell}^m(\cos \theta) e^{\pm im\phi} r^2 \sin \theta \, d\phi \, d\theta \, dr \quad (A5) \]

If the Z-axis is aligned along \( \vec{K} \), the only contribution to the integral over \( \phi \) is the exponential. Since the magnetic quantum number \( m \) is an integer,

\[
I_\phi = \int_0^{2\pi} e^{\pm im\phi} \, d\phi = \begin{cases} 0 & \text{for } m \neq 0 \\ 2\pi & \text{for } m = 0 \end{cases} \quad (A6)
\]

The integral over \( \theta \) is

\[
I_\theta = \int_0^{\pi} e^{iKr \cos \theta} P_k^m(\cos \theta) \sin \theta \, d\theta \quad (A7)
\]

The Legendre polynomial (ref. 17, p. 77) can be expressed as a Gegenbauer polynomial in the form

\[ P_k^m(\cos \theta) = C_k^{1/2}(\cos \theta) \quad (A8) \]

Let \( Z = Kr \) and \( \nu = 1/2 \), to put equation (A7) in the form of reference 17, page 77:

\[
\int_0^{\pi} e^{iZ \cos \theta} C_k^\nu(\cos \theta) \sin^{2\nu} \theta \, d\theta = \frac{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(2\nu + \ell)}{\ell! (2\nu)} i^{\ell} J_{\nu+\ell}(Z) \quad (A9)
\]

If the form of equation (A9) is used, equation (A7) becomes

\[
I_\theta = \left( \frac{2\pi}{Kr} \right)^{1/2} i^{\ell} J_{\ell+1/2}(Kr) \quad (A10)
\]

Incorporation of the terms involving \( r \) from equations (A5) and (A10) gives
\[
I_r = \int_0^\infty e^{-r/a_0} r L_{n+\ell}^{2\ell+1}(2r/na_0) e^{-r/na_0} j_{\ell+1/2}(Kr)^{3/2} \, dr
\]  
(A11)

Let

\[\xi = \frac{2r}{na_0}\]

and

\[\zeta = \frac{Kna_0}{2}\]

to put equation (A11) in the form

\[
I_r = \left(\frac{na_0}{2}\right)^{\ell+5/2} \int_0^\infty e^{-(1+n)\xi/2} \xi^{\ell+3/2} L_{n+\ell}^{2\ell+1}(\xi) J_{\ell+1/2}(\xi) d\xi
\]  
(A12)

The Laguerre polynomial has a generating function (ref. 18, p. 784), which after differentiation \(\alpha\) times with respect to \(\xi\) becomes

\[
\sum_{\beta = 0}^{\infty} \frac{L_{\alpha+\beta}(\xi) u^\beta}{(\alpha + \beta)!} = \frac{\xi u}{e^\xi} \frac{(-1)^\alpha e^{-\xi u}}{(1 - u)^{\alpha+1}}
\]  
(A13)

Employment of equation (A13) in equation (A12) gives

\[
\sum_{n = \ell+1}^{\infty} \frac{I_r u^{n-\ell-1}}{(n + \ell)!} = -\left(\frac{na_0}{2}\right)^{\ell+5/2} \frac{1}{(1 - u)^{2\ell+2}} \int_0^\infty e^{-\frac{[(n+1)-(n-1)u] \xi}{2(1-u)}} J_{\ell+1/2}(\xi) \xi^{\ell+3/2} \, d\xi
\]  
(A14)
Let
\[ b = \frac{(n + 1) - (n - 1)u}{2(1 - u)} \]
\[ \nu = \ell + \frac{1}{2} \]
\[ \mu = \ell + \frac{5}{2} \]
to put equation (A14) in the form of reference 16, page 32:

\[ \int_0^\infty e^{-b \xi} J_\nu(\xi \xi) \xi^{\mu - 1} d\xi = \frac{(\xi)^\nu}{b^\mu \Gamma(\nu + 1)} \binom{\mu + \nu}{2} \binom{\mu + \nu + 1}{2} \binom{\nu + 1}{\nu + 1} - \frac{\xi^2}{b^2} \] (A15)

which becomes

\[ \sum_{n=\ell+1}^{\infty} \frac{\Gamma(\nu + 1/2) \Gamma(\ell + 3)}{(n + 1) - (n - 1)u} \xi^{\ell+1/2} (1 - u)^{\ell+5/2} \]
\[ \left( \left[ \frac{2}{(n + 1) - (n - 1)u} \right]^{2\ell+3} \right) \frac{\Gamma(2\ell + 3)}{\Gamma(\ell + 3/2)} \]

\[ 2F_1 \left( \frac{\ell + 3}{2}, \frac{\ell + 2}{2}; \frac{-4\xi^2(1 - u)^2}{[n + 1] - (n - 1)u} \right) \] (A16)

The hypergeometric series of equation (A16) can be expressed as

\[ 2F_1 \left( \frac{\ell + 3}{2}, \frac{\ell + 2}{2}; \frac{-4\xi^2(1 - u)^2}{[n + 1] - (n - 1)u} \right) = \left\{ 1 + \frac{4\xi^2(1 - u)^2}{[n + 1] - (n - 1)u} \right\}^{-\ell-2} \] (A17)

Thus the integral over \( \xi \) becomes
\[
\sum_{n=\ell+1}^{\infty} \frac{I_n u^{n-\ell-1}}{(n+\ell)!} = \frac{[(n+1) - (n - 1)u](1 - u)A}{(1 - 2 \lambda v + v^2)^{\frac{\ell+2}{2}}} \tag{A18}
\]

where

\[
v = \omega u = u \left[ \frac{(n - 1)^2 + 4\xi^2}{(n + 1)^2 + 4\xi^2} \right]^{1/2} \tag{A19}
\]

\[
X = \frac{1}{\omega} \left[ \frac{n^2 - 1 + 4\xi^2}{(n + 1)^2 + 4\xi^2} \right] \tag{A20}
\]

\[
A = \frac{-2^{\ell+5/2} \xi^{\ell+1/2} \Gamma(2\ell + 3) \left( \frac{n\alpha_0}{2} \right)^{\ell+5/2}}{\Gamma(\ell + \frac{3}{2})} \tag{A21}
\]

Equating coefficients of \( u \) gives

\[
I_\ell = A(n + \ell)! \left[ (n + 1)\omega^{n-\ell-1}C_{n-\ell-1}^{\ell+2}(X) - 2n\omega^{n-\ell-2}C_{n-\ell-2}^{\ell+2}(X) + (n - 1)\omega^{n-\ell-3}C_{n-\ell-3}^{\ell+2}(X) \right] \tag{A22}
\]

where \( C_{n-\ell-1}^{\ell+2}(X) \) is Gegenbauer's polynomial defined by

\[
\sum_{n-\ell-1=0}^{\infty} C_{n-\ell-1}^{\ell+2}(X)u^{n-\ell-1} = (1 - 2uX + u^2)^{-(\ell+2)} \tag{12}
\]

Addition of \( I_\phi, I_\theta, \) and \( I_r \) and substitution into equation (A5) yield

\[
I_{0n}(K)dK = \frac{128\pi^5 m^2 e^4}{k^2 h^4 \lambda^4} \frac{dK}{K^3} \left| \psi_{n\ell m}(K) \right|^2 \tag{8}
\]

18
where

\[ \epsilon_{n\ell m}(K) = 2^{2\ell+3} n^{\ell+1} (2\ell + 1)^{1/2} (\ell + 1) [n - (\ell - 1)!]^{1/2} [(n + \ell)!]^{-1/2} \]

\[
(Ka_0)^\ell \left[ \frac{(n - 1)^2 + 4\xi^2}{(n + 1)^2 + 4\xi^2} \right]^{(n-\ell-3)/2} \left\{ (n + 1) \left[ (n - 1)^2 + 4\xi^2 \right] C_{n-\ell+1}^{\ell+2}(X) \right.
\]

\[
- 2n \left[ (n - 1)^2 + 4\xi^2 \right]^{1/2} \left[ (n + 1)^2 + 4\xi^2 \right]^{1/2} C_{n-\ell-2}^{\ell+2}(X)
\]

\[
+ (n - 1) \left[ (n + 1)^2 + 4\xi^2 \right] C_{n-\ell-3}^{\ell+2}(X) \right\} \] (9)
REFERENCES


