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RADIAL-FLOW HALL CURRENT
MAGNETOHYDRODYNAMIC GENERATOR

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SUMMARY

A magnetohydrodynamic generator of two-disk geometry operating in the Hall mode is considered. The generator working fluid flows radially between the two disks and a constant magnetic field is applied parallel to the axis of the disks. The conditions where the Mach number is small compared to one and where viscous effects can be neglected are analyzed. The power output, power-output density, and efficiency of this generator are determined and compared to the linear Hall generator. The calculations indicate that the linear generator has higher efficiency and power output for a given working fluid and entrance velocity.

INTRODUCTION

A magnetohydrodynamic (MHD) generator operating in the Hall mode (i.e., the Hall current is the load current) is considered. Generators of this type are particularly well suited for operation with large Hall parameters (ref. 1). A Hall generator with the Faraday electric field equal to zero can be shown to give a large amount of nonequilibrium ionization (ref. 2, eq. (23)). In linear generators the Faraday field is made to vanish by segmenting the electrodes and then shorting opposite pairs of segments. The short circuit for the Faraday current may lead to difficulties (ref. 3) so that a geometry having the Faraday field always equal to zero is of interest. Such a configuration is shown in figure 1 where the working fluid flows radially between the two disks with a constant magnetic field applied parallel to the disk axis. The design is functionally similar to that reported in reference 3.

This report analytically compares the performance of a disk and a linear Hall generator in terms of power density and efficiency. Emphasis is placed upon parametric
EQUATIONS DESCRIBING GENERATOR

The momentum equation for an inviscid fluid carrying a current and flowing in a magnetic field is (ref. 4)

$$\rho \frac{D\vec{u}}{Dt} + \text{grad} \, p = \vec{j} \times \vec{B} \quad (1)$$

(a list of symbols is given in appendix A). If the magnetic Reynolds number is small compared to 1 then the induced magnetic fields can be neglected in comparison to the applied magnetic field strength $\vec{B}$ (ref. 4). For this study $\vec{B}$ will be only in the $z$ direction. (See fig. 1 for coordinate sketch.) The velocity and current density vectors can be written:

$$\vec{u} = v\hat{r} + w\hat{\phi}$$
$$\vec{j} = j_r\hat{r} + j_\phi\hat{\phi}$$

so that equation (1) in component form becomes:

$$\rho \left( v \frac{dv}{dr} - \frac{w^2}{r} \right) + \frac{dp}{dr} = j_\phi B \quad (2)$$

$$\rho v \frac{1}{r} \frac{d}{dr} (rw) = -j_r B \quad (3)$$

For a flow with the Mach number small compared to one the continuity equation for axially symmetric flow requires:

2
\[
\frac{d}{dr} (rv) = 0
\]  \hspace{1cm} (4)

For a plasma with nonzero Hall parameters for ions and electrons and with the current flowing in a plane perpendicular to the magnetic field, Ohm's law becomes (ref. 5, eq. (49)):

\[
\mathbf{j} = \frac{1}{(1 + \beta_i \beta_1)} \left[ \sigma_0 (\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \beta_e \frac{\mathbf{B} \times \mathbf{j}}{B} \right]
\]

This equation can be solved for \( \mathbf{j} \) (ref. 5, eq. 50):

\[
\mathbf{j} = \sigma \left[ (1 + \beta_e \beta_1) (\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \beta_e \frac{\mathbf{B} \times (\mathbf{E} + \mathbf{u} \times \mathbf{B})}{B} \right]
\]

where

\[
\sigma = \frac{\sigma_0}{(1 + \beta_i \beta_1)^2 + \beta_e^2}
\]

For the coordinate system used herein, the equation for \( \mathbf{j} \) can be written in component form as shown:

\[
j_r = \sigma \left[ (1 + \beta_e \beta_1) (E_r + wB) + \beta_e vB \right] \hspace{1cm} (5)
\]

\[
j_\phi = \sigma \left[ (1 + \beta_e \beta_1)(-vB) + \beta_e (E_r + wB) \right] \hspace{1cm} (6)
\]

The requirement that \( \text{div} \ \mathbf{j} = 0 \) is

\[
\frac{d}{dr} (r j_r) = 0
\]

which yields

\[
r j_r = r_1 j_1 \hspace{1cm} (7)
\]

This value for \( r j_r \) may be substituted into equation (5) which, when solved for \( E_r \), yields
Using this value for $E_r$, the azimuthal current $j_\varphi$ may be obtained from equation (6)

$$j_\varphi = \frac{\beta_0 E_r - \sigma_0 v B}{1 + \beta_0^2}$$

Equations (4) and (3) can be solved for $v$ and $w$

$$v = v_1 \frac{r_1}{r}$$

$$w = \frac{j_1 B}{\rho v_1} \left( \frac{r_1^2 - r^2}{2r} \right)$$

Equations (7), (8), (9), (10), and (11) express all components of current, electronic field, and velocity as functions of radius in terms of initial conditions at $r = r_1$.

**POWER-OUTPUT DENSITY**

The equations in the preceding section describe a generator (positive power output), only for a certain range of initial current density, $j_1$. The smallest $j_1$ of interest is the open-circuit value, namely zero. The largest value is the short-circuit current, that is, the current which flows when the load voltage is zero. The load voltage can be expressed as

$$V = \int_{r_1}^{r_2} (-E_r) dr = \frac{r_1 \ln \frac{r_2}{r_1} \beta e v_1 B}{1 + \beta e^2} \left\{ \frac{1}{\sigma v_1 B_0 e} - \frac{j_1 [1 + (1 + \beta e^2)\delta f]}{\rho v_1 B_0 e} \right\}$$

where
It is evident from equation (12) that the short-circuit current is

\[
I_{sc} = \frac{\sigma v_1 B \beta_e}{1 + \delta f(1 + \beta_e \beta_1)}
\]

It is convenient to express the load voltage as a fraction of the open-circuit voltage. Let this fraction be called the load parameter and be denoted by \( K \), so that from equation (12)

\[
K = \frac{V}{V_{oc}} = 1 - \frac{j_1 \left[ 1 + \delta f(1 + \beta_e \beta_1) \right]}{\sigma v_1 B \beta_e}
\]

Thus the range on \( j_1 \) can be specified in terms of the load parameter as

\[
j_1 = \frac{\sigma v_1 B \beta_e (1 - K)}{1 + \delta f(1 + \beta_e \beta_1)} \quad 0 \leq K \leq 1
\]

The power output can be represented as the volume integral of \([\vec{j} \cdot (-\vec{E})]\)

\[
P = \int_{\text{vol}} [\vec{j} \cdot (-\vec{E})] \, d\tau = \int_{r_1}^{r_2} (2\pi h j_r)(-E_r) \, r \, dr
\]

With equations (7), (12), and (14), this power becomes
\[
P = 2\pi hr_1 j_1 V = \frac{\pi hr_1^2 \ln \left(\frac{r_2}{r_1}\right)}{(1 + \beta_e/\beta_1) \left[1 + \delta f(1 + \beta_e/\beta_1)\right]}
\]

Since the volume of the generator is \(\pi h(r_2^2 - r_1^2)\) the power density becomes

\[
\Pi = \frac{P}{\pi h(r_2^2 - r_1^2)} = \frac{\sigma v_1^2 B^2 \beta_e^2 K(1 - K)}{(1 + \beta_e/\beta_1) \left[1 + \delta f(1 + \beta_e/\beta_1)\right](1 + \lambda)}
\]

The maximum value for \(\Pi\) occurs for \(K = 1/2\) for the case of constant conductivity.

The effect of this constant conductivity restriction can be illustrated by comparison with results in reference 5. In figure 5 of that reference, the power density for a generator with a plasma whose properties are calculated with the theory of electron heating is shown to be a maximum for load parameter of about 0.42. This difference must be noted when the generator is operating under conditions such that electron heating is likely to be important.

**EFFICIENCY**

The efficiency of this generator may be defined as the ratio of the output power to the flow work per unit time done by the fluid. Since the only force considered herein is the electromagnetic-body force, the flow work per unit time is \([-\bar{u} \cdot (\bar{j} \times \bar{B})]\) so that the efficiency \(\eta\) becomes

\[
\eta = \frac{\int_{\text{vol}} \bar{j} \cdot (-\bar{E}) \, d\tau}{\int_{\text{vol}} [-\bar{u} \cdot (\bar{j} \times \bar{B})] \, d\tau} = \frac{\eta_0 K(1 - K)}{(\epsilon - \lambda) K^2 + \lambda K + 1}
\]

where

\[
\eta_0 = \frac{\beta_e^2 \left[1 + \delta f(1 + \beta_e/\beta_1)\right]}{(1 + \beta_e/\beta_1)^2 \left[1 + \delta f(1 + \beta_e/\beta_1)\right]^2 + \left[\beta_e \delta f(1 + \beta_e/\beta_1)\right]^2}
\]
\[ \lambda = \frac{1 + 3(1 + \beta e \beta_i) \delta f}{1 + (1 + \beta e \beta_i) \delta f} \eta_0 \]

\[ \epsilon = \frac{\lambda + \eta_0}{2} = \frac{1 + 2(1 + \beta e \beta_i) \delta f}{1 + (1 + \beta e \beta_i) \delta f} \eta_0 \] (17)

Differentiation of equation (16) shows that the efficiency is a maximum when

\[ K = \frac{1}{\sqrt{1 + \epsilon + 1}} \]

and has a value given by

\[ \eta_{\text{max}} = \frac{\eta_0}{\lambda + 2(\sqrt{1 + \epsilon + 1})} \] (18)

**SPECIFICATION OF MAGNETIC FIELD**

Although the parameters appearing in the expressions for the generator are arbitrary, there are some optimum values that can be specified. Because of ion slip there is a magnetic field which maximizes the efficiency and power density (ref. 6, figs. 5, 6, and 7). To examine this, introduce the following dimensionless variables:

\[ \delta_c = \frac{\sigma_0 \gamma_1}{2 \mu_e \mu_1 \rho \nu_1} \] (19a)

\[ \mu = \frac{\mu_e}{\mu_1} \] (19b)

\[ z = \beta e \beta_i = (\mu_e B)(\mu_1 B) \] (19c)

where \( \delta_c \) is a modified interaction parameter; \( \mu \) is the ratio of electron to ion mobility; and \( z \) is an ion slip term. The parameters \( \delta_c \) and \( \mu \) do not depend upon \( B \) since the plasma properties \( \mu_e, \mu_1, \) and \( \sigma_0 \) are assumed constant. Then by substitu-
tion in equation (15) the power density may be written as

\[ \Pi = \frac{2K(1 - K)\rho v^3 \mu \delta_c z^2}{r_1(1 + z)(1 + f) \left[ (1 + z)^2 + (\mu + \delta_c f)z + \delta_c fz^2 \right]} \] (20)

The value of \( z \) for which \( \Pi \) as well as the total power output is a maximum is determined from \( \partial \Pi / \partial z = 0 \)

\[ \delta_c fz(1 + z)(1 - z) + \mu z = (z + 1)^2(z - 2) \] (21)

The power output of the generator (but not the power density) also has a maximum with respect to variation in the radius ratio. This maximum occurs when

\[ \delta_c \left[ (y - 1) - 2f \right] z(z + 1) - \mu z = (1 + z)^2 \] (22)

where

\[ y = \left( \frac{r}{r_1} \right)^2 \]

Equations (21) and (22) determine conditions for which the power output is maximized. Thus, for specified \( \delta_c \) and \( \mu \) (which are parameters determined by the fluid properties, generator inlet size, and generator operating conditions) the optimum magnetic field and outside radius may be obtained from

\[ \delta_c = \frac{(z^2 - 1)}{z \left[ y - 1 - (z + 1)f \right]} \] (23)

\[ \mu = \left[ \frac{(z - 1)(y - 1 - 2f)}{y - 1 - (z + 1)f} - 1 \right] \frac{(z + 1)^2}{z} \] (24)

These functions are shown in figure 2. The resulting maximum power output (divided by the input kinetic-energy rate) is given by
The power-output density for a linear Hall generator is given as (ref. 7, eq. (14.43))

$$\frac{P}{2\pi hr_1 \rho v_1^3} = \frac{\delta \mu z \ln y}{4(1 + z)[1 + (1 + z)\delta f]}$$

and is shown in figure 3. The maximum efficiency, calculated from equation (18), is shown in figure 4. The ranges shown for the dimensionless variables are those that might occur in actual generators.

**COMPARISON WITH LINEAR HALL GENERATOR**

The power-output density for a linear Hall generator is given as (ref. 7, eq. (14.43))
It can be shown from equations (20) and (26) that, since $\delta_0 f \geq 0$ and $\beta_0 \beta_1 \geq 0$, the power density ratio $\Pi/\Pi' \leq 1$. This ratio is plotted in figure 5 using the results of equations (23) and (24). It may be seen that the ratio decreases with increasing values of the slip parameter.

The efficiency of the linear generator is (ref. 7, eq. (14.50))

$$\Pi_0 = \frac{\sigma K_0 (1 - K_0) v_1^2 \rho_0^2 B_0^2}{1 + \beta_0 \beta_1}$$  (26)
Figure 4. - Efficiency as function of modified interaction parameter for various mobility ratios.

The efficiency at this value of $K_\ell$ becomes

$$\eta_{\ell} = \frac{K_\ell (1 - K_\ell)}{(1 + \beta_e \beta_1)^2 + K_\ell}$$

This is equal to the disk-generator efficiency when $f = 0$, and has a maximum when the load parameter $K_\ell$ is

$$K_\ell = \frac{1}{\sqrt{1 + \frac{\beta_e^2}{(1 + \beta_e \beta_1)^2} + 1}}$$

The efficiency at this value of $K_\ell$ becomes

$$\eta_{\ell}^{\text{max}} = \frac{\beta_e^2}{(1 + \beta_e \beta_1)^2}$$
The ratio of efficiencies can be expressed as

\[
\frac{\eta}{\eta_0} = \frac{\eta_0}{\beta_e^2} \left[ \frac{\beta_e^2}{(1 + \beta_e \beta_i)^2} + 2 \left( \frac{\beta_e^2}{(1 + \beta_e \beta_i)^2} + 1 \right) \right] \lambda + 2 \left( \frac{\beta_e^2}{(1 + \beta_e \beta_i)^2} + 1 \right)
\]

and, since from equation (17)

\[
\eta_0 \leq \epsilon \leq \lambda \quad \text{and} \quad \epsilon \leq \frac{\beta_e^2}{(1 + \beta_e \beta_i)^2}
\]

then

\[
\frac{\eta}{\eta_0} \leq 1
\]

This ratio is plotted in figure 6, using the values from equations (23) and (24). The efficiency ratio is seen to be an increasing function of the slip parameter.

CONCLUDING REMARKS

The results of the calculations indicate that the disk generator is quite inferior in performance to the linear generator. For small values of the slip parameter the efficiency is much less and for large values of the slip parameter the power density is much less. However, it must be remembered that this comparison was made on the basis of maximum power output of the disk generator. Nevertheless, it appears that the disk-generator performance is so inferior that the greater benefits of the electron heating (as
indicated in reference 3, (figs. 6 and 10) are inadequate to make its performance competitive with that of the linear generator.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, May 14, 1965.
APPENDIX - SYMBOLS

\( \vec{B} \)  magnetic field strength  
\( \vec{E} \)  electric field strength  
\( f \)  function of radius ratio  
\( h \)  spacing between disks  
\( \vec{j} \)  current density  
\( K \)  load parameter  
\( P \)  power output  
\( p \)  pressure  
\( r \)  radial coordinate  
\( \hat{r}, \hat{\phi}, \hat{z} \)  unit vectors in coordinate directions  
\( \vec{u} \)  velocity  
\( V \)  load voltage  
\( v \)  radial velocity  
\( w \)  azimuthal velocity  
\( y \)  square of radius ratio \((r/r_1)^2\)  
\( z \)  magnetic-field parameter, eq. (19c)  
\( \beta_e, \beta_i \)  electron and ion Hall parameters  
\( \delta, \delta_c \)  interaction parameters, eq. (19a)  
\( \epsilon, \lambda, \eta_0 \)  efficiency parameters, eq. (17)  
\( \eta \)  efficiency  
\( \mu \)  mobility ratios, eq. (19b)  
\( \mu_e, \mu_i \)  electron and ion mobilities  
\( \Pi \)  power densities  
\( \rho \)  density  
\( \sigma, \sigma_0 \)  electrical conductivities  
\( d\tau \)  volume element  

Subscripts:

\( \ell \)  linear generator  
\( \text{oc} \)  open circuit  
\( \text{sc} \)  short circuit  
\( r \)  radial component  
\( 1 \)  inlet condition  
\( \phi \)  azimuthal component
REFERENCES


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