ON THE INTERACTION OF A SMALL NUMBER OF TRANSVERSE AND LONGITUDINAL MODES IN A PLASMA*

by

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ABSTRACT

We obtain equations for the time and space development of a small number of large amplitude transverse and longitudinal modes in a Vlasov plasma. The scale times and lengths are inversely proportional to the first power of the electric fields. For the special case of two transverse modes and a single resonantly driven longitudinal oscillation, we first classify systems according to the manner in which the size of the longitudinal field is limited, and then solve for the resulting mode evolution. As a result of the non-linear coupling, frequency conversion and polarization rotation effects appear. Qualitative differences are found for the behavior in one and two independent coordinates. In most (but not all) cases treated, the mode interactions cause changes in the electric field amplitudes rather than shifts in resonant frequencies and wave lengths.
I. **INTRODUCTION**

In this paper we study the resonant coupling of a small number of transverse and longitudinal waves in a Vlasov plasma of electrons embedded in a uniformly smeared out background of positive charge.

The problem differs from earlier problems treated by the quasi-linear theory of the Vlasov equation \(^1,2,3,4\) due to the differences in the time scales involved. In this problem, because of the presence of only a small number of modes, the rate of change of the modes is proportional to the strength of the amplitudes, while the spatially homogeneous part of the one-particle distribution function changes at a rate proportional to the square of the amplitudes. On the other hand, in earlier problems treated by quasi-linear theory, very many modes are present with random phases. Because of the random phase approximation, both the modes and the spatially homogeneous part of the one-particle distribution function vary at rates which are proportional to the square of the amplitudes.

We obtain a set of coupled, first-order partial differential equations for the amplitudes of the modes. The equations take into account simultaneously several mechanisms which may modify the longitudinal field: linear Landau damping, detuning, and nonlinear effects in space and time due to resonant mode coupling. In addition collisional damping effects are assumed to be additive to linear Landau damping effects. We confine our analysis to situations in which the transverse fields vary through mode-mode coupling, since otherwise resonant interaction effects are weak.
Similar equations (though not based on the Vlasov equation) have been derived by several authors\textsuperscript{5,6,7}. However, they usually neglect one or more of the modifying mechanisms, such as the Landau or collisional damping, and/or they consider only nonlinear effects in time, leaving out nonlinear effects in space.

In section (IIA) we define a physical system whose behavior is governed by the nonlinear Vlasov equation coupled with Maxwell's equations. In section (IIB) we seek a perturbation theoretic solution of this system of equations. In section (IIC) we obtain differential equations for the amplitudes of the electric field of the plasma modes which follow from the prescription that all secular terms cancel, and in section (IID) we particularize our considerations to the case of two transverse oscillations impinging on the plasma and the resulting driven longitudinal disturbance.

Then in section (IIIA) we obtain a set of conservation equations governing the energy exchange among the modes of (IID). In (IIIB) we determine limits on the size of the longitudinal disturbance of (IID), and the corresponding length and/or time scales for change of the transverse modes due to mode-mode coupling.

After this in IIIC, IIID, and IIIE we solve the differential equations for the amplitudes explicitly for several special cases which are grouped according to the dominant limiting mechanism on the longitudinal mode.
II. PLASMA AND ELECTROMAGNETIC FIELD EQUATIONS

A. THE NONLINEAR EQUATIONS

We assume that the plasma and the electromagnetic field are describable by the Vlasov equation coupled with the Maxwell equations:

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} (\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \]

\[ \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E} = 4\pi n_0 e (1 - \int f \, d\mathbf{v}) \]

\[ \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{B} = 0 \]

\[ \frac{\partial}{\partial \mathbf{x}} \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \frac{\partial}{\partial \mathbf{x}} \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \frac{4\pi n_0 e}{c} \int \mathbf{v} f \, d\mathbf{v} \]

where all of the symbols have their usual meaning.
B. CHOICE OF PERTURBATION EXPANSION

We wish to solve Eqs. (1) by means of an expansion in a small parameter $\epsilon$, where $\epsilon$ measures the departure of the system from the field-free quiescent state:

$$
\epsilon \sim \frac{|f - f_0|}{f_0},
$$

where $f_0(v)$ is the distribution function for the quiescent state.

According to the multiple time and spatial scale scheme\(^8,9\) we introduce into the problem a set of time scales, $t_0, t_1, t_2, \ldots$, and a set of spatial scales, $x_0, x_1, x_2, \ldots$, defined such that $t_0 = t$, $t_1 = \epsilon t$, $t_2 = \epsilon^2 t$, \ldots, and $x_0 = x$, $x_1 = \epsilon x$, $x_2 = \epsilon^2 x$, \ldots. Any function of $x$ and $t$ is assumed to be a function of $x_0, x_1, x_2, \ldots$, $t_0, t_1, t_2, \ldots$, respectively.

Let us seek solutions to Eqs. (1) which can written in the form:

$$
\begin{align*}
    f(x,v,t) &= f^{(0)}(v,x_1,t_1,\ldots) + \epsilon f^{(1)}(x_0,v,t_0,x_1,t_1,\ldots) + \\
    \mathbb{E}(x,t) &= \epsilon \mathbb{E}^{(1)}(x_0,t_0,x_1,t_1,\ldots) + \\
    \mathbb{B}(x,t) &= \epsilon \mathbb{B}^{(1)}(x_0,t_0,x_1,t_1,\ldots) + \\
\end{align*}
$$

(3)
The dependence of the functions on \( x_0 \) and \( t_0 \) has to be such that the Vlasov and Maxwell equations be satisfied to orders \( O(\varepsilon), O(\varepsilon^2), \text{etc.} \). The functional dependence on \( x_1, t_1, x_2, t_2, \text{etc.} \) gives us additional freedom of choice; and we can use it to eliminate secular terms from the equations to order \( O(\varepsilon^2), O(\varepsilon^3), \text{etc.} \), \text{i.e.} to as many orders as desired.

For the sake of simplicity, let us take \( f^{(o)}(\varepsilon) = f^{(o)}(|v|, \ldots) \) to be an isotropic function of velocity.

Substituting the expansion (3) into Eqs. (1), and writing all spatial and time derivatives as derivatives with respect to \( x_0, t_0, x_1, t_1, \text{etc.} \), we obtain the following equations:

\[
\varepsilon \left[ \frac{\partial f^{(1)}}{\partial t} + v \cdot \frac{\partial f^{(1)}}{\partial x} - \frac{e}{m} E^{(1)} \cdot \frac{\partial f^{(0)}}{\partial v} + \frac{\partial f^{(0)}}{\partial t} + v \cdot \frac{\partial f^{(0)}}{\partial x} \right]
\]

\[
+ \varepsilon^2 \left[ \frac{\partial f^{(2)}}{\partial t} + v \cdot \frac{\partial f^{(2)}}{\partial x} - \frac{e}{m} E^{(2)} \cdot \frac{\partial f^{(0)}}{\partial v'} \right]
\]

\[
- \frac{e}{m} \left( F^{(1)} + \frac{1}{c} v \times B^{(1)} \right) \cdot \frac{\partial f^{(1)}}{\partial v} + \frac{\partial f^{(1)}}{\partial t} + v \cdot \frac{\partial f^{(1)}}{\partial x} + \frac{\partial f^{(0)}}{\partial t} + v \cdot \frac{\partial f^{(0)}}{\partial x} \right] = 0 \tag{4}
\]
\[\varepsilon \left[ \frac{\partial}{\partial \mathbf{x}_0} \cdot \mathbf{E}^{(1)} + 4\pi n_0 e \int f^{(1)} \, d\mathbf{v} \right] + \varepsilon^2 \left[ \frac{\partial}{\partial \mathbf{x}_0} \cdot \mathbf{E}^{(2)} + 4\pi n_0 e \int f^{(2)} \, d\mathbf{v} + \frac{\partial}{\partial \mathbf{x}_1} \cdot \mathbf{E}^{(1)} \right] = 0, \quad (5)\]

\[\varepsilon \left[ \frac{\partial}{\partial \mathbf{x}_0} \cdot \mathbf{B}^{(1)} \right] + \varepsilon^2 \left[ \frac{\partial}{\partial \mathbf{x}_0} \cdot \mathbf{B}^{(2)} + \frac{\partial}{\partial \mathbf{x}_1} \cdot \mathbf{B}^{(1)} \right] = 0, \quad (6)\]

\[\varepsilon \left[ \frac{\partial}{\partial \mathbf{x}_0} \times \mathbf{E}^{(1)} + \frac{1}{c} \frac{\partial \mathbf{B}^{(1)}}{\partial t_0} \right] + \varepsilon^2 \left[ \frac{\partial}{\partial \mathbf{x}_0} \times \mathbf{E}^{(2)} + \frac{1}{c} \frac{\partial \mathbf{B}^{(2)}}{\partial t_0} \right]
+ \frac{\partial}{\partial \mathbf{x}_1} \times \mathbf{E}^{(1)} + \frac{1}{c} \frac{\partial \mathbf{B}^{(1)}}{\partial t_1} \right] = 0, \quad (7)\]

and

\[\varepsilon \left[ \frac{\partial}{\partial \mathbf{x}_0} \times \mathbf{B}^{(1)} - \frac{1}{c} \frac{\partial \mathbf{E}^{(1)}}{\partial t_0} + \frac{4\pi n_0 e}{c} \int v f^{(1)} \, d\mathbf{v} \right]
+ \varepsilon^2 \left[ \frac{\partial}{\partial \mathbf{x}_0} \times \mathbf{B}^{(2)} - \frac{1}{c} \frac{\partial \mathbf{E}^{(2)}}{\partial t_0} + \frac{4\pi n_0 e}{c} \int v f^{(2)} \, d\mathbf{v} \right]
+ \frac{\partial}{\partial \mathbf{x}_1} \times \mathbf{B}^{(1)} - \frac{1}{c} \frac{\partial \mathbf{E}^{(1)}}{\partial t_1} \right] = 0. \quad (8)\]
We shall look for steady state solutions on the "fast" scales, and shall assume all functions of \( x_0 \) and \( t_0 \) to be expanded in Fourier series with real wavenumbers and frequencies. Therefore we seek solutions of the form

\[
\left( \begin{array}{c}
f^{(v)} \\
E^{(v)} \\
B^{(v)}
\end{array} \right) = \frac{1}{VT} \sum_{k, \omega} \left( \begin{array}{c}
f^{(v)}(x_1, t_1, \ldots ; v) \\
E^{(v)}(x_1, t_1, \ldots) \\
B^{(v)}(x_1, t_1, \ldots)
\end{array} \right) e^{i(k \cdot x_0 - \omega t_0)}, \quad (9)
\]

where \( v = 1, 2, \ldots \). This essentially means that \( E^{(v)} \) is undamped on the scales specified by \( x_0 \) and \( t_0 \). Any damping, which may occur, proceeds at a rate of \( \mathcal{O}(\epsilon) \), and is incorporated in the "slow" time and space scale dependence of the Fourier amplitudes.

From Eq. (4), to \( \mathcal{O}(\epsilon) \) and \( \mathcal{O}(\epsilon^2) \), for \( k \neq 0 \) and \( \omega \neq 0 \) we obtain

\[
f_{k\omega}^{(1)} = \frac{e i}{m} \frac{\Re \left( \frac{\partial \mathbf{E}^{(1)}(0)}{\partial v} \right)}{\omega - k \nu + i\eta}, \quad (10)
\]

and
where \( n \) is a small positive number.

It will sometimes be convenient to consider the longitudinal and transverse components of \( E_{k \omega}^{(0)} \) separately. We shall denote them by the symbols \( E_{Lk \omega}^{(0)} \) and \( E_{Tk \omega}^{(0)} \), respectively.

Substituting Eqs. (10) and (11) into Eqs. (5) and (7), we obtain, for \( k \neq 0 \) and \( \omega \neq 0 \),

\[
\epsilon D_L(k, \omega) \left| E_{Lk \omega}^{(1)} \right| + \epsilon^2 D_L(k, \omega) \left| E_{Lk \omega}^{(2)} \right| = \epsilon^2 \left[ \frac{1}{k} \frac{\partial}{\partial x_1} \cdot E_{k \omega}^{(1)} + \frac{4\pi n_o e}{k} \int \frac{\partial f_{k \omega}^{(1)}}{\partial t_1} + \frac{\partial f_{k \omega}^{(1)}}{\partial x_1} \right] \frac{\partial f_{k \omega}^{(1)}}{\partial v} \cdot \frac{\partial f_{k \omega}^{(1)}}{\partial \omega} \times \omega - k \cdot v + i n \, dv
\]
\[- \frac{1}{\nu T} \frac{\omega^2}{k} \sum_{k', \omega'} \left\{ \frac{(E_{1 k', \omega'}^{(1)} + \frac{1}{c} \frac{\partial f}{\partial v} \times B_{1 k', \omega'}^{(1)}) \cdot \frac{\partial f_{k-k', \omega-k'}}{\partial v}}{\omega - k \cdot v + i \eta} \right\} \right\} \), (12)

where \( D_L(k, \omega) = 1 + \frac{\omega^2}{k^2} \int \frac{k \cdot \frac{\partial f(0)}{\partial v}}{\omega - k \cdot v + i \eta} \, dv \), the longitudinal dielectric function; and

\[
\epsilon \left[ D_T(k, \omega) E_{Tk\omega}^{(1)} + D_L(k, \omega) E_{Lk\omega}^{(1)} \right] + \epsilon^2 \left[ D_T(k, \omega) E_{Tk\omega}^{(2)} + D_L(k, \omega) E_{Lk\omega}^{(2)} \right]
\]

\[
+ \frac{1}{\nu T} \frac{\omega^2}{k} \sum_{k', \omega'} \left\{ \frac{v \left( E_{1 k', \omega'}^{(1)} + \frac{1}{c} \frac{v}{v} \times B_{1 k', \omega'}^{(1)} \right) \cdot \frac{\partial f_{k-k', \omega-k'}}{\partial v}}{\omega - k \cdot v + i \eta} \right\} \right\} 

- \frac{4 \pi n e}{\omega} \int \frac{v}{\omega - k \cdot v + i \eta} \left( \frac{\partial E_{Tk\omega}}{\partial t} + \frac{\partial E_{Lk\omega}}{\partial x_1} \right) \, dv + \frac{i}{\omega} \left( 1 + \frac{c^2 \ell^2}{\omega^2} \right) \frac{\partial E_{Tk\omega}}{\partial t}

+ \frac{i}{\omega} \frac{\partial E_{Lk\omega}}{\partial t} - \frac{i c^2}{\omega^2} k \left( \frac{\partial}{\partial x_1} \cdot E_{Tk\omega}^{(1)} \right) + \frac{2i c^2}{\omega^2} (k \cdot \frac{\partial}{\partial x_1}) E_{Tk\omega}^{(1)} 

- \frac{i c^2}{\omega^2} k \left( \frac{\partial}{\partial x_1} \times E_{Lk\omega}^{(1)} \right) \right] = 0 \), \quad (13)
where $D_T(k, \omega) = 1 - \frac{c^2 k^2}{\omega^2} - \frac{\omega^2}{\omega} \int \frac{f(0)}{\omega - k \cdot v + i \eta} dv$, the transverse dielectric function.

Since the quiescent state of the system is field-free, we require that $E^{(1)}_{k=0, \omega=0} = 0$. Hence the $k = 0$ and $\omega = 0$ component of Eq. (4) yields

$$\frac{\partial f(0)}{\partial t} + v \cdot \frac{\partial f(0)}{\partial x_1} = 0.$$  

We shall take $\frac{\partial f(0)}{\partial t} = \frac{\partial f(0)}{\partial x_1} = 0$. Therefore $f(0) = f(0)(|v|, x_2, t_2, x_3, t_3, \ldots)$.

C. SECULAR TERMS IN THE LONGITUDINAL AND TRANSVERSE FOURIER COMPONENTS

Eq. (12) shows that secular terms appear in the longitudinal fields when $D_L \sim O(\varepsilon)$. They can be eliminated by choosing the dependence of the $O(\varepsilon)$ longitudinal Fourier components on $x_1$ and $t_1$ such that the sum of the terms involving derivatives with respect to $x_1$ and $t_1$ cancel the secular term.

To perform the two integrations on the right side of Eq. (12), we eliminate $f^{(1)}_{k \omega}$ and $f^{(1)}_{k-k', \omega-\omega'}$ by means of Eqs. (10). We notice that the nonlinear term contains only the coupling of transverse components, there being
no terms involving the coupling of longitudinal components or the coupling of longitudinal with transverse components.

We also notice that the two integrals on the right side of Eq. (12) can be split up into two terms, a principal value integral of order $\varepsilon^2$ and a pole contribution which detailed estimate shows to be of order $\varepsilon^{5/2}$. Since the pole contribution is formally of higher order it will henceforth be neglected. This neglect corresponds to the neglect of non-linear Landau damping. Since the general effect of the non-linear Landau damping is the lessening of the total Landau damping from the linear value, by keeping the linear value we obtain a lower bound for mode coupling effects when Landau damping limits the size of the longitudinal components.

The principal value integrals are evaluated by expanding the integrands in a power series in $(V_0/(\omega/k))$. Only the lowest order non-vanishing terms are retained.

Having evaluated the integrals on the right side of Eq. (12) according to the procedure outlined above, we eliminate secular terms from Eq. (12) by requiring that for $k$ and $\omega$ for which $D_L(k,\omega) \sim 0(\varepsilon)$ the longitudinal component of the electric field satisfy the equation

$$
\frac{\partial E_{Lk\omega}^{(1)}}{\partial t} + \frac{V^2}{\omega} \left( k \cdot \frac{\partial}{\partial x_1} \right) E_{Lk\omega}^{(1)} + \frac{1}{2\omega} \frac{\omega^3}{\omega_p} D_L(k,\omega) E_{Lk\omega}^{(1)} = \frac{i\omega^3}{2k} I_{k\omega},
$$

(14)
where

$$I_{k\omega} = \frac{1}{\gamma T} \frac{ie}{\omega^2} \sum_{k',\omega'} \frac{1}{\omega' - \omega} \left\{ \frac{2}{\omega} (k \cdot E^{(1)}_{k',\omega'}) (k \cdot E^{(1)}_{k-k',\omega'-\omega'}) ight.$$  

$$+ \frac{1}{\omega' - \omega} (k \cdot E^{(1)}_{k',\omega'}) \left[ (k-k') \cdot E^{(1)}_{k-k',\omega'-\omega'} \right]$$  

$$- \frac{1}{\omega'} k \times (k' \times E^{(1)}_{k',\omega'}) \cdot E^{(1)}_{k-k',\omega'-\omega'} \right\}$$  

(15)

the mode coupling term.

Eq. (13) shows that secular terms may arise in the transverse fields when $D_T \sim O(\epsilon)$ or smaller. They can be eliminated by choosing the dependence of the transverse modes on $x_1$ and $t_1$ such that the sum of the terms in Eq. (13) involving derivatives with respect to $x_1$ and $t_1$ will cancel the secular terms. Proceeding in a manner analogous to that which led to Eq. (14), we eliminate secular terms from Eq. (13) by requiring that for $k$ and $\omega$ for which $D_T(k,\omega) \sim O(\epsilon)$ the transverse component of the electric field satisfy the equation

$$\frac{\partial E^{(1)}_{tk\omega}}{\partial t_1} + \frac{c^2}{\omega} (k \cdot \frac{\partial}{\partial x_1}) E^{(1)}_{tk\omega} = \frac{\omega^2}{21} \left( \frac{\omega}{k^2} k \cdot \frac{I_{k\omega}}{k_{k\omega}} - \frac{J_{k\omega}}{k_{k\omega}} \right),$$  

(16)
where

\[ J_{k\omega} = \frac{1}{VT} \sum_{m} \frac{\epsilon_{m} \left\{ \left( k \cdot E_{k-k',\omega-\omega'}^{(1)} \right) E_{k',\omega'}^{(1)} \right\}}{\omega^2 (\omega-\omega')} \]

\[ + \frac{(k \cdot E_{k',\omega'}^{(1)}) E_{k-k',\omega-\omega'}^{(1)}}{\omega^2 (\omega-\omega')} + \frac{\left[ (k-k') \cdot E_{k-k',\omega-\omega'}^{(1)} \right] E_{k',\omega'}^{(1)}}{\omega (\omega-\omega')^2} + \frac{\left( k \times E_{k',\omega'}^{(1)} \right) \times (k' \times E_{k',\omega'}^{(1)})}{\omega (\omega-\omega')} \]

(17)

and \( I_{k\omega} \) is given by Eq. (15).

**D. CHOICE OF LOWEST ORDER NON-VANISHING MODES**

Eq. (12) shows that when \( D_L \sim 0(1) \), \( E_{L,k\omega}^{(1)} = 0 \); \( E_{L,k\omega}^{(1)} \not= 0 \) only for those waves for which \( D_L \sim 0(\varepsilon) \) or smaller. Eq. (13) shows that \( E_{T,k\omega}^{(1)} \not= 0 \) only when \( D_T \sim 0(\varepsilon) \) or smaller.

For the work to follow we take the transverse \( 0(\varepsilon) \) field \( E_{T}^{(1)} \) to be the superposition of two plane waves inside the plasma, \( i.e. \)
\[ E_T^{(1)}(x_0, t_0, x_1, t_1, \ldots) = E_1(x_1, t_1, \ldots) \sin(k_1 \cdot x_0 - \omega_1 t_0 + \phi_1(x_1, t_1, \ldots)) \]

\[ + E_2(x_1, t_1, \ldots) \sin(k_2 \cdot x_0 - \omega_2 t_0 + \phi_2(x_1, t_1, \ldots)), \quad (18) \]

where \( D_T(k_1, \omega_1) = D_T(k_2, \omega_2) \approx 0 \), \( D_L(k_1 - k_2, \omega_1 - \omega_2) \approx 0(\epsilon) \), and \((k_1 \cdot E_1)

and \((k_2 \cdot E_2)\) are at most of \(0(\epsilon)\). In (18) we assume that \(\phi_1\) is the same for both polarizations of \(E_1\), and similarly for \(\phi_2\). This assumption can be made consistently for all cases for which we solve explicitly.

We note that the form of (18) allows in principle for externally induced variations in the amplitudes and phases of the transverse modes.

Moreover, fourier-analyzing (18), according to Eq. (9), and substituting into Eq. (15), we see that \(I_{kw} \neq 0\) in Eq. (14) only for \(\omega = \omega_1 - \omega_2\) and \(k = k_1 - k_2\), or \(\omega = -(\omega_1 - \omega_2)\) and \(k = -(k_1 - k_2)\). Therefore, with the particular choice of \(E_T^{(1)}\) made in (18), the most general form of

\[ E_L^{(1)}(x_0, t_0, x_1, t_1, \ldots) \], which is modified by transverse mode coupling, can be written in the form

\[ E_L^{(1)}(x_0, t_0, x_1, t_1, \ldots) = E_3(x_1, t_1, \ldots) \hat{k}_3 \cos(k_3 \cdot x_0 - \omega_3 t_0 + \phi_3) \]

\[ + E_4(x_1, t_1, \ldots) \hat{k}_3 \sin(k_3 \cdot x_0 - \omega_3 t_0 + \phi_3), \quad (19) \]

where \(\hat{k}_3 = \frac{k_3}{|k_3|}\), \(k_3 = k_1 - k_2\), \(\omega_3 = \omega_1 - \omega_2\), and \(\phi_3 = \phi_1 - \phi_2\).
Multiplying Eq. (14) by $\frac{1}{VT} e^{i(k'x_0 - \omega t)}$ and summing over $k$ and $\omega$, we obtain the following two equations:

\[
\frac{\partial E_3}{\partial t} + \frac{\omega}{\omega_3} k_3 \cdot \frac{\partial E_3}{\partial x_1} + \frac{1}{2} \gamma_L E_3 + E_4 \left( \frac{\partial \phi_3}{\partial t} + \frac{\omega}{\omega_3} k_3 \cdot \frac{\partial \phi_3}{\partial x_1} - \Delta \right) = \frac{e^{\omega P} k_3}{4m \omega_1 \omega_2} (E_1 \cdot E_2)
\]

(20)

and

\[
\frac{\partial E_4}{\partial t} + \frac{\omega}{\omega_3} k_3 \cdot \frac{\partial E_4}{\partial x_1} + \frac{1}{2} \gamma_L E_4 - E_3 \left( \frac{\partial \phi_3}{\partial t} + \frac{\omega}{\omega_3} k_3 \cdot \frac{\partial \phi_3}{\partial x_1} - \Delta \right) = 0
\]

(21)

where $\Delta$ and $\gamma_L$ are defined such that $D_L(k_3, \omega) = \frac{2\Delta}{\omega p} + \frac{i\gamma_L}{\omega_p} \circ O(\varepsilon)$. $\gamma_L$ is the linear Landau damping decrement, and $\Delta$ is the "detuning" factor to $O(\varepsilon)$.

Eqs. (20) and (21) show that the longitudinal modes can be limited by linear Landau damping, by "detuning", and by the coupling between transverse modes (which enter through the "slow" derivatives).

Since the presence of a collision term would introduce, in the linear approximation, a damping decrement which plays a role analogous to that of the Landau damping decrement, in the work below we replace $\gamma_L$ by $\gamma \left[ \Xi(\gamma_L + \gamma_c) \right]$ where $\gamma_c$ is the collisional damping decrement.
If we fourier-analyze both (18) and (19), according to Eq. (9), and substitute into Eq. (16), we obtain the following equations for $E_1$, $E_2$, $\phi_1$, and $\phi_2$:

\[
\frac{\partial E_1}{\partial t_1} + \frac{c}{\omega_1} \left( k_1 \cdot \frac{\partial}{\partial x_1} \right) E_1 = \frac{ek_3}{4m\omega_2} E_3 \left[ \frac{(k_1 \cdot E_2)k_1}{k_1^2} - E_2 \right],
\]

\[
\frac{\partial E_2}{\partial t_1} + \frac{c}{\omega_2} \left( k_2 \cdot \frac{\partial}{\partial x_1} \right) E_2 = -\frac{ek_3}{4m\omega_1} E_3 \left[ \frac{(k_2 \cdot E_1)k_2}{k_2^2} - E_1 \right],
\]

\[
E_1 \left( \frac{\partial f_1}{\partial t_1} + \frac{c}{\omega_1} k_1 \cdot \frac{\partial f_1}{\partial x_1} \right) = \frac{ek_3}{4m\omega_2} E_4 \left[ E_2 - \frac{(k_1 \cdot E_2)k_1}{k_1^2} \right],
\]

and

\[
E_2 \left( \frac{\partial f_2}{\partial t_1} + \frac{c}{\omega_2} k_2 \cdot \frac{\partial f_2}{\partial x_1} \right) = \frac{ek_3}{4m\omega_1} E_4 \left[ E_1 - \frac{(k_2 \cdot E_1)k_2}{k_2^2} \right].
\]

III. DEPENDENCE OF MODES ON $x_1$ and $t_1$

A. CONSERVATION LAWS

Let us notice that $\frac{c^2}{\omega_1} k_1$, $\frac{c^2}{\omega_2} k_2$, and $\frac{v^2}{\omega_3} k_3$ are, respectively, the group velocities of the two transverse modes and the longitudinal mode. We shall denote them by $V_{g1}$, $V_{g2}$, and $V_{g3}$.
If we eliminate the terms containing $\phi_3$ from Eq. (20) with the aid of Eq. (21), we can derive the following conservation equations from Eqs. (20), (22) and (23):

$$\frac{\partial}{\partial t_1} \left( \frac{E_1^2}{\omega_1} + \frac{E_2^2}{\omega_2} \right) + \frac{\partial}{\partial x_1} \cdot \left( \frac{E_1^2}{\omega_1} \mathbf{v}_{\text{gl}} + \frac{E_2^2}{\omega_2} \mathbf{v}_{\text{g2}} \right) = 0$$

and

$$\frac{\partial}{\partial t_1} \left[ \frac{E_1^2}{\omega_1} + \frac{(E_3^2 + E_4^2)}{\omega_3} \right] + \frac{\partial}{\partial x_1} \cdot \left[ \frac{E_1^2}{\omega_1} \mathbf{v}_{\text{g1}} + \frac{(E_3^2 + E_4^2)}{\omega_3} \mathbf{v}_{\text{g3}} \right] + \gamma \frac{(E_3^2 + E_4^2)}{\omega_3} = 0 \quad (26)$$

We will now seek to make the physical meaning of Eqs. (26) more explicit.

First we define $U_1$ to be the average with respect to $x_0$ of the sum of the electric, the magnetic, and the mechanical field energy densities of the first transverse wave, $U_2$ to be the corresponding average for the second transverse wave, and $U_3$ to be the corresponding average for the longitudinal wave.

The average mechanical field energy density of the first wave is of the form $\left\langle \frac{1}{2} n_0 m u_1^2 \right\rangle_{\text{av.}}$, where $u_1$ satisfies the equation

$$\frac{\partial u_1}{\partial t_0} = -\frac{e}{m} E_1 \sin(k_1 \cdot x_0 - \omega_1 t_0 + \phi_1).$$
Therefore \[ \langle \frac{1}{2} n_0 m u_1^2 \rangle_{av.} = \frac{1}{2} \frac{\omega_0^2}{\omega_1^2} \frac{E_1^2}{8\pi} \], and \( U_1 = \frac{1}{2} \left( \frac{E_1^2}{8\pi} + \frac{B_1^2}{8\pi} \right) \).

\[ \langle \frac{1}{2} n_0 m u_2^2 \rangle_{av.} = \frac{E_2^2}{8\pi} \]. Similarly \( U_2 = \frac{E_2^2}{8\pi} \) and \( U_3 = \frac{1}{2} \left( \frac{E_3^2 + E_4^2}{8\pi} \right) \), where \( u_3 \) satisfies the equation

\[ \frac{\partial u_3}{\partial t_0} = -\frac{e}{m} E_3 \cos(k_3 \cdot x_0 - \omega_3 t_0 + \phi_3) \]

\[ -\frac{e}{m} E_4 \sin(k_3 \cdot x_0 - \omega_3 t_0 + \phi_3) \] .

We now note that \( U_1 \), \( U_2 \), and \( U_3 \) satisfy a set of equations identical with Eqs. (26), with \( E_1^2 \) replaced by \( U_1 \), \( E_2^2 \) by \( U_2 \), and \( (E_3^2 + E_4^2) \) by \( U_3 \).

If we define \( N_1 = \left( \frac{U_1}{\hbar \omega_1} \right) \) to be the number density of quanta of both modes of polarizations of the first transverse modes; \( N_2 = \left( \frac{U_2}{\hbar \omega_2} \right) \), of the second transverse mode; and \( N_3 = \left( \frac{U_3}{\hbar \omega_3} \right) \), the number density of quanta of the longitudinal mode, then \( N_1 \), \( N_2 \) and \( N_3 \) satisfy the conservation equations:

\[ \frac{\partial}{\partial t_1} (N_1 + N_2) + \frac{\partial}{\partial x_1} \cdot (N_1 V_{g1} + N_2 V_{g2}) = 0 \]

and

\[ \frac{\partial}{\partial t_2} (N_1 + N_3) + \frac{\partial}{\partial x_1} \cdot (N_1 V_{g1} + N_3 V_{g3}) = -\gamma N_3 \] .

(27)
Eqs. (27) have a lucid interpretation in terms of a three-fluid model. We have three fluids of quanta streaming with fluid velocities \( V_{g1} \), \( V_{g2} \), and \( V_{g3} \), respectively. Because of the nonlinear coupling of modes, the quanta can be created and annihilated. Whenever a quantum of the higher-frequency transverse mode is annihilated, its energy is shared between a lower-frequency transverse quantum and a longitudinal quantum, which are simultaneously created; and vice versa, whenever a higher-frequency transverse quantum is created, a lower frequency transverse quantum and a longitudinal quantum are annihilated. At the same time, there is a dissipation process which destroys the longitudinal quanta and converts their energy into thermal energy.

Equations (27) differ from those derived by Sturrock\(^6,12,13\) in that they include the dissipation of the energy of the longitudinal mode by the processes of Landau and collisional damping.

B. RELATIVE IMPORTANCE OF THE LIMITING MECHANISMS

We present here expressions for estimating the relative strengths of the different mechanisms which limit the longitudinal mode. This is done by taking each mechanism separately, assuming that it alone is present, and estimating the strength of the longitudinal field. The most effective mechanism, in a particular situation, is the one which gives the smallest amplitude of the longitudinal mode.
Let $k_T$, $\omega_T$, and $E_T$ refer, respectively, to the characteristic wavenumber, frequency, and amplitude of the transverse field (remember that $\omega_T \propto c k_T$); and let $k_L$ and $E_L$ refer, respectively, to the wavenumber and amplitude of the longitudinal field. Let $\tau_N$ and $\lambda_N$ be, respectively, the time and spatial scales of variation due to mode-mode coupling.

Eq. (20)-(25), when written in dimensional form, become

$$\left(\frac{1}{\tau_N} + \frac{v^2 k_L}{\omega_p} \frac{1}{\lambda_N} + \frac{1}{2} \gamma + \Delta\right) E_L = \frac{ek_L}{4m} \frac{\omega}{\omega_T} \frac{\omega}{\omega_T} E_T^2$$

and

$$\left(\frac{1}{\tau_N} + \frac{c}{\lambda_N}\right) = \frac{ek_L}{4mc} E_L$$

For initial value problems

$$\frac{1}{\tau_N} = \frac{ek_L}{4mc} E_L = \frac{e}{4mc} \frac{\omega}{\omega_T} k_c \frac{c}{\omega_p} E_L$$

(29)

For boundary value problems

$$\frac{1}{\lambda_N} = \frac{ek_L}{4mc} E_L = \frac{e}{4mc} \frac{\omega}{\omega_T} k_c \frac{c}{\omega_p} E_L$$

(30)

When "detuning" is dominant,

$$E_L = \frac{e}{4mc} \frac{\omega}{\omega_T} k_c \frac{c}{\omega_p} E_T^2 \frac{E_T^2}{\Delta}$$

(31)

and when damping is dominant,

$$E_L = \frac{e}{2mc} \frac{\omega}{\omega_T} k_c \frac{c}{\omega_p} E_T^2$$

(32)
We have \( \gamma = \gamma_L + \gamma_c \). The Landau damping decrement, \( \gamma_L \), may be written in the form

\[
\gamma_L \approx \pi c \left( \frac{2e^2}{mc^2} \right)^{1/2} n_o^2 \left( \frac{ck_L}{\omega_p} \right)^3 (V_o/c)^{-3} \times \\
\times \exp \left[ -\frac{1}{2} \left( \frac{ck_L}{\omega_p} \right)^{-2} \left( \frac{V_o}{c} \right)^{-2} \right].
\]

The collision damping decrement, \( \gamma_c \), is given by\(^{14}\)

\[
\gamma_c \approx \frac{714}{c^3} \left( \frac{4\pi e^2}{m} \right)^2 \left( \frac{V_o}{c} \right)^{-3} n_o \ln \Lambda
\]

where \( \Lambda = 6\pi n_o L_D^3 \), with \( L_D = \left( k_B T / 4\pi n_o e^2 \right)^{1/2} \). Here \( k_B \) is Boltzmann's constant and \( T \) is the electron "temperature".

When resonant mode-coupling effects are dominant, and we are doing a pure initial value problem,

\[
E_L = \left( \frac{\omega_p}{\omega_T} \right)^{1/2} E_T.
\] (33)

When resonant mode-coupling effects are dominant, and we are doing a pure boundary value problem,

\[
E_L = \left( \frac{c}{V_o} \right) \left( \frac{\omega_p}{\omega_T} \right)^{1/2} \left( \frac{\omega_p}{ck_L} \right)^{1/2} E_T.
\] (34)

Eqs. (33) and (34) were obtained by neglecting \( \gamma \) and \( \Lambda \) in Eqs. (28) and then solving for \( E_L \) and \( \tau_N \) or \( \lambda_N \).
At this point we note that convective non-linearities may limit the longitudinal mode by detuning \(^{15,16}\), with an amplitude which can be written as:

\[ E_L \sim \left( m \pi \right)^{\frac{1}{3}} \frac{c}{2} \left[ \left( \frac{\omega}{\omega_p} \right)^2 \frac{E_T^{2/3}}{n_o^{1/6}} \right] \left( \frac{\omega}{\omega_T} \right)^{2/3} \left( \frac{\omega}{\omega_p} \right)^{2/3} n_o^{1/6} / \alpha^{1/3}. \]  

(35)

We have not derived this effect analytically since formally it is derived from second order terms in the expansion of (3).

Since

\[ \frac{ck_L}{\omega_p} \sim \left[ 1 + 4 \left( \frac{\omega_p}{\omega} \right)^2 \sin^2 \left( \frac{\theta}{2} \right) \right]^{\frac{1}{2}} \]

(with \( \theta \) the angle between the \( k \) vectors of the two transverse waves) by means of Eqs. (29)-(35) one concludes that \( \lambda_n \) and \( \tau_n \) are least for \( \omega_p/\omega_T \) approaching one from below. Under these conditions \( c k_L / \omega_p \sim 1 \). Then among the longitudinal limiting mechanisms, non-linear effects grow relatively stronger as \( E_T \) and \( v_T/c \) increase, and as \( n_o \) decreases.

We note that spatial resonant coupling limits the longitudinal mode less effectively than temporal resonant coupling. In fact, the ratio of the longitudinal field limited in time to that limited in space is proportional to \( \left( \frac{v_T}{c} \right) \left( \frac{c}{\omega_p/k_L} \right)^{\frac{1}{2}} \). Hence at zero temperature, spatial coupling is incapable of limiting the longitudinal mode. This last fact can also be seen in the work of Montgomery\(^7\) for a cold plasma, which does not yield solutions when only the spatial nonlinearity due to resonant mode coupling is presumed to limit the longitudinal mode.
C. DETUNING IS DOMINANT

In this case we neglect the derivatives with respect to \( x_1 \) and \( t_1 \) as well as the damping decrements in Eqs. (20) and (21). Since we fix the phase angles for the two polarizations of each transverse wave in (18) to be the same, we find that detuning alone is capable of limiting the longitudinal field only when no rotations of \( E_1 \) and \( E_2 \) occur. Therefore in this section we restrict ourselves to situations with \( \frac{(k_1 \cdot xE_2) \cdot xk_1}{k_1^2} \) parallel to \( E_1 \) and \( \frac{(k_2 \cdot xE_2) \cdot xk_2}{k_2^2} \) parallel to \( E_2 \). The resultant equations leave us the option of doing a pure boundary value problem, a pure initial value problem, or a mixed problem.

If we do a pure boundary value problem, we obtain the following solution to Eqs. (20)-(25):

\[
\begin{align*}
E_1(x_1) &= E_1(x_1=0), \\
E_2(x_1) &= E_2(x_1=0), \\
E_3 &= 0, \\
E_4 &= -\frac{\omega_p |k_3|}{4m\omega_1\omega_2} \frac{E_1 \cdot E_2}{\Delta}, \\
\phi_1 &= -\frac{e^2 k_3^2}{16m^2 c^2 \omega_2^2 k_1} \frac{(E_1 \cdot E_2)^2}{E_1^2 \Delta} k_1 \cdot x_1 + \phi_1(x_1=0), \\
and \\
\phi_2 &= -\frac{e^2 k_3^2}{16m^2 c^2 \omega_2^2 k_2} \frac{(E_1 \cdot E_2)^2}{E_2^2 \Delta} k_2 \cdot x_1 + \phi_2(x_1=0).
\end{align*}
\]
If we take \( k_1 \) parallel to \( k_2 \) and \( E_1 \) parallel to \( E_2 \), our results are essentially identical with those of Montgomery.

Had we done a pure initial value problem instead of a pure boundary value problem, our results would have been:

\[
\begin{align*}
E_1(t_1) &= E_1(t_1=0), \\
E_2(t_1) &= E_2(t_1=0), \\
E_3 &= 0, \\
E_4 &= -\frac{e\omega_p k_3}{4m\omega_1\omega_2} \frac{E_1 \cdot E_2}{\Delta},
\end{align*}
\]

\( \phi_1 = -\frac{e^2 k_3^2 \omega_p}{16m^2 \omega_1^2 \omega_2^2} \frac{(E_1 \cdot E_2)^2}{E_1 \Delta} t_1 + \phi_1(t_1=0), \)

and

\( \phi_2 = -\frac{e^2 k_3^2 \omega_p}{16m^2 \omega_1^2 \omega_2^2} \frac{(E_1 \cdot E_2)^2}{E_2 \Delta} t_1 + \phi_2(t_1=0). \)

We see from Eqs. (36) and (37) that the effects of mode-mode coupling can be interpreted to produce "wave-number shifts" in the case of a pure boundary value problem, and "frequency shifts" in the case of a pure initial value problem.
D. DAMPING IS DOMINANT

In this case we neglect the "slow" derivatives and the "detuning" terms in Eqs. (20) and (21). We immediately obtain that $E_4 = 0$ and

$$E_3 = \frac{e \omega p k_3}{2 \omega_1 \omega_2} \left( \frac{E_1 \cdot E_2}{\gamma} \right)$$

17. Because of Eqs. (24) and (25), $\phi_1$ and $\phi_2$ may be taken to be constants.

To simplify the mathematics, let us take $k_1$ parallel to $k_2$ and pointing in the direction of the positive x-axis, and let us do a one-dimensional problem. We again have the option of doing an initial value or a boundary value problem; we shall do a boundary value problem. Let $E_{10} = E_1(x_1=0)$ and $E_{20} = E_2(x_1=0)$. Two cases will be treated: (1) $E_{10}$ parallel to $E_{20}$, and (2) $E_{10}$ not parallel to $E_{20}$.

Case (1):

In this case there is no rotation of the directions of polarization. From Eqs. (20), (22), (23) and (26) we obtain the equations

$$\begin{align*}
\frac{\partial E_1}{\partial x_1} &= -\frac{e^2 k_3^2 \omega_p E_1 E_2}{8 \omega_1^2 \omega_2^2 k_1} \frac{E_1 E_2}{\gamma}, \\
\frac{\partial E_2}{\partial x_1} &= \frac{e^2 k_3^2 \omega_p E_1 E_2}{8 \omega_1^2 \omega_2^2 k_2} \frac{E_1 E_2}{\gamma},
\end{align*}$$

and

$$\begin{align*}
\frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2} = \frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2}.
\end{align*}$$

\[\text{(38)}\]
Eqs. (38) can be easily integrated, and have the solution

\[
\frac{E_1^2}{E_{10}^2} = \frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2}
\]

and

\[
\frac{E_2^2}{E_{20}^2} = \frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2} e^{2\Lambda x_1}
\]

where

\[
\Lambda = \frac{e^2 \omega_p k^2}{8m^2 c^2 k_1 k_2 \gamma} \left( \frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2} \right)
\]

The solution (39) shows that, as \( x \to \infty \), the higher-frequency transverse mode - and hence the longitudinal mode - is completely attenuated, and only the lower-frequency transverse mode remains.

**Case (2):**

Here we select \( E_{10} \) and \( E_{20} \) so that they form some arbitrary angle.

It is convenient to rewrite Eqs. (22) and (23) such that multiplicative constants do not appear explicitly. For this purpose, we define two vectors,
u and v, in the following way:

\[
\begin{align*}
\mathbf{u} &= \left( \frac{e^2 \omega_p k_3^2}{8m^2 c^2 \omega_1^2 \omega_2^2} \right)^{\frac{1}{2}} E_1, \\
\mathbf{v} &= \left( \frac{e^2 \omega_p k_3^2}{8m^2 c^2 \omega_3^2 \omega_2^2} \right)^{\frac{1}{2}} E_2.
\end{align*}
\]

Eqs. (20)-(23) can, therefore, be written in the form

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial x_1} &= - \mathbf{v} (u \cdot v) \\
\frac{\partial \mathbf{v}}{\partial x_1} &= \mathbf{u} (u \cdot v).
\end{align*}
\]

Eqs. (40) yield the conservation equations

\[
\begin{align*}
u_y^2 + v_y^2 &= u_{yo}^2 + v_{yo}^2 \\
u_z^2 + v_z^2 &= u_{zo}^2 + v_{zo}^2
\end{align*}
\]

The form of Eqs. (41) suggests that we parametrize Eqs. (40) in the following manner:

Let \( u_y = (u_{yo}^2 + v_{yo}^2)^{\frac{1}{2}} \cos \theta \),

\( v_y = (u_{yo}^2 + v_{yo}^2)^{\frac{1}{2}} \sin \theta \),

\( u_z = (u_{zo}^2 + v_{zo}^2)^{\frac{1}{2}} \cos \theta' \),

and \( v_z = (u_{zo}^2 + v_{zo}^2)^{\frac{1}{2}} \sin \theta' \).
Evaluating \( \frac{du}{dz} \) from Eqs. (40), then from Eqs. (42), then equating the two expressions and integrating, we obtain

\[
\theta' = \theta + \beta ,
\]

(43)

where \( \beta = \theta' - \theta \), \( \theta' \) and \( \theta \) being the values of \( \theta' \) and \( \theta \), respectively, at \( x = 0 \). Defining three new quantities, \( \phi \), \( \Lambda \), and \( \alpha \) by

\[
\phi = 2\theta + \alpha + \pi ,
\]

\[
\Lambda = \left\{ (u_y^2 + v_y^2) + (u_z^2 + v_z^2) \cos^2 \beta \right\}^{1/2},
\]

\[
\cos \alpha = \frac{(u_y^2 + v_y^2) + (u_z^2 + v_z^2) \cos 2\beta}{\Lambda},
\]

(44)

and

\[
\sin \alpha = \frac{(u_z^2 + v_z^2) \sin 2\beta}{\Lambda},
\]

Eqs. (40) reduce to only one equation,

\[
\frac{\partial \phi}{\partial x_1} = -\Lambda \sin \phi .
\]

(45)

Eq. (45) has the solution \( \tan \frac{\phi}{2} = e^{-\Lambda x_1} \tan \frac{\phi(x_1=0)}{2} \). But this implies that, when \( x \to \infty \), \( u \cdot v = -\frac{1}{2} \sin \phi = 0 \), and \( E_1 \) and \( E_2 \) become perpendicular to one another. The asymptotic values of the fields follow from (42) and (44).
We see from cases (1) and (2) that the characteristic length for the higher frequency transverse wave being damped out, or the polarizations of the two transverse waves rotating until they are perpendicular, is \( l/A \). Had we done a pure initial value problem, we would have obtained an analogous behavior of the waves in time, with the characteristic time \( 1/A \), where \( A \) is given by Eq. (44), but \( u \) and \( v \) are given by

\[
\begin{align*}
  u &= \left( \frac{e^{2 \omega_p k_3} k_2}{2^{1/2} 8 \pi\omega_1 \omega_2 \gamma} \right) E_1 \\
  v &= \left( \frac{e^{2 \omega_p k_3} k_2}{2^{1/2} 8 \pi\omega_1 \omega_2 \gamma} \right) E_2
\end{align*}
\]

E. RESONANT MODE COUPLING

In this case we neglect the detuning and the damping terms in Eqs. (20) and (21).

It can be shown that the fields are limited in time and in space. For example, in the case of a pure initial value problem, we obtain from Eqs. (27) that 

\[
(N_1(t_1) + N_2(t_1)) = (N_1(t_1=0) + N_2(t_1=0)) \quad \text{and} \quad (N_1(t_1) + N_3(t_1)) = (N_1(t_1=0) + N_3(t_1=0)).
\]

Since \( N_1, N_2, \) and \( N_3 \) are finite at \( t_1 = 0 \), they must be finite for \( t_1 > 0 \).
In the case of a pure boundary value problem the argument is more difficult. However, if we restrict ourselves to one-dimensional situations, with \( k_1, k_2 \) and \( k_3 \) parallel to one another, we obtain from Eqs. (27) that

\[
(N_1(x_1)\psi_{g1} + N_2(x_1)\psi_{g2}) = (N_1(x_1=0)\psi_{g1} + N_2(x_1=0)\psi_{g2}) \quad \text{and} \quad (N_1(x_1)\psi_{g1} + N_3(x_1)\psi_{g3}) = (N_1(x_1=0)\psi_{g1} + N_3(x_1=0)\psi_{g3}).
\]

Since \( N_1, N_2 \) and \( N_3 \) are finite at \( x_1 = 0 \), they must be finite for \( x_1 > 0 \).

We shall now do several pure boundary value problems. We shall assume that \( E_3 = E_4 = 0 \) on the boundary. It can be seen from Eqs. (20)-(25) that only \( E_3 \) can be built up from zero as we move away from the boundary, while \( E_4 \) remains zero. Hence \( \phi_1 \) and \( \phi_2 \) remain constant inside the plasma, and we have to solve only Eqs. (20), (22) and (23) for each boundary value problem.

To simplify the mathematics, let us define new field variable \( u, v, \) and \( w \) by

\[
\begin{align*}
u & \equiv \left( \frac{e^2 \omega p k_3}{16m^2 c^2 \omega_1 \omega_2 \omega_0^2} \right)^{\frac{1}{2}} E_1, \\
v & \equiv \left( \frac{e^2 \omega p k_3}{16m^2 c^2 \omega_2 \omega_1 \omega_0^2} \right)^{\frac{1}{2}} E_2, \\
w & \equiv \left( \frac{e^2 k_3^2}{16m^2 c^4 k_1 k_2^2} \right)^{\frac{1}{2}} E_3.
\end{align*}
\]
Let us also take $k_1$ and $k_2$ to be parallel or "almost" parallel to each other and to point in the positive $x$-direction. Let $u_0$ and $v_0$ be the constant values of $u$ and $v$ specified on the plane $x = 0$. Eqs. (20), (22) and (23) then become

\[
\begin{align*}
\frac{\partial u}{\partial x_1} &= -w v, \\
\frac{\partial v}{\partial x_1} &= w u, \\
\frac{k_3}{k_3} \cdot \frac{\partial w}{\partial x_1} &= u \cdot v.
\end{align*}
\]

Eqs. (47) yield the conservation equations (41). They can be parametrized by means of Eqs. (42), with the relationship between $\theta$ and $\theta'$ being given by Eq. (43). If we also make use of the definitions (44), Eqs. (47) will reduce to the form

\[
\begin{align*}
\frac{\partial \phi}{\partial x_1} &= 2w, \\
\frac{k_3}{k_3} \cdot \frac{\partial w}{\partial x_1} + \frac{1}{2} \lambda \sin \phi &= 0.
\end{align*}
\]

Case (1):

$k_3$ is parallel to $k_1$ and $k_2$. We let $w(x_1=0) = 0$ and $\theta_0 \equiv \theta(x_1=0)$. In this case Eqs. (48) yield the equation
Eq. (49) can be integrated to give

\[
\int_{\phi(x_1=0)}^{\phi/2} \frac{d(\phi/2)}{\sqrt{\sin^2 \frac{1}{2} \phi - \sin^2 \frac{1}{2} x_1}} = \pm \sqrt{\Lambda} x_1.
\]

If we define the number \( k \), with \( |k| < 1 \), by the equation

\[
k^2 = \sin^2 \frac{1}{2} \phi(x_1=0) = \cos^2(\theta_0 + \frac{\alpha}{2}),
\]

then, by means of the substitution

\[
\sin \phi = \frac{1}{k} \sin \left( \frac{\phi}{2} \right),
\]

the integration of Eq. (49) can be written in the form

\[
\int_{0}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \pm \sqrt{\Lambda} x_1 + \kappa \quad \text{where} \quad \kappa = \int_{0}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},
\]

elliptic integral of the first kind

Therefore \( u, v, \) and \( w \) can be expressed in terms of elliptic functions:

\[
\begin{align*}
    u_y &= (u_{y_0}^2 + v_{y_0}^2)^{1/2} \left[ k \cos \frac{\alpha}{2} \sn(\sqrt{\Lambda} x_1 + \kappa, k) + \sin \frac{\alpha}{2} \dn(\sqrt{\Lambda} x_1 + \kappa, k) \right], \\
    v_y &= (u_{y_0}^2 + v_{y_0}^2)^{1/2} \left[ \cos \frac{\alpha}{2} \dn(\sqrt{\Lambda} x_1 + \kappa, k) - k \sin \frac{\alpha}{2} \sn(\sqrt{\Lambda} x_1 + \kappa, k) \right], \\
    u_z &= (u_{z_0}^2 + v_{z_0}^2)^{1/2} \left[ k \cos(\frac{\alpha}{2} - \beta) \sn(\sqrt{\Lambda} x_1 + \kappa, k) + \sin(\frac{\alpha}{2} - \beta) \dn(\sqrt{\Lambda} x_1 + \kappa, k) \right], \\
    v_z &= (u_{z_0}^2 + v_{z_0}^2)^{1/2} \left[ \cos(\frac{\alpha}{2} - \beta) \dn(\sqrt{\Lambda} x_1 + \kappa, k) - k \sin(\frac{\alpha}{2} - \beta) \sn(\sqrt{\Lambda} x_1 + \kappa, k) \right],
\end{align*}
\]

and

\[
w = - k\sqrt{\Lambda} \cn(\sqrt{\Lambda} x_1 + \kappa, k).
\]
Eqs. (50) show that the amplitudes of the modes are oscillatory functions of $x_1$.

In the case of a one-dimensional pure initial value problem we would obtain results which are identical with Eq. (50); but with $x_1$ replaced by $t_1$; the $u$, $v$, and $w$ defined by

$$u = \left( \frac{e^2 \omega p k_3^2}{16m^2 \omega_1 \omega_2} \right)^{\frac{1}{2}} E_1,$$

$$v = \left( \frac{e^2 \omega p k_3^2}{16m^2 \omega_1 \omega_2} \right)^{\frac{1}{2}} E_2,$$

$$w = \left( \frac{e^2 k_3^2}{16m^2 \omega_1 \omega_2} \right)^{\frac{1}{2}} E_3;$$

and the $u_0$ and $v_0$ being the values of $u$ and $v$ at $t = 0$. ($E_3(t_1=0) = 0$).

Case (2):

We consider the special two dimensional situation of Diagram 1.
The resonant plasma region boundaries on the horizontal lines $aa'$ and $BB'$. The lines $AD'$ and $BC'$, both parallel to $k_1/|k_1|$, denote the bounds of the region within which the mode $(\omega_1, k_1)$ is localized. The lines $AD$ and $BC$ bear a similar relation with respect to the mode $(\omega_2, k_2)$. We assume

\[ \frac{|k_1 \times k_2|}{|k_1||k_2|} \ll 1 \]  \hspace{1cm} (51)

\[ \frac{k_1 \cdot k_3}{|k_1||k_3|} \ll 1 \]  \hspace{1cm} (52)

\[ \frac{\partial E_1}{\partial y} \bigg|_{(x=d)} \left( \frac{E_1}{E_1(y=0)} \right) \ll 1 \]  \hspace{1cm} (53)

\[ \frac{\partial E_1}{\partial y} \bigg|_{(x=0)} \left( \frac{E_1}{E_1(y=0)} \right) \geq 1 \]  \hspace{1cm} (54)

On using (20) and (22), conditions (53) and (54) become

\[ \left( \frac{e\omega_p k_3}{4m\omega_1 \omega_2 v_0^2} \right) \left( \frac{\omega_3}{k_3} \right) \left( \frac{ek_3}{4m\omega_3 c^2} \right) \left( \frac{\omega_1}{k_1} \right) E_1 E_2 L \frac{|k_1 \times k_2|}{|k_1||k_2|} \ll 1 \]  \hspace{1cm} (55)

\[ \left( \frac{e\omega_p k_3}{4m\omega_1 \omega_2 v_0^2} \right) \left( \frac{\omega_3}{k_3} \right) \left( \frac{ek_3}{4m\omega_3 c^2} \right) \left( \frac{\omega_1}{k_1} \right) E_1 E_2 D \geq 1 \]  \hspace{1cm} (56)
Inequality (56) insures the occurrence of mode coupling effects in the volume of diagram 1.

In view of (56) we may look for solutions to Eqs. (48) of the form \( \phi = \phi(\xi) \), where \( \xi \equiv (4\Lambda x_1 y_1)^{1/2} \). With this substitution, Eqs. (48) becomes

\[
\frac{d^2\phi}{d\xi^2} + \frac{1}{\xi} \frac{d\phi}{d\xi} + \sin \phi = 0 .
\]  

(57)

Eq. (57) is difficult to solve. However, we can get some idea of the behavior by considering small values of \( \phi_0 \) \((\phi(x_1=0, y_1=0))\). Then \( \sin \phi \approx \phi \), and the solution of Eq. (57) is of the form \( \phi = \phi_0 J_0(\xi) \) for \( \xi \ll (\phi_0)^{-2} \). Here \( J_0(\xi) \) is the Bessel function of the first kind of order zero, so that \( \phi \) is an oscillating function of \( \xi \) which approaches zero for \( \xi \gg 1 \).

The restriction that \( \phi \) be small implies that if \( u_o \) is parallel to \( v_o \), \( \|u_o\| \) is taken to be small; however, when \( u_o \) is not parallel to \( v_o \), \( u_o \) and \( v_o \) can have arbitrary magnitudes, but have to be almost perpendicular.

The asymptotic limit of \( \phi \) implies that \( w(\xi \to \infty) = 0 \); also \( u(\xi \to \infty) = 0 \) when \( u_o \) is parallel to \( v_o \), and \( u \cdot v = 0 \) when \( u_o \) is not parallel to \( v_o \).
CONCLUSIONS

We have established the connection between Sturrock's mode-coupling equations\textsuperscript{6} which were derived by means of a derivative expansion technique, and the quasi-linear solution of the non-linear Vlasov equation and Maxwell's equations, generalizing the former to include dissipation, finite temperature and spatial variation effects. For the system of two transverse modes and the resulting driven longitudinal mode we have determined and treated non-linear frequency conversion and polarization rotation effects. We note that in the absence of detuning and damping limitations, for problems involving a single coordinate there is a characteristic evolution which is basically oscillatory in that it is describable in terms of elliptic functions; whereas even for a relatively simple two-dimensional problem the evolution of the transverse modes is characterized by an oscillation of decreasing magnitude which tends towards the same limit as when damping alone controls the longitudinal oscillations. From an experimental viewpoint the mode coupling effects are optimized by the choice of \( \left( \frac{\omega_p}{\omega_T} \right) \) of order unity.

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17. An identical expression for $E_3$ was obtained by N. M. Kroll, A. Ron, and N. Rostoker, Phys. Rev. Letters 13, 83 (1964). However, their $E_1$ and $E_2$ were constants, because they neglected resonant mode coupling.

18. The definition and properties of the elliptic integrals and functions can be found, for example, in E. Jahnke and F. Emde, Tables of Functions, Dover Publications, New York.