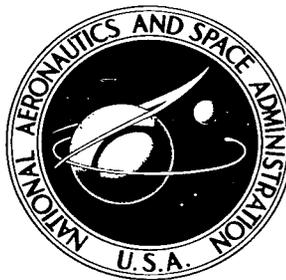


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SUMMARY

The role of the electrical conductivity in the stability of magnetohydrodynamic channel flow with parallel magnetic field is examined through exact numerical solution of the pertinent sixth-order system of disturbance equations throughout an extended range of magnetic Reynolds numbers. The results obtained indicate that the conductivity of the fluid acts as a stabilizing agent, as long as it is small, and as a destabilizing agent, if it is large. It is concluded that the conductivity reverses its role as a stabilizing agent at a magnetic Reynolds number of order unity. Examination of the contributions to a disturbance energy balance equation shows that the conductivity acts as a destabilizing agent by setting up a time-independent Maxwell stress that has the same sign as the vorticity of the basic flow.

INTRODUCTION

The hydromagnetic stability of laminar flows of electrically conducting fluids has been analyzed by many authors. In particular, it has been shown that, when a uniform magnetic field is imposed in the direction of the laminar flow, the flow is always more stable than in the absence of a magnetic field. This alignment of the magnetic field is especially significant and will be the configuration examined herein, since the mean motion of the fluid is not affected by the imposed magnetic field; therefore, the net effect of the magnetic field on the stability of a given velocity distribution can be investigated.

Previous treatments of this problem have not made clear the role of the conductivity of the fluid in the stability phenomenon. Previous investigators have considered two extreme limiting cases of the hydromagnetic stability problem; namely, the stability of a fluid of very low and very high electrical conductivity.

Stuart (ref. 1) treated the case of a fluid at a very low conductivity in a very strong magnetic field. His results indicated that the magnetic field

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stabilized the motion, but his treatment of the problem involved several simplifying assumptions, primarily the assumed smallness of the magnetic Reynolds number. These assumptions precluded any investigation of the effect of the conductivity alone on the stability phenomenon. Velikhov (ref. 2), on the other hand, treated the case of a fluid of very high electrical conductivity. His results showed that the minimum critical Reynolds numbers were markedly greater than at the low values of the conductivity examined by Stuart. This would lead one to believe that the stabilizing effect of the magnetic field is enhanced as the conductivity is increased. However, Velikhov found this trend reversed at the large values of the conductivity investigated; that is, an increase at these high values of the conductivity was destabilizing, and the minimum critical Reynolds number for infinite magnetic Reynolds number is below that obtained for large but finite magnetic Reynolds number. However, with Velikhov's simplifying assumptions, primarily the assumed largeness of the magnetic Reynolds number, again no indication could be obtained of when the conductivity reversed its role as a stabilizing agent.

In the present report, the role of the conductivity is clarified by examining the stability problem between the limiting cases of very low and very high conductivity. This is accomplished by solving the complete disturbance equations numerically without invoking the simplifying assumptions made in earlier treatments of this problem.

SYMBOLS

Vector components with a numerical index refer to dimensional quantities; vector components with a literal index refer to nondimensional quantities.

A	Alfven number, $\sqrt{B^2/\rho U_m^2 \mu_0}$
B	reference magnetic induction
b_1, b_2	disturbance magnetic induction
c	$c_r + ic_i$
c_i	time amplification factor of disturbance wave
c_r	phase speed
E	electric field
\mathcal{E}	disturbance energy
J	electric current density
k	wave number
L	reference length

N	magnetic interaction parameter, $\sigma B^2 L / \rho U_m$
P	pressure
Pr_m	magnetic Prandtl number, $\sigma \mu_0 \nu$
p	disturbance pressure
Re	Reynolds number, $U_m L / \nu$
Re_m	magnetic Reynolds number, $\sigma \mu_0 U_m L$
s	disturbance vorticity amplitude, $\phi'' - \alpha^2 \phi$
t	time
U	velocity
U_m	reference velocity
u_1, u_2	disturbance velocity components
x_1, x_2	Cartesian coordinates, dimensional
x, y	Cartesian coordinates, nondimensional
α	dimensionless wave number, $L k_1$
ξ	disturbance vorticity
θ	disturbance electric current density amplitude, $\psi'' - \alpha^2 \psi$
μ_0	permeability (vacuum value)
ν	fluid kinematic viscosity
Π	$P + (\vec{B} \cdot \vec{B}) / 2\mu_0$
π	disturbance total pressure amplitude
ρ	fluid density
σ	electrical conductivity
Φ	disturbance stream function
ϕ	disturbance stream function amplitude
Ψ	disturbance vector potential
ψ	disturbance vector potential amplitude

Subscripts:

i imaginary part

r real part

c differentiation with respect to c

cr critical

FORMULATION OF PROBLEM

The general equations of magnetohydrodynamics are the equations of electrodynamics for moving media and the Navier-Stokes equations modified to include the electromagnetic body force. For an incompressible fluid with a scalar conductivity the governing equations are (ref. 3)

$$\rho \left(\frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \nabla \vec{U} \right) = - \nabla P + \vec{J} \times \vec{B} + \rho \nu \nabla^2 \vec{U} \quad (1)$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (2)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad (3)$$

$$\vec{J} = \sigma (\vec{E} + \vec{U} \times \vec{B}) \quad (4)$$

$$\nabla \cdot \vec{U} = 0 \quad (5)$$

$$\nabla \cdot \vec{B} = 0 \quad (6)$$

The kinematic viscosity coefficient ν is taken to be a constant, and the excess charge density and displacement currents are neglected in the preceding equations in accordance with the usual magnetohydrodynamic approximations.

The steady flow under consideration is the flow with parabolic velocity profile between perfectly conducting parallel planes in the presence of a constant imposed magnetic field parallel to the fluid velocity.

The solution of the steady equations that satisfies the imposed conditions is

$$\vec{U} : [U_1(x_2), 0, 0]$$

$$\vec{B} : [B_1, 0, 0]$$

where B_1 is the constant imposed magnetic field and

$$U_1 = U_m \left(1 - \frac{x_2^2}{L^2} \right) \quad (7)$$

where $2L$ is the spacing between the planes and U_m is the velocity at the centerline ($x_2 = 0$). It is now proposed to consider the equations governing small two-dimensional disturbances $\vec{u} : (u_1, u_2, 0)$, $\vec{b} : (b_1, b_2, 0)$, and p , which are superimposed on the steady-state solution. There is no loss in generality in assuming that the disturbances are two-dimensional in nature, since, for the problem under consideration, the motion is always more stable for three-dimensional than for two-dimensional disturbances as pointed out by Stuart (ref. 1). This is the analogue of Squire's theorem (ref. 4) established in the ordinary incompressible parallel flow stability theory. Periodic sinusoidal disturbances are considered in which \vec{u} , \vec{b} , and p are all functions of x_2 multiplied by $\exp[ik_1(x_1 - c_1 t)]$, where c_1 is the complex phase velocity and k_1 is the wave number in the x_1 -direction. Since k_1 is always positive, the disturbances are amplified or damped according to whether the imaginary part of c_1 is positive or negative. If the imaginary part of c_1 is zero, the disturbance is neutrally stable. The steps leading to the disturbance equations are carried out in the appendix. This procedure consists of superimposing \vec{u} , \vec{b} , and p onto the steady-state solution, substituting them into the general equations (1) to (6), and linearizing the equations with respect to the small disturbances \vec{u} , b , and p . Elimination of the variables u_1 , b_1 , and p leads to two simultaneous ordinary differential equations for b_2 and u_2 . If $U_m \phi(x_2)$ and $B_1 \psi(x_2)$ denote quantities proportional to the amplitude functions of u_2 and b_2 , respectively, the following nondimensional simultaneous equations are found to govern ϕ and ψ

$$\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi = i\alpha \text{Re} [(U_x - c)(\phi'' - \alpha^2 \phi) - U_x'' \phi - A^2(\psi'' - \alpha^2 \psi)] \quad (8)$$

$$\psi'' - \alpha^2 \psi = i\alpha \text{Re}_m [(U_x - c)\psi - \phi] \quad (9)$$

where $x = x_1/L$, $y = x_2/L$, $c = c_1/U_m$, $\alpha = Lk_1$, $\text{Re} = U_m L/\nu$ is the Reynolds number, $\text{Re}_m = \sigma \mu_0 U_m L$ is the magnetic Reynolds number, and $A^2 = B_1^2 / \rho U_m^2 \mu_0$ is the Alfvén number squared. The primes denote differentiation with respect to y . It is to be noted here that, in the absence of a magnetic field ($A^2 = 0$), equation (8) reduces to the ordinary Orr-Sommerfeld equation.

Two new parameters enter into the consideration of the stability of the flow of an electrically conducting fluid; namely, the magnetic Reynolds number, which is proportional to the conductivity of the fluid, and the Alfvén number, which is proportional to the strength of the imposed magnetic field.

Equations (8) and (9) and the boundary conditions (appropriate for perfectly conducting planes which bound the flow)

$$y = \pm 1 : \phi = \phi' = \psi = 0 \quad (10)$$

constitute an eigenvalue problem, which, for α , Re , Re_m , and A^2 fixed, consists of determining c in order to satisfy the boundary conditions.

Earlier treatments of the eigenvalue problem have considered the limiting

cases $Re_m \ll 1$ (ref. 1) or $Re_m \gg 1$ (ref. 2). In both limiting cases, the governing equations can be simplified somewhat. The resulting simplified equations are more amenable to solution than the coupled equations (8) and (9). However, if it is proposed to examine the behavior of the solutions of equations (8) and (9) between the limiting values of $Re_m \ll 1$ and $Re_m \gg 1$, the complete set of equations must be considered.

Simplified Disturbance Equations

The assumption underlying the simplified treatment of the eigenvalue problem in reference 1 will now be examined. For many electrically conducting fluids or slightly ionized gases used in laboratory experiments, Re_m is usually very small. However, the assumption $Re_m \ll 1$ is by itself not sufficient to derive Stuart's form of the disturbance equations. Stuart's simplified disturbance equation can be obtained from equations (8) and (9) by following the method of Tatsumi (ref. 3); namely, elimination of ϕ , making the transformation $\theta = \psi'' - \alpha^2\psi$, and performing the limiting process $Re_m \rightarrow 0$ but A^2Re_m finite. These steps result in the following equation for θ :

$$\theta'''' - 2\alpha^2\theta'' + \alpha^4\theta = i\alpha Re \left[(U_x - c)(\theta'' - \alpha^2\theta) - U_x''\theta + i\alpha A^2 Re_m \theta \right] \quad (11)$$

The boundary conditions for equation (11) follow from equations (8) and (9) and equation (10) if the limiting form of these equations is considered. The boundary conditions are

$$y = \pm 1 : \theta = \theta' = 0 \quad (12)$$

The transformation and subsequent limiting process clearly show the nature of the assumptions underlying this simpler eigenvalue problem. In terms of physical properties, the assumptions are the flow of a fluid of low conductivity and, in addition, under a very strong magnetic field. It is to be noted here that the quantity $A^2Re_m = \sigma B_1^2 L / \rho U_m \equiv N$ is referred to as the magnetic interaction parameter. When this parameter is negligibly small, equation (11) is identical to the Orr-Sommerfeld equation. Other properties of this extreme limiting case are that the two parameters A^2 and Re_m have been collapsed into a single parameter, $N = A^2Re_m$, and that the order of the coupled differential system (eqs. (8) and (9)) has been reduced from six to four. Also the six boundary conditions (eq. (10)) have been used in conjunction with equation (9) to formulate the four boundary conditions (eq. (12)) for the case of the infinitely conducting wall.

Numerical Solution of Eigenvalue Problem

The numerical solution of the eigenvalue problems formulated will now be considered. The general method used is quite analogous to the method established in reference 5 for solving the Orr-Sommerfeld equation; therefore, it will be sufficient to give only a brief account of the steps required to generalize the established method in order to handle this problem. Furthermore, since equation (11) is a limiting form of equations (8) and (9), the present account will be concerned with the more general system (eqs. (8) and (9))

together with the boundary conditions of equation (10).

Before describing the numerical method, a simplification of the problem, which halves the range of integration, should be pointed out. This simplification is based on the observation that the disturbances in equations (8) and (9) can be separated into even and odd functions, since the variable coefficients in these equations, namely U_x and U_x'' , are even functions of y . In ordinary hydrodynamic stability theory, the even solution, which has the simpler flow pattern, usually gives a lower minimum critical Reynolds number and will be the solution examined herein. This simplification enables one to consider only even solutions in half of the channel through introduction of appropriate symmetry conditions at $y = 0$. The new boundary conditions for equations (8) and (9) are

$$y = 0 : \varphi' = \varphi''' = \psi' = 0 \quad (13)$$

$$y = 1 : \varphi = \varphi' = \psi = 0 \quad (14)$$

The approach to the eigenvalue problem for fixed α , Re , A^2 , and Re_m used herein is to find values of $c = c_r + ic_i$ (eigenvalues) for which equations (8) and (9) have solutions (eigenfunctions) that satisfy the boundary conditions equations (13) and (14).

In order to find an eigenvalue, the following iterative procedure is carried out. A trial solution of equations (8) and (9) is obtained by numerical integration. In order to integrate numerically, additional boundary values are specified at both boundaries $y = 0$ and $y = 1$, and a value of c is specified. Then both integrals are stepped in toward the middle $y = 1/2$. The additional boundary values and the eigenvalue c have to be adjusted in order to match the solutions in the middle. After a trial integration, subsequent boundary values and eigenvalues are automatically calculated by the Newton-Raphson technique of finding successive approximations.

The various elements that enter into the preceding iterative procedure will now be put forward. Instead of solving equations (8) and (9) as they stand, a system of second-order differential equations is formulated by introducing the disturbance vorticity amplitude function $s = \varphi'' - \alpha^2\varphi$. In terms of the vorticity amplitude, the system (eqs. (8) and (9)) goes over to

$$\varphi'' = \alpha^2\varphi + s \quad (15)$$

$$\psi'' = \alpha^2\psi + i\alpha Re_m [(U_x - c)\psi - \varphi] \quad (16)$$

$$s'' = \alpha^2s + i\alpha Re [(U_x - c)s - U_x''\varphi - A^2(\psi'' - \alpha^2\psi)] \quad (17)$$

In order to obtain a system of equations in which derivatives of the dependent variables do not appear on the right side, ψ'' is eliminated from equation (17) by means of equation (16) giving

$$s'' = \alpha^2s + i\alpha Re [(U_x - c)(s - i\alpha N\psi) - (U_x'' - i\alpha N)\varphi] \quad (18)$$

where

$$N = A^2 \text{Re}_m$$

Equations (15), (16), and (18) are solved subject to the boundary conditions (eqs. (13) and (14)) that employ the definition of the vorticity amplitude function.

For a forward solution (y increasing) starting at $y = 0$, the initial values are specified according to the following table

$y = 0 :$	φ	φ'	s	s'	ψ	ψ'
	1	0	$s(0)$	0	$\psi(0)$	0

For a backward solution (y decreasing) starting at $y = 1$, the initial values are

$y = 1$	φ	φ'	s	s'	ψ	ψ'
	0	0	$s(1)$	$s'(1)$	0	$\psi'(1)$

The additional boundary values that have to be specified in order to integrate numerically are those enumerated in the second row of each table. The condition $\varphi(0) = 1$ is a normalizing condition and fixes the size of the whole solution. With the preceding boundary values specified and a value of c specified, trial solutions of equations (15), (16), and (18) are obtained. Next, the process of matching at a common point is carried out. The matching point was taken to be $y = 1/2$ to equalize the numerical errors that grow in proportion to the number of integration steps taken. At the matching point, the solution must be continuous. The requirements of continuity lead to six conditions that have to be satisfied at $y = 1/2$. The six quantities $S(0)$, $\psi(0)$, $s(1)$, $s'(1)$, $\psi'(1)$, and c have to be adjusted in order to satisfy these conditions. Let the difference between the forward and backward solutions evaluated at $y = 1/2$ be designated by $g_i (i = 1 \dots 6)$ and set up a correspondence between the differences and the various functions according to the following table:

φ	φ'	s	s'	ψ	ψ'
g_1	g_2	g_3	g_4	g_5	g_6

Now, the g_i depend on the adjustable parameters (the additional boundary conditions and c). The Newton-Raphson method is used to fulfill the condition of continuity at the matching point (i.e., all $g_i = 0$).

For ease in writing, the various adjustable parameters are designated according to the following scheme:

$s(0)$	$\psi(0)$	$s(1)$	$s'(1)$	$\psi'(1)$	c
z_1	z_2	z_3	z_4	z_5	z_6

If the chosen values z_i produce a solution that gives all g_i as approximately zero, a better approximation is obtained by starting with $z_i + \Delta z_i$. The quantities Δz_i are solutions of the equations

$$0 = g_i + \sum_{j=1}^5 \frac{\partial g_i}{\partial z_j} \Delta z_j \quad (i = 1 \dots 6) \quad (19)$$

The partial derivatives $\partial g_i / \partial z_j$ are obtained by solving additional initial-value problems. All the partial derivatives with respect to the additional boundary conditions $z_i (i = 1 \dots 5)$ can be obtained by solving equations (15), (16) and (18) with appropriate initial conditions since these equations are linear. For example, derivatives with respect to $s(0)$ of the variables in the forward integration are obtained by solving equations (15), (16), and (18) starting at $y = 0$ with initial conditions given according to the following table:

$y = 0 :$	φ	φ'	s	s'	ψ	ψ'
	0	0	1	0	0	0

For the backward integration, the variables are taken to be independent of $s(0)$. For derivatives with respect to c , a different system of equations has to be solved. This system is obtained by partial differentiation of the terms in equations (15), (16), and (18) with respect to c . The quantities

$$\frac{\partial \varphi}{\partial c} \equiv \varphi_c, \quad \frac{\partial s}{\partial c} \equiv s_c, \quad \text{and} \quad \frac{\partial \psi}{\partial c} \equiv \psi_c$$

satisfy the system of equations

$$\varphi_c'' = \alpha^2 \varphi_c + s_c \quad (20)$$

$$\psi_c'' = \alpha^2 \psi_c + i\alpha \text{Re}_m [(U_x - c)\psi_c - \varphi_c - \psi] \quad (21)$$

$$s_c'' = \alpha^2 s_c + i\alpha \text{Re} [(U_x - c)(s_c - i\alpha N \psi_c) - (U_x'' - i\alpha N)\varphi_c - (s - i\alpha N \psi)] \quad (22)$$

The appropriate initial conditions for equations (20) to (22) are homogeneous initial conditions for both the forward and backward integrations.

After obtaining a trial solution and the various partial derivatives, equation (19) can be solved for the corrections. Successive applications of this procedure should converge to an eigenvalue c .

The procedure just outlined for finding an eigenvalue c for α, Re, A^2 ,

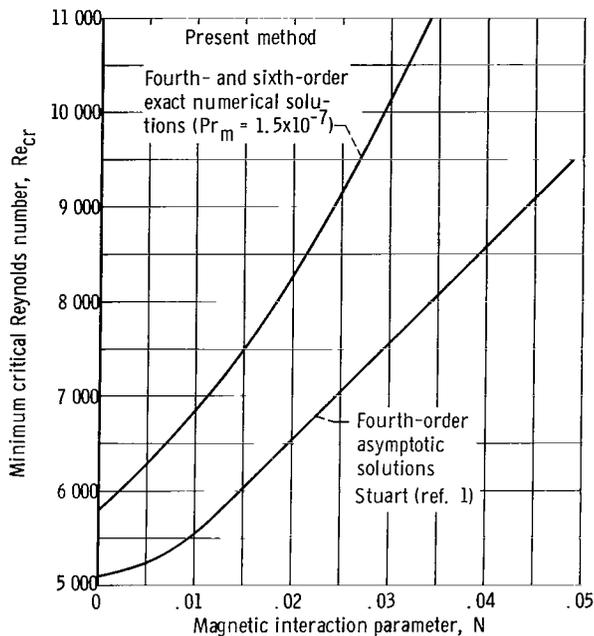


Figure 1. - Minimum critical Reynolds number.

and Re_m fixed was programed for solution on the IBM 7094 computer located at the Lewis Research Center.

In reference 5, the question of the accuracy and rate of convergence of the method is discussed with regard to solutions of the Orr-Sommerfeld equation. Also, several simplifications are pointed out there that reduce the amount of labor that is required in the application of the method.

Carrying out the foregoing procedure for a system of second-order equations enables one to obtain exact numerical solutions of equations (8) and (9), and equation (11).

SOLUTIONS

Solutions for $Re_m \ll 1$

A comparison will now be made of the exact numerical solutions of equations (8) and (9), the sixth-order system; the exact numerical solutions of equation (11), the fourth-order system; and Stuart's asymptotic solution of the fourth-order system. This comparison is made under the conditions that should satisfy the assumptions made by Stuart in going from the sixth-order system to the fourth-order system. These conditions would correspond to the flow of mercury with a magnetic Prandtl number Pr_m of 1.5×10^{-7} . For a Reynolds number of 10^4 , the value of the magnetic Reynolds number would be about 1.5×10^{-3} .

Figure 1 shows the minimum critical Reynolds number as a function of the interaction parameter N corresponding to the solutions of the three eigenvalue problems. The significance of the minimum critical Reynolds number is that all disturbances will be damped in flows with Reynolds numbers below the minimum critical value. The exact numerical solutions of the sixth- and fourth-order systems yield identical results. Thus, it can be inferred that Stuart's truncation from the sixth-order to the fourth-order system is justified when $Re_m \ll 1$.

Also shown in figure 1 are Stuart's results for this case. The difference in the values of Re_{cr} at the same value of the interaction parameter is due only to the difference in the methods used to solve equation (11). The results obtained by Stuart differ from those obtained by the present method by as much as 30 percent. Furthermore, the value of Re_{cr} obtained by the present method for $N = 0$, in which case equation (11) reduces to the Orr-Sommerfeld equation, agrees with the value of Re_{cr} , which Thomas (ref. 6) obtained by solving the Orr-Sommerfeld equation.

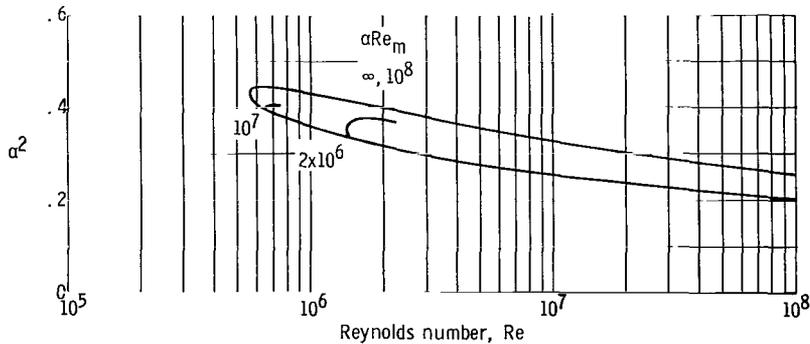


Figure 2. - Neutral stability diagrams for very large magnetic Reynolds numbers (from ref. 2). Alfven number, 0.08.

Solutions for Re_m Not Small

The solution of the complete eigenvalue problem corresponding to the differential equations (8) and (9) (i.e., the sixth-order system) and boundary conditions will now be discussed for the case where Re_m is not restricted to be small. The case $Re_m \gg 1$ was solved by Velikhov by obtaining asymptotic solutions of equations (8) and (9) at large values of αRe and αRe_m . The results obtained by Velikhov for an Alfven number, $A = 0.08$, are shown in figure 2. As may be seen, the minimum critical Reynolds numbers are considerably higher than those for $Re_m \ll 1$ shown in figure 1. Hence, it could be concluded that the stabilizing effect is enhanced as Re_m is increased. However, as can be seen from figure 2, the higher Re_m yield the lower Re_{cr} . As pointed out by Tatsumi (ref. 3), it seems probable that there exists some value of Re_m above which the role of conductivity as a stabilizing agent is reversed. From the various factors considered subsequently and the results of the next section on the balance of disturbance energy, it appears that the conductivity reverses its role as a stabilizing agent at values of Re_m of order unity.

The question of reversal of the stabilizing effect of conductivity is examined by fixing Re and α and determining the variation of c_i for increasing Re_m in numerical solutions of equations (8) and (9). A destabilizing effect of increasing Re_m will be evidenced by c_i increasing. This would be equivalent to Velikhov's calculated decrease of Re_{cr} as Re_m is increased. Velikhov, however, made his calculations after passing to the limit of large Re_m and hence could not give an indication of when the conductivity reversed its role as a stabilizing agent. In the present study, on the other hand, the entire range of Re_m from $Re_m \ll 1$ to $Re_m \gg 1$ has been investigated by varying the magnetic Prandtl number.

The results of the present calculations are shown in figure 3. These calculations were carried out at a fixed point in the α, Re plane; that is, $\alpha = 1, Re = 9000$. Since Re was held constant, each value of Re_m shown in figure 3 corresponds to a different magnetic Prandtl number, Pr_m ($Pr_m = Re_m/Re = \mu\sigma\nu$). Scales of both quantities Re_m and Pr_m are shown

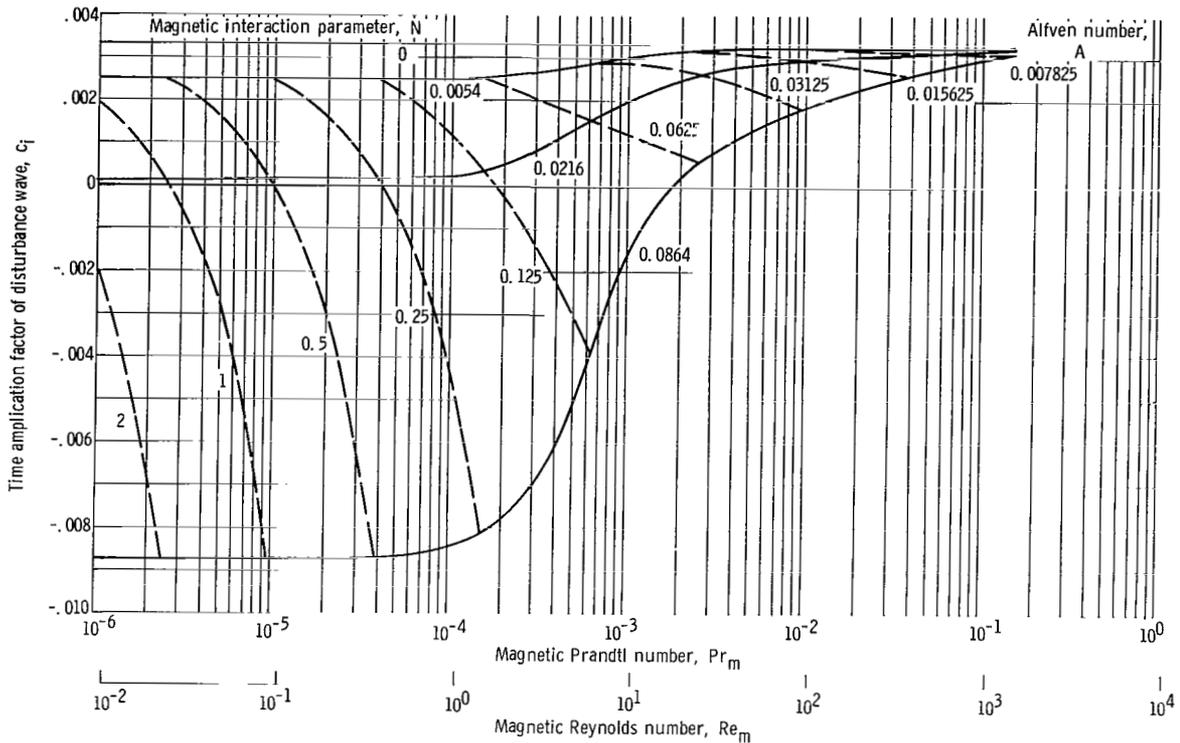


Figure 3. - Effect of increasing magnetic Reynolds number. Nondimensional wave number, 1; Reynolds number, 9000.

as abscissae in figure 3, where c_1 , the time amplification factor is shown as the ordinate. Curves are shown both for constant values of the magnetic interaction parameter N , and for the Alfvén number A , which is directly proportional to the magnetic field strength.

The results presented in figure 3 may be interpreted in the following manner: Imagine, for example, a flow in a given channel where the fluid viscosity and flow velocity are fixed and the flow is subjected to disturbances of a given wavelength. This corresponds to a fixed point in the α, Re plane. In this example, the conductivity of the fluid and the strength of the imposed magnetic field can be altered at will. The point $\alpha = 1, Re = 9000$ is an unstable point in the absence of a magnetic field, as shown by the curve $A = 0, N = 0$ in figure 3.

To be noted immediately is that, for an electrically conducting fluid, the presence of any magnetic field tends to reduce the amplification rate below that obtained in the absence of a magnetic field. This is true regardless of whether the magnetic Reynolds number is large or small. For a given magnetic Reynolds number, the damping rate relative to the nonmagnetic case increases monotonically with an increase in magnetic field strength; for a given magnetic field strength (represented by constant A in our example) the damping rate increases monotonically with increasing interaction parameter. The magnitudes of these variations, however, depend on whether the magnetic Reynolds number is small or large.

For a constant interaction parameter N , the variation of amplification rate c_i with magnetic Reynolds number is negligible until Re_m increases past order unity, whence it increases sharply relative to the value for $Re_m \ll 1$ and gradually approaches the value in the nonmagnetic case as Re_m becomes very large. The present results for $Re_m \ll 1$ are consistent with those obtained from the fourth-order system (eq. (11)). It is seen that c_i depends only on the single parameter N when $Re_m \ll 1$, but varies also with Re_m when Re_m exceeds order unity. As Re_m increases at constant N , the growth rates increase, but are always less than in the nonmagnetic case.

For constant A , which in our example is interpreted as constant magnetic field, increasing the conductivity always gives damping. However, there is substantial increase in the damping rate with the magnetic Reynolds number when Re_m is small; for large magnetic Reynolds number, an increase in Re_m tends to increase the damping only slightly.

A more detailed physical description of the nature of the stability problem as Re_m increases past unity may be obtained by examining the terms in a disturbance energy balance. This examination is made subsequently.

The general situation here in which two transport properties enter, namely, the electrical conductivity and the viscosity, is quite analogous to the situation in which the viscosity coefficient alone enters. In ordinary hydrodynamic stability, it is known that viscous mechanisms are stabilizing (the dissipation) as long as the viscosity is large enough (Re small enough). It is only when the viscosity becomes small enough (Re large enough) that viscous mechanisms come into play which destabilize the resulting motion by providing a means of transferring energy to the disturbance motion through the Reynolds stress. In magnetohydrodynamic stability, the role played by the resistivity (the reciprocal of the conductivity) is analogous to the role played by the viscosity in ordinary hydrodynamic stability; that is, at small Re_m (high resistivity) the conductivity σ is primarily stabilizing because of the predominance of the joule heating. At large Re_m , the conductivity tends to be destabilizing under certain conditions because there is a mechanism that can transfer energy to the disturbance motion.

DISTURBANCE ENERGY BALANCE

Formulation of a disturbance energy balance and subsequent examination of the various contributions to the rate of change of disturbance energy gives an insight into the mechanics of the disturbance motion and enables one to identify the various processes that are operating.

The starting point for the derivation of the disturbance energy balance equation are equations (A9) to (A12). Multiplying the terms in equations (A9) by u_x , (A10) by u_y , (A11) by $A^2 b_x$, and (A12) by $A^2 b_y$, adding, and performing some algebraic manipulations yield (for $B_x = 1$):

$$\begin{aligned} \frac{D}{Dt} \mathcal{E} = & - (u_x u_y - A^2 b_x b_y) \frac{dU_x}{dy} - \frac{1}{Re} \zeta^2 - \frac{A^2}{Re_m} j^2 + \left\{ A^2 \frac{\partial}{\partial x} (b_x u_x + b_y u_y) \right. \\ & \left. - \frac{\partial}{\partial x} (u_x \pi) - \frac{\partial}{\partial y} (u_y \pi) + \frac{1}{Re} \left[\frac{\partial}{\partial x} (u_y \zeta) - \frac{\partial}{\partial y} (u_x \zeta) \right] + \frac{A^2}{Re_m} \left[\frac{\partial}{\partial x} (b_y j) - \frac{\partial}{\partial y} (b_x j) \right] \right\} \end{aligned} \quad (23)$$

where

$$\mathcal{E} \equiv \frac{u_x^2 + u_y^2}{2} + A^2 \frac{b_x^2 + b_y^2}{2}$$

and

$$\frac{D\mathcal{E}}{Dt} \equiv \frac{\partial \mathcal{E}}{\partial t} + U_x \frac{\partial \mathcal{E}}{\partial x}$$

The quantity \mathcal{E} is the dimensionless kinetic energy of the fluid plus the energy of the magnetic field, all referred to the kinetic energy of the mean flow. The quantity $\zeta = \partial u_y / \partial x - \partial u_x / \partial y$ is the dimensionless vorticity of the disturbance flow, and $j = \partial b_y / \partial x - \partial b_x / \partial y$ is the dimensionless curl of the disturbance magnetic field, which by Amperes law is equal to the disturbance electrical current density.

Equation (23) gives the time rate of increase of the disturbance energy (per unit volume) of a fluid element that moves with the basic flow. The terms in this equation are to be integrated over a cell that extends across the channel in the y -direction and along the channel for a distance of wavelength. All terms in the braces in equation (23) vanish either because a disturbance quantity vanishes at the boundaries of the flow for the integration across the channel or because of periodicity in the direction along the channel. The equation giving the time rate of increase of energy of the disturbance motion is therefore:

$$\begin{aligned} \frac{D}{Dt} \int_{x=0}^{2\pi/\alpha} \int_{y=-1}^1 \mathcal{E} \, dx \, dy = & \int_{x=0}^{2\pi/\alpha} \int_{y=-1}^1 dx \, dy \left(-u_x u_y \frac{dU_x}{dy} \right. \\ & \left. + A^2 b_x b_y \frac{dU_x}{dy} - \frac{1}{Re} \zeta^2 - \frac{A^2}{Re_m} j^2 \right) \end{aligned} \quad (24)$$

Of course, it is the real part of each disturbance amplitude that is required in equation (24).

It is appropriate at this time to identify the various factors on the right side of equation (24), which contribute to the rate of change of the disturbance energy. As in ordinary hydrodynamic stability theory, there appear the Reynolds stress term and the viscous dissipation term, the first and third

terms, respectively, on the right side of equation (24). As is well known, the dissipation term is always stabilizing, and the Reynolds stress term can be either stabilizing or destabilizing. The new terms that appear in the stability of a conducting fluid are the second and fourth terms. The fourth term $(-A^2 j^2 / \text{Re}_m)$ can be identified as joule heating and appears in the energy balance equation in such a way that this term will always be stabilizing. The quantity $b_x b_y$ in the second term can be identified as a component of the Maxwell stress tensor of electromagnetic theory. The question of whether this new term is stabilizing or destabilizing can be answered by solving the eigenvalue problem and computing the eigenfunctions.

The disturbance quantities on the right side of equation (24) can be expressed in terms of the complex amplitude functions ϕ and ψ by means of the relations

$$u_x = \text{Rl} \left\{ \phi'(y) \exp[i\alpha(x - ct)] \right\} \quad (25)$$

$$u_y = \text{Rl} \left\{ -i\alpha\phi(y) \exp[i\alpha(x - ct)] \right\} \quad (26)$$

$$b_x = \text{Rl} \left\{ \psi'(y) \exp[i\alpha(x - ct)] \right\} \quad (27)$$

$$b_y = \text{Rl} \left\{ -i\alpha\psi(y) \exp[i\alpha(x - ct)] \right\} \quad (28)$$

where Rl denotes the real part of the complex quantity. The integration with respect to x can be performed in equation (24) after substitution of equations (25) to (28). For neutral disturbances ($c_i = 0$)

$$\begin{aligned} \int_{y=-1}^1 \int_{x=0}^{2\pi/\alpha} \mathcal{E} \, dx \, dy = 2\pi \int_{y=0}^1 dy \left\{ -(\phi_r' \phi_i - \phi_r \phi_i') \frac{dU_x}{dy} \right. \\ \left. + A^2 (\psi_r' \psi_i - \psi_r \psi_i') \frac{dU_x}{dy} - \frac{1}{\alpha \text{Re}} \left[(\phi_r'' - \alpha^2 \phi_r)^2 + (\phi_i'' - \alpha^2 \phi_i)^2 \right] \right. \\ \left. - \frac{A^2}{\alpha \text{Re}_m} \left[(\psi_r'' - \alpha^2 \psi_r)^2 + (\psi_i'' - \alpha^2 \psi_i)^2 \right] \right\} \quad (29) \end{aligned}$$

The range of integration is taken from $y = 0$ to 1 since all the integrands are even functions. For other than neutral disturbances, it is only necessary to multiply the right side of equation (29) by factor $\exp(2\alpha c_i t)$ in order to obtain the time rate of change of disturbance energy.

In view of the conclusion drawn in the previous section concerning the role played by the conductivity as a stabilizing agent, it is of interest to compare the various terms that contribute to the disturbance energy balance when Re_m is small and when it is large. Figure 4 shows the distribution across half the channel of the electromagnetic terms in the energy balance equation for the two cases $\text{Re}_m = 0.00135$ and 88.5 ($N = 0.0216$ for both). The area under these curves represents the net contribution to the rate of increase of the disturbance energy. For $\text{Re}_m = 0.00135$, the Maxwell stress is negligible and

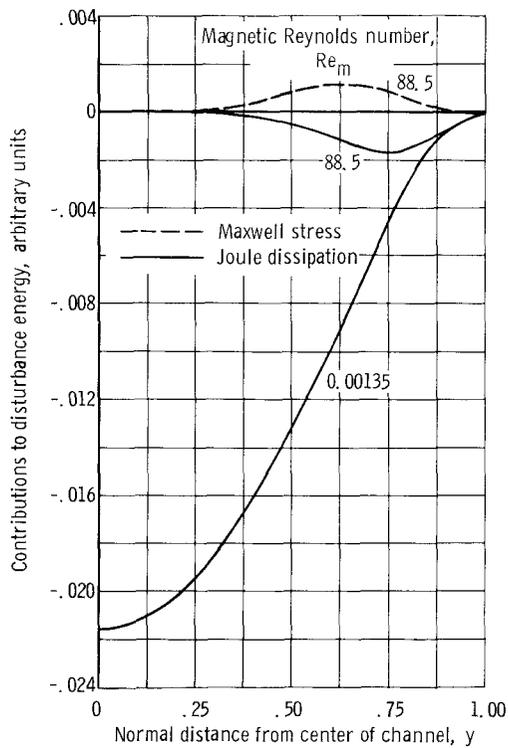


Figure 4. - Maxwell stress and Joule dissipation. Nondimensional wave number, 1; Reynolds number, 9000; magnetic interaction parameter, 0.0216.

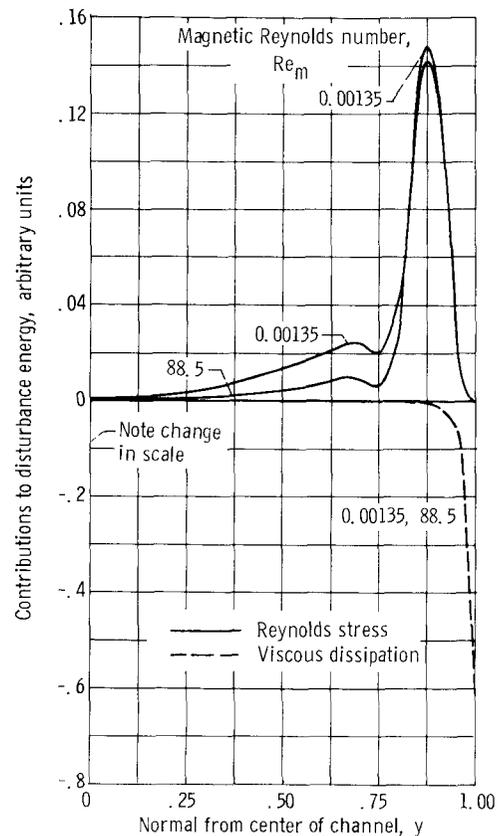


Figure 5. - Reynolds stress and viscous dissipation. Nondimensional wave number, 1; Reynolds number, 9000; magnetic interaction parameter, 0.0216.

is not plotted, but the joule dissipation contributes substantially to the stabilization of the motion (the area under the curve being negative in this case). For $Re_m = 88.5$, the Maxwell stress becomes significant and is destabilizing. Of course, the joule heating is again stabilizing, but the contribution is much less than in the previous case.

Figure 5 shows the distribution across half the channel of the hydrodynamic terms in the energy balance equation for the two cases. The distribution of the viscous dissipation energy is very closely the same for the two Re_m values, but, for the Reynolds stress energy distributions, there is less area under the curve for $Re_m = 88.5$ than for $Re_m = 0.00135$. However, this decrease in production of disturbance energy by the Reynolds stress is offset by the destabilizing effects of the decrease in the joule dissipation and the increase of disturbance energy by the Maxwell stress (fig. 4). The net effect of increasing the conductivity is then a consequence of the balance between the Reynolds stress mechanism on the one hand and the joule dissipation and Maxwell stress mechanism on the other hand. This net effect can be determined by estimating the areas under the curves in figures 4 and 5. This comparison leads to the result that, at a fixed point in the α, Re plane, the net effect of increasing the conductivity N (the magnetic interaction parameter) constant is destabilizing in agreement with the results presented in figure 3. As the Re_m

is raised, the behavior of the various terms in the energy balance equation under the conditions mentioned previously can be summarized as follows: Reynolds stress: stabilizing in that there is less production of disturbance energy; viscous dissipation: negligible change; Maxwell stress: destabilizing, production of disturbance energy occurs only at high values of Re_m ; joule dissipation: destabilizing, the amount of energy that can be dissipated is reduced as the conductivity is raised. If N is not held constant but is allowed to increase as Re_m increases, the flow is again stabilized as observed by Velikov (ref. 2).

CONCLUSIONS

The stability of plane magnetohydrodynamic channel flow with parallel magnetic field has been reexamined through exact numerical integration of the pertinent sixth-order system of disturbance equations, subject to appropriate boundary conditions. The results confirm that Stuart's reduction of the problem to a fourth-order disturbance equation is valid for magnetic Reynolds number small compared with 1. However, Stuart's asymptotic values are about 30 percent below the present numerical results. For magnetic Reynolds numbers of order 1 or greater, there are significant changes in the stability characteristics. This is borne out by a calculation of the various viscous and magnetic contributions to the rate of change of disturbance energy.

It is shown that the resistivity enters this problem in two ways: (1) it sets up a time-independent Maxwell stress that augments the disturbance energy when it is of the same sign as the vorticity of the basic flow, and (2) through joule dissipation the disturbance energy is decreased. For small resistivity ($Re_m \gg 1$) the former dominates, leading to a net augmentation of disturbance energy, while for large resistivity ($Re_m \ll 1$) the dissipative effect is dominant.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, September 22, 1965.

APPENDIX - DERIVATION OF DISTURBANCE EQUATIONS

Although the disturbance equations used herein appear in the literature (e.g., ref. 3), they must be displayed in component form in order to formulate a disturbance energy balance equation. Since this form appears in the process of deriving the disturbance equations, a brief derivation of these equations follows.

The starting point for the derivation is equations (1) to (6). Elimination of \vec{J} from equations (1) and (3) results in

$$\rho \left(\frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \nabla \vec{U} \right) = -\nabla \Pi + \frac{1}{\mu_0} \vec{B} \cdot \nabla \vec{B} + \rho \nu \nabla^2 \vec{U} \quad (A1)$$

where

$$\Pi = P + \frac{\vec{B} \cdot \vec{B}}{2\mu_0}$$

Since two-dimensional disturbances are being considered, only a two-dimensional magnetic field need be considered. A second equation is obtained by the elimination of \vec{E} and \vec{J} from equations (2) to (4) with the aid of equations (5) and (6) and results in

$$\frac{\partial \vec{B}}{\partial t} + \vec{U} \cdot \nabla \vec{B} = \vec{B} \cdot \nabla \vec{U} + \frac{1}{\sigma \mu_0} \nabla^2 \vec{B} \quad (A2)$$

The stability investigation is limited to mean flows that satisfy equation (7).

The disturbance equations are obtained by introducing a two-dimensional disturbance into equations (5), (6), (A1), and (A2); that is, let

$$\vec{U} \rightarrow \vec{U} + \vec{u} : (U_1 + u_1, u_2, 0)$$

$$\vec{B} \rightarrow \vec{B} + \vec{b} : (B_1 + b_1, b_2, 0)$$

$$P \rightarrow P + p$$

where the lower-case quantities are small perturbations of the basic quantity. After this is accomplished and after subtracting out the basic flow, the disturbance equations are

$$\rho \left(\frac{\partial u_1}{\partial t} + u_2 \frac{dU_1}{dx_2} + U_1 \frac{\partial u_1}{\partial x_1} \right) = - \frac{\partial \pi}{\partial x_1} + \frac{1}{\mu_0} B_1 \frac{\partial b_1}{\partial x_1} + \rho \nu \nabla^2 u_1 \quad (A3)$$

$$\rho \left(\frac{\partial u_2}{\partial t} + U_1 \frac{\partial u_2}{\partial x_1} \right) = - \frac{\partial \pi}{\partial x_2} + \frac{1}{\mu_0} B_1 \frac{\partial b_2}{\partial x_1} + \rho \nu \nabla^2 u_2 \quad (A4)$$

$$\frac{\partial b_1}{\partial t} + U_1 \frac{\partial b_1}{\partial x_1} = b_2 \frac{dU_1}{dx_2} + B_1 \frac{\partial u_1}{\partial x_1} + \frac{1}{\sigma\mu_0} \nabla^2 b_1 \quad (A5)$$

$$\frac{\partial b_2}{\partial t} + U_1 \frac{\partial b_2}{\partial x_1} = B_1 \frac{\partial u_2}{\partial x_1} + \frac{1}{\sigma\mu_0} \nabla^2 b_2 \quad (A6)$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \quad (A7)$$

$$\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} = 0 \quad (A8)$$

where

$$\pi = p + \frac{\vec{B} \cdot \vec{b}}{\mu_0}$$

It is convenient to cast the disturbance equations into nondimensional form by making the following replacement of variables:

$$x_1 \rightarrow xL, \quad x_2 \rightarrow yL, \quad t \rightarrow \frac{L}{U_m} t$$

$$U_1 \rightarrow U_x U_m, \quad B_1 \rightarrow B_x B$$

$$u_1 \rightarrow u_x U_m, \quad u_2 \rightarrow u_y U_m, \quad b_1 \rightarrow b_x B, \quad b_2 \rightarrow b_y B$$

$$\pi \rightarrow (\rho U_m^2) \pi$$

where L , U_m , and B are fixed reference dimensional quantities. The disturbance equations in nondimensional form are

$$\frac{\partial u_x}{\partial t} + u_y \frac{dU_x}{dy} + U_x \frac{\partial u_x}{\partial x} = - \frac{\partial \pi}{\partial x} + A^2 B_x \frac{\partial b_x}{\partial x} + \frac{1}{Re} \nabla^2 u_x \quad (A9)$$

$$\frac{\partial u_y}{\partial t} + U_x \frac{\partial u_y}{\partial x} = - \frac{\partial \pi}{\partial y} + A^2 B_x \frac{\partial b_y}{\partial x} + \frac{1}{Re} \nabla^2 u_y \quad (A10)$$

$$\frac{\partial b_x}{\partial t} + U_x \frac{\partial b_x}{\partial x} = b_y \frac{dU_x}{dy} + B_x \frac{\partial u_x}{\partial x} + \frac{1}{Re_m} \nabla^2 b_x \quad (A11)$$

$$\frac{\partial b_y}{\partial t} + U_x \frac{\partial b_y}{\partial x} = + B_x \frac{\partial u_y}{\partial x} + \frac{1}{Re_m} \nabla^2 b_y \quad (A12)$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \quad (\text{A13})$$

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0 \quad (\text{A14})$$

where

$$A^2 = \frac{B^2}{\rho U_m^2 \mu_0}$$

is the Alfvén number squared

$$\text{Re} = \frac{U_m L}{\nu}$$

is the Reynolds number, and

$$\text{Re}_m = \sigma \mu_0 U_m L$$

is the magnetic Reynolds number.

Besides these basic nondimensional quantities, certain other combinations of them are often used; namely,

$$\text{Pr}_m = \sigma \mu_0 \nu$$

the magnetic Prandtl number and

$$N = \frac{\sigma B^2 L}{\rho U_m}$$

the magnetic interaction parameter. The following relations apply:

$$\text{Re}_m = \text{Pr}_m \cdot \text{Re}$$

and

$$N = A^2 \text{Re}_m$$

The number of dependent variables can be reduced by introducing a stream function for the disturbance velocity and a vector potential for the magnetic field. Let

$$u_x = \frac{\partial \Phi}{\partial y} \quad (\text{A15})$$

$$u_y = - \frac{\partial \Phi}{\partial x} \quad (\text{A16})$$

$$b_x = \frac{\partial \Psi}{\partial y} \quad (A17)$$

$$b_y = -\frac{\partial \Psi}{\partial x} \quad (A18)$$

Substitution of equations (A15) to (A18) into equations (A9) to (A12) leads to the following disturbance equations:

$$\phi_{yt} - \frac{dU_x}{dy} \phi_x + U_x \phi_{xy} = -\frac{\partial \pi}{\partial x} + A^2 B_x \psi_{yx} + \frac{1}{Re} (\phi_{yxx} + \phi_{yyy}) \quad (A19)$$

$$\phi_{xt} + U_x \phi_{xx} = +\frac{\partial \pi}{\partial y} + A^2 B_x \psi_{xx} + \frac{1}{Re} (\phi_{xxx} + \phi_{xyy}) \quad (A20)$$

$$\psi_{yt} + U_x \psi_{xy} = -\frac{dU_x}{dy} \psi_x + B_x \phi_{xy} + \frac{1}{Re_m} (\psi_{yxx} + \psi_{yyy}) \quad (A21)$$

$$\psi_{xt} + U_x \psi_{xx} = B_x \phi_{xx} + \frac{1}{Re_m} (\psi_{xxx} + \psi_{xyy}) \quad (A22)$$

Elimination of π between equations (A19) and (A20) by cross differentiations leads to

$$\begin{aligned} \phi_{yyt} + \phi_{xxt} + U_x (\phi_{xyy} + \phi_{xxx}) - \frac{d^2 U_x}{dy^2} \phi_x = A^2 [B_x (\psi_{xyy} + \psi_{xxx})] \\ + \frac{1}{Re} (\phi_{yyyy} + 2\phi_{xxyy} + \phi_{xxxx}) \end{aligned} \quad (A23)$$

If the form of the disturbance is taken to be

$$\Phi = \phi(y) \exp[i\alpha(x - ct)] \quad (A24)$$

$$\Psi = \psi(y) \exp[i\alpha(x - ct)] \quad (A25)$$

equation (A23) becomes

$$\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi = i\alpha Re \left\{ (U_x - c)(\phi'' - \alpha^2 \phi) - U_x'' \phi - A^2 [B_x (\psi'' - \alpha^2 \psi)] \right\} \quad (A26)$$

where the primes denote differentiation with respect to y , and equations (A21) and (A22) become

$$\psi'' - \alpha^2 \psi = i\alpha Re_m [(U_x - c)\psi - B_x \phi] \quad (A27)$$

Equations (A21) and (A22) lead to a single independent equation since equation (A21), for the assumed form of the disturbance, can be obtained from equation

tion (A22) by differentiation with respect to y .

Equations (A26) and (A27) are the final form assumed by the disturbance equations. Since B_1 is a constant, it is appropriate to replace B_x by unity; that is, the reference quantity B is taken to be B_1 . This final step yields equations (8) and (9).

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