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ELECTRODYNAMICS OF MOVING ANISOTROPIC MEDIA: THE FIRST-ORDER THEORY +

by

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Abstract

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Minkowski's theory of moving media is extended hereby to the anisotropic case. The corresponding Maxwell-Minkowski equations have been derived under the condition that the velocity of the moving medium is small compared to the velocity of light. The theory is applied to derive the characteristic equation for the index of refraction in a drifting magneto-ionic medium. The result verifies the previous one obtained by Epstein. Two different formulations based upon the convection current model and the polarization model are discussed in detail from the point of view of the transform method.

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Introduction:

The foundation of the electrodynamics of moving media was laid by Minkowski[1] and based upon the special theory of relativity. For a moving isotropic medium, the complete treatment is given by Sommerfeld[2]. When the velocity of the moving isotropic medium is small compared to the velocity of light, the resultant wave equations can be greatly simplified[3]. Several techincal problems arising from these wave equations have already been investigated[4, 5, 6, 7].

In this report we shall present an extension of Minkowski's theory to moving anisotropic media, the theory is finally applied to a drifting magneto-ionic medium to determine the normal modes of plane waves which can exist in such a medium. The characteristic equation for the index of refraction so obtained by the first order relativistic transform method is the same as the one previously derived by Epstein[8] using the EHPMv formulation, originally adopted by Unz[9], and the equation of motion of a bound electron in a drifting magneto-ionic medium. Finally, the equivalence between the convection current model and the polarization current model will be discussed from the point of view of the transform method.

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First-order Relativistic Transformation of Field Vectors:

In this section, we shall first review the relativistic transformation of the field vectors defined in Maxwell's equations as first recognized by Einstein and later elaborated by Minkowski. Two inertial systems (x, y, z) and (x', y', z',) are in relative motion. The primed system is assumed to be moving with a velocity $\bar{v}$ with respect to the unprimed system. The time variable in the two systems will be denoted, respectively, by t and t'. Maxwell's equations and the equation of continuity defined in the systems are:

$$\nabla' \times \bar{E}' = -\frac{\partial \bar{B}'}{\partial t'} \tag{1}$$

$$\nabla' \times \bar{H}' = \bar{J}' + \frac{\partial \bar{D}'}{\partial t'} \tag{2}$$

$$\nabla' \cdot \bar{J}' = -\frac{\partial \rho'}{\partial t'} \tag{3}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{4}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \tag{5}$$

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} \tag{6}$$

If the velocity v is small compared to the velocity of light, the first-order relativistic transformation between the two sets of field vectors, resulting from neglecting terms of the order of $(v/c)^2$, is given by[10]

$$\bar{D}' = \bar{D} + \frac{1}{c^2}\, \bar{v} \times \bar{H} \tag{7}$$

$$\bar{E}' = \bar{E} + \bar{v} \times \bar{B} \tag{8}$$

$$\bar{B}' = \bar{B} - \frac{1}{c^2}\, \bar{v} \times \bar{E} \tag{9}$$

$$\bar{H}' = \bar{H} - \bar{v} \times \bar{D} \tag{10}$$

$$\bar{J}' = \bar{J} - \rho \bar{v} \tag{11}$$

$$\rho' = \rho - \frac{1}{c^2} \bar{v} \cdot \bar{J} \tag{12}$$

where

$$c^2 = 1/\mu_o \epsilon_o . \tag{13}$$

If two material field vectors $\bar{P}$ and $\bar{M}$ are used, such that one writes

$$\bar{D} = \epsilon_o \bar{E} + \bar{P} \tag{14}$$

$$\bar{B} = \mu_o (\bar{H} + \bar{M}) \tag{15}$$

and similarly for the primed quantities, then the first-order relativistic transformation of the material field vectors can be deduced from (7 - 10).

They are

$$\bar{P}' = \bar{P} - \frac{1}{c^2}\,\bar{v} \times \bar{M} \tag{16}$$

$$\bar{M}' = \bar{M} + \bar{v} \times \bar{P} \tag{17}$$

<u>The First-order Constitution Relations between the Field Vectors defined in the Unprimed System</u>:

Let us assume that constitutive relations for an <u>anisotropic</u> medium in the primed system are known. In the most general case, they are:

$$\bar{D}' = \bar{\bar{\epsilon}}' \cdot \bar{E}' \tag{18}$$

$$\bar{B}' = \bar{\bar{\mu}}' \cdot \bar{H}' \tag{19}$$

$$\bar{J}' = \bar{\bar{\sigma}}' \cdot \bar{E}', \tag{20}$$

where the sign "=" is used to denote a dyadic. If the medium is dispersive, it is understood that $\bar{\bar{\epsilon}}'$, $\bar{\bar{\mu}}'$, and $\bar{\bar{\sigma}}'$ would be functions of the frequency defined in the primed system, which will be denoted by ω'. By substituting (7 - 11) into (18 - 20), we obtain

$$\bar{D} + \frac{1}{c^2}\,\bar{v} \times \bar{H} = \bar{\bar{\epsilon}}' \cdot (\bar{E} + \bar{v} \times \bar{B}) \tag{21}$$

$$\bar{B} - \frac{1}{c^2}\,\bar{v} \times \bar{E} = \bar{\bar{\mu}}' \cdot (\bar{H} - \bar{v} \times \bar{D}) \tag{22}$$

$$\bar{J} - \rho\,\bar{v} = \bar{\bar{\sigma}}' \cdot (\bar{E} + \bar{v} \times \bar{B}) \tag{23}$$

By eliminating $\bar{D}$ or $\bar{B}$ from (21 - 22) and neglecting terms of the order of $(v/c)^2$, we have

$$\bar{D} = \bar{\bar{\epsilon}}' \cdot \left[\bar{E} + \bar{v} \times (\bar{\bar{\mu}}' \cdot \bar{H})\right] - \frac{1}{c^2}\,\bar{v} \times \bar{H} \tag{24}$$

$$\bar{B} = \bar{\bar{\mu}}' \cdot \left[\bar{H} - \bar{v} \times (\bar{\bar{\epsilon}}' \cdot \bar{E})\right] + \frac{1}{c^2}\,\bar{v} \times \bar{E} \tag{25}$$

$$\bar{J} = \bar{\bar{\sigma}} \cdot \left[\bar{E} + \bar{v} \times (\bar{\bar{\mu}} \cdot \bar{H}) \right] + \rho \bar{v} \tag{26}$$

Since $\nabla \cdot \bar{D} = \rho$, (26) may be written in the form

$$\bar{J} = \bar{\bar{\sigma}}' \cdot \left[\bar{E} + \bar{v} \times (\bar{\bar{\mu}}' \cdot \bar{H}) \right] + \bar{v} \nabla \cdot (\bar{\bar{\epsilon}}' \cdot \bar{E}) \tag{27}$$

Again, terms of the order of $(v/c)^2$ have been neglected in deriving (27). Equations (24), (25), and (27) contain the first-order constitutive relations between the field quantities defined in the unprimed system in terms of the contitutive parameters known or given in the primed system. By substituting these equations into (4 - 5), we obtain the Maxwell-Minkowski wave equations for the field vectors $\bar{E}$ and $\bar{H}$ in a moving anisotropic medium which represent a normal extension of Minkowski's theory of moving media to the anisotropic case.

<u>The Magneto-ionic Theory for Drifting Plasma</u>:

In this section, we shall apply the above formulation to derive the characteristic equation for the index of refraction for a plane wave propagating in a drifting magneto-ionic medium. The convection current model will be treated first. In such a model the plasma is assumed

to consist of free electrons with a drift velocity $\bar{v}$. The primed system is attached to the drifting electrons and the unprimed system is fixed to an observer. According to the well-known theory of stationary magneto-ionic media, the relation between the convection current $\bar{J}$ and a harmonically oscillating electirc field $\bar{E}'$ with frequency ω' can be written in the form

$$\bar{\bar{r}}' \cdot \bar{J}' = \bar{E}' \tag{28}$$

where the dyadic $\bar{\bar{r}}'$ is given by

$$\bar{\bar{r}}' = \frac{j}{X' \omega' \epsilon_0} \left[(1 - j Z') \bar{\bar{I}} + j \bar{\bar{Y}}' \right] \tag{29}$$

The parameters contained in (29) are defined as follows:

$$X' = \frac{N|e|^2}{m\omega'^2 \epsilon_0} = \left(\frac{\omega\rho}{\omega'} \right)^2 ,$$

$\omega\rho$ = plasma frequency,

$$Z' = \frac{\omega_0'}{\omega'} ,$$

ω'_c = collision frequency,

$$\bar{\bar{Y}} = \begin{bmatrix} 0 & -Y'_z & Y'_y \\ Y'_z & 0 & -Y'_x \\ -Y'_y & Y_x & 0 \end{bmatrix}$$

$$\bar{Y}' = Y'_x \hat{x} + Y'_y \hat{y} + Y'_z \hat{z} = \frac{\mu_o |e| \bar{H}_o}{m\omega'}$$

$$= \frac{\bar{\omega}_m}{\omega'},$$

ω_m = gyromagnetic frequency.

The dyadic $\bar{\bar{r}}'$ is clearly the reciprocal of $\bar{\bar{\sigma}}'$ previously introduced in (20), i.e.,

$$\bar{\bar{\sigma}}' \cdot \bar{\bar{r}}' = \bar{\bar{I}} \tag{30}$$

In the convection current model, the medium is considered to be unpolarized and unmagnetized, hence,

$$\bar{\bar{\epsilon}}' = \epsilon_o \bar{\bar{I}} \tag{31}$$

$$\bar{\bar{\mu}}' = \mu_o \bar{\bar{I}} \tag{32}$$

Equations (24), (25), and (27), therefore, reduce to

$$\bar{D} = \epsilon_o \bar{E} \tag{33}$$

$$\bar{B} = \mu_o \bar{H} \tag{34}$$

$$\bar{\bar{r}}' \cdot \left[\bar{J} - \bar{v}\nabla \cdot (\mu_o \bar{E}) \right] = \bar{E} + \mu_o \bar{v} \times \bar{H} \tag{35}$$

On account of (33 - 34), (4 - 5) become, for a harmonically oscillating field,

$$\nabla \times \bar{E} = - j\omega\mu_o \bar{H} \tag{36}$$

$$\nabla \times \bar{H} = \bar{J} + j\omega\epsilon_o \bar{E} \tag{37}$$

Equations (35 - 37) are the three basic equations relating $\bar{E}$, $\bar{H}$, and $\bar{J}$ in a drifting magneto-ionic medium. They have been derived by applying Minkowski's first-order relativistic transform method.

To derive the characteristic equation for the index of refraction, let us assume a harmonically oscillating plane wave to be propagating in the z-direction in the unprimed system so that all the field components have a dependence of $e^{j(\omega t - kz)}$. The wave number k, the index of refraction n, the phase velocity V_p and the angular frequency ω are related as follows:

$$k = \frac{\omega}{V_p} = \frac{n\omega}{c} \tag{38}$$

A a result of the assumption of a plane wave propagating in the z-direction, (36 - 37) can be written as

$$-jk\,\hat{z} \times \bar{E} = -j\omega\mu_o \bar{H} \tag{39}$$

$$-jk\,\hat{z} \times \bar{H} = \bar{J} + j\omega\epsilon_o \bar{E} \tag{40}$$

Eliminating $\bar{H}$ between (39) and (40), we obtain

$$\bar{J} = -j\omega\epsilon_0 \left[(1 - n^2)\, \bar{E} + n^2\, E_z\, \hat{z} \right] \tag{41}$$

Substituting the expressions for $\bar{H}$ and $\bar{J}$ as given by (39) and (41) into (35), we obtain a homogeneous equation for $\bar{E}$

$$-j\omega\, {}_0\bar{\bar{r}}' \cdot \left[(1 - n^2)\, \bar{E} + (n^2\, \hat{z} - n\, \bar{\beta})\, E_z \right] \tag{42}$$

$$= \bar{E} + n\, \bar{\beta} \times (\hat{z} \times \bar{E})$$

where

$$\bar{\beta} = \bar{v}/c\ . \tag{43}$$

To find the characteristic equation for n, the primed quantities contained in $\bar{\bar{r}}'$ must first be converted into explicit functions of n. Because of the Doppler shift, the relationship between ω' and ω, accurate to the order of v/c, is given by

$$\omega' = \omega\left(1 - \frac{k}{\omega} v_z\right) \tag{44}$$

$$= \omega\left(1 - \frac{n}{c} v_z\right)$$

$$= \omega\left(1 - n\beta_z\right)$$

The parameters contained in $\bar{\bar{r}}'$, therefore, can be written as:

$$X' = \left(\frac{\omega_p}{\omega'}\right)^2 = \frac{1}{(1 - n\beta_z)^2}\left(\frac{\omega_p}{\omega}\right)^2 = \frac{X}{(1 - n\beta_z)^2} \tag{45}$$

$$Z' = \frac{\omega_c}{\omega'} = \frac{1}{1 - n\beta_z}\left(\frac{\omega_c}{\omega}\right) = \frac{Z}{1 - nb_z}, \tag{46}$$

$$\bar{Y}' = \frac{\bar{\omega}_m}{\omega'} = \frac{1}{1 - n\beta_z}\left(\frac{\bar{\omega}_m}{\omega}\right) = \frac{\bar{Y}}{1 - n\beta_z}. \tag{47}$$

hence,

$$-j\omega\epsilon_o \bar{\bar{r}}' = \frac{1 - n\beta_z}{X}\left[\left(1 - j\frac{Z}{1 - n\beta_z}\right)\bar{\bar{I}} + j\frac{\bar{\bar{Y}}}{1 - n\beta_z}\right] \tag{48}$$

$$= \frac{1}{X}\left(1 - n\beta_z - jZ\right)\bar{\bar{I}} + j\bar{\bar{Y}}$$

On account of (48), the explicit expression for (42) is given by

$$\left[(1 - n\beta_z - jZ)\bar{\bar{I}} + j\bar{\bar{Y}}\right] \cdot \left[(1 - n^2)\bar{E} + (n^2 \hat{z} - n\bar{\beta})E_z\right] \tag{49}$$

$$= \bar{X}\left[\bar{E} + n\bar{\beta} \times (\hat{z} \times \bar{E})\right]$$

Equation (49) can be written in the form

$$\bar{\bar{A}} \cdot \bar{E} = 0 \tag{50}$$

Where the element of the dyadic $\bar{\bar{A}}$ are found to be

$$A_{xx} = (n^2 - 1)(1 - n\beta_z - jZ) + (1 - n\beta_z)\,X \tag{51}$$

$$A_{xy} = -j(n^2 - 1)\,Y_z$$

$$A_{xz} = (1 - n\beta_z - jZ)n\beta_x - jn\beta_y Y_z - j(1 - n\beta_z)Y_y$$

$$A_{yx} = j(n^2 - 1)Y_z$$

$$A_{yy} = (n^2 - 1)(1 - n\beta_z - jZ) + (1 - n\beta_z)X$$

$$A_{yz} = (1 - n\beta_z - jZ)n\beta_z + jn\beta_x Y_z + j(1 - n\beta_z)Y_x$$

$$A_{zx} = -j(n^2 - 1)Y_y + n\beta_x X$$

$$A_{zy} = j(n^2 - 1)Y_x + n\beta_y X$$

$$A_{zz} = -(1 - n\beta_z)(1 - n\beta_z - jZ) - jn\beta_x Y_y + jn\beta_y Y_x + X$$

Equation (50) is the same as the one previously obtained by Epstein after correcting the original work of Unz[9]. The alternative derivation presented here appears to provide a better view as to how the original Appleton-Hartree matrix was transformed as a result of the motion. It also establishes a criterion that the characteristic equation for n derivable from (50) is only accurate to the order of v/c. This characteristic equation is, of course, obtained by letting the determinent of A be equal to zero.

The Polarization Current Model:

In the book by Ratcliffe[11], the Appleton-Hartree equation for a stationary magneto-ionic medium was derived by considering the medium

to be made of polarized matter or bound electrons. We shall call such a model "the Polarization Current Model" in contrast to the convection current model presented in the previous section. In the polarization current model, one had

$$\bar{J}' = 0 \tag{52}$$

$$\bar{M}' = 0 \tag{53}$$

and

$$\bar{P}' = \epsilon_0 \bar{\bar{\chi}}' \cdot \bar{E}' \tag{54}$$

where, numerically, the dyadic susceptibility $\bar{\bar{\chi}}'$ is related to $\bar{\bar{\sigma}}'$ defined by (30) in the following way

$$j\omega' \epsilon_0 \bar{\bar{\chi}}' = \bar{\bar{\sigma}}' \tag{55}$$

As a result of using $\bar{P}'$, (24 - 25) become

$$\bar{D} = \bar{\bar{\epsilon}}' \cdot (\bar{E} + \mu_0 \bar{v} \times \bar{H}) - \frac{1}{c^2} \bar{v} \times \bar{H} \tag{56}$$

$$\bar{B} = \mu_0 \left[\bar{H} - \bar{v} \times (\bar{\bar{\epsilon}}' \cdot \bar{E}) \right] + \frac{1}{c^2} \bar{v} \times \bar{E} \tag{57}$$

where $$\bar{\bar{\epsilon}}' = \epsilon_0 (\bar{\bar{I}} + \bar{\bar{\chi}}') \tag{58}$$

Two alternative expressions for (56 - 57) are

$$\bar{D} = \epsilon_0 \bar{E} + \epsilon_0 \bar{\bar{\chi}}' \cdot (\bar{E} + \mu_0 \bar{v} \times \bar{H}) \tag{59}$$

$$\bar{B} = \mu_0 \bar{H} - \mu_0 \bar{v} \times (\epsilon_0 \bar{\bar{\chi}}' \bar{E}) \tag{60}$$

If we denote

$$\epsilon_0 \bar{\bar{\chi}}' \cdot (\bar{E} + \mu_0 \bar{v} \times \bar{H}) = \bar{P} \tag{61}$$

then $$\bar{D} = \epsilon_0 \bar{E} + \bar{P} \tag{62}$$

$$\bar{B} = \mu_0 (\bar{H} - \bar{v} \times \bar{P}) \tag{63}$$

Within the order of accuracy of v/c, the apparent difference between (60) and (63) is of no consequence. The basic equations involved in the polarization current model are summarized below:

$$\nabla \times \bar{E} = j\omega\mu_0(\bar{H} - \bar{v} \times \bar{P}) \tag{64}$$

$$\nabla \times \bar{H} = j\omega(\epsilon_0 \bar{E} + \bar{P}) \tag{65}$$

$$\bar{P} = \epsilon_0 \bar{\bar{\chi}}' \cdot (\bar{E} + \mu_0 \bar{v} \times \bar{H}) \tag{66}$$

These equations play a role similar to (35 - 37) in the convection current model. As far as the result is concerned it will be shown that two models do yield the same answer as far as the characteristic equation for n is concerned. Equations (64 - 66) may be transformed into another form by introducing a field vector $\bar{H}_p$ defined by

$$\bar{H}_p = \bar{H} - \bar{v} \times \bar{P} \tag{67}$$

that gives

$$\nabla \times \bar{E} = -j\omega\mu_0 \bar{H}_p \tag{68}$$

$$\nabla \times \bar{H}_p = j\omega(\epsilon_0 \bar{E} + \bar{P}) - \nabla \times (\bar{v} \times \bar{P}) \tag{69}$$

$$\bar{P} = \epsilon_0 \bar{\bar{\chi}}' \cdot (\bar{E} + \mu_0 \bar{v} \times \bar{H}_p) \tag{70}$$

A term of the order of $(v/c)^2$ is discarded in converting (66) into (70).

Equations (68) and (69) are identical in form to the two equations used by Unz[9] who originally investigated that problem based upon the so-called EHPMv formulation. Equation (70) can be transformed to the force equation interpreted by Epstein.[8] Thus, if we write (70) in the form

$$j\omega' \bar{P} = \bar{\bar{\sigma}}' \cdot (\bar{E} + \mu_o \bar{v} \times \bar{H}_p) \tag{71}$$

or

$$j\omega' \bar{\bar{r}}' \cdot \bar{P} = (\bar{E} + \mu_o \bar{v} \times \bar{H}_p) \tag{72}$$

Using $\bar{\bar{r}}'$ given by (29) and substituting it into (72) we may reconstruct the differential equation corresponding to (72). We consider

$$\frac{-1}{\epsilon_o X'} \left[(1 - jZ') \bar{\bar{I}} + j\bar{\bar{Y}}' \right] \cdot \bar{P} = \bar{E} + \mu_o \bar{v} \times \bar{H}_p \tag{73}$$

which is the same as

$$\frac{-m\omega'^2}{N|e|^2} \left[\left(1 - j\frac{\omega_c'}{\omega'}\right) \bar{P} + j\frac{\mu_o |e| \bar{H}_o}{m\omega'} \times \bar{P} \right] = \bar{E} + \mu_o \bar{v} \times \bar{H}_p \tag{74}$$

If we let

$$\bar{P} = - N e \bar{R}_p \tag{75}$$

then, (74) is equivalent to

$$m \frac{\partial^2 \bar{R}_p}{\partial t'^2} + m\omega_c' \frac{\partial \bar{R}_p}{\partial t'} = - e (\bar{E} + \mu_o \frac{\partial \bar{R}_p}{\partial t'} \times \bar{H}_o + \mu_o \bar{v} \times \bar{H}_p) \tag{76}$$

Since t' is defined in the primed system, the partial derivation of a function with respect to t' is the same as the material derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + (\bar{v} \cdot \nabla)$ evaluated in the unprimed system as a result of the first-order Lorentz transformation, i.e., neglecting terms of the order of $(v/c)^2$. Equation (76), therefore, is identical to the force equation considered by Epstein after re-interpreting the work of Unz. As we have mentioned earlier, as far as the characteristic equation for n is concerned both the convection -current model and the polarization model provide the same answer. However, it is obvious from this discussion that there are several delicate concepts involved in the polarization model. It appears that these concepts and their acceptance can best be understood and justified with the aid of Minkowski's theory.

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