TECHNICAL REPORT

TECHNICAL DYNAMICS FOR STABILITY ANALYSIS

VOLUME I - DERIVATION OF EQUATIONS

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Contract No. NAS3-231
This document covers the derivation of a set of linearized dynamic equations for use in missile stability analyses. The equations are presented in matrix form and represent the transfer function from engine command signal to gyro output signals. The derivation includes the effects of propellant sloshing, elastic deformation of the vehicle structure, and the dynamics of gimbaled engines. The effects of fixed thrusting engines and the effects of inertia corrections to the bending data have also been considered. The vehicle structure is taken as a multi-branched beam under the influence of bending deflections, shear deflections, rotary inertia, and axial accelerations.

The results are presented in terms of a set of generalized coordinates representing rigid body motion, bending deflections, engine deflections, and slosh mass deflections. The equations are transformed into normal coordinates and the results are also presented in terms of a set of orthogonal combined modes.
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This document covers the derivation and programming of a set of linearized dynamic equations for use in missile stability analysis. The work presented here is essentially an expansion of the equations and digital computer program described in Reference 1. Since the complete derivation is presented in detail, this document supersedes Reference 1. The revision was undertaken in order to increase the flexibility of the computer program with regard to the types of missiles to be studied and to incorporate changes in the nomenclature and data output found desirable as a result of the use of the previous program. The major changes incorporated are as follows:

(a) Bending data derived for multi-branched beams may be used.
(b) The output, including punched cards for the system matrix, may be obtained in either generalized coordinates or normal coordinates (combined mode representation.)
(c) A plotting option is provided so that either the input or the output modes may be machine plotted.
(d) Inertia corrections can be made at any of twenty locations on the missile independent of slosh tank locations.
(e) Gyro slope data is punched in the system matrix and provisions are made to mix the signals from two rate gyro locations.
(f) Input bending data need not be normalized to total missile mass.
(g) The equations and the program have been arranged to accept a massless engine for use with secondary injection type thrust vector control.
(h) In order to utilize the ability to reduce truncation errors with combined mode representation, the program has been arranged so that higher frequency combined modes may be dropped from the punched output.
(i) Changes may be made in bending data which has been introduced into the program from column binary bending decks.
When studying the effects of variations in damping or performing a limit cycle study using non-linear damping it is necessary to obtain output data for various values of damping. Since damping has no effect on the input or output modes, and is only introduced during the calculations for the punched output, an option is provided to allow a number of output decks to be punched from a single run. This option provides a series of output decks for various values of damping with a minimum of machine time.

An option is also provided to punch out the eigenvector or modal matrix on cards for use in the root analysis program used in limit cycle studies.

This document has been issued in two volumes. Volume I covers the formulation of the dynamic equations and the system matrix. Volume II describes the programming of the equations for solution by 7090 digital computers and contains detailed instructions for the use of the program. Section I of Volume I contains an introduction and a statement of the problem to be solved. In Section II the dynamic equations and system matrix are formulated in generalized coordinates. These equations and system matrix are then transformed into normal coordinates in Section III.

The development of the equations in this document includes all linear terms consistent with the following assumptions:

(a) All physical parameters of the missile such as mass, inertia, and thrust are considered constant. Although these parameters actually vary slowly with time, such variations will have negligible effect on the short time missile response and stability.

(b) Input bending modes are determined for free-free end conditions with engine and sloshing mass rigidity attached and include the effects of bending, shear deflections, rotary inertia and axial acceleration.

(c) Aerodynamic forces normal to the missile axis are assumed to be independent of the local structural bending slope and are considered to vary linearly with the angle of attack developed by the underformed elastic axis of the missile.
(d) Aerodynamic forces along the missile axis are assumed to be independent of local bending and angle of attack.

(e) The dynamic equations are developed for motions in the missile yaw plane, defined as a plane containing the missile velocity vector and a perpendicular to the local vertical. The trim conditions on $\alpha$, $\delta$, and $\theta$ will be zero in this plane. The equations are also applicable to the pitch plane if the trimmed values of these variables may be neglected. The trimmed values for a gravity turn are negligible. These conditions are usually approached with the approximate gravity turns employed in practice.

I - 2 MISSILE STABILITY

At any instant of flight time, a missile in its normal or trimmed condition may be considered a system in dynamic equilibrium. All external forces are steady forces in equilibrium with inertia forces. In accordance with D'Alembert's principle, and the internal kinetic and potential energies are constant. Under these conditions the missile is an $M$th degree of freedom closed dynamic system and when displaced or perturbed from its normal or trimmed condition, the resulting motion with respect to the trimmed condition will be divergent for an unstable missile or will damp out for a stable missile. A stable missile must be stable for the slightest displacements from the equilibrium position and a stability analysis is concerned with the stable and unstable tendencies of the system at the equilibrium position. Therefore, the analysis may be based upon the assumption of small displacements for which the effects of second and higher order terms may be neglected.

In the absence of external driving forces the motion of the system can be represented by $M$ linearized homogeneous second order differential equations in $M$ independent variables. When expressed in LaPlace notation these equations become $M$ homogeneous linear algebraic equations in the $M$ independent variables where the coefficients are linear combinations of the LaPlace operator, $s$, and $s^2$. In matrix form these equations may be written as

$$R \mathbf{c} = 0$$

I-2.1
where the $M \times M$ coefficient matrix $[R]$ is a function of $s$ and is referred to as the system matrix. The column matrix $[Q]$ is a column of the independent generalized coordinates of the system. The system matrix $[R]$ can be written as

$$[R] = \begin{bmatrix} s^2 [A] + s [B] + [C] \end{bmatrix}$$


The system of dynamic equations expressed in Laplace notation I-2.1 represents an eigenvalue problem which has $2M$ eigenvalues or roots found by equating the determinant of $[R]$ to zero. The determinant of $[R]$ is a $2M^{th}$ degree polynomial in $s$ and when equated to zero may be written as the product of $M$ quadratics

$$\prod_{i=1}^{M} (s^2 + A_i s + B_i) = 0$$

The roots may be real or complex. However, all complex roots appear in conjugate pairs and a plot of the roots in the $s$ plane is symmetrical about the real axis. Corresponding to each eigenvalue or root, there is an associated eigenvector having $M$ constant components. These components are the elements of the column matrix $[Q]$ which satisfies equation I-2.1 when the corresponding value of $s$ is substituted in the $[R]$ matrix. The ratios of the components of the associated eigenvector then represents the ratios of the generalized variables present in the motion for that particular root.

The system has $2M$ roots each with an associated eigenvector to represent the relative motion associated with that root. However, as stated previously, the system has $M$ degrees of freedom and, therefore, $M$ natural modes of motion which are orthogonal to each other or completely uncoupled. Each of these orthogonal or uncoupled modes of motion can be represented dynamically by a single degree of freedom system as shown in Figure I-1.

![Figure No. I-1](image)

**Single Degree of Freedom System**
where

\[ z = \text{normal coordinate for mode} \]

\[ m = \text{normalized mass for mode} \]

\[ k = \text{modal spring constant} = \omega^2 \]

\[ \omega = \text{modal frequency} \]

\[ d = \text{modal damping} \]

\[ mz + dz + kz = 0 \]

\[ (ms^2 + ds + k) z = 0 \]

\[ s^2 = \frac{d}{2m} + \sqrt{\frac{d^2}{4m^2} - \frac{k}{m}} \]

\[ \epsilon = \sigma + \lambda \]

then

\[ z = e^{\sigma t} \left[ Ae^{\lambda t} + Be^{-\lambda t} \right] \]

In the case of under-damped modes where the damping ratio is less than unity

\[ \lambda = j\omega. \]

Substituting for \( \lambda \) reduces equation I-2.5 to

\[ z = e^{\sigma t} \left[ a \cos \omega t - b \sin \omega t \right] \]

Therefore, each complex pair of roots produces a single mode of oscillatory motion defined in terms of the generalized variables by the components of the associated eigenvector. It should be noted that the negative frequency represented by \(-j\omega\) results from the mathematical possibility of negative rotation of the amplitude vector generating sinusoidal functions. Since the components of the eigenvectors are complex numbers showing both amplitude and phase relationships between the generalized variables, the eigenvectors corresponding to a pair of conjugate roots will be identical except that the signs of the imaginary parts will be opposite. When the eigenvector is used to study the motion resulting from a pair of complex roots the eigenvector corresponding to positive frequency should be used.
In the case of over-damped modes where the damping ratio is greater than unity \( \lambda \) is real, and equation I-2.5 can be written

\[
z = e^{\sigma t} \left[ (A+B) \cosh \lambda t + (A-B) \sinh \lambda t \right]
\]

Therefore, each over-damped mode represents the motion of two real roots. The relative magnitudes of the generalized variables in the resultant motion can be described by a linear combination of the eigenvectors corresponding to the two roots. The only other type of mode which may occur is a damped mode for which the modal spring \( k \) is zero.

Then

\[
\lambda = 0
\]

and

\[
s = 0
\]

\[
s = 2\sigma
\]

equation I-2.5 reduces to

\[
z = A e^{2\sigma t} + B
\]

Therefore, each mode of this type represents the motion corresponding to two roots one of which is at the origin. The motion corresponding to the root at the origin is simply \( B = \) equal to a constant, and since the motion has been taken as the motion with respect to the nominal or trimmed condition, this constant must be zero. It may be seen that in order for a root to appear at the origin, one of the equations of motion must contain \( s \) in all terms such that \( s \) may be divided out of the equation. Since the motion resulting from these roots is zero, this operation is justified, and the \( M \) equations of motion need not all be second degree in \( s \) in which case the number of roots may be less than \( 2M \). However, the number of orthogonal natural modes of motion will always be \( M \).

Any free or driven motion of the system must be a linear combination of the \( M \) natural modes of motion. In any such combination representing the free motion of the system, the stable natural modes of motion will damp out in accordance with their damping ratios, reducing the motion to a combination of any unstable modes present. Therefore, a stability analysis of the system for any form of excitation reduces to the study of the stability
of each of the $N$ natural modes of motion. A simple stable or unstable type of analysis for an existing missile could be performed by formulating the dynamic equations, obtaining the roots of the system matrix and noting the root locations in the $s$ plane. However, such an analysis does not lend itself to the determination of stability margins, the study of system modifications, the design of new systems, or limit cycle studies. A considerable reduction in computer expense together with a greater insight into the physical aspects of the problem will result if the analysis is performed in a number of steps. In general these steps consist of the following:

(a) The free-free uncoupled bending modes and the rigid tank fluid sloshing parameters are obtained from separate computer programs.

(b) The bending and slosh data together with trajectory and engine data are introduced into a set of homogeneous dynamic equations representing the missile dynamics. These equations in LaPlace notation are expressed as a coefficient matrix where the coefficients are functions of the LaPlace operator, $s$.

(c) The coefficient matrix expressing missile dynamics is then expanded to the system matrix by the addition of the autopilot and guidance parameters.

(d) The system matrix is used to perform a number of different types of stability studies. It may be used to obtain the system roots and associated eigenvectors which in turn are used to indicate the stability of each mode of motion or the amplitudes of limit cycles. The system matrix is also used to obtain the poles and zeros of the open loop transfer function of the system. These data are in turn used to obtain open loop frequency response generally used in bending and rigid body studies or they may be used to obtain closed loop root loci, generally used in fluid slosh studies.
The equations formulated in Volume I of this document and the digital computer program described in Volume II are used in performing the work outlined in step (b). The input data consists of the bending, slosh, trajectory, and engine data, while the output is the missile dynamics coefficient matrix in the form of punched cards.

I - 3 SYSTEM MATRIX

A general block diagram for a missile control system is shown in Figure I-2. As noted in Section I-2, the system has M degrees of freedom and will have M natural modes of motion when $\theta_c$ is zero. Since the missile mass is considered as a distributed mass in the calculation of bending modes there are an infinite number of elastic modes. Likewise, integration over the total fluid to obtain the equivalent slosh parameters produces an infinite number of fluid modes for each fluid tank. It is therefore necessary to limit the number of bending and slosh modes which may be incorporated into the analysis in order that M remain a finite number. The effects of limiting the number of bending and slosh modes considered is discussed in Section II and it is assumed here that M has been reduced to a finite number.

In Section I-2 it was assumed the dynamics of the system was expressed by M second order linear differential equations in M generalized independent variables. Using LaPlace notation, these equations become M second degree linear algebraic equations in the same variables where the coefficients were linear functions of $s$ and $s^2$.

The $M \times M$ square coefficient matrix for these equations was defined as the system matrix. However, the dynamic equations and the system matrix may be expanded by the addition of a number of dependent variables and an equal number of auxiliary equations. Therefore, the system matrix for the M degree of freedom system may be a square matrix of any order greater than M. The total number of roots of the system will remain $2M$ less any roots at the origin, and the system will have M orthogonal natural modes of motion independent of the order of the system matrix.
That portion of the system matrix representing the equations formulated in this document and referred to as the missile dynamics matrix in Section I-2, corresponds to the transfer functions between the gyro angles $\theta_p$ and $\theta_R$ and the engine command signal, $\delta_c$ (Figure I-2). This matrix, which appears in the program output both in printed form and as a punched deck of cards, is not a square matrix. The number of columns will be one greater than the number of rows, therefore, the matrix represents the coefficients of a series of homogeneous equations for which the number of variables is one greater than the number of equations. The addition of an equation relating the engine command signal to the gyro outputs will produce the square system matrix from which the closed loop roots may be determined.

In practice the command signal, $\delta_c$, is related to the gyro outputs by the addition of a number of auxiliary equations and dependent variables. The final equation relates the output variable to the input variable thereby closing the loop producing the square system matrix.

In order to determine the open loop characteristics of the system with the loop opened at point such as F in Figure I-2, the dependent variable $\theta_F^*$ is introduced together with an auxiliary equation relating $\theta_F^*$ to $\theta_F$. This auxiliary equation should always be represented by the last row in the system matrix. The non-square matrix obtained by omitting the last row of the system matrix then represents the transfer function $\frac{\theta_F^*}{\theta_F}$ which is defined as the system transfer function.

The auxiliary equation relating $\theta_F^*$ to $\theta_F$ may be written

\[ a \theta_F^* + b \theta_F = 0 \]  

therefore, the last row of the system matrix will have $a$ in the column representing $\theta_F^*$ and $b$ in the column representing $\theta_F$. Substituting -1 for either $a$ or $b$ and +1 for the other will make $\theta_F^*$ equal to $\theta_F$ and the solution of the system matrix will give the closed loop roots and the corresponding modes of motion for the system.
Letting $b$ equal unity and $a$ equal zero reduces equation I-3.1 to

\[ \Theta_F^* = 0 \]

and \( \Theta_F \) is arbitrary. The resulting motion when the input is zero must be the natural modes of the system, therefore, solutions of the system matrix will give the open loop poles and the corresponding modes of motion.

Since the system also has \( M \) degrees of freedom for the open loop condition there will be \( 2M \) roots and \( M \) natural modes of motion.

Letting $a$ equal unity and $b$ equal zero reduces equation I-3.1 to

\[ \Theta_F^* = 0 \]

and \( \Theta_F^* \) is arbitrary. The solution of the system matrix under these conditions will give the roots and modes of motion required to produce zero output for an arbitrary input. These roots are defined as the open loop zeros of the system.

The system transfer function may be written

\[
\frac{\Theta_F}{\Theta_F^*} = (K_R)_N \frac{(s - z_i)}{(s - p_j)}
\]

where \((K_R)_N\) = nominal gain of system

\[ (s - z_i) = \text{product of} \ (s - z_i) \]

\[ (s - p_j) = \text{product of} \ (s - p_j) \]

\[ z_i = \text{open loop zero} \]

\[ p_j = \text{open loop pole} \]

For the closed loop condition the transfer function is equal to unity and equation I-3.2 becomes

\[
\frac{(s - p_j)}{(s - z_i)} = (K_R)_N
\]

I-3.3
Letting the system gain become a variable, then for the open loop condition 
\( K_R \) is zero and I-3.3 becomes
\[
\prod (s - p_j) = 0
\]
and the open loop poles are the system roots. However, if \( K_R \) becomes
infinite equation I-3.3 becomes
\[
\prod (s - z_i) = 0
\]
and it may be seen that the open loop zeros are also the roots of the closed
loop system when the system gain is infinite. As the system gain is varied
from zero to infinity the closed loop roots will move from the open loop
poles to the open loop zeros passing through the nominal closed loop roots
when the gain is equal to the nominal gain. As the gain approaches infinity
in an \( M^{th} \) degrees of freedom system, the system may have less than \( M \) degrees
of freedom and less than \( M \) natural modes of motion. In this case the roots
must approach zeros which are at infinity. Solution of the system matrix
for the zeros of the system will produce only finite zeros and the difference
between the number of zeros found and \( 2M \) represents the number of zeros at
infinity.
II-1 COORDINATE SYSTEM

It is assumed that at some time, \( t \), the missile in its nominal or trimmed condition flies along an inertial axis, \( H \), such that the missile pitch plane contains the local vertical and the missile yaw plane makes an angle \( \gamma \) with the local vertical, see figure II-1.1. The center of gravity of the missile is displaced a distance, \( Q_H \), along the \( H \) axis measured from a second inertial axis, \( L \), in the yaw plane perpendicular to the \( H \) axis.

![Figure II-1.1](image-url)
It is further assumed that the missile is displaced or perturbed in the yaw plane and acts as a system of mass elements, \( dm \), each of which has relative motion with respect to the total system center of gravity. The displacement in the yaw plane of the total system center of gravity at time, \( t \), is defined by the coordinates \( Q_H \) and \( Q_L \) measured along the \( H \) and \( L \) axes, see figures II-1.2.

The \( R \) and \( P \) axes are non-rotating in inertial space and are translating in the yaw plane with the total system \( c.g. \) at a velocity \( V_{cg} \) making an angle \( \Lambda \) with the \( R \) and \( H \) axes. The \( R \) and \( P \) axes remain parallel to the \( H \) and \( L \) axes. The yaw plane coordinates of the mass element \( dm \) in inertial space are \( H \), \( L \), and \( \Lambda \)

where

\[
\Lambda = \text{rotational displacement of } dm \text{ with respect to } R
\]

\[
H = Q_R + R
\]

\[
L = Q_L + P
\]

The component of the acceleration of gravity in the yaw plane is \( g \cos \gamma \) directed along the negative \( H \) axis.

When the missile is in its nominal or trimmed condition, \( Q_L \), \( \Lambda \),

and \( \Lambda \) are zero, \( R \) and \( P \) are constant, and \( Q_H \) is increasing along the \( H \) axis. The dynamics of the system is such that the magnitudes of these coordinates have oscillatory components when the missile has been displaced or perturbed. The oscillatory components are damped in a stable system and divergent in an unstable system.
The motion of the mass elements, \( dm \), with respect to the system C.G. is defined with respect to a third set of coordinate axes defined in the yaw plane such that the origin is at the center of gravity and the axes are angularly displaced by an angle \( \Theta \) with respect to the R and P axes, see figure II-1.3.

![Diagram](image)

**FIGURE II - 1.3**

The displacements of the element of mass \( dm \) with respect to the \( \bar{I} - \bar{J} \) axes are \( x \), \( y \) and \( \psi \).

\( \bar{\bar{\omega}} \), \( \Theta \), and \( \alpha \) are taken as positive for rotation about the - \( \bar{k} \) axes while \( \Lambda \) and \( \psi \) are taken as positive for rotation about the + \( \bar{k} \) axis. It should be noted that no restrictions have been placed upon the selection of \( \Theta \), which is arbitrary. The only restriction placed upon the \( \bar{I} - \bar{J} \) coordinate system is that the origin be at the total c.g. of the system. This restriction requires that

\[
\int_{\text{TM}} x\,dm = 0 \quad \int_{\text{TM}} y\,dm = 0
\]

where \( \int_{\text{TM}} \) represents integration over the total mass or total missile.
From equation II-1.1

\[ \int \dot{x} \, dm = 0 = \text{total linear momentum along the } \bar{i} \text{ axis resulting from motion with respect to c.g.} \]

\[ \int \dot{y} \, dm = 0 = \text{total linear momentum along the } \bar{j} \text{ axis resulting from motion with respect to c.g.} \]

\[ \int \dot{x} \, dm = 0 = \text{net inertia force in the } \bar{i} \text{ direction resulting from motion with respect to c.g.} \quad \text{II-1.2} \]

\[ \int \dot{y} \, dm = 0 = \text{net inertia force in the } \bar{j} \text{ direction resulting from motion with respect to c.g.} \]

The angular momentum of the mass element \( dm \) about the \( \bar{k} \) axis due to motion with respect to the c.g. is

\[ d \dot{\eta}_{xy} = (\dot{y}x - \dot{x}y + r^2 \ddot{y}) \, dm \]

\[ \text{where } r = \text{radius of gyration of the element of mass, } dm. \]

From the geometry of the system:

\[ R = x \cos \Theta + y \sin \Theta \]

\[ P = y \cos \Theta - x \sin \Theta \]

\[ \Lambda = \dot{y} - \Theta \]

\[ \gamma = \Theta + \alpha \]

\[ H = Q_H + x \cos \Theta + y \sin \Theta \]

\[ L = Q_L + y \cos \Theta - x \sin \Theta \]

\[ \text{II-1.4} \]
Placing the origin of the \( \mathbf{I} - \mathbf{J} \) coordinate system at the c.g. makes the net linear momentum in any direction, resulting from motion with respect to the c.g., equal to zero. Therefore, the sum of the external forces in any direction must equal the rate of change of linear momentum of the total mass located at the total c.g. This effect uncouples the motion of the center of gravity from the motion of the mass elements with respect to the c.g.

Likewise the sum of the external moments about the c.g. equals the time rate of change of angular momentum about the c.g. and it would be desirable to define the angle \( \theta \) such that the rotation of the \( \mathbf{I} - \mathbf{J} \) axes is a function of the external torques only and independent of the motion of the mass elements with respect to the c.g.

Letting \( \mathbf{\dot{m}}_T \) = angular momentum of total mass about c.g.

\[
\mathbf{\dot{m}}_T = (x - \dot{x}) \mathbf{i} + (y - \dot{y}) \mathbf{j} + (z - \dot{z}) \mathbf{k}
\]

substituting from equations II-1.4

\[
\mathbf{\dot{m}}_T = (\dot{x}x - \dot{y}y + r^2\dot{\psi}) \mathbf{i} + \dot{\theta}(x^2 + y^2 + r^2) \mathbf{j}
\]

\[
\mathbf{\dot{m}}_T = \int_{TM} (\dot{x}x - \dot{y}y + r^2\dot{\psi}) \mathbf{i} - \dot{\theta} \int_{TM} (x^2 + y^2 + r^2) \mathbf{j}
\]

The first integral is the angular momentum of the total mass resulting from motion with respect to the c.g. and equals \( \mathbf{\dot{m}}_{xy} \) from equation II-1.3. The second integral is the total mass moment of inertia about the c.g. which is defined as I.

\[
\mathbf{\dot{m}}_T = \mathbf{\dot{m}}_{xy} - I \dot{\theta}
\]

Torque = \( \mathbf{\tau}_T = \frac{d}{dt} (I \dot{\theta}) \)

Equation II-1.6 shows that it is desirable to define \( \theta \) such that

\[
\mathbf{\dot{m}}_{xy} = 0
\]
9 will then always take a value such that

\( \gamma_{xy} = \text{constant} \) \hspace{1cm} \text{II-1.8}

and

\[ \sum (\text{External torques}) = -\frac{d}{dt} (I\dot{\omega}) \] \hspace{1cm} \text{II-1.9}

It will be noted that the rotation of the \( \bar{I} - \bar{J} \) axes, \( \theta \), is still coupled to the motion with respect to the c.g. by the term \( I\dot{\omega} \) which results from the non-linear centrifugal and coriolis forces. However, for first order linearization, \( \frac{d}{dt}(I\dot{\omega}) = I\ddot{\omega} \) and the motions are uncoupled. Furthermore, for a linear oscillatory system \( \gamma_{xy} \) must be a sine function which can only have the constant value of zero.

The displacement of the total system, can be defined by the displacement of the c.g., \( Q_H \) and \( Q_L \), the rotation of the \( \bar{I} - \bar{J} \) coordinate system, \( \theta \), and the displacement of the system with respect to the \( \bar{I} - \bar{J} \) axes. The motion of the system with respect to the \( \bar{I} - \bar{J} \) axes will have \( N \) degrees of freedom and can be described by \( N \) generalized coordinates \( Q_h \). The coordinates \( x, y, \) and \( \psi \) for each element of mass can then be defined in terms of the \( Q_h \)'s and the constraints of the system such that

\[ x = x(Q_h)'s \]
\[ y = y(Q_h)'s \]
\[ \psi = \psi(Q_h)'s \] \hspace{1cm} \text{II-1.10}

where equations II-1.10 need not be linear.

The total system then has \( N + 3 \) degrees of freedom and can be described by the generalized coordinates \( Q_H, Q_L, \theta, \) and \( N \) coordinates \( Q_h \). It has already been indirectly shown that the dynamic equations associated with the \( Q_H, Q_L, \) and \( \theta \) coordinates are

\[ \sum P_H = M_H = \text{sum of external force along } H \text{ axes} \]
\[ \sum P_L = M_L = \text{sum of external forces along } L \text{ axes} \]
\[ -\frac{d}{dt} (I\ddot{\omega}) = \text{sum of external moments about c.g.} \]
It will also be shown that the motion of the system with respect to the $\vec{i} - \vec{j}$ axes can be defined in terms of $N$ normal coordinates $q_n$. The dynamic equations derived in terms of the generalized coordinates $Q_n$ will then be transformed in terms of the normal coordinates $q_n$.

The displacements of the mass element $dm$ in inertial space are given by equation II-1.4 as

$$
H = Q_H + x \cos \Theta + y \sin \Theta \\
V = Q_L + y \cos \Theta - x \sin \Theta \\
\Lambda = \vec{U} - \Theta
$$

The coordinates of the $\vec{i} - \vec{j}$ axes are $Q_H$, $Q_L$, and $\Theta$ which are considered as generalized coordinate and are only a function of time. The coordinates of the mass element $dm$ with respect to the $\vec{i} - \vec{j}$ axes are $x$, $y$, and $\vec{U}$. The motion with respect to the $\vec{i} - \vec{j}$ axes has not been defined in terms of the constraints of the system. Therefore, $x$, $y$, and $\vec{U}$ cannot be considered generalized coordinates and the general dynamic relationships discussed in this section will apply to any system of mass elements.

II-2.1 VELOCITIES

Differentiating equations II-1.4 with respect to time

$$
\dot{H} = \dot{Q}_H + \dot{x} \cos \Theta - x \dot{\Theta} \sin \Theta + y \dot{\Theta} \cos \Theta \\
\dot{V} = \dot{Q}_L + \dot{y} \cos \Theta - y \dot{\Theta} \sin \Theta - \dot{x} \sin \Theta - x \dot{\Theta} \cos \Theta \\
\dot{\Lambda} = \dot{\Theta} + \dot{\Theta}
$$

The total velocity of the mass element, $dm$ with respect to the fixed $H$ and $L$ axes, expressed as components along the rotating $\vec{i} - \vec{j}$ axes is

$$
V = \vec{i} \left[ \dot{H} \cos \Theta - \dot{V} \sin \Theta \right] + \vec{j} \left[ \dot{H} \sin \Theta + \dot{V} \cos \Theta \right]
$$

Substituting for $\dot{H}$ and $\dot{V}$

$$
V = \vec{i} \left[ (\dot{Q}_H \cos \Theta - \dot{Q}_L \sin \Theta) + \dot{x} + y \dot{\Theta} \right] + \vec{j} \left[ (\dot{Q}_H \sin \Theta + \dot{Q}_L \cos \Theta) + \dot{y} - x \dot{\Theta} \right]
$$
since $V_{cg}$ is the velocity of c.g. at time $t$

$$\dot{Q}_H = V_{cg} \cos \alpha = V_{cg} \cos (\theta + \alpha)$$
$$\dot{Q}_L = -V_{cg} \sin \alpha = V_{cg} \sin (\theta + \alpha)$$  

Therefore

$$v = \bar{I} \left[ V_{cg} \cos \alpha + \dot{x} + \dot{\theta} y \right] + J \left[ -V_{cg} \sin \alpha + \dot{y} - \dot{\theta} x \right]$$  

II-2.2 KINETIC ENERGY

The kinetic energy of the mass element $dm$ is

$$d\tau = \frac{1}{2} v^2 \, dm + \frac{1}{2} r^2 (\ddot{r} - \dot{\phi})^2 \, dm$$

Substituting from equations II-2.2 and using II-1.2

$$\tau = \frac{1}{2} (\dot{Q}_H^2 + \dot{Q}_L^2) \int_T \frac{dm}{M} + \frac{1}{2} \dot{\phi}^2 \int_T (x^2 + y^2 + r^2) \, dm + \frac{1}{2} \int_T (x^2 + y^2 + r^2 \ddot{\phi}^2) \, dm$$

$$\dot{\phi} \int_T (\dot{y} - \dot{x} + \dot{\phi}^2) \, dm$$

$$\tau = \frac{1}{2} M (\dot{Q}_H^2 + \dot{Q}_L^2) + \frac{1}{2} I \ddot{\phi}^2 + \tau_{xy} - \dot{\phi} \gamma_{xy}$$

$$\tau = \frac{1}{2} M \dot{v}_{cg}^2 + \frac{1}{2} I \dot{\phi}^2 + \tau_{xy} - \dot{\phi} \gamma_{xy}$$  

where $\tau_{xy}$ is the kinetic energy of the system due to motion with respect to the $\bar{I} - \bar{J}$ axes and is only a function of the $Q_H$'s and their derivatives. By equation II-1.6, $\gamma_{xy}$ is a constant. The generalized momentum associated with each generalized coordinate is then

$$\frac{\dot{\tau}}{\dot{Q}_H} = M \dot{Q}_H$$
$$\frac{\dot{\tau}}{\dot{Q}_L} = M \dot{Q}_L$$
$$\frac{\dot{\tau}}{\dot{\phi}} = I \ddot{\phi} - \gamma_{xy}$$  

II-2.6
\[ \frac{\partial \tau}{\partial Q_i} = \frac{\partial \tau}{\partial Q_n} \]  

The generalized inertia force associated with each generalized coordinate is then

\[ \frac{d}{dt} \left( \frac{\partial \tau}{\partial Q_H} \right) = \alpha Q_H \]

\[ \frac{d}{dt} \left( \frac{\partial \tau}{\partial Q_L} \right) = \alpha Q_L \]

\[ \frac{d}{dt} \left( \frac{\partial \tau}{\partial \theta} \right) = \frac{d}{dt} \left( \frac{\partial \tau}{\partial Q_n} \right) \]

The generalized force associated with each generalized coordinate as a result of non-linearities are

\[ \frac{\partial \gamma}{\partial Q_H} = 0 \]

\[ \frac{\partial \tau}{\partial Q_L} = 0 \]

\[ \frac{\partial \tau}{\partial \theta} = 0 \]

\[ \frac{\partial \tau}{\partial Q_n} = \frac{\partial \tau}{\partial Q_n} \]

For a linear system the last equation of II-2.8 is also zero.

II-2.3 POTENTIAL ENERGY AND DISSIPATION FUNCTION

The potential energy of the system will be divided into two types, the potential energy resulting from the positions of the mass elements in the gravity field, and the internal potential energy stored in elastic deformation of the structure.

\[ dV_g = Eg \cos \gamma \, dm \]
substituting from II-1.4 and using II-1.2
\[ V_g = M g \cos \gamma \]  

The potential energy due to gravity is taken as zero at the origin of the inertial axes L-H and the acceleration of gravity is assumed constant for all mass elements.

It is assumed that the internal potential energy \( V_I \) is a function of the x's and y's corresponding to the positions of the mass elements with respect to the \( \hat{I} - \hat{J} \) axes and is not a function of the generalized coordinates \( Q_H', Q_L', \) and \( \theta \).

\[ V_I = V_I(Q_n \text{'s}) \]  

The rate at which energy is dissipated in the system is defined as \( 2F_D \). If it is assumed that all of the energy dissipated results from relative motion of the mass elements with respect to the \( \hat{I} - \hat{J} \) axes, the dissipation function, \( F_D' \), is then independent of the generalized coordinates \( Q_H', Q_L', \) and \( \theta \).

\[ F_D = F_D(Q_n \text{'s}) \]  

THE LAGRANGE EQUATIONS

The general form of Lagrange's equations of motions are

\[ \frac{d}{dt} \left( \frac{\partial \Gamma}{\partial \dot{Q}} \right) - \frac{\partial \Gamma}{\partial Q} + \frac{\partial V}{\partial Q} + \frac{\partial V}{\partial \dot{Q}} + \frac{\partial F_D}{\partial \dot{Q}} = P_Q \]  

where

\[ P_Q = \gamma \left[ P_H \frac{\partial H}{\partial Q} + P_L \frac{\partial L}{\partial Q} \right] \]

\( P_H \) = Component of external force \( P \) along \( H \)
\( P_L \) = Component of external force \( P \) along \( L \)

The external force \( P \) can be defined by

\[ P = \hat{I} P_x + \hat{J} P_y \]
then

\[ P_H = P_x \cos \theta + P_y \sin \theta \]
\[ P_L = P_y \cos \theta - P_x \sin \theta \]

and

\[ P_\mathbf{Q} = \frac{\nabla}{\nabla} \left[ \left( P_x \cos \theta + P_y \sin \theta \right) \frac{\partial P_H}{\partial \mathbf{Q}} + \left( P_y \cos \theta - P_x \sin \theta \right) \frac{\partial P_L}{\partial \mathbf{Q}} \right] \]

The point of application of \( P \) is \( H_p \) and \( L_p \) defined by equation II-1.4 as

\[ H_p = Q_H + x_p \cos \theta - y_p \sin \theta \]
\[ L_p = Q_L + y_p \cos \theta - x_p \sin \theta \]

then

\[ \frac{\partial H_p}{\partial Q_H} = 1 \]
\[ \frac{\partial L_p}{\partial Q_H} = 0 \]
\[ \frac{\partial H_p}{\partial Q_L} = 0 \]
\[ \frac{\partial L_p}{\partial Q_L} = 1 \]
\[ \frac{\partial H_p}{\partial \theta} = -x_p \sin \theta + y_p \cos \theta \]
\[ \frac{\partial L_p}{\partial \theta} = -y_p \sin \theta - x_p \cos \theta \]
\[ \frac{\partial H_p}{\partial Q_n} = -\cos \theta \frac{\partial P_y}{\partial Q_n} + \sin \theta \frac{\partial P_y}{\partial Q_n} \]
\[ \frac{\partial L_p}{\partial Q_n} = \cos \theta \frac{\partial P_y}{\partial Q_n} - \sin \theta \frac{\partial P_x}{\partial Q_n} \]
\[
\begin{align*}
P_{QH} &= \cos\theta \sum_P P_x + \sin\theta \sum_P P_y \\
P_{QL} &= \cos\theta \sum_P P_y - \sin\theta \sum_P P_x \\
P_\theta &= -\sum_P \left[ P_y x_p - P_x y_p \right] = \text{moment of external forces about c.g.} \\
P_{Qn} &= \sum_P \left[ x \frac{\partial x_p}{\partial q_n} + y \frac{\partial y_p}{\partial q_n} \right]
\end{align*}
\]
From equations II-2.7, II-2.8, II-2.9, II-2.10, II-2.11 and II-2.15 the Lagrange equations of motion for coordinates \(Q_H\), \(Q_L\), \(\theta\), and \(Q_n\) are

\[
\begin{align*}
M_{QH} + Mg \cos\gamma &= \cos\theta \sum_P P_x + \sin\theta \sum_P P_y \\
M_{QL} &= \cos\theta \sum_P P_y - \sin\theta \sum_P P_x \\
\frac{d}{dt} (\dot{Q}_H) &= -\sum_P \left[ P_y x_p - P_x y_p \right] \\
\frac{d}{dt} (\ddot{Q}_H) &= \dddot{Q}_H - \frac{\dot{Q}_H}{M} - g \cos\gamma
\end{align*}
\]
\(Q_H\) and \(Q_L\) are the accelerations of the total center of gravity in the direction of the nominal flight path and normal to the nominal flight path. From equations II-2.16

\[
\dot{Q}_H = \cos\theta \sum_P \frac{P_x}{M} + \sin\theta \sum_P \frac{P_y}{M} - g \cos\gamma
\]
also by differentiating equations II-2.3

\[
\begin{align*}
\dddot{Q}_H &= -v_{cg} (\dot{\theta} + \alpha_0) \sin(\theta + \alpha_0) + \dot{v}_{cg} \cos(\theta + \alpha_0) \\
\dddot{Q}_L &= -v_{cg} (\dot{\theta} + \alpha_0) \cos(\theta + \alpha_0) - \dot{v}_{cg} \sin(\theta + \alpha_0)
\end{align*}
\]
The second equation of II-2.18 is the so-called normal force equation and the first will be referred to as the axial force equation. The equations of motion can now be written:

\[ \ddot{Q}_h = -V_{c_g}(\dot{\theta} + \dot{\alpha}_o) \sin(\dot{\theta} + \dot{\alpha}_o) + V_{c_g}(\dot{\theta} + \dot{\alpha}_o) = \cos^2 \theta \sum \frac{\dot{X}}{p} + \sin \theta \sum \frac{\dot{Y}}{p} - g \cos \gamma \]  

II-2.19

\[ \ddot{Q}_l = -V_{c_g}(\dot{\theta} + \dot{\alpha}_o) \cos(\dot{\theta} + \dot{\alpha}_o) - V_{c_g}(\dot{\theta} + \dot{\alpha}_o) = \sin \theta \sum \frac{\dot{X}}{p} + \cos \theta \sum \frac{\dot{Y}}{p} \]  

II-2.20

\[ \frac{d}{dt} (I \dot{\theta}) = - \sum \left[ P \frac{d}{dt} \frac{X}{p} - P \frac{d}{dt} \frac{Y}{p} \right] \]  

II-2.21

\[ \frac{d}{dt} \left( \frac{\partial \dot{X}}{Q_n} - \frac{\partial \dot{Y}}{Q_n} + \frac{\partial \dot{Y}}{Q_n} + \frac{\partial F_D}{Q_n} = \sum \left[ P_x \frac{\partial X}{Q_n} + P_y \frac{\partial Y}{Q_n} \right] \right) \]  

II-2.22

From the fact that the motion of the center of gravity is independent of the motion with respect to the c.g., and the definition of \( \Theta \), equations II-2.19, II-2.20, and II-2.21 could have been simply derived from Newton's equations. Also equation II-2.22 is Lagrange's equations for the motion of the total mass with respect to the \( \dot{I} - \dot{J} \) axes and is independent of \( Q_h \), \( Q_l \), and \( \Theta \).

The acceleration of the center of mass can also be expressed as components along the \( \dot{I} \) and \( \dot{J} \) axes, \( a_x \) and \( a_y \), where

\[ a_x = Q_h \cos^2 \theta - Q_l \sin \theta \]  

II-2.23

\[ a_y = Q_l \cos \theta + Q_h \sin \theta \]

substituting for \( Q_h \) and \( Q_l \), equations II-2.19 and II-2.20 become

\[ a_x = -V_{c_g}(\dot{\theta} + \dot{\alpha}_o) \sin \theta + V_{c_g}(\dot{\theta} + \dot{\alpha}_o) = \sum \frac{\dot{X}}{p} - \cos \theta \cos \gamma \]  

II-2.24

\[ a_y = -V_{c_g}(\dot{\theta} + \dot{\alpha}_o) \cos \theta - V_{c_g}(\dot{\theta} + \dot{\alpha}_o) = \sum \frac{\dot{Y}}{p} - \sin \theta \cos \gamma \]  

II-2.25

It should be noted that a linear system and small angles have not been assumed. The only restrictions placed on the system are that the origin of the \( \dot{I} - \dot{J} \) axes be at the total c.g. and \( \gamma_{xy} \) = constant.
In Section II-2 general dynamic equations were derived for a system of mass elements having relative motion with respect to a set of rotating body axes, \( \text{i - j} \). The system was assumed to have \( N \) degrees of freedom described by \( N \) generalized coordinates \( Q_n \). In this section the system of mass elements will be defined in terms of a missile configuration and the \( N \) generalized coordinates will be established.

The missile body is assumed to consist of a relative slender flexible body incorporating \( R \) fluid tanks which may be only partially filled. A number of thrusting engines are assumed to be attached to the missile body such that they may or may not be rotated with respect to the body center line. In addition provisions are provided to make inertia corrections at various points on the missile body. These inertia corrections in effect consist of the addition of a quantity of matter which has no mass but has a mass moment of inertia \( \Delta I \).

II-3.1 SLOSH REPRESENTATION

Translational or rotational motion of each partially filled fluid tank will result in relative motion between the fluid and tank, thus producing dynamic forces or the tank. The moving fluid represents an infinite degree of freedom system having an infinite number of modes of motion. It has been shown (references 2, 3, and 4) that equivalent tank loads for each fluid mode will result from a spring mass analogy. The analogy consists of placing a cap or diaphragm at the free fluid surface to confine the fluid and prevent sloshing and attaching a portion of the fluid mass to the structure by a simple spring. Equations for the calculation of the equivalent sloshing mass, natural frequency of the spring mass combination, and the required attach station on the fluid tank are derived in Reference 4 for tanks of arbitrary shape. The equations for tanks with axial symmetry have been programmed for computer computation. A description of this program and instructions for its use are contained in Reference 5. Results obtained from the slosh program show that the slosh masses associated with fluid modes above the first mode are small and in general these modes may be neglected.
For the calculation of the mass moment of inertia of the missile about its center of gravity and the computation of bending modes, it is necessary to consider the moment of inertia of the fluid in the capped tank. The equation for finding the mass moment of inertia about the fluid c.g. for a fluid in a cylindrical tank is derived in Reference 3. These data are presented as a correction factor, \( k_I \), to be applied to the moment of inertia that would result if the fluid were frozen as a solid. This correction factor \( k_I \) has been plotted against tank aspect ratio (length divided by diameter) in Figure II-3.1. The frozen or solid inertia of a fluid mass \( m_f \) in a cylindrical tank of radius \( R_T \) and length \( L_T \) is

\[
I_{\text{solid}} = m_f \left[ \frac{R_T^2}{4} + \frac{L_T^2}{12} \right]
\]

II-3.1

The free fluid inertia \( I_f \) is then

\[
I_f = k_I I_{\text{solid}}
\]

II-3.2

The mass moment of inertia that results when the fluid is considered as a filament concentrated along the missile centerline for a distance \( L \) is given by

\[
I_{\text{FIL}} = m_f \frac{L^2}{12}
\]

II-3.3

When the missile inertia or bending modes are based upon a filament of fluid, the correction factor that must be applied at the fluid c.g. is

\[
\Delta I = I_f - I_{\text{FIL}}
\]

II-3.4

where \( \Delta I \) may be positive or negative.

It should be noted that the slosh mass amplitude \( \rho_j \) used in this analysis is the amplitude of motion of the equivalent slosh mass of the spring mass analogy. \( \rho_j \) does not represent the slosh amplitude of the fluid in the tank. The fluid amplitude is obtained by applying a correction factor to \( \rho_j \). This correction is obtained from the output data of the sloshing program.
II - 3.2 BENDING MODES

The free-free flexible missile body moving in the yaw plane represents an infinite degree of freedom system consisting of three zero frequency or rigid body modes and an infinite number of elastic or bending modes. However, in an analysis of this type only a finite number, \( T \), of the modes can be considered and it is assumed that the elastic deformation is represented by a finite collection of the first \( T \) bending modes in order of increasing modal frequency. Experience has shown that the system being studied is relatively loosely coupled, that is, the resulting system natural frequencies are near the input bending, slosh, and engine frequencies and the resulting modes consist primarily of the corresponding bending, slosh, and engine motions. Therefore, if the \( T^{th} \) bending frequency is well above the highest slosh and engine input frequency, the driving functions for the higher modes above the \( T^{th} \) mode will be small and the truncation errors for all but the highest mode, \( T^{th} \), should be negligible and the above assumption is considered justified. The output data for the \( T^{th} \) mode should never be used in a stability analysis. The input data should always include at least one mode above the highest mode to be studied.

All elastic deflections are taken with respect to the undeformed elastic axis (UEA). The UEA is defined as the body center line in the absence of elastic deformation and is assumed to be a straight line. The center of mass for every beam element for all beam branches must lie on the UEA in the absence of elastic deformation. Bending mode calculations are made assuming that all mass elements are distributed along the center line with radii of gyration, \( r \). The location of a point on the body is defined as \( x_h \) measured forward along the UEA from the total system center of gravity, see Figure II-3.2.

For the purpose of this analysis it is assumed that the bending modes have been calculated for a multi-branched beam taking into account, shear deflection,
rotary inertia, and axial acceleration, and that all engine and slosh masses are rigidly attached to the missile. The translational deflection of a beam element at \( x_i \) perpendicular to the USA is taken as \( u \) and the total slope of the beam center line at \( x_i \) with respect to the USA is \( u' \). The total slope \( u \) is the sum of the slope due to bending deformation and the slope due to shear deformation. The bending slope is defined as \( \psi \) and is the rotation of the beam element at \( x_i \) since shear deformation produces no rotation. The translational deflection in the direction of the USA is neglected. The deflection \( u \) and slopes \( u' \) and \( \psi \) at any point \( x_i \) are defined by the following equations:

\[
\begin{align*}
    u &= \sum_i \phi_i b_i \\
    u' &= \sum_i \phi'_i b_i \\
    \psi &= \sum_i \lambda_i b_i
\end{align*}
\]

where

- \( \phi_i \) = deflection of the \( i^{th} \) bending mode at \( x_i \)
- \( \phi'_i \) = total slope of the \( i^{th} \) bending mode at \( x_i \)
- \( \lambda_i \) = bending slope of the \( i^{th} \) bending mode at \( x_i \)
- \( b_i \) = \( i^{th} \) bending coordinate

**FIGURE II-3.3**

Bending Deflection Due \( i^{th} \) Mode
The bending deflection due to the \( i \)th mode is shown in figure II-3.3. For multi-branched beams \( \phi_i', \phi_i \) and \( \lambda_i \) will have values for each branch present at \( x_h \). The normalized mass for the \( i \)th mode is given by

\[
\int_{TM} (\phi_i'^2 + r^2 \lambda_i^2)dm = M_i \tag{II-3.7}
\]

where the integration is the sum of the integrals carried out over each branch of the beam. It is desirable to normalize each mode such that \( M_i \) is the total missile mass. Since the mode shapes normalized to mass \( M_i \) can be normalized to the mass \( M \) by multiplying \( \phi_i', \lambda_i \) and \( \phi_i \) by \( \sqrt{k_i} \) were

\[
k_i = \frac{M}{M_i} \tag{II-3.8}
\]

it will be assumed that all bending modes are normalized to total missile mass.

For the \( i \)th mode shape shown in Figure II-3.3 the values of \( \phi_i', \phi_i \) and \( \lambda_i \) are functions of \( x_h \) only. \( \phi_i \) is the deflection when the bending coordinate is unity, and is non dimensional. Since \( \phi_i' \) and \( \lambda_i \) are the slopes when the bending coordinate is unity they have the dimension of \( \text{ft} \). The bending coordinate \( b_i \) is then defined at the value of \( x_h \) where \( \phi_i \) is unity for a mode shape normalized to total mass. The \( i \)th bending mode represents the maximum deflections at each point along the beam for sinusoidal motion since \( b_i \) is a sine function of time \( t \) and the modal frequency. Therefore, for the input mode shown in figure II-3.3 the deflection and slopes could be multiplied by \(-1\) and a new input mode slope would be defined which would shift the phase of the final \( b_i \) by \( 180 \) degrees. Either input mode can be used, however, in order to be consistent a positive input mode is defined as that phase of the mode which results in a positive deflection at the aft end of the missile.

Since the input modes are orthogonal

\[
\int_{TM} (\phi_i'\phi_i' + r^2 \lambda_i'\lambda_i')dm = 0 \quad t \neq i \tag{II-3.9}
\]

also since the bending modes produce no translation of the c.g. and no rotation of the USA the linear momentum of the total system and the angular momentum about the c.g. must be zero. Therefore
As previously noted, this analysis assumes that the bending modes have been calculated for a multi-branched beam taking into account, shear deflection, rotary inertia, and axial acceleration, and that all engines and slosh masses are rigidly attached to the beam. If the effects of the beam column loads due to axial acceleration are to be neglected, no changes in the input data are required. If the effects of shear deformation are neglected in the bending modes, the total slope is made equal to the bending slope. If the inertial energy stored in the beam elements due to rotation are to be neglected, the radius of gyration of each beam element becomes zero in all equations. However, the engine input data and the dynamic equations are based upon a mass moment of inertia about the gimbal point, $I_{Ge}$, for each gimballed engine and the rotary inertia of these engines cannot be neglected. The moment of inertia of the engine about its gimbal point is

$$ I_{Ge} = M_{Ge} \int_{Ge} r^2 \, dm. $$

where $M_{Ge}$ = mass of $e^{th}$ engine

$\int_{Ge}$ = distance from gimbal to c.g. of $e^{th}$ engine

Therefore, when using bending data which neglects the effects of rotary inertia, it is necessary to introduce an inertia correction $\Delta I_e$ at each gimballed engine c.g. such that

$$ \Delta I_e = \int_e r^2 \, dm = I_{Ge} - M_{Ge} \int_{Ge} r^2 $$
It is again noted that the engines and slosh masses are assumed to be rigidly attached to the beam when the bending mode are determined. It is important that the engine mass distribution be such that the static moment of the mass about the engine gimbal point be equal to $M_{e} \int_{e}$ used as gimballed engine input data; and in addition each slosh mass must appear in the mass distribution as a concentrated mass at the spring mass attach point. Any other distribution of these masses will introduce errors in the analysis the magnitudes of which are a function of the effect of the distribution on the modal deflections and slopes at the attach points.

II-3.3 ENGINES

Two types of thrusting engines are considered, fixed engines, and gimballed engines. Fixed engines are those engines which are fixed or locked to the missile such that the thrust vector is always tangent to the body center line at the point of attachment. These engines do not produce modes of oscillation but do provide thrust forces which drive other modes and must therefore be treated separately from the gimballed engines each of which does produce an additional mode of oscillation. The $e$th gimballed engine is considered free to rotate about its gimbal point such that the engine center line makes an angle $\delta_{e}$ with the body center line. It should be noted that a gimballed engine which is locked during some portion of flight time by locking the engine actuator or by making the command signal zero, must be considered gimballed at all times of flight since engine angles will result from the flexibility of the actuating mechanism. Although only fixed and gimballed engines are considered in the derivation of the dynamic equations, a third type of engine is of importance. For secondary injection, jet vane or jetavator thrust vector control systems the thrust vector is deflected through an angle but the engine mass remains fixed to the missile center line. This type of engine is essentially a fixed engine with controlled thrust deflections or a massless gimballed engine engine. The engine does not produce a mode of oscillation but is the controlled driving force for all other modes.
No difficulties are encountered when a gimbaled engine mass and inertia of zero are introduced into the equations derived in this section. However, when using the combined mode representation covered in section III, a massless engine produces a mass matrix which is not positive and definite and which will be rejected by the eigenvalue subroutine used. A program subroutine provided to handle this case is discussed in Volume II.

II-3.4 MASS MOMENT OF INERTIA

In the introduction to this section it was noted that provisions are included to make inertia corrections at various points along the missile. These corrections are introduced primarily to compensate for the fluid inertia in the propellant tanks as discussed in section II-3.1. Equation II-3.5 gives the inertia correction that must be applied at the fluid center of gravity when the bending modes are based upon the fluid mass distributed along the missile center line. For the general type of bending data normally used, rotating inertia is taken in account and the fluid inertia correction can best be introduced in the inertia distribution, in which case no inertia corrections are required in the dynamic equations. In the case of bending data neglecting rotating inertia it is desirable to introduce the fluid inertia correction from Equation II-3.5. In addition if the inertia distribution for the non-fluid portion of the structure is known the effect of rotating inertia can be introduced. In any case where rotating inertia is neglected it is necessary to introduce the rotary inertia of all gimbaled engines since the derivation of the dynamic equation assumes this inertia to be present. The inertia corrections are not system parameters which affect the dynamics of the system but should be considered correction factors introduced to correct the bending data. The mass moment of inertia of the system is

\[ I = \int_{TM} (x^2 + y^2 + r^2) \, dm \]  

II-3.11

expressed in terms of the coordinates along the \( i \) - \( j \) axes. In the previous derivation (reference 1) the inertia corrections were added to the value of \( I \) introduced into the equations. However, for the present derivation it is assumed that the value of \( I \) inputed has already been corrected for fluid inertia and the self inertia of the beam elements.
II-3.5 NOTATION

Input data pertaining to the engines, slosh masses, bending modes, and inertia corrections are designated by the following notation:

<table>
<thead>
<tr>
<th>GIMBALED</th>
<th>SLOSHING</th>
<th>BENDING</th>
<th>FIXED</th>
<th>INERTIA</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENGINES</td>
<td>MASSES</td>
<td>MODES</td>
<td>ENGINE</td>
<td>CORRECTIONS</td>
</tr>
</tbody>
</table>

Numbering 1 thru P 1 thru R 1 thru T 1 thru F 1 thru K

General Subscript e j i k

Specific Subscript p r t

Attach point $x_{ne}$ $x_{nj}$ ($x_{ni}=0$) $x_{nf}$ $x_{nk}$

Thrust $T_{ge}$

Deflection $\delta_{e}$ $\rho_{j}$ $b_{i}$

The subscript $h$ used for the attach points indicates that the coordinate is measured along the UEA and not the $i$ axis. The specific subscripts $p$, $r$, $t$ have been introduced since general expressions contain summations of $e$, $j$, and $i$ and must be differentiated with respect to specific engines, slosh masses and bending modes. For example, the general expression for kinetic energy contains the term $\frac{1}{2} \sum_{e=1}^{P} \frac{1}{2} m_{e} \delta_{e}^{2}$. Therefore the general expression for the $p^{th}$ engine equations will contain the term

$$\frac{d}{dt} \left( \frac{1}{2} \delta_{p} \right) \sum_{e=1}^{P} \frac{1}{2} m_{e} \delta_{e}^{2} = m_{p} \delta_{p} = m_{e} \delta_{e}$$

The subscripts $p$, $r$, and $t$ will appear in the $p^{th}$ engine equation, the $r^{th}$ sloshing equation, and the $t^{th}$ bending equation.

The motion of the mass element $dm$ with respect to the $i - j$ axes is the sum of the rigid body motion due to motion of the UEA, and the elastic deflection, due to motion with respect to the UEA. The rigid body motion of the UEA is a function of the engine angles $\delta_{e}$ and the sloshing deflections $\rho_{j}$ and also the bending coordinate $b_{i}$ when inertia corrections are applied to the bending modes. The bending deflections are only a function of $b_{i}$. Therefore, the motion of each element of mass can be defined in terms of the $P$ values of $\delta_{e}$, the $R$ values of $\rho_{j}$ and the $T$ values of $b_{i}$. The total number of generalized coordinates $N$ is then
These \( N \) generalized coordinates, \( \omega_n \), are then defined as follows: the first \( P \) coordinates are the engine deflections, \( \delta_e \) in radians, for the \( P \) gimbaled engines in the order of input, the next \( R \) coordinates are the sloshing mass deflections \( \rho \) in feet, for the \( R \) sloshing masses in the order of input; the last \( T \) coordinates are the input bending mode coordinates, \( \varphi_i \) in feet for the \( T \) input bending modes in the order of input.

\[
\begin{align*}
\{q_n\} &= \begin{bmatrix} \delta_e \\ \rho_1 \\ \rho_2 \\ \vdots \\ \rho_R \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_T \end{bmatrix} \\
&= \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \vdots \\ \delta_P \\ \rho_1 \\ \rho_2 \\ \vdots \\ \rho_R \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_T \end{bmatrix} \\
&= \begin{bmatrix} q_1 \\ \cdots \\ q_P \\ q_{P+1} \\ \cdots \\ q_{P+R} \\ q_{P+R+1} \\ \cdots \\ q_{P+R+T} \end{bmatrix}
\end{align*}
\]

In this analysis the braces \( \{ \} \) will be used as a symbol denoting a column matrix and the subscript \( T \) will be used to denote a transposed matrix, therefore \( \{ \}^T \) is a row matrix composed of the elements of the \( \{ \} \) column matrix.

The resulting motion with respect to the \( I - J \) axes when \( b_i \) is unity and all other generalized coordinates are zero has become known as the \( i \)th input bending mode. Likewise the resulting motion when \( \delta_e \) is unity and all other coordinates are zero is defined as the \( e \)th input engine mode, and the resultant motion when \( \rho_j \) is unity and all other coordinates are zero is defined as the \( j \)th input slosh mode. These so called input modes or non-orthogonal modes are not true modes of the system. However, for a linear system superposition is valid and the output modes are a linear
combination of these modes. These input modes are numbered from 1 thru N in the same order as their corresponding generalized coordinate. The motion of the missile with respect to the \( I - j \) axes can then be defined by \( N \) dynamic equations in \( N \) generalized coordinates. This motion is a linear combination of \( N \) so-called input modes. The subscripts \( m \) and \( n \) will be used to designate the \( m^{th} \) and \( n^{th} \) of these coordinates, equations, and modes. Where possible \( n \) will be used for the coordinates and \( m \) for these equations and input modes.

It will be noted that with the definition of \( x_{ni} = 0 \), there is an attach point associated with each input mode and each generalized coordinate. Therefore, an attach point can be defined for each input mode or coordinate

\[
\begin{align*}
\{x_m\} &= \begin{bmatrix} x_1 \\ \vdots \\ x_P \\ \vdots \\ x_{P+1} \\ \vdots \\ x_{P+R} \\ \vdots \\ x_{P+R+n} \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} x_{n1} \\ \vdots \\ x_{nm} \end{bmatrix} \\
\end{align*}
\]

Likewise for each fixed engine and each inertia correction

\[
\begin{align*}
\{x_{IF}\} &= F \times I = \{x_{nF}\} \\
\{x_{IK}\} &= K \times I = \{x_{nK}\}
\end{align*}
\]

The only input bending data required in the dynamic equations of the system are the deflection and bending slope at each of the attach points. Therefore, the required bending data can be defined by a system of subscripts where the first subscript denotes the attach station and the second subscript the mode or coordinate causing the deflection. For example
\( \lambda_{ei} \) = bending slope at \( e \) th engine attach point due to \( i \) th bending mode.

\( \lambda_{ji} \) = bending slope at \( j \) th slosh attach point due to \( i \) th bending mode.

\( \lambda_{Ffi} \) = bending slope at \( f \) th fixed engine due to \( i \) th bending mode.

\( \lambda_{Iki} \) = bending slope at \( k \) th inertia correction due to \( i \) th bending mode.

The same subscripts are used in connection with the deflection \( \phi \). However, in each case the slopes and deflections must be taken for the correct beam branch at each attach point. By further defining

\( \lambda_{ee}, \lambda_{ei}, \lambda_{ije}, \lambda_{ij}, \lambda_{iie}, \lambda_{iij}, \lambda_{FFe}, \lambda_{IKe}, \lambda_{IkJ}, \lambda_{Fj} \),

and the corresponding deflections all equal to zero, general symbols for the bending slopes and deflections can be defined as

\[ \lambda_{mn} = \text{bending slope at } m \text{ th attach point due to } n \text{ th mode} \]

\[ \lambda_{nm} = \text{bending slope at } n \text{ th attach point due to } m \text{ th mode} \]

\[ \phi_{mn} = \text{deflection at } m \text{ th attach point due to } n \text{ th mode} \]

\[ \phi_{nm} = \text{deflection at } n \text{ th attach point due to } m \text{ th mode} \]

\[ \lambda_{Ffn} = \text{bending slope at } f \text{ th fixed engine attach point due to } n \text{ th mode} \]

\[ \phi_{Ffn} = \text{deflection at } f \text{ th fixed engine attach point due to } n \text{ th mode} \]

\[ \lambda_{Ikn} = \text{bending slope at } k \text{ th inertia correction due to } n \text{ th mode} \]

\[ \phi_{Ikn} = \text{deflections at } k \text{ th inertia correction due to } n \text{ th mode} \]

**II-4 MOTION WITH RESPECT TO \( \tilde{1} - \tilde{j} \) AXES**

In Section II-2 the dynamic equations were derived in terms of the \( x, y \) and \( z \) coordinates of each element of mass with respect to the \( \tilde{i} - \tilde{j} \) axes. In section II-3 a missile configuration was developed for which the motion with respect to the \( \tilde{i} - \tilde{j} \) axes can be described by \( N \) generalized coordinates \( Q_n \). The motion of the missile with respect to the \( \tilde{i} - \tilde{j} \) axes will be developed in this section. The development is based upon first order linearization.
II-4.1 DISPLACEMENT

The displacement of the missile configuration with respect to the 
1 - 2 axes is shown in Figure II-4.1.

The x and y displacements of the origin on the UEA are defined as \( x_0 \) and \( v \) and the rotation of the UEA as \( \omega \). The elastic displacements with respect to the USA are \( u \) and \( \psi \) defined by equations II-3.6. The

coordinates of a point on the beam are

\[
\begin{align*}
\bar{x}_b &= x_0 + x_h \cos \omega - u \sin \omega \\
\bar{y}_b &= v + x_h \sin \omega + u \cos \omega \\
\psi_b &= w + \psi
\end{align*}
\]

Coordinates of the \( j \)th Slosh Mass Attach Point

\[
\begin{align*}
x_j &= x_0 + x_{hj} \cos \omega - u_j \sin \omega - \rho_j \sin (\psi_j + \omega) = x_{bj} - \rho_j \sin (\psi_j + \omega) \\
y_j &= v + x_{hj} \sin \omega + u_j \cos \omega + \rho_j \cos (\psi_j + \omega) = y_{bj} + \rho_j \cos (\psi_j + \omega) \\
\bar{\psi}_j &= v + \psi_j
\end{align*}
\]
Coordinates of the \( k^{th} \) Inertia Correction Attach Point

\[
x_k = x_o + x_hk \cos w - u_k \sin w = x_{bk}
\]

\[
y_k = v + x_hk \sin w + u_k \cos w = y_{bk}
\]

\[
\psi_k = \psi + \psi_{Ik}
\]

Coordinates of a Point on \( e^{th} \) Engine

\[
x_E = x_{BE} + 2z_e \sin \frac{\delta_e}{2} \sin (w + \psi_e + \frac{\delta_e}{2})
\]

\[
y_E = y_{BE} + 2z_e \sin \frac{\delta_e}{2} \cos (w + \psi_e + \frac{\delta_e}{2})
\]

\[
\psi_e = \psi + \psi_e + \delta_e
\]

\( \psi_e \) is the bending rotation of the beam at the gimbal of the \( e^{th} \) engine.

The angle required is the rotation of the rigid engine as a result of elastic deformation of the beam. The use of \( \psi_e \) for this angle assumes that the engine rotation is due to bending deflections only. If the engine actuator supporting structure is such that engine rotation results from shear deflection in addition to bending deflection, a small error will be introduced which may be corrected by modification of the input bending data.

The value of \( \psi_e \) may be determined from the fact that the origin of \( i - j \) axes remains at the center of gravity.
Therefore

\[ \int_{TM} x \, dm = 0 \]

\[ \int_{b} x_b \, dm + \sum_{j} \int_{j} x_j \, dm + \sum_{e} \int_{e} x_e \, dm = 0 \]

substituting from II-4.1, II-4.2 and II-4.3

\[ \int_{e} z_e \, dm = m_{Ge} l_{Ge} \]

\[ m_{Ge} = \text{mass of } e^{th} \text{ engine} \]

\[ l_{Ge} = \text{distance from gimbal to c.g. of } e^{th} \text{ engine} \]

\[ x_0 = \frac{1}{M} \sum_{j} [m_j \rho_j \sin (\psi_j + \omega)] - \frac{1}{M} \sum_{e} 2m_{Ge} l_{Ge} \sin \frac{\delta_e}{2} \sin (\psi_e + \delta_e) \]

The displacement of the origin of the UEA in the direction of the \( \hat{1} \) axes is the sum of second order terms and \( \delta \) is therefore neglected in linearization of the system and

\[ x_0 = 0 \]

The value of \( \nu \) may also be determined from the fact that the origin of \( \hat{1} - \hat{J} \) axes remains at the center of gravity.

\[ \int_{TM} y d\omega = 0 \]

\[ \int_{b} y_b \, dm + \sum_{j} \int_{j} y_j \, dm + \sum_{e} \int_{e} y_e \, dm = 0 \]

again substituting for \( y_b, y_j \) and \( y_e \)

\[ \nu = -\sum_{j} \frac{m_j}{M} \rho_j \cos (\psi_j + \omega) + \sum_{e} \frac{m_{Ge} l_{Ge}}{M} \sin \frac{\delta_e}{2} \cos (\psi_e + \delta_e) \]
assuming small angles

\[
v = -\frac{m}{j} \frac{1}{M} \rho_j + \sum_e \frac{m_{Ge} e}{M} \delta_e\]

Let \( v_j = -\frac{m}{4} \)

\[
v = \frac{m_{Ge} e}{M} \]

\( v_i = 0 \)

then \( v = \sum_e v_\delta_e + \sum_j v_j \rho_j + \sum_i v_i b_i \)

\[
v = \begin{bmatrix} v_e \end{bmatrix}^T \begin{bmatrix} \delta_e \end{bmatrix} + \begin{bmatrix} v_j \end{bmatrix}^T \begin{bmatrix} \rho_j \end{bmatrix} + \begin{bmatrix} v_i \end{bmatrix}^T \begin{bmatrix} b_i \end{bmatrix}
\]

\[
v = \begin{bmatrix} v_e \end{bmatrix}^T \begin{bmatrix} \delta_e \end{bmatrix} + \begin{bmatrix} v_j \end{bmatrix}^T \begin{bmatrix} \rho_j \end{bmatrix} + \begin{bmatrix} v_i \end{bmatrix}^T \begin{bmatrix} b_i \end{bmatrix}
\]

where \( \begin{bmatrix} v_e \end{bmatrix}^T \begin{bmatrix} \delta_e \end{bmatrix} = \begin{bmatrix} v_e \end{bmatrix} \begin{bmatrix} \delta_e \end{bmatrix} \)

\[
\begin{bmatrix} v_i \end{bmatrix}^T \begin{bmatrix} b_i \end{bmatrix}
\]

where \( v_e, v_j, \) and \( v_i \) are constants for each input mode or coordinate.

For the linear system the coordinates given by equations II-4.1 thru II-4.4 become Coordinate of a Point on Beam

\[
\begin{align*}
x_b &= x_h \\
y_b &= v + x_h w + u \\
\psi_b &= w + \psi
\end{align*}
\]
Coordinates of the \( j \)th Slosh Mass Attach Point

\[
x_j = x_{hj}
\]
\[
y_j = v + x_{hj}w + u_j + \rho_j = x_{bj} + \rho_j
\]  
\[
\psi_j = w + \psi_j
\]  

Coordinates of the \( k \)th Inertia Corrections Attach Points

\[
x_k = x_{hk} = x_{bk}
\]
\[
y_k = v + x_{hk}w + u_{ik} = y_{bk}
\]
\[
\psi_k = w + \psi_{ik} = \psi_{bk}
\]

Coordinates of a point on the \( e \)th Engine

\[
x_E = x_{he} - z_e
\]
\[
y_E = v + x_{ew}w + u_E - z_e \delta_e = y_{be} - z_e \delta_e
\]
\[
\psi_e = w + \psi_e + \delta_e
\]

Coordinate of Attach Point of \( f \)th Fixed Engine

\[
x_{Ff} = x_{hf}
\]
\[
y_{Ff} = v + x_{Ff}w + u_{Ff} = y_{df}
\]
\[
\psi_{Ff} = w + \psi_{Ff}
\]

Equations II-4.9 thru II-4.13 show that \( x \) coordinate along the \( \bar{1} \) axis is equal to the \( x_h \) coordinate along the UEA for all input parameters. Therefore the column matrix defined by equation II-3.14 also represents the \( x \) coordinate along \( \bar{1} \) axis for each attach point.

Also since \( x_h \) at any point is a constant and not a function of time, all of the mass elements of the missile have no motion in the direction of the \( \bar{1} \) axis and all motion is in the direction of the \( j \) axis. The value of \( w \) can be found from the fact that the angular momentum about the c.g. due to motion with respect to \( \bar{1} - j \) axes is zero.
\[ M_{xy} = 0 \text{ for linear system} \]

\[ M_{xy} = \int_{TM} (\dot{y}_x - \dot{y}_y) \, dm \quad + \quad \int_{TM} r^2(\dot{w} + \dot{w}_y) \, dm \quad + \quad \sum_{e} \int_{e} r^2 \hat{e} \, dm \quad + \quad \sum_{k} \Delta I_k (\dot{\psi}_k + \dot{w}) \quad (II-4.14) \]

Substituting for \( x, y, \dot{x} \) and \( \dot{y} \)

\[ M_{xy} = \dot{\psi} \left[ \int_{TM} (x^2 + r^2) \, dm \quad + \quad \sum_{k} \Delta \dot{I}_k \right] + \int_{TM} (\dot{u}x + r^2 \dot{\psi}) \, dm \quad + \quad \sum_{j} m_j \ddot{x}_j \]

\[ + \quad \sum_{e} \hat{e} \left[ \int_{e} (z^2 + r^2) \, dm - x_e l Ge Ge \right] + \sum_{k} \Delta I_k \dot{\psi}_k \]

since

\[ \int_{TM} (x^2 + r^2) \, dm \quad + \quad \sum_{k} \Delta \dot{I}_k = I \]

\[ \int_{TM} (\dot{u}x + r^2 \dot{\psi}) \, dm = 0 \]

\[ \int_{e} (z^2 + r^2) \, dm = I_{Ge} = \text{mass moment of inertia of } e^{th} \text{ engine about gimbal} \]

\[ M_{xy} = \dot{\psi} I + \sum_{j} m_j \ddot{x}_j + \sum_{e} \left[ I_{Ge} - x_e l Ge Ge \right] \hat{e} + \sum_{k} \Delta I_k \dot{\psi}_k \]

since \( M_{xy} = 0 \) for a linear system

\[ \dot{\psi} = - \sum_{e} \frac{I_{Ge} - x_e l Ge Ge}{I} \hat{e} - \sum_{j} \frac{m_j x_j}{I} \dot{\psi}_j - \sum_{k} \frac{\Delta I_k}{I} \psi_k \]

\[ \dot{\psi}_k = \sum_{i} \lambda_{iki} \dot{b}_i \]

\[ \ddot{\psi} = \sum_{e} \frac{I_{Ge} - x_e l Ge Ge}{I} \hat{e} - \sum_{j} \frac{m_j x_j}{I} \dot{\psi}_j - \sum_{k} \frac{\dot{\psi}_k}{I} \lambda_{iki} \quad (II-4.15) \]

Define \( \psi_e = - \frac{I_{Ge} - x_e l Ge Ge}{I} \]

\[ \ddot{\psi}_j = - \frac{m_j x_j}{I} \quad (II-4.16) \]

\[ \ddot{\psi}_i = - \sum_{k} \frac{\dot{\psi}_k}{I} \lambda_{iki} \]
then
\[ \dot{v} = \sum_e w_e \delta_e + \sum_j w_j \dot{\rho}_j + \sum_i v_i \dot{b}_i \]
defining \( w = 0 \) for \( \delta_e = \rho_j = b_i = 0 \)

\[ w = \sum_e w_e \delta_e + \sum_j w_j \rho_j + \sum_i w_i b_i \]

\[ v = \begin{bmatrix} v_e \\ v_j \\ v_i \end{bmatrix} = \begin{bmatrix} \delta_e \\ \rho_j \\ b_i \end{bmatrix} \]

where
\[ \{ w \} = N \times 1 \]

column matrix
\[ \begin{bmatrix} w_e \\ w_j \\ w_i \end{bmatrix} \]

where \( w \) is a constant

II-4.2 KINETIC ENERGY WITH RESPECT TO \( \mathbf{i} - \mathbf{j} \) AXES

\[ i_{xy} = i_{xy} \text{ (translation)} + i_{xy} \text{ (rotation)} \]

\[ i_{xy} \text{ (trans)} = \frac{1}{2} \int_b (x_b^2 + y_b^2) \, dm + \frac{1}{2} \sum_j \int_j (x_j^2 + y_j^2) \, dm + \frac{1}{2} \sum_e \int_e (x_e^2 + y_e^2) \, dm \]

\[ i_{xy} \text{ (rot)} = \frac{1}{2} \int_b r^2(\dot{y} + \dot{\omega})^2 \, dm + \frac{1}{2} \sum_j \int_j r^2(\psi_j + \omega)^2 \, dm + \frac{1}{2} \sum_e \int_e r^2(\psi_e + \psi_s)^2 \, dm + \frac{1}{2} \sum_k \Delta I_k (\psi_k + \omega)^2 \]

\[ x_b = 0 \]

\[ \dot{y}_b = \dot{v} + x\dot{w} + \dot{u} \]

\[ \dot{x}_j = 0 \]

\[ \dot{y}_j = \dot{y}_j + \dot{b}_j \]
\[x_E = 0\]
\[y_E = y_{p2} - z_e \delta_e\]
\[x_e = x_e - z_e\]
\[u_E = u_e - \psi_e^2\epsilon_e\]

Substituting

\[2 \tau_{xy} \text{ (trans.)} = \dot{\psi}^2 \mathcal{M} + \dot{\psi}^2 \int_{TM} x^2 \, dm + \int_{TM} \dot{u}^2 \, dm + 2 \dot{\psi} \int_{TM} x \dot{u} \, dm + \sum_j m_j \left( \dot{r}_j^2 + 2 \rho_j [\dot{\psi} + x_j \dot{\psi} + \dot{u}_j] \right)\]
\[+ \sum_e \left[ \dot{\psi}_e \int_{e} \dot{z}_e \, dm - 2 \dot{\psi} \delta_e m_ge \mathcal{L}_e - 2 \dot{\psi} \dot{w}_e \mathcal{L}_e m_e Ge + 2 \dot{\psi} \int_{e} \dot{z}_e \, dm \right]\]
\[2 \tau_{xy} \text{ (rot)} = \int_{TM} \dot{r}^2 \dot{\psi} \, dm + 2 \dot{\psi} \int_{TM} \dot{r}^2 \, dm + \dot{\psi} \int_{TM} \dot{r}^2 \, dm + \sum_e \int_{e} \dot{r}^2 \left[ 2 \dot{\psi}_e + 2 \dot{\psi}_e \dot{\psi}_e \right] \, dm\]
\[+ \sum_k \Delta I_k \left[ \dot{\psi}_k^2 + 2 \dot{\psi}_k \dot{\psi}_k + \dot{\psi}_k^2 \right]\]

Adding

\[\tau_{xy} = -\frac{1}{2} \mathcal{M} \dot{\psi}^2 + \frac{1}{2} \dot{I}_c^2 + \frac{1}{2} \sum_i \mathcal{M}_i \dot{\psi}_i^2 + \frac{1}{2} \sum_j \mathcal{M}_j \rho_j [\dot{\psi} + x_j \dot{\psi} + \dot{u}_j]\]
\[+ \frac{1}{2} \sum_e I_e \dot{\psi}_e^2 + \sum_e \delta_e \left[ \psi_e (I_e - x_e \mathcal{L}_e m_e Ge) + \dot{\psi}_e m_e \mathcal{L}_e Ge u_e - m_e Ge \dot{\psi}_e \right]\]
\[+ \sum_k \Delta I_k \left[ \frac{1}{2} \dot{\psi}_k^2 + \dot{\psi}_k^2 \right]\]

Since

\[\frac{m_j}{M} > \frac{v_j}{v}\]
\[x_j = \frac{I}{M} \frac{v}{v_j}\]
\[I_e = x_e \mathcal{L}_e m_e Ge = - \frac{I}{v}\]
\[\mathcal{L}_e Ge = \frac{M v}{v_e}\]
\[ \frac{\text{d}}{\text{dt}} \left( \frac{\partial^2 \text{I}_XY}{\partial \dot{Q}_m^2} \right) = \frac{\partial \text{I}_X}{\partial \dot{Q}_m} \dot{Q}_m + \frac{\partial \text{I}_Y}{\partial \dot{Q}_m} \dot{Q}_m + \frac{\partial \text{F}_D}{\partial \dot{Q}_m} = P_m \]

From equation II-2.22, dynamic equation associated with each of the generalized coordinates \( Q_m \) is the Lagrange Equation for motion with respect to the \( i - \bar{i} \) axes

\[ \frac{1}{M} \frac{\text{d}}{\text{dt}} \left( \frac{\partial \text{I}_XY}{\partial \dot{Q}_m} \right) + \frac{1}{M} \frac{\partial \text{I}_Y}{\partial \dot{Q}_m} + \frac{1}{M} \frac{\partial \text{F}_D}{\partial \dot{Q}_m} = \frac{P_m}{M} \]

The second term is zero for a linear system therefore the Lagrange equation divided by the total system mass becomes

\[ \frac{1}{M} \frac{\text{d}}{\text{dt}} \left( \frac{\partial \text{I}_XY}{\partial \dot{Q}_m} \right) + \frac{1}{M} \frac{\partial \text{I}_Y}{\partial \dot{Q}_m} + \frac{1}{M} \frac{\partial \text{F}_D}{\partial \dot{Q}_m} = \frac{P_m}{M} \]

Equation II-4.20 represents a series of \( N \) dynamic equations one for each \( Q_m \). In matrix form II-4.20 becomes

\[ \begin{bmatrix} \frac{1}{M} \frac{\text{d}}{\text{dt}} \left( \frac{\partial \text{I}_XY}{\partial \dot{Q}_m} \right) \\ \frac{1}{M} \frac{\partial \text{I}_Y}{\partial \dot{Q}_m} \\ \frac{1}{M} \frac{\partial \text{F}_D}{\partial \dot{Q}_m} \end{bmatrix} = \begin{bmatrix} \frac{P_m}{M} \end{bmatrix} \]

The first column matrix represents the inertia forces associated with the \( Q_m \) coordinates. Where \( Q_m \) equals \( \delta_p \), the inertia force associated with the \( p \)th engine is

\[ \frac{1}{M} \frac{\text{d}}{\text{dt}} \left( \frac{\partial \text{I}_XY}{\partial \delta_p} \right) = \frac{\partial \text{I}_X}{\partial \delta_p} \delta_p - \sum_{i} \delta_i \left[ \frac{\partial \text{I}_Y}{\partial \delta_p} \delta_i + \frac{\partial \text{F}_D}{\partial \delta_p} \delta_i \right] - \sum_{j} \delta_j \left[ \frac{\partial \text{I}_Y}{\partial \delta_p} \delta_j + \frac{\partial \text{F}_D}{\partial \delta_p} \delta_j \right] \]

\[ \frac{1}{M} \frac{\text{d}}{\text{dt}} \left( \frac{\partial \text{I}_XY}{\partial \phi_p} \right) = \frac{\partial \text{I}_X}{\partial \phi_p} \phi_p + \sum_{i} \delta_i \left[ \frac{\partial \text{I}_Y}{\partial \phi_p} \delta_i + \frac{\partial \text{F}_D}{\partial \phi_p} \delta_i \right] \]

\[ \frac{1}{M} \frac{\text{d}}{\text{dt}} \left( \frac{\partial \text{I}_XY}{\partial \lambda_p} \right) = \frac{\partial \text{I}_X}{\partial \lambda_p} \lambda_p - \sum_{i} \delta_i \left[ \frac{\partial \text{I}_Y}{\partial \lambda_p} \delta_i + \frac{\partial \text{F}_D}{\partial \lambda_p} \delta_i \right] \]
The inertia force associated with the $r^{th}$ sloshing coordinate $\phi_r$ is

$$\frac{1}{M} \frac{d}{dt} \left( \frac{\partial^2 \phi_r}{\partial \zeta r} \right) = -v_r \frac{\partial \phi_r}{\partial \zeta r} - \sum_e \delta_e \left[ \frac{I}{M} w_e r + v_e r \right] - \sum_j b_j \left[ \frac{I}{M} w_j r + v_j r \right]$$

$$- \sum_i b_i \left[ \frac{I}{M} w_i r + v_i r \right] - \sum_i b_i \left[ \frac{I}{M} w_i r + v_i r \right]$$

$$\text{II-4.23}$$

The inertia force associated with the $t^{th}$ bending coordinate $\phi_t$ is

$$\frac{1}{M} \frac{d}{dt} \left( \frac{\partial^2 \phi_t}{\partial \zeta t} \right) = \frac{b_t}{M} - \sum_e \delta_e \left[ \frac{I}{M} w_e t + v_e t \right] - \sum_j b_j \left[ \frac{I}{M} w_j t + v_j t \right]$$

$$- \sum_i b_i \left[ \frac{I}{M} w_i t + v_i t \right] + \sum_j \delta_j \left[ \sum_k \frac{\Delta I_k}{M} \sum_{l_k} \sum_{l_t} \phi_r \phi_t \right]$$

$$\text{II-4.24}$$

Equations II-4.22, II-4.23, and II-4.24 can be written as

$$\frac{1}{M} \frac{d}{dt} \left( \frac{\partial^2 \phi_r}{\partial \zeta r} \right) = \left\{ b_p \right\}_T \left\{ \frac{\partial \phi_r}{\partial \zeta r} \right\}_T - \left\{ \frac{I_{Gp}}{M} \frac{\partial \phi_p}{\partial \zeta r} \right\}_T - \left\{ \frac{I_{Gp}}{M} \frac{\partial \phi_p}{\partial \zeta r} \right\}_T$$

$$\text{II-4.22}$$

$$\frac{1}{M} \frac{d}{dt} \left( \frac{\partial^2 \phi_t}{\partial \zeta t} \right) = \left\{ B_r \right\}_T \left\{ \frac{\partial \phi_t}{\partial \zeta t} \right\}_T - \left\{ \frac{I_{Gp}}{M} \frac{\partial \phi_t}{\partial \zeta t} \right\}_T - \left\{ \frac{I_{Gp}}{M} \frac{\partial \phi_t}{\partial \zeta t} \right\}_T$$

$$\text{II-4.23}$$

$$\frac{1}{M} \frac{d}{dt} \left( \frac{\partial^2 \phi_t}{\partial \zeta t} \right) = \left\{ B_t \right\}_T \left\{ \frac{\partial \phi_t}{\partial \zeta t} \right\}_T - \left\{ \frac{I_{Gp}}{M} \frac{\partial \phi_t}{\partial \zeta t} \right\}_T - \left\{ \frac{I_{Gp}}{M} \frac{\partial \phi_t}{\partial \zeta t} \right\}_T$$

$$\text{II-4.24}$$

Defining $B_m$ where $B_m = \frac{I_{Ge}}{M}$, $B_j = -v_j$, $B_t = 1.0$

$C_m$ where $C_m = \frac{I_{Ge}}{M}$, $C_j = 0$, $C_i = 0$
then \[
\frac{1}{M} \frac{d}{dt} \frac{\partial^2 Y}{\partial t^2} = [B] \{q_n\}
\]

and in Laplace notation
\[
\begin{cases}
\frac{1}{M} \frac{d}{dt} \frac{\partial^2 Y}{\partial t^2} = s \{q_n\} \\
\end{cases}
\]

where \([B]\) is an \(N \times N\) square matrix \([B_{mn}]\)

- \(m = \text{row}\)
- \(n = \text{column}\)

\[
B_{mn} = \frac{\partial \mathcal{B}(m,n)}{\partial \mathcal{B}_m} - \left[ \frac{\mathcal{I}_n}{M} v_m v_n + v_m v_n \right] + C_m \lambda_{mn} C_n \lambda_{nm} - v_m \phi_m
\]

\[
- v_n \phi_{mn} + \sum_k \frac{A_{km} I_k}{M} \lambda_{km} \lambda_{kn}
\]

The \(B\) matrix is the dynamic coupling matrix where \(MB_{mn}\) gives the generalized force along \(Q_m\) resulting from unit acceleration of \(Q_n\), which is equal to the generalized force along \(Q_n\) resulting from unit acceleration of \(Q_m\). Therefore,

\[
B_{mn} = B_{nm}
\]

and \([B]\) must be a symmetric matrix.

**II-4.3 INTERNAL POTENTIAL ENERGY**

In considering the power balance of the system, rate of energy input equals rate of energy absorbed, the energy of the engine actuators must be taken into account. When the actuator moment, \(M_{ae}\), is in the same direction as the actuator velocity, \(\dot{\delta}_{ae}\), the work rate or power \(M_{ae} \dot{\delta}\) is positive and the actuator acts as an external force increasing the system energy. However, when the directions are opposite in sign the work rate or power is negative and the actuator acts as a damper decreasing the energy of the system. Since the actuator can introduce energy and absorb energy it also appears as a spring storing potential energy. Although each approach will produce the same dynamic equations, the energy of the actuator will be introduced as potential energy in this analysis.
\[ K_{ae} = \text{Torsional spring constant between e}^{\text{th}} \text{ engine and its actuator in pound-feet per radian.} \]

\[ M_{oe} = \text{Actuator moment of the e}^{\text{th}} \text{ actuator equal to reaction torque from the e}^{\text{th}} \text{ engine.} \]

\[ M_{oe} = K_{ae} (\delta_{ae} - \delta_e) \]

\[ V_{ie1} = \frac{1}{2} M_{oe} (\delta_{ae} - \delta_e) = \frac{1}{2} K_{ae} (\delta_{ae} - \delta_e)^2 \]

For a given engine angle, \( \delta_e \), the magnitude of \( \delta_{ae} \) determines the potential energy stored in the engine spring \( K_{ae} \).

It is also assumed that due to fluid lines etc. there is a torsional spring, \( K_{ne} \), between the e\textsuperscript{th} engine and the missile body. The energy stored in this spring is

\[ V_{ie2} = \frac{1}{2} K_{ne} \delta_e^2 \]

The total internal potential energy stored by the engine springs, due to both engine deflection and actuator deflection is

\[ V_{ie} = \sum_{e} \left[ \frac{1}{2} K_{ae} (\delta_{ae} - \delta_e)^2 + \frac{1}{2} K_{ne} \delta_e^2 \right] \]

The total internal potential energy stored by the engines, slosh springs, and elastic deformation is

\[ V_1 = V_{ie} + \frac{1}{2} \sum_j k_j \rho_j^2 + \frac{1}{2} \sum_i k_i \beta_i^2 \]
but

\[ k_j = m_j \omega_j^2 \]

\[ k_1 = M \omega_1^2 \]

Therefore,

\[ V_I = \frac{1}{2} \sum_e K_{ae} (\delta_{ae} - \delta_e)^2 + \frac{1}{2} \sum_e K_{ne} \delta_e^2 + \frac{1}{2} \sum_j m_j \omega_j^2 \rho_j^2 + \frac{1}{2} M \sum_{\omega_1} b_1^2 \]

II-4.28

The second term of equation II-4.21 then becomes:

For the \( p \)th engine

\[ \frac{1}{M} \frac{\partial V_I}{\partial \rho_p} = \frac{K_{ap} + K_{np}}{M} b_p - \frac{K_{sp}}{M} \delta_{sp} \]

II-4.29

For the \( r \)th slosh mass

\[ \frac{1}{M} \frac{\partial V_I}{\partial \rho_r} = -V_r \omega_r^2 \rho_r \]

II-4.30

For the \( t \)th bending mode

\[ \frac{1}{M} \frac{V_I}{b_t} = \omega_t^2 b_t \]

II-4.31

Defining \( N \times 1 \) column matrices \( \{ A_m \}, \{ k_m \} \) and \( \{ \delta_{en} \} \) such that

\[ \begin{align*}
\{ A_m \} &= \begin{cases}
\{ A_e = \frac{K_{ae} + K_{ne}}{M} \}
\end{cases} \\
\{ A_j \} &= \begin{cases}
\{ A_j = -V_{rj} \omega_r^2 \}
\end{cases} \\
\{ A_1 \} &= \begin{cases}
\{ A_1 = \omega_1^2 \}
\end{cases} \\
\{ k_e \} &= \begin{cases}
\{ k_e = \frac{K_{ae}}{M} \}
\end{cases} \\
\{ k_j \} &= \begin{cases}
\{ k_j = 0 \}
\end{cases} \\
\{ k_1 \} &= \begin{cases}
\{ k_1 = 0 \}
\end{cases}
\end{align*} \]

II-4.32 II-4.33
\[
\begin{bmatrix}
0a1 \\
0a2 \\
\vdots \\
0aP
\end{bmatrix}
\]

then

\[
\frac{1}{M} \frac{\partial^2 Y}{\partial Q_m^2} = [A] \{Q\} - [k] \{\delta_{an}\}
\]

where

\[
A_{mn} = A_{nm} = A_\ell \delta(m,n)
\]

\[
k_{mn} = k_\ell \delta(m,n)
\]

The A matrix is the static coupling matrix where \(A_{mn}\) gives the generalized force along \(Q_m\) resulting from unit deflection of \(Q_n\), which is equal to the generalized force along \(Q_n\) resulting from unit deflection of \(Q_m\). Therefore \([A]\) must be symmetric, however the system is statically uncoupled and

\[
A_{mn} = A_{nm} = 0 \text{ when } n \neq m
\]

hence \([A]\) is a diagonal matrix.

II-4.4 DISSIPATION FUNCTION

The dissipation function \(F_D\) appearing in the third term of equation II-4.21 is defined as one half of the rate at which energy is dissipated in the system. It is assumed that all damping forces result from bending, slosh, and engine velocities such that

\[
F_D = \frac{1}{2} \sum_e D_{Ge} \dot{e}_e^2 + \frac{1}{2} \sum_j D_{sj} \dot{\xi}_j^2 + \frac{1}{2} \sum_i D_{bi} \dot{b}_i^2
\]

where \(D_{Ge}\) = engine damping in pound feet per radian

\(D_{sj}\) = slosh damping in pounds per feet per second

\(D_{bi}\) = bending damping in pound per feet per second.
but
\[ D_{s_j} = 2 m_j \omega_j \zeta_j \]
\[ D_{b_1} = 2M \omega_1 \zeta_1 \]

where \( \zeta_j \) is the damping ratio associated with the \( j \) slosh mass and \( \zeta_1 \) is the damping ratio associated with the \( 1 \) bending mode

then
\[ F_D = \frac{1}{2} \sum_e D_{Ge} \delta_e^2 + \sum_j \omega_j \zeta_j \rho_j + M \omega_1 \zeta_1 \rho_1^2 \]

Since the engine damping is usually given as torque per deflection rate as found from tests, the engine damping will be left as \( D_{Ge} \). It should be noted that \( \omega_j \) is not the frequency of the \( j \)th slosh spring and mass when attached to the missile body but is the frequency when the spring mass is attached to a fixed point. \( \zeta_j \) equal to unity does not represent critical damping for the missile system when the \( j \)th sloshing mode only is present but represents critical damping for the \( j \)th tank when rigidly held fixed. Likewise \( \omega_1 \) is not the frequency of the input bending mode when inertia corrections have been made.

The third term of equation II-4.21 then becomes for \( p \)th engine
\[ \frac{1}{M} \frac{\partial F_D}{\partial \delta_p} = \frac{D_{Op}}{M} \dot{\delta}_p \]

For the \( r \)th slosh mass
\[ \frac{1}{M} \frac{\partial F_D}{\partial \delta_r} = -2\nu \omega_r \zeta_r \dot{\delta}_r \]

For the \( t \)th bending mode
\[ \frac{1}{M} \frac{\partial F_D}{\partial \delta_i} = 2\nu \omega_t \zeta_t \dot{b}_t \]
Defining a $N \times 1$ column matrix \( \{D\}_m \) such that
\[
\begin{bmatrix}
D_e = \frac{D Ge}{M} \\
D_j = -2v_j a_j \zeta_j \\
D_i = 2v_i \zeta_i
\end{bmatrix}
\]
then
\[
\begin{aligned}
& \begin{bmatrix}
\frac{1}{M} & F_D \\
Q_m 
\end{bmatrix} = s \begin{bmatrix}
D
\end{bmatrix} \begin{bmatrix}
Q_n
\end{bmatrix}
\end{aligned}
\]
where
\[
D_{mn} = D_m \delta(m, n)
\]
The \( [D] \) matrix is the damping matrix where \( M_{Dmn} \) gives the generalized force along \( Q_m \) resulting from unit velocity of \( Q_n \) which is equal to the generalized force along \( Q_n \) resulting from unit velocity \( Q_m \). Therefore, \( [B] \) must be symmetric. However, for the coordinate system used \( [D] \) is a diagonal matrix.

From equations II-4.25, II-4.35 and II-4.42 equation 4.21 can be written, in La Place transform notation as
\[
s^2 \begin{bmatrix}
B
\end{bmatrix} \begin{bmatrix}
\xi_m
\end{bmatrix} + \begin{bmatrix}
A
\end{bmatrix} \begin{bmatrix}
\xi_n
\end{bmatrix} + s \begin{bmatrix}
D
\end{bmatrix} \begin{bmatrix}
\zeta_m
\end{bmatrix} + \begin{bmatrix}
K
\end{bmatrix} \begin{bmatrix}
\delta_m
\end{bmatrix} + \begin{bmatrix}
P_m
\end{bmatrix} \begin{bmatrix}
M
\end{bmatrix} \begin{bmatrix}
Q_m
\end{bmatrix}
\]

II-4.5 INPUT FREQUENCIES

The undamped and undriven motion that results when a single generalized coordinate \( Q_m \) is given a magnitude of unity and all other generalized coordinates are made zero, has been defined as an input mode. Since the system is linear the damped driven modes represented by equation II-4.43 will be linear combinations of these input modes.

For studying the frequency shift of the system, it is necessary to know the natural frequencies of these input modes. For a single coordinate undamped and undriven mode, the \( [D] \), \( [K] \) and \( \begin{bmatrix}
P_m \\
M
\end{bmatrix} \) matrices become zero and the \( [B] \) and \( [A] \) matrices are \( 1 \times 1 \) matrices. Equation II-4.43 becomes
where

\[ s^2 \frac{B_{mm}}{A_{mm}} Q_m + \frac{A_{mm}}{B_{mm}} Q_m = 0 \]

\[ \left( s^2 \frac{B_{mm}}{A_{mm}} + \frac{A_{mm}}{B_{mm}} \right) Q_m = 0 \]

then

\[ s^2 = -\frac{A_{mm}}{B_{mm}} \]

\[ \omega_{\text{input}} = \left[ \frac{A_{mm}}{B_{mm}} \right]^{1/2} \]

but

\[ A_{mm} = A_m \]

\[ B_{mm} = B_m - \left[ \frac{I}{M} v_m^2 + \frac{V_m^2}{m} \right] + \sum_k \frac{\Delta I_k}{M} \lambda_{km}^2 \]

\[ \omega_{\text{INPUT}} = \left[ \frac{A_m}{B_m - \frac{I}{M} v_m^2 + \frac{V_m^2}{m} + \sum_k \frac{\Delta I_k}{M} \lambda_{km}^2} \right]^{1/2} \]

The slosh mass, \( m_j \), has an associated sloshing frequency of \( \omega_j \) calculated by the sloshing program. This frequency is equal to the square root \( \sqrt{\frac{1}{m_j}} \) and is the natural frequency of the spring mass system when attached to a stationary body. In the slosh input mode the spring mass is attached to a moving missile body and the input natural frequency as given by equation II-4.44 will be higher than the square root of \( \sqrt{\frac{1}{m_j}} \). Likewise the engine input frequency, calculated from equation II-4.44, will be higher than \( \sqrt{\frac{K_{ae} + K_{ne}}{I_{Ge}}} \). In the case of the input bending modes the input frequency is equal to the input modal frequency unless inertia corrections have been made. The addition of \( \Delta I \)'s will produce higher input bending frequencies.
II-5 EXTERNAL FORCES

The external forces are assumed to result from the thrust of the gimballed engines, \( T_{Ge} \), the thrust of the fixed engines, \( T_{Ff} \), and the aerodynamic forces \( P_a \). The external forces will be expressed by components along the \( \overline{I-J} \) axes in the form

\[
P = I P_x + J P_y
\]

The external forces due to the gimballed engine \( e \) are

\[
P_{xe} = T_{Ge} \cos (w + \psi_e + \delta_e) \approx T_{Ge}
\]

\[
P_{ye} = T_{Ge} \sin (w + \psi_e + \delta_e) \approx T_{Ge} (w + \psi_e + \delta_e)
\]

\[
x_{pe} = x_e
\]

\[
y_{pe} = v + x_e w + u_e
\]

The external forces due to the fixed engine \( f \) are

\[
P_{xf} = T_{Ff} \cos (w + \psi_{Ff}) \approx T_{Ff}
\]

\[
P_{yf} = T_{Ff} \sin (w + \psi_{Ff}) \approx T_{Ff} (w + \psi_{Ff})
\]

\[
x_{pf} = x_{Ff}
\]

\[
y_{pf} = v + x_{Ff} w + u_{Ff}
\]
The aerodynamic forces are obtained from the pressure distribution on the surface of the missile body. The net aerodynamic force on an element of the missile $dx_h$ in length, measured along the UEA, is found by integrating the pressure over the external surface of that element. This net force is resolved into components $dP_N$ normal to the UEA and $dP_D$ along the UEA. These forces are defined by

$$dP_N = \alpha C_{Nx} \, dx \, q \, S$$
$$dP_D = dD + \alpha C_{Ax} \, dx \, q \, S$$

where the axial or drag forces are measured positive aft and

$$\alpha = (\alpha_0 + \omega) = \text{angle of attack of the UEA}$$
$$q = \text{dynamic pressure}$$
$$S = \text{reference area}$$

$C_{Nx}$ = distributed non-dimensional force coefficient
$C_{Ax}$ = distributed non-dimensional force coefficient

$dD$ = drag force independent of angle of attack

Defining

$$N_{\alpha x} = C_{Nx} \, q \, S$$
$$A_{\alpha x} = C_{Ax} \, q \, S$$

then

$$dP_N = \alpha N_{\alpha x} \, dx$$
$$dP_D = dD + \alpha A_{\alpha x} \, dx$$

The distributed normal and axial force coefficients $N_{\alpha x}$ and $A_{\alpha x}$ are functions of $x_h$ and are the force per unit length per radian of angle of attack.
For the purpose of this analysis, it is assumed that axial force is not a function of the angle of attack ($\alpha_x = 0$) and

$$dP_D = dD$$

$$P_D = \int_{TM} dD = D$$

where $D$ is the drag force in pounds and is considered a constant at a given time of flight.

It is further assumed that the normal force distribution is not a function of elastic deformation and is independent of the local bending slope.

The external aero forces due to drag are then

$$P_{xD} = -D \cos w \approx -D$$

$$P_{yD} = D \sin w \approx Dw$$

$$x_{PD} = \xi_p$$

$$y_{PD} = v + \xi_p w$$

where $\xi_p$ is the $x_n$ coordinate of the center of pressure defined below.

The external aero forces due to normal force are then

$$dP_{xN} = -dP_N \sin w \approx 0$$

$$dP_{yN} = dP_N \cos w \approx dP_N$$

$$x_{PN} = x_n$$

$$y_{PN} = v + x_n w + u$$

The total normal force $P_N$ is then

$$P_N = \int_{TM} dP_N = \alpha \int_{TM} N_{\alpha x} dx = \alpha N_{\alpha x}$$
where $N_\alpha$ is the total normal force coefficient

Taking moments of the aero forces about the origin on the UEA

$$\frac{dM_A}{dN} = x_h \alpha P_N$$

$$M_A = \int_{TM} x_h dP_N = \alpha \int_{TM} x_h N_{\alpha h} dx = \frac{1}{N_\alpha} \int_{TM} x_h N_{\alpha h} dx$$

where $\int_{TM}$ is the location of the center of pressure

$$\int_{TM} x_h N_{\alpha h} dx$$

Then for the aerodynamic forces

$$\frac{P_{xa}}{M} = -\frac{D}{M}$$

$$\frac{P_{ya}}{M} = \frac{N_\alpha}{M} \alpha - \frac{Dw}{M}$$

$$P_{xa} x_p = \int_{TM} dP_N x_h - Dw \int_{TM} x_h N_{\alpha h} dx = (N_\alpha \alpha - Dw) \int_{TM} x_h N_{\alpha h} dx$$

$$P_{xa} y_p = -D(v + \int_{TM} x_h N_{\alpha h} dx)$$

II-5.1 FORCING FUNCTION FOR AXIAL ACCELERATION EQUATION

The axial acceleration equation II-2.24 requires the forcing function

$$\frac{1}{M} \sum_{P} P_{xa}$$

$$\frac{1}{M} \sum_{P} P_{xa} = \sum_{e} \frac{T_{Ge}}{M} + \sum_{f} \frac{T_{FF}}{M} - \frac{D}{M}$$

$$\frac{1}{M} \sum_{P} P_{xa} = \frac{T_{T} - D}{M} = \beta_x$$

where $T_{T} = \sum_{e} T_{Ge} + \sum_{f} T_{FF} = \text{total thrust}$
II-5.2 FORCING FUNCTION FOR NORMAL ACCELERATION EQUATION

The forcing function associated with normal acceleration equation II-2.25 is

\[ \frac{1}{M} \sum_P P_y = \frac{T_{ge}}{M} (w + y'e + \delta_e) + \sum \frac{T_{ff}}{M} (w + y'f) + \frac{N}{M} \alpha - \frac{Dw}{M} \]

\[ \frac{1}{M} \sum_P P_y = \frac{N}{M} \alpha + w_b + \sum_i \left[ \frac{T_{ge}}{M} \lambda_{e1} + \frac{T_{ff}}{M} \lambda_{f1} \right] b_i + \sum \frac{T_{ge}}{M} \delta_e \]

Define

\[ \sum \frac{T_{ge}}{M} \lambda_{e1} + \sum \frac{T_{ff}}{M} \lambda_{f1} = \alpha_{11} \]

or

\[ \sum \frac{T_{ge}}{M} \lambda_{en} + \sum \frac{T_{ff}}{M} \lambda_{fn} = \alpha_{ln} \]

\[ \alpha_{ln} \] is a constant for each bending mode and is zero when \( n \) not equal to \( i \).

Then

\[ \frac{1}{M} \sum_P P_y = \frac{N}{M} \alpha + \left( w_b + \alpha_{ln} + \frac{T_{gn}}{M} \right) \begin{bmatrix} q \end{bmatrix} \begin{bmatrix} n \end{bmatrix} \]

II-5.6

\[ \frac{1}{M} \sum_P P_y = \frac{N}{M} \alpha + \left( a_n \right) \begin{bmatrix} q \end{bmatrix} \begin{bmatrix} n \end{bmatrix} \]

II-5.7

where

\[ a_n = \beta \cdot w + \alpha_{ln} + \frac{T_{gn}}{M} \]

II-5.8

where \( \frac{T_{gn}}{M} \) is zero for all modes except engine modes.

II-5.3 FORCING FUNCTION FOR MOMENT EQUATION

The forcing function for the moment equation II-2.21 is the net moment of the external forces about the system center of gravity,

\[ \sum_P [P_y y_p - P_x y_p] \]

\[ \sum_P [P_y y_p - P_x y_p] = \sum_P [P_y x_p] - \sum_P [P_x y_p] \]

II-5.9
The total external moment will be found in two parts, the moment due to \( y \) components and the moment due to \( x \) components of the forces.

\[
\sum P_y x_p = \sum_e x_e (w + \mu_e + B_e) + \sum_i \frac{T_{Ge}}{M} (w + \mu_{Fr}) + (\bar{H} - Dw) \lambda_p
\]

dividing by total mass \( M \)

\[
\frac{1}{M} \sum P_y x_p = \frac{\sum T_{Ge}}{M} x_e + \frac{\sum T_{Fr}}{M} x_f - \frac{\sum D_f}{M} + \sum_i \left[ \left( \frac{\sum T_{Ge}}{M} x_e \lambda_{ei} + \frac{\sum T_{Fr}}{M} x_f \lambda_{Fr1} \right) b_i + \sum \frac{T_{Ge}}{M} x_e b_e \right]
\]

Define \( \beta_T = \text{constant} \)

\[
\frac{\sum T_{Ge}}{M} x_e + \frac{\sum T_{Fr}}{M} x_f - \frac{D_f}{M} = \alpha_{21}
\]

or \( \alpha_{2n} \) is a constant for each bending mode and is zero when \( n \) not equal to \( i \)

then

\[
\frac{1}{M} \sum P_y x_p = \frac{\sum T_{Ge}}{M} x_e + \frac{\sum T_{Fr}}{M} x_f - \frac{D_f}{M} + \left\{ \beta_T w_n + \alpha_{2n} + \frac{T_{Gn}}{M} x_n \right\} T \{ Q_n \}
\]

II-5.11

for the \( x \) components the moment is

\[
\sum P_x x_p = \sum_e x_e (v + x_e w + u_e) + \sum_i \frac{T_{Fr}}{M} (v + x_f w + u_f) - D(v + w)\]

dividing by \( M \)

\[
\frac{1}{M} \sum P_x x_p = \beta x + \beta_T w + \sum_i \left[ \left( \frac{T_{Ge}}{M} \phi_{ei} + \frac{T_{Fr}}{M} \phi_{Fr} \right) b_i \right]
\]
Define
\[ \sum_{e} \frac{T_{Ge}}{M} \phi_{ei} + \sum_{f} \frac{T_{Fr}}{M} \phi_{Ff} = \alpha_{3i} \]

or
\[ \sum_{e} \frac{T_{Ge}}{M} \phi_{en} + \sum_{f} \frac{T_{Fr}}{M} \phi_{Ffn} = \alpha_{3n} \]  \hspace{1cm} \text{II-5.12}

\( \alpha_{3n} \) is a constant for each bending mode and is zero when \( n \) not equal to \( i \).

Therefore,
\[ \frac{1}{M} \sum_{P} P x y_p = \left\{ \beta x_n + \beta_{2n} + \alpha_{3n} \right\} T \left\{ q_n \right\} \]  \hspace{1cm} \text{II-5.13}

From equation II-5.9, II-5.11, and II-5.13 the total external moment is
\[ \sum_{P} P x y_p - P x y_p = N \alpha^p \alpha + M \left\{ \frac{T_{Un}}{M} x_n - \beta x_n + \alpha_{2n} - \alpha_{3n} \right\} T \left\{ q_n \right\} \]

dividing by \( I \)
\[ \frac{1}{I} \sum_{P} \left[ P x y_p - P x y_p \right] = \frac{N \alpha^p I}{I} \alpha + M \left\{ \frac{T_{Un}}{M} x_n - \beta x_n + \alpha_{2n} - \alpha_{3n} \right\} T \left\{ q_n \right\} \]  \hspace{1cm} \text{II-5.14}

\[ \frac{N \alpha^p I}{I} = \mu \alpha \]  \hspace{1cm} \text{II-5.15}

Let \( M \left\{ \frac{T_{Un}}{M} x_n - \beta x_n + \alpha_{2n} - \alpha_{3n} \right\} = v_n \)  \hspace{1cm} \text{II-5.16}

\[ \frac{1}{I} \sum_{P} \left[ P x y_p - P x y_p \right] = \mu \alpha + \left\{ \mu \right\} T \left\{ q_n \right\} \]  \hspace{1cm} \text{II-5.17}

**II-5.4 FORCING FUNCTION FOR \( m \)th MODAL EQUATION**

The forcing function associated with the \( m \)th generalized coordinate is given by equation II-2.22 as
\[ \frac{P_m}{M} \sum_{P} \left[ \frac{P x y_p}{M} \frac{\partial x_p}{\partial q_m} + \frac{P y y_p}{M} \frac{\partial y_p}{\partial q_m} \right] \]
Since \( x_{pe}, x_{pr}, \) and \( x_{pa} \) as given by equations II-5.1, II-5.2, and II-5.4, are all constants

\[
\frac{\gamma x_p}{\theta m} = 0
\]

for all coordinates and all driving forces result from components of force in the y direction. Therefore,

\[
\frac{P_m}{M} = \frac{1}{M} \sum_{P} \frac{\gamma y_p}{\theta m}
\]

\[
\frac{P_m}{M} = \sum_{e} \frac{P y_e}{M} \frac{\partial}{\partial \theta m} (v + x_e v + u_e) + \sum_{f} \frac{P y_f}{M} \frac{\partial}{\partial \theta m} (v + x_f v + u_f)
\]

\[
- \frac{Dv}{M} \frac{\partial}{\partial \theta m} w + \int_{TM} \left( \frac{N}{\alpha x} \frac{\partial}{\partial \theta m} (v + x_n w + u) \right)
\]

\[
\frac{P_m}{M} = \sum_{e} \frac{P y_e}{M} (v_m + x_e v_m + \phi_{em}) + \sum_{f} \frac{P y_f}{M} (v_m + x_f v_m + \phi_{pfm})
\]

\[
- \frac{Dv}{M} (v_m + \int_{p} w_m) + v_m \frac{N}{\alpha} + \int_{TM} \frac{N}{\alpha} \frac{\partial}{\partial \alpha} \frac{N}{\alpha} \frac{\partial}{\partial \alpha} M_{\alpha} \frac{\partial}{\partial \alpha} M_{\alpha}
\]

\[
\frac{P_m}{M} = v_m \sum_{P} \frac{P y}{M} + x_m \sum_{P} \frac{P y}{M} + \frac{\alpha}{M} \int_{TM} \phi_{m} M_{\alpha} \frac{\partial}{\partial \alpha} M_{\alpha} + \sum_{e} \frac{P y_e}{M} \phi_{em} + \sum_{f} \frac{P y_f}{M} \phi_{pfm}
\]

\[
\sum_{P} \frac{P y}{M} \text{ is the sum of the forces in the y direction divided by M and is given by equation II-5.7.} \sum_{P} \frac{P x}{M} \text{ is the external moment due to the y components divided by M and is given by equation II-5.11. Letting the last two terms of equation II-5.18 equal } X_m
\]
Defining \[ N_{\alpha m} = \int_{TM} \phi_m \alpha x \, dx \]

\[ \frac{P_m}{M} = \frac{N}{M} (v_m + I_p v_m) \alpha + \frac{N_{\alpha m}}{M} \alpha + \left\{ \begin{array}{l}
\nu_m \alpha_n + \beta_n w_m \nu_n + w_m^{2n} + \beta_m w_m \nu_n + w_m \frac{T_{Gn}}{M} x_n \end{array} \right\} \{ q_n \} + X_m \]

II-5.19

where

\[ X_m = \sum_{e} \frac{T_{Ge}}{M} \psi_e \phi_{em} + \sum_{f} \frac{T_{Fr}}{M} \psi_f \phi_{Ffm} \]

\[ X_m = \sum_{e} \frac{T_{Ge}}{M} (w + \psi_e + \delta_e) \phi_{em} + \sum_{f} \frac{T_{Fr}}{M} (w + \psi_f) \phi_{Ffm} \]

\[ X_m = w \alpha_m + \sum_{e} \left[ \sum_{i} \frac{T_{Ge}}{M} \lambda_{ei} \phi_{em} + \sum_{f} \frac{T_{Fr}}{M} \lambda_{Ffi} \phi_{Ffm} \right] b_i + \sum_{n} \frac{T_{Gn}}{M} \phi_{nm} n \]

Define

\[ \left[ \sum_{e} \frac{T_{Ge}}{M} \lambda_{em} \phi_{em} + \sum_{f} \frac{T_{Fr}}{M} \lambda_{Ffm} \phi_{Ffm} \right] = \alpha_{4mn} \]

II-5.20

\( \alpha_{4mn} \) is a constant for each combination of \( m \) and \( n \) and is zero unless both \( m \) and \( n \) represent bending modes.

then

\[ X_m = \left\{ w_m \alpha_m + \alpha_{3m} + \frac{T_{Gn}}{M} \phi_{em} \right\} T \{ q_n \} \]

then from II-5.19

\[ \frac{P_m}{M} = \frac{N}{M} (v_m + I_p v_m) \alpha + \frac{N_{\alpha m}}{M} \alpha + \left\{ \begin{array}{l}
\nu_m \alpha_n + \beta_n w_m \nu_n + w_m^{2n} + \beta_m w_m \nu_n + w_m \frac{T_{Gn}}{M} x_n + w_n \alpha \end{array} \right\} \{ q_n \} \]

\[ + \alpha_{4mn} + \frac{T_{Gn}}{M} \phi_{nm} \}

II-5.21

\[ \left\{ \frac{P_m}{M} \right\} = \left\{ J_m \right\} \alpha + \left\{ P \right\} \{ q_n \} \]

II-5.22

where

\[ J_m = \frac{N}{M} (v_m + I_p v_m) + \frac{N_{\alpha m}}{M} \]

II-5.23
II-6  DYNAMIC EQUATIONS IN GENERALIZED COORDINATES

Combining the results of sections II-2, II-4, and II-5, the dynamic equations representing the motion of the displaced or perturbed missile can be written in terms of generalized coordinates.

II-6.1  AXIAL ACCELERATION EQUATION

The axial acceleration along the nominal flight path and along the \( \overline{I} \) axes are given by equations II-2.19 and II-2.24 as

\[
\dot{Q}_H = -V_{cg} (\dot{\phi} + \alpha_o) \sin(\phi + \alpha_o) + \dot{V}_{cg} \cos(\phi + \alpha_o) = \cos \phi \sum \frac{P_x}{M} + \sin \phi \sum \frac{P_y}{M} - g \cos \gamma \\
\dot{a}_x = -V_{cg} (\dot{\phi} + \alpha_o) \sin \alpha_o + \dot{V}_{cg} \cos \alpha_o = \sum \frac{P_x}{M} - \cos \phi \ g \cos \gamma
\]

For small angles and dropping non-linear terms these accelerations become

\[
\dot{Q}_H = \dot{V}_{cg} = \sum \frac{P_x}{M} - g \cos \gamma \\
\dot{a}_x = \dot{V}_{cg} = \sum \frac{P_x}{M} - g \cos \gamma
\]

from equation II-5.5

\[
\dot{Q}_H = \dot{a}_x = \dot{V}_{cg} = \beta_x - g \cos \gamma = \text{a constant} \tag{II-6.1}
\]

Therefore for the linear system under consideration the acceleration along the nominal flight path is equal to the acceleration along the \( \overline{I} \) axis which is a constant.

II-6.2  NORMAL ACCELERATION EQUATION

The normal accelerations perpendicular to the nominal flight path and perpendicular to the \( \overline{I} \) axis are given by equations II-2.20 and II-2.25 as

\[
P_{mn} = V_n \ a_n + B_{mn} \ n + w \ a_2 n + V_n \ \frac{T_{Gn}}{M} \ x + V_n \ \frac{T_{Gm}}{M} \ y
\]

II-5.24
\[ Q_L = -v_{cg} (\dot{\theta} + \dot{\alpha}_o) \cos(\theta + \alpha_o) - v_{cg} \sin(\theta + \alpha_o) = \cos \theta \sum \frac{P}{M} - \sin \theta \sum \frac{P}{M} \]

\[ a_y = -v_{cg} (\dot{\theta} + \dot{\alpha}_o) \cos \alpha_o - v_{cg} \sin \alpha_o = \sum \frac{P}{M} - \theta g \cos \gamma \]

For small angles and dropping non-linear terms, where \( v_{cg} \) is a constant,
these accelerations become

\[ Q_L = -v_{cg} (\dot{\theta} + \dot{\alpha}_o) - v_{cg} (\dot{\theta} + \dot{\alpha}_o) = \sum \frac{P}{M} - \theta \sum \frac{P}{M} \]

\[ a_y = -v_{cg} (\dot{\theta} + \dot{\alpha}_o) - v_{cg} \alpha_o = \sum \frac{P}{M} - \theta g \cos \gamma \]

From \( \text{II-5.5} \) and \( \text{II-5.6} \)

\[ \sum \frac{P}{M} = b_x \]

\[ \sum \frac{P}{M} = \frac{N}{M} \alpha + \{a_y\} T \{q_n\} \]

\[ Q_L = -v_{cg} (\dot{\theta} + \dot{\alpha}_o) - v_{cg} (\dot{\theta} + \dot{\alpha}_o) = \frac{N}{M} \alpha - \theta x \{a_y\} T \{q_n\} \] \( \text{II-6.2} \)

\[ a_y = -v_{cg} (\dot{\theta} + \dot{\alpha}_o) - v_{cg} \alpha_o = \frac{N}{M} \alpha - \theta g \cos \gamma + \{a_y\} T \{q_n\} \] \( \text{II-6.3} \)

These equations show that the accelerations normal to the nominal flight path and normal to the \( \overline{1} \) axis are not equal in the linear system but

\[ a_y = \ddot{q}_L + \theta \dot{v}_{cg} = \ddot{q}_L + a_x \theta \] \( \text{II-6.4} \)

In addition equation \( \text{II-6.2} \) shows that the velocity of the system center of gravity normal to the nominal flight path is given by

\[ \dot{q}_L = -v_{cg} (\dot{\theta} + \dot{\alpha}_o) \] \( \text{II-6.5} \)
By equations II-6.2 and II-6.3 the normal accelerations can be expressed in terms of either \( \alpha_o \) or \( \alpha \) and the remaining coordinates, where \( \alpha \) is the angle of attack of the USA and \( \alpha_o \) is the angle of attack of the \( \overline{T} \) axis, therefore

\[
\alpha = \alpha_o + \omega = \alpha_o + \left\{ \frac{v}{n} \right\}_T \left\{ q \right\}_n
\]

II-6.6

Since \( \alpha \) is the more convenient variable, \( \alpha_o \) is replaced by \((\alpha-\omega)\) in equations II-6.2 and II-6.3. In addition \( \dot{v}_{cg} \) is replaced by the constant \( a_x = \beta_x - g \cos \gamma \). Then in terms of the LaPlace operator, \( s \), these equations become

\[
\ddot{q}_L = -(v_{cg} s + a_x)(\theta + \alpha) + (v_{cg} s + a_x) \left\{ \frac{q}{n} \right\}_T \left\{ q \right\}_n = \frac{N}{M} \alpha \theta - \frac{a_x}{n} \left\{ y \right\}_T \left\{ q \right\}_n
\]

II-6.7

\[
a_y = -v_{cg} s \theta - (v_{cg} s + a_x)\alpha + (v_{cg} s + a_x) \left\{ \frac{y}{n} \right\}_T \left\{ q \right\}_n = \frac{N}{M} \alpha \theta - \frac{a_x}{n} \left\{ y \right\}_T \left\{ q \right\}_n
\]

II-6.8

Using either equation II-6.7 or II-6.8 the dynamic equation in terms of the variable \( \alpha \) is

\[
(v_{cg} s + \frac{N}{M} + a_x)\alpha + (v_{cg} s - g \cos \gamma)\theta + \left\{ a_y \right\}_T \left\{ q \right\}_n = 0
\]

II-6.9

Since it is convenient to know the normal acceleration in the system, it is desirable to retain either \( q_L \) or \( a_y \) and convert either equation II-6.7 or II-6.8 into two equations retaining \( \alpha \) as a dependent variable. In the previous analysis of reference 1, \( a_y \) the acceleration along the \( \overline{T} \) axis was used. However, in this present analysis \( q_L \), the acceleration normal to the nominal trajectory, will be retained. Equation II-6.7 then becomes

\[
\ddot{q}_L = -(v_{cg} s + a_x)(\theta + \alpha) + (v_{cg} s + a_x) \left\{ \frac{q}{n} \right\}_T \left\{ q \right\}_n
\]

II-6.10

\[
\ddot{q}_L = \frac{N}{M} \alpha - \beta_x \theta + \left\{ a_y \right\}_T \left\{ q \right\}_n
\]

II-6.11
Equation II-6.10 is the so-called normal force equation and II-6.11 is the normal acceleration equation. The normal force equation can then be written as

\[ (V_{cg} s + a_x)\alpha + \ddot{\dot{Q}}_L + (V_{cg} s + a_x) \theta - s \left\{ N_{Vn} \right\}_n^{T} \left\{ q_n \right\}_n + \left\{ N_{an} \right\}_n^{T} \left\{ q_n \right\} = 0 \]  

II-6.12

where \( N_{Vn} = V_{cg} w_n \)

\( N_{an} = a_x w_n \)

The normal acceleration equation can be written as

\[ -\frac{N}{M} \alpha + \ddot{Q}_L + a_x \theta - \left\{ a_{yn} \right\}_n^{T} \left\{ q_n \right\} = 0 \]

II-6.13

II-6.3 **MOMENT EQUATION**

The moment equation II-2.21 is

\[ \frac{d}{dt} (\ddot{\theta}) = -\sum_{F} [P_{y,F} P_{x,F}] \]

For small angles and a linear system

\[ \ddot{\theta} = -\frac{1}{I} \sum_{F} [P_{y,F} P_{x,F}] \]

From equation II-5.17

\[ \ddot{\theta} = -\mu_{\alpha} \alpha - \left\{ \mu_{n} \right\}_n^{T} \left\{ q_n \right\} \]

in terms of the LaPlace operator the moment equation becomes

\[ \mu_{\alpha} \alpha + \epsilon^2 \theta + \left\{ \mu_{n} \right\}_n^{T} \left\{ q_n \right\} = 0 \]

II-6.14

II-6.4 **MODAL EQUATIONS**

From equation II-4.43 the modal equations can be written as

\[ s^2 \left[ B \right] \{ q_n \} + \left[ A \right] \{ q_n \} + s \left[ D \right] \{ q_n \} = \left[ k \right] \{ f_{an} \} + \left\{ \frac{P_{m} M}{M} \right\} \]
From equation II-5.21

\[
\begin{pmatrix}
  P_m \\
  P_M
\end{pmatrix} = \begin{pmatrix}
  J_m \\
  J_M
\end{pmatrix} + [F] \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix}
\]

Finally the modal equations become

\[
s^2 \begin{bmatrix}
  S \\
  S
\end{bmatrix} \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} + s \begin{bmatrix}
  D \\
  D
\end{bmatrix} \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} + \begin{bmatrix}
  A \\
  A
\end{bmatrix} \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} = \begin{bmatrix}
  S_n \\
  S_n
\end{bmatrix} + \begin{bmatrix}
  J_m \\
  J_m
\end{bmatrix} \alpha + [F] \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix}
\]

II-6.15

II-6.5 \textbf{SUMMARY OF EQUATIONS}

The dynamic equations derived for a missile configuration considering flexible body, sloshing, and engine dynamics are:

\textbf{Axial Acceleration equation II-6.1}

\[
V_{cg} = a_x = \beta_x - g \cos \gamma
\]

II-6.16

\textbf{Normal Force Equation II-6.12}

\[
(V_{cg} s + a_x) \alpha + Q_L + (V_{cg} s + a_x) \theta - s N_{yn} T \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} = 0
\]

II-6.17

\textbf{Normal Acceleration Equation II-6.13}

\[
-\frac{N}{M} \alpha + Q_L + \beta \theta - s \begin{pmatrix}
  a_{yn} \\
  a_{yn}
\end{pmatrix} T \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} = 0
\]

II-6.18

\textbf{Normal Equation II-6.14}

\[
\mu \alpha + s^2 \theta + \begin{pmatrix}
  a_{yn} \\
  a_{yn}
\end{pmatrix} T \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} = 0
\]

II-6.19

\textbf{Modal Equations II-6.15}

\[
s^2 \begin{bmatrix}
  B \\
  B
\end{bmatrix} \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} + s \begin{bmatrix}
  D \\
  D
\end{bmatrix} \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} + \begin{bmatrix}
  A \\
  A
\end{bmatrix} \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix} = \begin{bmatrix}
  S_n \\
  S_n
\end{bmatrix} + \begin{bmatrix}
  J_m \\
  J_m
\end{bmatrix} \alpha + [F] \begin{pmatrix}
  q_n \\
  q_N
\end{pmatrix}
\]

II-6.20

where \(n\) has values from 1 thru \(N\)

As pointed out in section II-4, the system has \(N + 3\) degrees of freedom. The motion of the system is described by the above \(N + 4\) equations in the \(N + 4\) variables \(\alpha, a_x, \theta, Q_L\) and \(N\) values of \(q_n\). However, the variable
α can be expressed as a linear combination of \( Q_L \), \( \Theta \) and the \( q_n \)'s by equation II-6.18, therefore, \( \alpha \) is a dependent variable and is retained along with equation II-6.18 as a matter of convenience as discussed in section II-6.2. The \( N + 3 \) coordinates which define the motion of the linear system are then \( V_{cg} \), \( Q_L \), \( \Theta \) and the \( N \) values of \( q_n \). For the linear system the degree of freedom defined by \( Q_H = V_{cg} = a_x = \) constant, equation II-6.16, is uncoupled from the other coordinates and consists of a constant acceleration along the \( Q_H \) axis. Since the force along the \( Q_H \) axes is constant, it is independent of \( Q_H \) and \( Q_H \) and there are no spring or damping forces along the axes and the value of \( Q_H \) and \( Q_H \) at time \( t \) are arbitrary. The constant, \( a_x \), has been substituted for \( V_{cg} \) or \( V_{cg} \) and \( V_{cg} \) has been arbitrarily assigned to \( Q_H \) in the normal force equation II-6.17, therefore, the axial translational mode of motion has been eliminated from the system of equations. Elimination of equation II-6.16 reduces the system to \( N + 2 \) degrees of freedom equivalent to defining the missile motion with respect to a set of axis translating along the \( Q_H \) axes with velocity \( V_{cg} \). The L inertial axes is then coincident with the P axes and the coordinate \( Q_H \) is zero.

II-7 MISSILE DYNAMICS IN GENERALIZED COORDINATES

The dynamic equations given in section II-6 together with an engine hydraulic equations and an instrument equation, are the data required for the missile dynamics block of figure I-2. The missile dynamics represents the transfer function from delta command, \( \delta_c \), to the gyro outputs, \( \Theta_p \) and \( \Theta_R \). Figure II-7.1 contains a block diagram showing the effective flow of the data in the dynamic equations. It is assumed that in a system containing \( P \) gimbaled engines, the command signals for each engine or group of engines may be different. The engine hydraulic equations which give the engine actuator position \( \delta_{ae} \) corresponding to a given delta command, can be of any form but will usually appear in the form

\[
(K_1s + K_2) \delta_e + (K_3s + K_4) \delta_{ae} + \delta_{ce} = 0
\]

II-7.1 requiring a feedback of engine positions \( \delta_e \).
FIGURE II - 7.1
MISSILE DYNAMICS IN GENERALIZED COORDINATES
The engine actuator position, and the angle of attack feedback drive the modal equations II-6.15 producing the modal coordinates \( Q_n \). These coordinates in turn drive the normal acceleration and moment equations together with the equation

\[
v = \sum_{n} v_n Q_n
\]

to produce the rigid body rotation of the \( i - j \) axes, \( \Theta \), the rigid body rotation of the UBA with respect to the \( \bar{i} - \bar{j} \) axes, \( w \), and the angle of attack \( \alpha \). In addition the engine deflections are feedback to the engine hydraulic equations and the bending coordinates are combined with the bending data to produce the bending slopes at each gyro location. These bending slopes are combined with the rigid body rotations to produce the gyro signals.

\[
\Theta_p = \Theta - w - \psi_P
\]

\[
\Theta_{RI} = \Theta - w - \psi_{RI}
\]

\[
\Theta_{RII} = \Theta - w - \psi_{RII}
\]

\[
\Theta_R = \Theta_{RI} + \alpha_R \Theta_{RII}
\]

It is assumed that one position and two rate gyros are used and the net rate signal is a ratioed sum of the two rate gyros.

That portion of the system matrix representing missile dynamics is shown in Figure II-7.2. The first row of the matrix represents the normal force equation II-6.12. The second row is the normal acceleration equation II-6.13, and the third row is the moment equation II-6.14. The next \( N \) rows are the \( N \) modal equations II-6.15. The system shown is assumed to have three gimbaled engines, therefore, columns 11, 12, and 13 represent the three engine actuator positions. Row 11 and column 14 introduce the rigid body rotation with respect to the \( \bar{i} - \bar{j} \) axes were

\[
v = \begin{bmatrix} v_n \\ T \end{bmatrix} \begin{bmatrix} Q_n \end{bmatrix}
\]
### Figure II - 7.2

**System Matrix in Generalized Coord**

<table>
<thead>
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<th>1</th>
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<td>$V_{cg} ; a + a_x$</td>
<td>$a_y$</td>
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<td>$-s ; R_{v2}$</td>
<td>$-s ; R_{v3}$</td>
<td>$-s ; R_{vN}$</td>
<td>$-s ; R_{vN}$</td>
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</tr>
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<td>$\mu_3$</td>
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<td>$\mu_N$</td>
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<td>$K_2 s^{N+1}$</td>
<td>$K_2 s^{N+1}$</td>
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<td>$K_5 s^{N+1}$</td>
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</tbody>
</table>

$$s^2 \begin{bmatrix} B \end{bmatrix} + s \begin{bmatrix} D \end{bmatrix} \cdot [A] - [P]$$
<table>
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<tr>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
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<td>$\nu$</td>
<td>$\varphi_\theta$</td>
<td>$\varphi_\theta$</td>
<td>$\varphi_\theta$</td>
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<td>$-x_3$</td>
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</tbody>
</table>
The next four rows and columns introduce the instrument equations II-7.2 while the last three rows represent the engine hydraulic equations II-7.1. For the system shown it is assumed that a single command signal, column 19 controls the actuators of the first two gimbaled engines and the third engine is restrained by making its command signal zero.

The input to the missile dynamics is $\delta_c$ of column 19 while the output is the gyro signals of columns 15 and 18. The system matrix is completed by adding rows and columns representing the autopilot which operates on the gyro signals to produce the command signal.

The matrix representing missile dynamics, as shown in Figure II-7.2, will have a number of columns one greater than the number of rows. Adding an additional row to the matrix by placing unity in the delta command column, $\delta_c$, will make the input $\delta_c$ zero and the output arbitrary. The values of $s$ which make the determinant of the resulting square matrix zero represent the natural frequencies of the missile dynamic system, and are the open loop poles of the system due to missile dynamics. The eigenvector associated with each pole or natural frequency represents the mode shape for that pole or frequency expressed as a linear combination of the input modes.
In Section II the dynamic equations were derived for a missile configuration considering flexible body, slung lines, and engine dynamics. These equations of motion are:

Normal Force Equation
\[ (v_{cg} s + a_x)\alpha + Q_L + (v_{cg} s + a_x)\theta - s \left( N_{rn} \right)_T \left( Q_n \right) - \left( N_{an} \right)_T \left( Q_n \right) = 0 \]  
III-1.1

Normal Acceleration Equation
\[ -\frac{N}{M} \alpha + Q_L + \beta \theta - s \left( a_{yn} \right)_T \left( Q_n \right) = 0 \]  
III-1.2

Moment Equation
\[ \mu_\theta + s^2 \theta + \left( \mu_n \right)_T \left( Q_n \right) = 0 \]  
III-1.3

Modal Equations
\[ s^2 \left[ B \right] \left( Q_n \right) + s \left[ D \right] \left( Q_n \right) + \left[ A \right] \left( Q_n \right) = \left[ k \right] \left( \theta_{am} \right) + \left( g \right)_m \alpha + \left[ P \right] \left( Q_n \right) \]  
III-1.4

where \( n \) has values from 1 through \( N \).

As noted in Section II-6.5, the system has \( N + 2 \) degrees of freedom with respect to a translating set of inertial axes. The motion of the system is described by the above \( N + 3 \) equations in the variables \( \alpha, \theta, Q_L \) and the \( N \) values of \( Q_n \), where \( Q_K \) and Equation III-1.2 are a dependent variable and equation. The variables \( \theta \), and the \( Q_n \)'s are generalized coordinates and \( \alpha \) is a function of \( \theta \), the \( Q_n \)'s and the rate of change of the generalized coordinate \( Q_L \).

\[ \alpha = - \left[ \theta + \left( \frac{Q_L}{v_{cg}} - \left( \nu \right)_T \left( Q_n \right) \right) \right] \]  
III-1.5

The generalized coordinate \( Q_L \) is the displacement of the missile center of gravity normal to the nominal flight path. The generalized coordinate \( \theta \) is the rotation of the body axes, \( \overline{1-\overline{3}} \), and the \( N \) coordinates, \( Q_n \) define the motion of the missile with respect to the \( \overline{1-\overline{3}} \) axes. The displacement of the center of gravity along the nominal flight path has
been eliminated from the equations by assuming the reference axes to be translating along the flight path at a velocity $V_{cg}$.

For each of the $N$ coordinates $Q_n$ there is a so-called input mode as defined in Section II-4. Each input mode is the amplitude of motion resulting when the corresponding coordinate $Q_n$ has an amplitude of unity and all other coordinates are zero. The input modes when $Q_n$ represents an engine deflection, a sloshing deflection, and a bending deflection are shown in Figure III-1. The input engine modes are normalized to an engine deflection of 1 radian and the motion consists of a rigid body translation, $v_e$, and a rigid body rotation, $\psi_e$, of the UEA. The input slosh modes are normalized to a slosh mass deflection of 1 foot and the motion consists of a rigid body translation, $v_j$, and a rigid body rotation, $\psi_j$, of the UEA. The input bending modes are normalized to a mass equal to the total mass of the missile for motion with respect to the UEA. The rigid body translation $v_i$ is zero and the rigid body rotation, $\psi_i$, exists only when inertia corrections have been made. Since a bending coordinate, $b_i$, of unity represents motion equivalent to a normal mode, $b_i$ is measured at the point where the deflection is unity. The modal deflections and slopes are in ft. per ft. and radians per ft. since $b_i$ is in feet.

As noted in Section II-4 the modes defined as input modes are not true modes of the system since each can only exist if all other coordinates are held fixed. However, since the system is linear, superposition is valid and any free or driven motion of the system is a linear combination of the motions associated with the input modes and may be defined in terms of normal coordinates which are a linear combination of the input coordinates $Q_n$. It was also noted in Section II-7 that the modes of oscillation associated with the open loop natural frequencies or poles are a linear combination of the input modes. Likewise, the modes of oscillation associated with the closed loop natural frequencies or roots of the complete autopilot configuration are a linear combination of the input modes. Therefore, the motion associated with the closed loop roots is a linear combination of the motion associated with the open loop poles. From the above it can be seen that
\[ \delta_e = 1 \text{ radian} \]

**INPUT ENGINE MODE**

**INPUT SLOSH MODE**

**INPUT BENDING MODE**

FIGURE III-1
INPUT MODES
the modes for the closed loop roots, or the final motion of the system can be expressed in terms of a linear combination of any set of modes and their coordinates which are in turn a linear combination of the input modes. These new modes will be referred to in this analysis as "combined modes." It may be seen that the modes corresponding to the open loop poles of the system are in effect combined modes each with a normal coordinate which is a linear combination of the input coordinates $Q_n^i$. The final motion of the system corresponding to the closed loop roots may then be expressed in terms of a linear combination of these normal coordinates.

Combined mode representation in connection with missile dynamic equations consists of defining a set of orthogonal modes in terms of a linear combination of the so-called input modes defined in Section II-4 and shown in Figure III-1. Replacing the input modes by this set of orthogonal combined modes produces a set of missile dynamic equations in a form which is independent of missile configuration with regard to propellant tank and engine arrangement. By using this representation the final stability equations retain the same form regardless of the missile system configuration. Also, as noted in Section II-3, the highest input bending frequency should be above the highest engine or slosh mode frequency in order to reduce truncation errors due to the use of a finite number of bending modes. This criterion usually results in the use of from three to five bending modes. The use of additional modes increases the size of the system matrix and requires additional computer time to obtain the solutions. With the use of combined mode representation, a larger number of bending modes may be used in the calculation of the combined modes and the dropping of the higher frequency orthogonal combined modes from the system equations will tend to reduce truncation errors.

III-2 SELECTION OF COMBINED MODES

In Section III-1 the principle of superposition in a linear system was used to show that the motion of the missile may be described in terms of a linear combination of a set of combined modes and the dynamic equations can be expressed in terms of their normal coordinates which in turn are a linear combination of the coordinates $Q_n$. Let the normal coordinates of the combined
modes be represented by \( q_n \), \( n \) equals 1 through \( N \). Combined mode representation consists of making a linear coordinate transformation from the generalized coordinates \( Q_n \) to the normal coordinates \( q_n \) and defining the motion in terms of the combined modes instead of the input modes. Considering the modal equation represented by \( III-1.4 \),

\[
s^2 \left[ B \right] \left[ q_n \right] + s \left[ D \right] \left[ q_n \right] + \left[ A \right] \left[ q_n \right] = \left[ k \right] \left[ \delta_{nm} \right] + \left[ \delta_{im} \right] \alpha + \left[ p \right] \left[ q_n \right]
\]

The left side of the equation is a function of only the \( q_n \)'s and when equated to zero represents a eigenvalue problem, the solution of which gives the natural frequencies and mode shapes for the system when the actuator position, aerodynamic forces, and thrust forces are zero. The right side of the equation then represents the driving forces for these new combined modes.

The matrix \( [p] \left[ q_n \right] \) represents the spring forces resulting from engine thrust and is also only a function of the \( q_n \)'s. This term may be transferred to the left side of the equation and the combined modes then become the undriven damped modes with engine thrust forces present. Likewise, the aerodynamic forces \( \{J_m\} \alpha \) can be made functions of the \( q_n \)'s only by using Equations \( III-1.1 \), \( III-1.2 \), and \( III-1.3 \) to express \( \alpha \) as a function of \( q_n \)'s. Transferring this term to the left side of the equation will result in combined modes for undriven, damped modes with engine thrust and aerodynamic forces present. The only driving force remaining on the right side is \( \left[ k \right] \left[ \delta_{nm} \right] \) due to engine actuator positions. These mode shapes and frequencies require only the addition of the engine hydraulic equation to yield the system open loop poles and associated mode shapes. Calculation of the combined modes with the loop opened in this manner would require a considerable expenditure of computer time approaching that required to obtain open loop poles. The use of combined modes is only justified if the modes can be obtained quickly with a small expenditure of computer time. This criterion requires that the aerodynamic forces \( \{J_m\} \alpha \) and the thrust spring forces \( \left[ p \right] \left[ q_n \right] \) remain on the right side of Equation \( III-1.4 \) as driving forces. In addition, the damping forces represented by \( s \left[ D \right] \left[ q_n \right] \) make the constants which express the combined modes as a linear
combination of the input modes, complex numbers which greatly increases the
required computer time. The more suitable form of Equation III-1.4 for com-
bined mode calculations is then

\[ s^2 [B] \{q_n\} + [A] \{q_n\} = -s [D] \{q_n\} + [k] \{\delta_m\} + \{y_m\} + [F] \{Q_n\} \]

Equation III-2.1

In this form the homogeneous equation

\[ s^2 [B] \{q_n\} + [A] \{q_n\} = 0 \]

Equation III-2.2

may be solved by the AX = \lambda BX computer subroutine (Reference 6). This
subroutine is extremely fast but requires that the [A] and [B] matrix be
symmetric and that the [B] matrix be positive and definite. The solution of
Equation III-2.2 will give the undamped natural frequencies of the system when
the actuator positions, the aerodynamic forces, and the thrust forces are all
zero. The [P] matrix representing the thrust spring forces cannot be placed
on the left side of the equation since it is not a symmetric matrix. In the
previous analysis, Reference 1, the symmetric components of the [P] matrix
were placed on the left side and included in the combined modes; however, for
this analysis all of the effects of the thrust forces are considered driving
forces.

III-3 COORDINATE TRANSFORMATION

Solution of Equation III-2.2 will result in \(N\) values of \(s^2\) where

\[ s^2 = -\omega_a^2 \]

and \(N\) associated eigenvectors \(\{e_{na}\}\). Where:

\[ \omega_a \] = natural frequency of the \(a\) \th combined mode

\[ \{e_{na}\} \] = a column matrix of the values of \(\{Q_n\}\) obtained from III-2.2

when \(s^2 = -\omega_a^2\).

Then \(e_{na}\) is the amount of \(Q_n\) present in the \(a\) \th combined mode. The eigenvectors
\(\{e_{na}\}\) obtained from the subroutine are normalized such that

\[ \{e_{na}\}^T [B] \{e_{na}\} = 1 \]

Equation III-3.1
and since the modes are orthogonal
\[
\{e_{na}\}^T [B] \{e_{nb}\} = 0
\]  
III-3.2

Let
\[
[E] = \begin{bmatrix} \{e_1\} & \{e_2\} & \ldots & \{e_{na}\} & \ldots & \{e_{N}\} \end{bmatrix} = \begin{bmatrix} e_{na} \end{bmatrix}
\]  
III-3.3

when [E] is a N x N square matrix, the columns of which are the eigenvectors found in the solution of Equation III-2.2. This [E] matrix is defined as the normalized modal matrix of Equation III-2.2. From III-3.1, III-3.2, and III-3.3
\[
[E]^T [B] [E] = [I]
\]  
III-3.4

Since the values of B and A in Equation III-2.2 are the masses and spring constants divided by the total mass of the system, Equations III-3.1 and III-3.4 are equivalent to normalizing the combined modes to a mass equal to the total mass. Since \{e_{na}\} is the solution of III-2.2 when \(s^2 = \Omega_a^2\)
\[
[A] \{e_{na}\} = \Omega_a^2 [B] \{e_{na}\}
\]  
from which \([A][E] = [B][E][\Omega^2] \)  
III-3.5

where \(\Omega^2\) is a diagonal matrix of the values of \(\Omega_a^2\), premultiplying III-3.5 by \([E]^T\)
\[
[E]^T [A][E] = [E]^T [B][E][\Omega^2]
\]  
using III-3.4
\[
[E]^T [A][E] = [\Omega^2]
\]  
III-3.6

Since \(e_{na}\) is the amount of \(Q_n\) present in the \(n^{th}\) combined mode and \(q_a\) is the amount of the \(a^{th}\) combined mode present,
\[
Q_n = \sum_a e_{na} q_a = \{n^{th}\text{ row of } [E]\} \{q_a\}
\]

then
\[
\{Q_n\} = [E] \{q_a\}  \quad \text{III-3.7}
\]
Substituting this coordinate transformation into Equation III-1.4

\[
s^2 \left[ B \right]^{T} \left[ E \right] \{q_a\} + \left[ A \right] \left[ E \right] \{q_a\} = -s \left[ D \right] \left[ E \right] \{q_a\} + \left[ \alpha \right] \{ \delta_{am} \} + \left[ J_m \right] \alpha + \left[ P \right] \left[ E \right] \{q_a\}
\]

premultiplying by \([E]_T\)

\[
s^2 \left[ E \right]_T \left[ B \right] \left[ E \right] \{q_a\} + \left[ E \right]_T \left[ A \right] \left[ E \right] \{q_a\} = -s \left[ E \right]_T \left[ D \right] \left[ E \right] \{q_a\} + \left[ E \right]_T \left[ k \right] \{ \delta_{am} \} + \left[ E \right]_T \left[ J_m \right] \alpha + \left[ E \right]_T \left[ P \right] \left[ E \right] \{q_a\}
\]

from III-2.6 and III-2.8

\[
s^2 \left[ B \right] \left[ E \right] \{q_a\} + \left[ E \right]_T \left[ A \right] \left[ E \right] \{q_a\} = -s \left[ E \right]_T \left[ D \right] \left[ E \right] \{q_a\} + \left[ E \right]_T \left[ k \right] \{ \delta_{am} \} + \left[ E \right]_T \left[ J_m \right] \alpha \]

\[
\left[ s^2 + \gamma_a \right] \{q_a\} = -s \left[ E \right]_T \left[ D \right] \left[ E \right] \{q_a\} + \left[ E \right]_T \left[ k \right] \{ \delta_{am} \} + \left[ E \right]_T \left[ J_m \right] \alpha
\]

\[
+ \left[ E \right]_T \left[ P \right] \left[ E \right] \{q_a\}
\]

III-3.8

where

\[
\left[ s^2 + \gamma_a \right] \text{ is a diagonal matrix } = \left( s^2 + \gamma_a^2 \right) \left[ I \right]
\]

Let \( \left[ P \right] = \left[ E \right]_T \left[ D \right] \left[ E \right] = \left[ D_{ab} \right] = \text{ damping matrix in normal coordinates where } D_{ab} \text{ is } 1/M \text{ times the damping force associated with the } q_a \text{ coordinate due to unit velocity of the } q_b \text{ coordinate.}
\]

Let \( \left[ k \right] = \left[ E \right]_T \left[ k \right] = \left[ k_{ab} \right] \)

III-3.9

where \( k_{ab} \) is \( 1/M \) times the driving force for the \( a^{th} \) combined mode due to unit deflection of the \( b^{th} \) engine actuator.

Let \( \left[ J \right] = \left[ E \right]_T \left[ J \right] \)

III-3.10

where \( J_{ca} \) is \( 1/M \) times the driving force for the \( a^{th} \) combined mode due to a unit angle of attack.
Let \( [P_c] = [E_T]^T [P] [E] = [P_{ab}] \) III-3.12

where \( P_{ab} \) is \( 1/M \) times the thrust spring force associated with the \( a^{th} \) normal coordinate due to unit deflection of the \( b^{th} \) normal coordinate.

Then the modal equation in normal coordinates becomes

\[
[I] s^2 + [\gamma]^2 + s [D_c]^{-1} [P_c] \{q_a\} = [k_c] \{\hat{5}_{am}\} + \{J_{ca}\} \alpha
\]

III-3.13

### III-4 DYNAMIC EQUATIONS IN NORMAL COORDINATES

Substituting the coordinate transformation \( \{q_0\} = [E] \{q\} \) into the normal force equation III-1.1

\[
(V_{cg} s + a_x)\chi + Q_L + (V_{cg} s + a_x)\varnothing - s \{N_{vn}\}_T [E] \{q\}_s - \{N_{an}\}_T [E] \{q\}_a = 0
\]

or

\[
(V_{cg} s + a_x)\chi + Q_L + (V_{cg} s + a_x)\varnothing - s \{N_{cva}\}_T \{q\}_s - \{N_{caa}\}_T \{q\}_a = 0
\]

III-4.1

where

\[
\{N_{cva}\} = [E_T]^T \{N_{vn}\}
\]

III-4.2

\[
\{N_{caa}\} = [E_T]^T \{N_{an}\}
\]

III-4.3

The normal acceleration equation III-1.2 becomes

\[
\begin{align*}
\frac{N}{M} \chi + \ddot{Q}_L + \beta_x \varnothing - \{a_{yn}\}_T [E] \{q\}_a &= 0 \\
\frac{N}{M} \chi + \ddot{Q}_L + \beta_x \varnothing - \{a_{yca}\}_T \{q\}_a &= 0
\end{align*}
\]

III-4.4

where

\[
\{a_{yca}\} = [E_T]^T \{a_{yn}\}
\]

III-4.5

The moment equation III-1.3 becomes:
\[ u_i^\omega + s^2\omega + \{u_n\}_T [E] \{q_n\} = 0 \]
\[ u_i^\omega + s^2\omega + \{u_{ca}\}_T \{q_{ca}\} = 0 \]  

III-4.6

where
\[ \{u_{ca}\} = [E]_T \{u_n\} \]  

III-4.7

the modal equation is given by III-3.13.

III-5  COMBINED MODES

The combined modes selected in Section III-2 are defined by the solutions to Equation III-2.2
\[ s^2 [B] \{q_B\} + [A] \{q_A\} = 0 \]

The resulting N eigenvalues of \[ \Omega_B \] and the corresponding N eigenvectors \[ \{e_{na}\} \] represent the N combined mode frequencies and mode shapes. The order of numbering the combined modes is arbitrary and is a function of the manner in which the roots of Equation III-2.2 are numbered. Since the equation is to be solved by means of the AX = \[ \lambda BX \] computer subroutine, it is convenient to order the roots and therefore the combined modes in the order which the computer subroutine finds the roots. This process is not arbitrary since in a loosely coupled system the routine tends to find roots in an order corresponding to the frequencies represented by the data along the main diagonals of the A and B matrices. As a result, the first combined mode will usually have a frequency close to the first input engine frequency and will contain the largest first engine amplitude. Likewise, the second combined mode will have a frequency close to and contain large second engine amplitudes, etc. This correspondence between the input mode frequencies and the combined mode frequencies has resulted in a tendency to name the combined in the same order as the input modes, first engine mode, first slosh mode, second slosh mode, etc. However, it should be noted that this correspondence of input and output frequencies may not hold for tightly coupled modes such as two slosh or engine modes with input frequencies close together. In fact,
a rerun of the program with slight changes in parameters may cause two such
modes to exchange position in the output. This process has no effect upon
the equations or their solutions once the combined modes are numbered but
does lead to some confusion if the combined modes are associated with a
corresponding input mode.

As noted in Section III-3, the eigenvector \( \{ e_{na} \} \) is the solution
to Equation III-2.2 when \( s^2 = \Omega^2 \). Therefore, \( e_{na} \) is the amount of \( Q_n \)
present in the \( a^{th} \) combined mode, and letting \( \{ n_3 \} \) equal \( \{ e_{na} \} \) defines
the \( a^{th} \) combined mode.

From Equation III-3.3
\[
[E] = \begin{bmatrix}
\{ e_{n1} \} & \{ e_{n2} \} & \{ e_{n3} \} & \ldots & \{ e_{n5} \} & \ldots & \{ e_{nB} \} \\
\end{bmatrix}
\]
and
\[
[E]_T = \begin{bmatrix}
\{ e_{n1} \}_T \\
\{ e_{n2} \}_T \\
\{ e_{na} \}_T \\
\{ e_{n3} \}_T \\
\{ e_{n4} \}_T \\
\end{bmatrix}
\]
the \( a^{th} \) column of the modal matrix \([E]\) gives the composition of the \( a^{th} \)
combined mode in terms of the normalized input modes. Likewise, the \( a^{th} \)
row of \([E]_T \) also gives the composition of the \( a^{th} \) combined mode.

It was noted in Section II-5.2 that the deflections and slopes of
an input bending mode could be multiplied by \(-1\) thereby defining a new input
bending mode and shifting the phase of the final \( b_1 \) by 180 degrees. In
order to be consistent, a positive input bending mode was defined as that
phase of the mode which results in a positive deflection at the aft end of
the missile. Likewise, the positive input engine and slosh modes were de-
defined in Section III-1 Figure III-1 as that phase of the motion resulting
from positive engine and slosh deflections. Since \( e_{na} \) is the amount of the
\( n^{th} \) input mode present in the \( a^{th} \) combined mode, a value of \( e_{na} \) of \( +.7 \) for
example indicates that the mode shape of the \( a^{th} \) combined mode contains \( .7 \)
times the normal $n^{th}$ input mode. Defining the opposite phase of the input mode would result in an $e_{n\alpha}$ of $-0.7$ and the resulting combined mode would be the same. Therefore, it may be seen that the definition of a positive phase for the input modes is not required in order to determine the combined modes or to perform a stability analysis but has been done in order to provide a consistent basis upon which to compare the phase relationship between the input modes for different missile configurations or for different times of flight. It may also be seen that the phase of the $a^{th}$ combined mode and therefore the phase of $q_a$ can be changed by 180 degrees if the $a^{th}$ column of $[E]$ or the $a^{th}$ row of $[E]^T$ is multiplied by a factor of $-1.0$. It is then desirable to define a positive phase of the combined modes or in effect a positive phase of each $q_a$. Since the engine actuator positions represent the external driving forces for the dynamic equations, and the first inputed engine is usually the main control engine, the deflection of the first input engine provides a convenient common reference for the combined modes. The positive phase of each combined mode is then taken as that phase for which the first engine deflection is positive. This phase relationship is accomplished by multiplying the columns of the normalized modal matrix $[E]$ by plus or minus unity such that all of the elements of the first row of the matrix are positive numbers. The $a^{th}$ combined mode then represents the motion with respect to the $i-j$ axes when the normal coordinate $q_a$ is equal to $+1.0$.

Referring to Figure III-5.1

![Figure III-5.1](image-url)
The deflection of the $a^{th}$ combined mode at station $\xi$ is
\[ \phi_c^{a} \xi = v_{ca} + \frac{\xi c g - \xi}{12} w_{ca} + u_{c}^{a} \xi a \] III-5.1

The total slope of the $a^{th}$ combined mode at station $\xi$ is
\[ \phi_{c}^{a} \xi = w_{ca} + u_{c}^{a} \xi a \] III-5.2

The bending slope of the $a^{th}$ combined mode at station $\xi$ is
\[ \lambda_{c}^{a} \xi a = w_{ca} + u_{c}^{a} \xi a \] III-5.3

From Equation II-4.8
\[ v = \{v_{m}\}_T \{q_{n}\} \]
\[ v_{ca} = \{v\}_T \{e\} \] III-5.4

or \[ \{v_{ca}\} = [E]_T \{v_{m}\} \] III-5.5

From Equation II-4.18
\[ w = \{w\}_T \{q_{n}\} \]
\[ w_{ca} = \{w\}_T \{e_{na}\} \] III-5.6

or \[ \{w_{ca}\} = [E]_T \{w_{m}\} \] III-5.7

from II-3.6
\[ u_{c} = \sum_{i} \phi_{i}^{a} b_{i} = \{\phi_{i}^{a} n\}_T \{q_{n}\} \] III-5.8
\[ u_{c}^{a} = \sum_{i} \phi_{i}^{a} b_{i} = \{\phi_{i}^{a} n\}_T \{q_{n}\} \] III-5.9
\[ \psi = \sum_{i} \lambda_{i} v_{i} = \{\lambda_{i} n\}_T \{q_{n}\} \] III-5.10
The magnitude of the engine deflections and sloshing deflections associated with each combined mode can be obtained from the modal matrix $[E] = [e_{na}]$ since $e_{na}$ is a amplitude of the $n^{th}$ generalized coordinate in the $a^{th}$ combined mode.

In combined mode representation the motion of the system is described in terms of the normal combined mode coordinates $q_a$. However, a feedback of engine deflections is required for the engine hydraulic equations, therefore, each engine deflection must be expressed in terms of the normal coordinates $q_a$. From Equation III-3.7
III-6 MISSILE DYNAMICS IN NORMAL COORDINATES

In Section II-2, the dynamic equations were derived for a system of masses in terms of the translational motion of the system center of gravity, the rotation of a set of $\mathbf{i}-\mathbf{j}$ body axes, and the motion of the masses with respect to the $\mathbf{i}-\mathbf{j}$ body axes. The system of masses was defined in terms of a missile configuration in Section II-3, and the motion with respect to the $\mathbf{i}-\mathbf{j}$ axes was derived in Section II-4 in terms of a set of $N$ generalized coordinates, $\mathbf{q}_n$. The generalized coordinates used where those for which the motion was statically uncoupled. Missile dynamics in terms of these generalized coordinates is presented in Section II-7.

The motion of the system with respect to the $\mathbf{i}-\mathbf{j}$ axes has been redefined in Section III in terms of the normal modes of the undamped, undriven system and the associated normal coordinates. The dynamic equations for the system transformed into normal coordinates are given in Section III-4. These equations together with an engine hydraulic equation and an instrument equation are the data required for the missile dynamics block of Figure I-2. The missile dynamics represents the transfer function from delta command, $\delta_c$, to the gyro outputs, $\mathbf{q}_P$ and $\mathbf{q}_R$. Figure III-6.1 contains a block diagram showing the effective flow of the data in the dynamic equations in normal coordinates. It is assumed that in a system containing $P$ gimbaled engines, the command signals for each engine or group of engines may be different.

The engine hydraulic equations will again be of the form of equation II-7.1 requiring a feedback of engine positions $\delta_e$. The engine actuator positions together with the angle of attack feedback drive the modal equations III-3.13 producing the normal coordinates $\mathbf{q}_a$. These coordinates in turn drive the
normal force, normal acceleration and moment equations to produce the rigid body rotation of the \( \mathbf{i} - \mathbf{j} \) axes, \( \theta \), and the angle of attack \( \alpha \). In addition the normal coordinates operate with equation III-5.18 to produce the engine deflection to be feedback to the engine hydraulic equations and are combined with the combined mode bending data to produce the bending slopes at each gyro location. These bending slopes are combined with the rigid body rotations to produce the gyro signals. Since the rigid body rotation due to motion with respect to the \( \mathbf{i} - \mathbf{j} \) axes, \( v \), is included in the combined mode slopes this term does not appear in the instrument equations when in normal coordinates. Equations II-7.2 then become

\[
\begin{align*}
\theta_P &= \theta - \psi_p \\
\theta_{RI} &= \theta - \psi_{RI} \\
\theta_{RIII} &= \theta - \psi_{RIII} \\
\theta_R &= \theta_{RI} + a_R \theta_{RIII}
\end{align*}
\]

That portion of the system matrix representing missile dynamics in normal coordinates is shown in Figure III-6.2. The first row of the matrix represents the normal force equation III-4.1. The second row is the normal acceleration equation III-4.4, and the third row is the moment equation III-4.6. The next \( N \) rows are the \( N \) modal equations III-3.13. The system shown is assumed to have three gimbaled engines, therefore, columns 11, 12 and 13 and rows 11, 12 and 13 represent the engine angles found from equation III-5.18 while columns 14, 15 and 16 represent the three engine actuators. The next four rows and columns introduce the instrument equations III-6.1 while the last three rows represent the engine hydraulic equations II-7.1. For the system shown it is assumed that a single command signal, column 21, controls the actuators of the first two gimbaled engines and the third engine is restrained by making its command signal zero.

The input to the missile dynamics is \( \delta_c \) of column 21 while the output is the gyro signals of columns 17 and 20. The system matrix is completed by adding rows and columns representing the autopilot which operates on the gyro signals to produce the command signal.
The table and graph represent a system matrix in normal coordinates. The matrix is partitioned into sections labeled from 1 to 20. Each section contains terms that include variables such as $a$, $s$, $N$, and $\Omega$. The graph shows a polynomial relationship $s^2 [I] + [\Omega^2] - s[\mathbf{p}_c] - [\mathbf{p}_c]$. The polynomial is $s^2 + [\Omega^2] - s[\mathbf{p}_c] - [\mathbf{p}_c]$. The matrix contains various parameters and functions, including $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$, $\Delta_5$, $\Delta_6$, $\Delta_7$, $\Delta_8$, $\Delta_9$, $\Delta_{10}$, $\Delta_{11}$, $\Delta_{12}$, $\Delta_{13}$, $\Delta_{14}$, $\Delta_{15}$, $\Delta_{16}$, $\Delta_{17}$, $\Delta_{18}$, $\Delta_{19}$, $\Delta_{20}$, and $\Delta_{21}$. The table includes elements such as $\mu_c$, $\beta_c$, $\gamma_c$, and $\chi_c$. The elements are arranged in a structured manner, indicating the relationships and interactions within the system.
<table>
<thead>
<tr>
<th></th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>EQUATION</th>
</tr>
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<tr>
<td>$b_2$</td>
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<td>$b_a2$</td>
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<td>$o_R$</td>
<td>$o_R$</td>
<td>$o_R$</td>
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<td>$-k_{21}$</td>
<td>$-k_{22}$</td>
<td>$-k_{23}$</td>
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<td>$-k_{33}$</td>
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<td>$-k_{a2}$</td>
<td>$-k_{a3}$</td>
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</tbody>
</table>
The matrix representing missile dynamics, as shown in figure III-6.2, will have a number of columns one greater than the number of rows. Adding an additional row to the matrix by placing unity in the delta command column, $\delta_c$, will make the input $\delta$ zero and the output arbitrary. The values of $s$ which make the determinate of the resulting square matrix zero represent the natural frequencies of the missile dynamic system, and are the open loop poles of the system due to missile dynamics, the eigenvectors associated with each pole or natural frequencies represents the mode shape for that pole or frequency expressed as a linear combination of the combined modes.
GLOSSARY

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$a$</td>
<td>subscript denoting $a^{th}$ combined mode</td>
</tr>
<tr>
<td>$a_r$</td>
<td>coefficient used in combining rate gyro outputs</td>
</tr>
<tr>
<td>$a_x$</td>
<td>acceleration of c.g. along i axis</td>
</tr>
<tr>
<td>$a_y$</td>
<td>acceleration of c.g. along j axis</td>
</tr>
<tr>
<td>$b_i$</td>
<td>subscript denoting bending</td>
</tr>
<tr>
<td>$b_{i_i}$</td>
<td>generalized bending coordinate for $i^{th}$ bending mode</td>
</tr>
<tr>
<td>$c$</td>
<td>subscript denoting combined mode data</td>
</tr>
<tr>
<td>$c_e$</td>
<td>subscript denoting $e^{th}$ gimbaled engine</td>
</tr>
<tr>
<td>$f$</td>
<td>subscript denoting $f^{th}$ fixed engine</td>
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<td>$g$</td>
<td>acceleration of gravity</td>
</tr>
<tr>
<td>$h$</td>
<td>subscript referring to the UEA</td>
</tr>
<tr>
<td>$i$</td>
<td>subscript denoting $i^{th}$ bending mode</td>
</tr>
<tr>
<td>$i_I$</td>
<td>rotating reference axis</td>
</tr>
<tr>
<td>$j$</td>
<td>subscript denoting $j^{th}$ sloshing mass</td>
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<tr>
<td>$j_I$</td>
<td>rotating reference axis</td>
</tr>
<tr>
<td>$k$</td>
<td>subscript denoting $k^{th}$ inertia correction</td>
</tr>
<tr>
<td>$k_I$</td>
<td>fluid inertia correction factor</td>
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<tr>
<td>$k_i$</td>
<td>equivalent bending spring constant</td>
</tr>
<tr>
<td>$k_j$</td>
<td>slosh spring constant</td>
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<tr>
<td>$[K]$</td>
<td>spring matrix</td>
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<tr>
<td>$L_{Ge}$</td>
<td>distance from gimbal to center of gravity of $e^{th}$ gimbaled engine</td>
</tr>
<tr>
<td>$L_p$</td>
<td>distance from c.g. to center of pressure</td>
</tr>
<tr>
<td>$m$</td>
<td>subscript denoting $m^{th}$ generalized coordinate</td>
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<td>total fluid mass in tank</td>
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<tr>
<td>$m_j$</td>
<td>equivalent slosh mass</td>
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<tr>
<td>$n$</td>
<td>subscript denoting $n^{th}$ generalized coordinate</td>
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<td>$p$</td>
<td>subscript denoting $p^{th}$ gimbaled engine</td>
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<td>$q_a$</td>
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<tr>
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<td>radius of gyration of element of mass $dm$</td>
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<tr>
<td>$r$</td>
<td>subscript denoting $r^{th}$ slosh mass</td>
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<td>$s$</td>
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<td>time</td>
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<tr>
<td>$t$</td>
<td>subscript denoting $t^{th}$ bending mode</td>
</tr>
</tbody>
</table>
(Cont'd)

GLOSSARY

u - translational deflection of beam element at xₙ due to elastic deformation
u' - total slope at xₙ due to elastic deformation
v - displacement of UEA along j axis
w - rotation of UEA in I-J plane
x - coordinate along I axis
y - coordinate along j axis
z - coordinate on e-th gimbaled engine measured aft from gimbal

[A] - normalized spring matrix
[B] - normalized mass matrix
D - total aerodynamic drag

[D] - damping matrix
D.Ge - gimbal damping for e-th engine (ft-lbs per radian)
D.sj - damping for j-th slosh mass (lbs. per ft/sec.)
D.bi - damping j-th bending mode (lbs. per ft./sec.)
E - subscript denoting a point on a gimbaled engine
F - number of fixed engines
F - subscript denoting a fixed engine

F.D - dissipation function
H - inertial axis
I - mass moment of inertia of system about its c.g.
I.Ge - mass moment of inertia of e-th engine about its gimbal point

[J] - matrix representing aerodynamic forces
K - number of inertia corrections
K.se - spring constant for e-th engine
L - inertial axis
L.F - length of fluid tank
M - total mass of system
M.I - normalized mass of i-th bending mode
M.Ge - mass of e-th gimbaled engine
M.Ke - moment at gimbal for e-th gimbaled engine
N - number of degrees of freedom with respect to I-J axes
(Cont'd) GLOSSARY

$N_x$ - normal force coefficient (lbs per radian)

$N_{PD}$ - distributed normal force coefficient

$N_{vn}$ - normal force due to $n^{th}$ coordinate as a result of the velocity

$N_{an}$ - normal force due to $n^{th}$ coordinate as a result of aerodynamics

P - axis parallel to L axis but translating with c.g.

P - number of gimbaled engines

P - external force when used with subscript

Q - generalized coordinate when used with subscript

R - axis parallel to H axis but translating with c.g.

R - number of fluid tanks

$R_T$ - fluid tank radius

T - number of bending modes

$T_{Ge}$ - thrust of $e^{th}$ gimbaled engine

$T_{Pr}$ - thrust of $r^{th}$ fixed engine

$T_T$ - total thrust of all engines

T - used as subscript to denote the transpose of a matrix

$V_{cg}$ - velocity of center of mass at time t

$V_G$ - potential energy due to gravity

$V_F$ - internal potential energy

$\alpha$ - angle of attack of UEA

$\beta_0$ - angle of attack of $I$ axis

$\beta_T$ - thrust constant $T_T - D_T$

$\gamma$ - angle between local vertical and nominal trajectory

$\delta_e$ - deflection of $e^{th}$ gimbaled engine

$\delta_{ae}$ - position of actuator on $e^{th}$ gimbaled engine

$\Theta$ - rotation of $I-J$ axis in inertial space

$\Theta_P$ - position gyro angle

$\Theta_R$ - rate gyro angle

$\Delta_{ae}$ - engine angle of $e^{th}$ engine due to $a^{th}$ combined mode

$\zeta$ - perturbation of $V_{cg}$

$\omega_a$ - frequency of $a^{th}$ combined mode
(Cont'd)

GLOSSARY

\[ \mathcal{L} \] - frequency matrix
\[ \omega \] - rotation of mass element \ dm \ with respect to the R axis
\[ \psi \] - rotation of mass element \ dm \ with respect to the R axis
\[ \chi \] - rotation of mass element \ dm \ with respect to the USA
\[ \mathcal{M} \] - momentum
\[ \mathcal{W} \] - Kinetic energy
\[ \Delta I \] - inertia correction
\[ \rho \] - slosh amplitude in feet
\[ \zeta \] - damping ratio
\[ \phi \] - modal deflection
\[ \phi' \] - total modal slope
\[ \lambda \] - modal bending slope
\[ \omega \] - input modal frequency
\[ \mu_\alpha \] - rotational acceleration per unit angle of attack
\[ u_n \] - rotational acceleration per unit \ Q_n
\[ \sigma \] - real part of root
\[ \text{UEA} \] - undeformed elastic axis
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