

TECHNICAL MEMORANDUM X-53365

BOUNDARY VALUE PROBLEMS ASSOCIATED WITH OPTIMIZATION THEORY

By

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ABSTRACT

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Optimization theory is applied to the physics of trajectory problems to yield a boundary value formulation. Three practical numerical methods for obtaining solutions to boundary value problems are discussed, and a detailed explanation of the application of the most efficient method to a sample problem is included. Numerical methods are needed because the application of optimization theory to trajectory problems generally results in nonlinear differential equations connecting the boundary conditions that are not all specified at the same value of the independent variable. The methods discussed are adaptable to the solution of boundary value problems other than the ones associated with trajectory problems or optimization theory.

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AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

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SUMMARY

Optimization theory is applied to the physics of trajectory problems to yield a boundary value formulation. Three practical numerical methods for obtaining solutions to boundary value problems are discussed, and a detailed explanation of the application of the most efficient method to a sample problem is included. Numerical methods are needed because the application of optimization theory to trajectory problems generally results in nonlinear differential equations connecting the boundary conditions that are not all specified at the same value of the independent variable. The methods discussed are adaptable to the solution of boundary value problems other than the ones associated with trajectory problems or optimization theory.

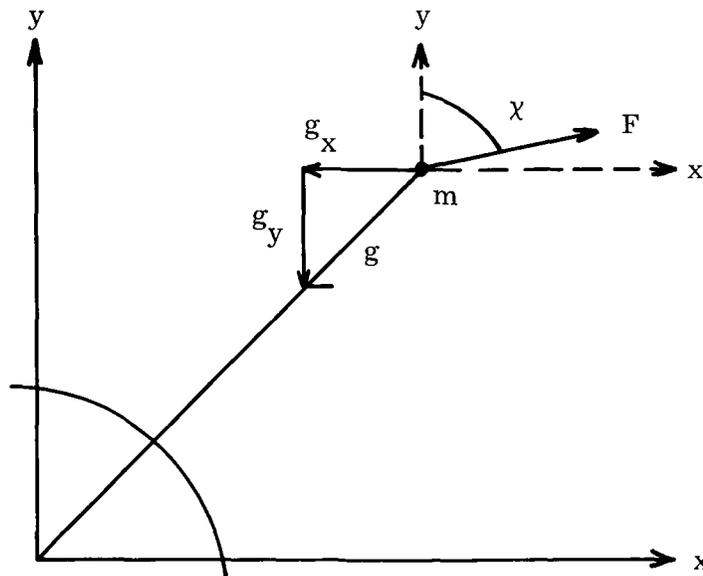
INTRODUCTION

The calculation of trajectories or flight paths for rocket-powered vehicles and projectiles has long been of interest to mathematicians, physicists, and engineers. With the advent of modern steerable rocket-powered vehicles the problem has become especially interesting because of the multiplicity of paths that can occur. Obviously, for a particular vehicle, there should be some path that allows the attainment of a desired destination most economically. More generally, the vehicle must be steered so that it achieves a certain mission in an optimum fashion, where optimum usually means most economically although other considerations may be added to the definition. The determination of an optimum path can be accomplished by the application of optimization theory to the physics of the problem. The next section of this report will be concerned with an explanation of this procedure with the aid of a simple example.

GENERAL DISCUSSION

Formulation of the Boundary Value Problem

From elementary physics it is known that a force acting on a point mass produces an acceleration. Locate the force and point mass in a two-dimensional Cartesian coordinate system with its origin at the center of a gravitational field and the most simple mathematical approximation to the motion of a rocket-powered vehicle is obtained. To be more specific the following diagram is used.



The diagram aids in the expression of the differential equations of motion.

$$\ddot{x} = (F/m) \sin \chi - g_x$$

$$\ddot{y} = (F/m) \cos \chi - g_y$$

The equations are the sums of the accelerations in the x and y directions; no aerodynamic accelerations are considered. F is assumed to remain constant for the duration of the powered flight, and the mass of the rocket is assumed to diminish at a constant rate (i. e., fuel is burned at a constant rate). Then,

$$m(t) = m_0 - \dot{m}(t - t_0)$$

where \dot{m} is the constant rate of change of mass with respect to time. Also, the accelerations caused by gravity can be expressed as functions of the position of the vehicle because gravity fields are usually assumed to be potential fields. The potential per unit mass outside a large spherical mass is given by the expression

$$V = \frac{\mu}{R}$$

where $R = \sqrt{x^2 + y^2}$ and μ is a constant characteristic of a particular gravity field. Then from the physics of potential fields

$$g_x = - \frac{\partial V}{\partial x} = \frac{\mu x}{R^3}$$

$$g_y = - \frac{\partial V}{\partial y} = \frac{\mu y}{R^3}$$

The preceding relationships define all of the expressions in the equations for \ddot{x} and \ddot{y} except the angle χ . Chi (χ) is called the steering angle or the control angle, and it is to be determined at every instant of the powered flight so that the thrust (F) will be pointed in an optimum direction to minimize some specified performance parameter. If a steering angle program $\chi(t)$ and starting conditions ($x_0, y_0, \dot{x}_0, \dot{y}_0, t_0$) are given, then the equations of motion (\ddot{x} and \ddot{y}) can be integrated numerically to describe the flight path; and $x(t), y(t), \dot{x}(t), \dot{y}(t)$ are determined and tabulated at intervals of time. The symbols \dot{x} and \dot{y} are the first integrals of the equations of motion.

Because many different steering angle programs could be specified for this example, the problem is to find a particular program that minimizes the fuel consumed to move the vehicle from the starting conditions to the desired end conditions. For example, if the mission is to achieve a circular orbit at a specified altitude (R_f), the following conditions must be satisfied at the termination of the thrusting period for the rocket-powered vehicle:

$$R_f = \sqrt{x^2 + y^2}$$

$$v_f = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{\mu}{R_f}}$$

$$0 = x\dot{x} + y\dot{y}$$

$$\text{or } G_1 = x^2 + y^2 - R_f^2 = 0$$

$$G_2 = \dot{x}^2 + \dot{y}^2 - v_f^2 = 0$$

$$G_3 = x\dot{x} + y\dot{y} = 0$$

The second expression ($v_f = \sqrt{\frac{\mu}{R_f}}$) is determined by equating the force caused by gravity to the centrifugal force, due to the rocket's velocity. The third expression states that the velocity vector must be perpendicular to the radius vector (i. e., their dot product is zero).

A more abstract notation will now be used for convenience in explaining the conditions that must be satisfied by the optimum steering angle program. The differential equations of motion are written in the following first-order form:

$$\dot{x}_i = f_i(\bar{x}, \bar{u}, t)$$

where

$$i = 1, 2, \dots, n$$

$$\bar{x} = x_1, x_2, \dots, x_n \quad (\text{state variables})$$

$$\bar{u} = u_1, u_2, \dots, u_r \quad (\text{control variables}) .$$

For the example problem the substitutions required are:

$$x_1 = x$$

$$x_2 = y$$

$$x_3 = \dot{x}$$

$$x_4 = \dot{y}$$

$$x_5 = m$$

and $u_1 = \chi$.

Then $\dot{x}_1 = x_3$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = \left(\frac{F}{x_5} \right) \sin u_1 - \frac{\mu(x_1)}{(\sqrt{x_1^2 + x_2^2})^3}$$

$$\dot{x}_4 = \left(\frac{F}{x_5} \right) \cos u_1 - \frac{\mu(x_2)}{(\sqrt{x_1^2 + x_2^2})^3}$$

$$\dot{x}_5 = -\dot{m} .$$

The quantity to be optimized (maximized or minimized) is written as follows:

$$S = \sum_{i=1}^n c_i x_i(t_f)$$

where the c_i 's are arbitrary constants and t_f is the time at which the thrust is terminated and the mission conditions are to be satisfied.

For the example problem, choose

$$c_1 = c_2 = c_3 = c_4 = 0, \text{ and } c_5 = 1 .$$

Then $S = m(t_f)$ and a maximization of S will cause the fuel used to be a minimum. Also, the starting conditions and mission conditions are written in a functional form:

$$F_\alpha (\bar{x}^0, t^0) = 0 ,$$

where $\bar{x}^0 = (x_1^0, x_2^0, \dots, x_n^0)$

$$\alpha = 1, 2, \dots, k \leq n + 1 .$$

$$G_\beta (\bar{x}_f, t_f) = 0 .$$

where $\bar{x}_f = (x_1^f, x_2^f, \dots, x_n^f)$

$$\beta = 1, 2, \dots, l \leq n + 1$$

For the example system of differential equations, the functional form of the boundary conditions can be written explicitly.

$$F_1 = x_1(t_0) - x_0 = 0$$

$$F_2 = x_2(t_0) - y_0 = 0$$

$$F_3 = x_3(t_0) - \dot{x}_0 = 0$$

$$F_4 = x_4(t_0) - \dot{y}_0 = 0$$

$$F_5 = x_5(t_0) - m_0 = 0$$

$$F_6 = t(t_0) - t_0 = 0$$

$$G_1 = (x^2 + y^2) - R^2(t_f) = 0$$

$$G_2 = (\dot{x}^2 + \dot{y}^2) - v^2(t_f) = 0$$

$$G_3 = x\dot{x} + y\dot{y} = 0 .$$

Now the flight path optimization problem can be stated completely in the more abstract notation. The statement of the problem is as follows: Determine a $\bar{u}(t)$ that satisfies the system of differential equations

$$\begin{aligned} \dot{\bar{x}}_i &= f_i(\bar{x}, \bar{u}, t) & i &= 1, 2, \dots, n \\ \bar{x} &= x_1, x_2, \dots, x_n \\ \bar{u} &= u_1, u_2, \dots, u_r \end{aligned}$$

with boundary conditions

$$\begin{aligned} F_\alpha(\bar{x}^0, t^0) &= 0 & \alpha &= 1, 2, \dots, l \leq n + 1 \\ G_\beta(\bar{x}_f, t_f) &= 0 & \beta &= 1, 2, \dots, m \leq n + 1 \end{aligned}$$

and that maximizes or minimizes the quantity

$$S = \sum_{i=1}^n c_i x_i(t_f) \quad .$$

To determine a $\bar{u}(t)$ that satisfies the problem statement, an adjoint system of differential equations must be defined

$$\lambda_i^{\dot{}} = - \sum_{j=1}^n \lambda_j \left[\frac{\partial f_j(\bar{x}, \bar{u}, t)}{\partial x_i} \right] \quad \begin{array}{l} i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \end{array} \quad .$$

Now the total system of differential equations ($\dot{\bar{x}}_i$ and $\dot{\lambda}_i$) can be integrated numerically to yield $x_i(t)$ and $\lambda_i(t)$ if the initial conditions (x_i^0 , λ_i^0 , and t^0) are known and a steering function $u_k(t)$ is specified in some manner. The optimum steering functions $u_k(t)$ can be specified by defining

$$H = \sum_{j=1}^n \lambda_j f_j(\bar{x}, \bar{u}, t) \quad .$$

Then another necessary condition to be satisfied for a minimization (maximization) of the quantity S is that the function H be a maximum (minimum) with respect to u at every $t_0 \leq t \leq t_f$. Actually, the maximization or minimization of H usually allows the relations $\frac{\partial H}{\partial u_k} = 0$ ($k = 1, 2, \dots, r$) to be solved to give a steering function u_k that depends on the x_i 's and λ_i 's at every $t_0 \leq t \leq t_f$.

Also, the second order terms in the series expansion for H in terms of u_k about the u_k that makes the first partials zero must be examined to assure a maximization or minimization of H with respect to u_k . This examination results in a condition which requires that the eigenvalues or the principle minors of the matrix of second partial derivatives of H with respect to u_k be all positive for H a minimum and alternate in signs (starting with a negative) for H a maximum. The example problem in Appendix II shows how this condition can be satisfied for two control variables.

The final necessary conditions to be discussed are concerned with the boundary conditions. The x_i^0 , λ_i^0 , t_0 , and t_f are chosen to satisfy the following relationships:

$$F_\alpha(\bar{x}^0, t^0) = 0$$

$$\lambda_i(t_0) = \sum_{\alpha=1}^l p_\alpha \left(\frac{\partial F_\alpha}{\partial x_i} \right)_{t_0}$$

$$H(t_0) + \sum_{\alpha=1}^l p_\alpha \left(\frac{\partial F_\alpha}{\partial t} \right)_{t_0} + \sum_{i=1}^n c_i \dot{x}_i(t_0) = 0$$

$$G_\beta(\bar{x}_f, t_f) = 0$$

$$\lambda_i(t_f) = -c_i - \sum_{\beta=1}^m \rho_\beta \left(\frac{\partial G_\beta}{\partial x_i} \right)_{t_f}$$

$$H(t_f) - \sum_{\beta=1}^m \rho_\beta \left(\frac{\partial G_\beta}{\partial t} \right)_{t_f} = 0$$

The boundary conditions are actually only $n + 1$ independent relationships in x_i , λ_i , and t at t_0 and t_f because the multipliers p_α and ρ_β can be eliminated by using l of the $n + l + 1$ relations at t_0 to solve for the p_α 's and m of the $n + m + 1$ relations at t_f to solve for the ρ_β 's. When the p_α 's and ρ_β 's are eliminated, the necessary conditions to be satisfied by the optimization problem can be written in a more compact system of equations

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \quad i = 1, 2, \dots, n$$

$$\lambda_i = - \frac{\partial H}{\partial x_i}$$

$$0 = \frac{\partial H}{\partial u_k} \quad k = 1, 2, \dots, r$$

$$H \text{ a maximum for a minimization of } S = \sum_{i=1}^n c_i x_i(t_f)$$

and

$$H \text{ a minimum for a maximization of } S = \sum_{i=1}^n c_i x_i(t_f)$$

with boundary conditions

$$D_j(\bar{x}_0, \bar{\lambda}_0, \bar{u}_0, t_0) = 0 \quad j = 1, 2, \dots, n+1$$

$$E_j(\bar{x}_f, \bar{\lambda}_f, \bar{u}_f, t_f) = 0 \quad .$$

A derivation of the preceding necessary conditions is given in Appendix I.

The next section of this report will be concerned with three of the numerical techniques that may be used to determine the \bar{x}_0 , $\bar{\lambda}_0$, t_0 , and t_f that will satisfy the $2n + 2$ relations D_j and E_j .

Methods for Solution

Newton's Method with Numerical Derivatives. - The simplest and most straightforward method of satisfying the D_j and E_j relations is to evaluate numerically the effects of small changes in x_{i0} , λ_{i0} , t_0 , and t_f on the D_j and E_j relations. This is done by computing what is called a trial with estimated values for the unknowns x_{i0} , λ_{i0} , t_0 , and t_f . Naturally the D_j and E_j relations will not be zero, but if the unknowns x_{i0} and λ_{i0} are changed, small amounts one at a time, the changes can be used to determine the correct values for x_{i0} , λ_{i0} , t_0 , and t_f . The initial trials and the $2n$ small changes in the unknowns allow a numerical determination of the partial derivative of D_j and E_j with respect to x_{i0} and λ_{i0} .

To show how this can be done let D_j^e , E_j^e , \dot{D}_j^e , and \dot{E}_j^e at t_0 and t_f be the symbols used to denote the values of D_j , E_j , \dot{D}_j , and \dot{E}_j at t_0 and t_f in the trial computation. Then let $D_j^{\Delta x_{i0}}$ and $E_j^{\Delta x_{i0}}$, at t_0 and t_f be the symbols used to denote the values of D_j and E_j at t_0 and t_f in the computation of the trial

associated with the small change Δx_{i0} in the single unknown x_{i0} . This allows the following expressions to be written:

$$\frac{\partial D_j(t_0)}{\partial x_{i0}} \approx \frac{D_j^{\Delta x_{i0}} - D_j^e}{\Delta x_{i0}}$$

$$\frac{\partial E_j(t_f)}{\partial x_{i0}} \approx \frac{E_j^{\Delta x_{i0}} - E_j^e}{\Delta x_{i0}}$$

The other partial derivatives $\frac{\partial D_j(t_0)}{\partial x_{i0}}$, $\frac{\partial E_j(t_f)}{\partial x_{i0}}$, $\frac{\partial D_j(t_0)}{\partial \lambda_{i0}}$, and $\frac{\partial E_j(t_f)}{\partial \lambda_{i0}}$ can be determined by similar independent small changes in the x_{i0} and λ_{i0} . $\dot{D}_j(t_0)$ and $\dot{E}_j(t_0)$ are also computed for the estimated values of the unknowns x_{i0} , λ_{i0} , t_0 , and t_f .

Newton's iteration formula can then be used in the following form:

$$0 = \begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix} + \begin{bmatrix} \frac{\partial D_j(t_0)}{\partial x_{i0}} & \frac{\partial D_j(t_0)}{\partial \lambda_{i0}} & \dot{D}_j^e(t_0) & 0 \\ \frac{\partial E_j(t_f)}{\partial x_{i0}} & \frac{\partial E_j(t_f)}{\partial \lambda_{i0}} & 0 & \dot{E}_j^e(t_f) \end{bmatrix} \begin{bmatrix} \Delta x_{i0} \\ \Delta \lambda_{i0} \\ \Delta t_0 \\ \Delta t_f \end{bmatrix}$$

This formula is simply the linear terms of a Taylor Series expansion in the $2n + 2$ variables x_{i0} , λ_{i0} , t_0 , and t_f .

Solving the matrix equation for Δx_{i0} , $\Delta \lambda_{i0}$, Δt_0 , and Δt_f gives the corrections that are needed for another trial.

$$\begin{bmatrix} \Delta x_{i0} \\ \Delta \lambda_{i0} \\ \Delta t_0 \\ \Delta t_f \end{bmatrix} = - \begin{bmatrix} \frac{\partial D_j(t_0)}{\partial x_{i0}} & \frac{\partial D_j(t_0)}{\partial \lambda_{i0}} & \dot{D}_j^e(t_0) & 0 \\ \frac{\partial E_j(t_f)}{\partial x_{i0}} & \frac{\partial E_j(t_f)}{\partial \lambda_{i0}} & 0 & \dot{E}_j^e(t_f) \end{bmatrix}^{-1} \begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix}$$

The entire process can now be repeated with the preceding solution as new estimated values for x_{i0} , λ_{i0} , t_0 , and t_f . Convergence of the process to values of x_{i0} , λ_{i0} , t_0 , and t_f that yields zero values for D_j and E_j can be greatly enhanced by a simple modification. Instead of using zero in Newton's formula, the quantity $K \begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix}$ is used. Then the expressions for Δx_{i0} , $\Delta \lambda_{i0}$, Δt_0 , and Δt_f become

$$\begin{bmatrix} \Delta x_{i0} \\ \Delta \lambda_{i0} \\ \Delta t_0 \\ \Delta t_f \end{bmatrix} = - (1-K) \begin{bmatrix} \frac{\partial D_j(t_0)}{\partial x_{i0}}, & \frac{\partial D_j(t_0)}{\partial \lambda_{i0}}, & \dot{D}_j^e(t_0), & 0 \\ \frac{\partial E_j(t_f)}{\partial x_{i0}}, & \frac{\partial E_j(t_f)}{\partial \lambda_{i0}}, & 0, & \dot{E}_j^e(t_f) \end{bmatrix}^{-1} \begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix}.$$

The constant K must be chosen in the range $0 \leq K < 1$. It is usually chosen to be zero unless the vector $\begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix}$ is not decreased on a particular iteration cycle. Then K can be increased as close to one as is necessary to assure that the computed values of Δx_{i0} , $\Delta \lambda_{i0}$, Δt_0 , and Δt_f produce a decrease in $\begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix}$.

Another modification to this procedure is called the Secant method. Its main advantage is that it does not require $2n$ integrations of the system of differential equations for each iteration cycle. After the first iteration cycle only one additional integration of the system of differential equations is used to modify the partial derivative matrix. A more complete explanation of this procedure can be found in Reference 1.

Newton's Method with Integrated Partial Derivatives. - The next method of solving the boundary value problem also uses Newton's iteration formula, but the partial derivative matrix is computed in a more accurate and efficient manner. A rigorous justification for the procedure is found in Reference 2, although most of the steps are intuitively obvious. The method involves chain rule differentiation of the boundary conditions. The system of equations which results is:

$$\frac{\partial D_j(t_0)}{\partial x_{i0}} = \left[\frac{\partial D_j(t_0)}{\partial x_1} \right] \left[\frac{\partial x_1(t_0)}{\partial x_{i0}} \right] + \left[\frac{\partial D_j(t_0)}{\partial \lambda_1} \right] \left[\frac{\partial \lambda_1(t_0)}{\partial x_{i0}} \right] + \left[\frac{\partial D_j(t_0)}{\partial u_k} \right] \left[\frac{\partial u_k(t_0)}{\partial x_{i0}} \right]$$

$$\frac{\partial D_j(t_0)}{\partial \lambda_{i0}} = \left[\frac{\partial D_j(t_0)}{\partial x_1} \right] \left[\frac{\partial x_1(t_0)}{\partial \lambda_{i0}} \right] + \left[\frac{\partial D_j(t_0)}{\partial \lambda_1} \right] \left[\frac{\partial \lambda_1(t_0)}{\partial \lambda_{i0}} \right] + \left[\frac{\partial D_j(t_0)}{\partial u_k} \right] \left[\frac{\partial u_k(t_0)}{\partial \lambda_{i0}} \right]$$

$$\frac{\partial E_j(t_f)}{\partial x_{i0}} = \left[\frac{\partial E_j(t_f)}{\partial x_1} \right] \left[\frac{\partial x_1(t_f)}{\partial x_{i0}} \right] + \left[\frac{\partial E_j(t_f)}{\partial \lambda_1} \right] \left[\frac{\partial \lambda_1(t_f)}{\partial x_{i0}} \right] + \left[\frac{\partial E_j(t_f)}{\partial u_k} \right] \left[\frac{\partial u_k(t_f)}{\partial x_{i0}} \right]$$

$$\frac{\partial E_j(t_f)}{\partial \lambda_{i0}} = \left[\frac{\partial E_j(t_f)}{\partial x_1} \right] \left[\frac{\partial x_1(t_f)}{\partial \lambda_{i0}} \right] + \left[\frac{\partial E_j(t_f)}{\partial \lambda_1} \right] \left[\frac{\partial \lambda_1(t_f)}{\partial \lambda_{i0}} \right] + \left[\frac{\partial E_j(t_f)}{\partial u_k} \right] \left[\frac{\partial u_k(t_f)}{\partial \lambda_{i0}} \right]$$

$$l = 1, 2, \dots, n \quad .$$

The only unknowns that appear on the right-hand side of the preceding equations

are the matrices $\left[\frac{\partial x_1}{\partial x_{i0}} \right]$, $\left[\frac{\partial \lambda_1}{\partial x_{i0}} \right]$, $\left[\frac{\partial u_k}{\partial x_{i0}} \right]$, $\left[\frac{\partial x_1}{\partial \lambda_{i0}} \right]$, $\left[\frac{\partial \lambda_1}{\partial \lambda_{i0}} \right]$, and $\left[\frac{\partial u_k}{\partial \lambda_{i0}} \right]$ at t_0 and t_f .

The matrices $\left[\frac{\partial u_k}{\partial x_{i0}} \right]$ and $\left[\frac{\partial u_k}{\partial \lambda_{i0}} \right]$ can be determined at any time $t_0 \leq t \leq t_f$

from chain rule differentiation of the third equation in the system of equations

$$0 = \frac{\partial H}{\partial u_k} \quad .$$

Differentiation yields:

$$0 = \left[\frac{\partial^2 H}{\partial u_m \partial x_1} \right] \left[\frac{\partial x_1}{\partial x_{i0}} \right] + \left[\frac{\partial^2 H}{\partial u_m \partial \lambda_1} \right] \left[\frac{\partial \lambda_1}{\partial x_{i0}} \right] + \left[\frac{\partial^2 H}{\partial u_m \partial u_k} \right] \left[\frac{\partial u_k}{\partial x_{i0}} \right]$$

$$0 = \left[\frac{\partial^2 H}{\partial u_m \partial x_1} \right] \left[\frac{\partial x_1}{\partial \lambda_{i0}} \right] + \left[\frac{\partial^2 H}{\partial u_m \partial \lambda_1} \right] \left[\frac{\partial \lambda_1}{\partial \lambda_{i0}} \right] + \left[\frac{\partial^2 H}{\partial u_m \partial u_k} \right] \left[\frac{\partial u_k}{\partial \lambda_{i0}} \right]$$

$$m = 1, 2, \dots, r \quad .$$

Therefore,

$$\begin{aligned} \begin{bmatrix} \frac{\partial u_k}{\partial x_{i0}} \end{bmatrix} &= - \begin{bmatrix} \frac{\partial^2 H}{\partial u_m \partial u_k} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \frac{\partial^2 H}{\partial u_m \partial x_1} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial x_{i0}} \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 H}{\partial u_m \partial \lambda_1} \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda_1}{\partial x_{i0}} \end{bmatrix} \right\} \\ \begin{bmatrix} \frac{\partial u_k}{\partial \lambda_{i0}} \end{bmatrix} &= - \begin{bmatrix} \frac{\partial^2 H}{\partial u_m \partial u_k} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \frac{\partial^2 H}{\partial u_m \partial x_1} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial \lambda_{i0}} \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 H}{\partial u_m \partial \lambda_1} \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda_1}{\partial \lambda_{i0}} \end{bmatrix} \right\} \end{aligned}$$

Now the matrices $\begin{bmatrix} \frac{\partial x_1}{\partial x_{i0}} \end{bmatrix}$, $\begin{bmatrix} \frac{\partial \lambda_1}{\partial x_{i0}} \end{bmatrix}$, $\begin{bmatrix} \frac{\partial x_1}{\partial \lambda_{i0}} \end{bmatrix}$, and $\begin{bmatrix} \frac{\partial \lambda_1}{\partial \lambda_{i0}} \end{bmatrix}$ at t_0 and t_f

must still be determined. The matrices at t_0 are determined from the initial conditions.

$$x_1(t_0) = x_{i0}$$

$$\lambda_1(t_0) = \lambda_{i0} \quad .$$

Differentiation yields:

$$\begin{bmatrix} \frac{\partial x_1(t_0)}{\partial x_{i0}} \end{bmatrix} = [I]$$

$$\begin{bmatrix} \frac{\partial x_1(t_0)}{\partial \lambda_{i0}} \end{bmatrix} = [0]$$

$$\begin{bmatrix} \frac{\partial \lambda_1(t_0)}{\partial x_{i0}} \end{bmatrix} = [0]$$

$$\begin{bmatrix} \frac{\partial \lambda_1(t_0)}{\partial \lambda_{i0}} \end{bmatrix} = [I] \quad .$$

The matrices at t_f are determined by integrating the matrix differential equations obtained by differentiation of the first and second equations of the system of equations

$$\dot{x}_1 = \frac{\partial H}{\partial \lambda_1}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1}$$

Differentiation yields:

$$\left[\frac{\partial \dot{x}_1}{\partial x_{i0}} \right] = \frac{d}{dt} \left[\frac{\partial x_1}{\partial x_{i0}} \right] = \left[\frac{\partial^2 H}{\partial \lambda_1 \partial x_1} \right] \left[\frac{\partial x_1}{\partial x_{i0}} \right] + \left[\frac{\partial^2 H}{\partial \lambda_1 \partial u_k} \right] \left[\frac{\partial u_k}{\partial x_{i0}} \right]$$

$$\left[\frac{\partial \dot{\lambda}_1}{\partial x_{i0}} \right] = \frac{d}{dt} \left[\frac{\partial \lambda_1}{\partial x_{i0}} \right] = - \left[\frac{\partial^2 H}{\partial x_1 \partial x_1} \right] \left[\frac{\partial x_1}{\partial x_{i0}} \right] - \left[\frac{\partial^2 H}{\partial x_1 \partial \lambda_1} \right] \left[\frac{\partial \lambda_1}{\partial x_{i0}} \right] - \left[\frac{\partial^2 H}{\partial x_1 \partial u_k} \right] \left[\frac{\partial u_k}{\partial x_{i0}} \right]$$

$$\left[\frac{\partial \dot{x}_1}{\partial \lambda_{i0}} \right] = \frac{d}{dt} \left[\frac{\partial x_1}{\partial \lambda_{i0}} \right] = \left[\frac{\partial^2 H}{\partial \lambda_1 \partial x_1} \right] \left[\frac{\partial x_1}{\partial \lambda_{i0}} \right] + \left[\frac{\partial^2 H}{\partial \lambda_1 \partial u_k} \right] \left[\frac{\partial u_k}{\partial \lambda_{i0}} \right]$$

$$\left[\frac{\partial \dot{\lambda}_1}{\partial \lambda_{i0}} \right] = \frac{d}{dt} \left[\frac{\partial \lambda_1}{\partial \lambda_{i0}} \right] = - \left[\frac{\partial^2 H}{\partial x_1 \partial x_1} \right] \left[\frac{\partial x_1}{\partial \lambda_{i0}} \right] - \left[\frac{\partial^2 H}{\partial x_1 \partial \lambda_1} \right] \left[\frac{\partial \lambda_1}{\partial \lambda_{i0}} \right] - \left[\frac{\partial^2 H}{\partial x_1 \partial u_k} \right] \left[\frac{\partial u_k}{\partial \lambda_{i0}} \right]$$

The initial conditions for the preceding system of matrix differential equations $\left[\frac{\partial x_1(t_0)}{\partial x_{i0}} \right]$, $\left[\frac{\partial x_1(t_0)}{\partial \lambda_{i0}} \right]$, $\left[\frac{\partial \lambda_1(t_0)}{\partial x_{i0}} \right]$, and $\left[\frac{\partial \lambda_1(t_0)}{\partial \lambda_{i0}} \right]$ have already been

determined. Also, $\left[\frac{\partial u_k}{\partial x_{i0}} \right]$ and $\left[\frac{\partial u_k}{\partial \lambda_{i0}} \right]$ have already been determined for

every $t_0 \leq t \leq t_f$; therefore the equations can be integrated to yield the desired results at t_f . The desired results at t_f are values for the matrices

$\left[\frac{\partial x_1}{\partial x_{i0}} \right]$, $\left[\frac{\partial \lambda_1}{\partial x_{i0}} \right]$, $\left[\frac{\partial x_1}{\partial \lambda_{i0}} \right]$, and $\left[\frac{\partial \lambda_1}{\partial \lambda_{i0}} \right]$. These matrices are substituted

into the formulas for $\frac{\partial E_j(t_f)}{\partial x_{i0}}$ and $\frac{\partial E_j(t_f)}{\partial \lambda_{i0}}$. Then all of the elements are again available for the use of Newton's iteration formula in the following form

$$\begin{bmatrix} \Delta x_{i0} \\ \Delta \lambda_{i0} \\ \Delta t_0 \\ \Delta t_f \end{bmatrix} = - \begin{bmatrix} \frac{\partial D_j(t_0)}{\partial x_{i0}} & \frac{\partial D_j(t_0)}{\partial \lambda_{i0}} & \dot{D}_j^e(t_0) & 0 \\ \frac{\partial E_j(t_f)}{\partial x_{i0}} & \frac{\partial E_j(t_f)}{\partial \lambda_{i0}} & 0 & \dot{E}_j^e(t_f) \end{bmatrix}^{-1} \begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix}.$$

The computations for the partial derivatives can be performed along each trial trajectory, and the results are much more accurate than the numerical differentiation procedure used in the first method described for satisfying the boundary conditions. The modification to increase the chances for convergence suggested in the explanation of the first method is still very helpful with the new, more accurate partial derivatives.

Modified Newton-Raphson Operator Method. -The final method to be discussed is a modification of the quasilinearization or Newton-Raphson operator method. The modified method is similar to the steepest descent procedure in that prespecified control functions are needed to start the iteration cycle, but the derivation is such that no backward integration is required. The first step in the explanation of the procedure is the linearization of the system of equations by using the first-order terms in a Taylor Series expansion in all the variables about the initial trial trajectory,

$$\begin{aligned} \dot{x}_i &= \dot{x}_i^e + \left[\frac{\partial^2 H}{\partial \lambda_i \partial x_1} \right] (\Delta x_1) + \left[\frac{\partial^2 H}{\partial \lambda_i \partial u_k} \right] (\Delta u_k) \\ \dot{\lambda}_i &= \dot{\lambda}_i^e - \left[\frac{\partial^2 H}{\partial x_i \partial x_1} \right] (\Delta x_1) - \left[\frac{\partial^2 H}{\partial x_i \partial \lambda_1} \right] (\Delta \lambda_1) - \left[\frac{\partial^2 H}{\partial x_i \partial u_k} \right] (\Delta u_k) \\ 0 &= \left(\frac{\partial H}{\partial u_m} \right)_e + \left[\frac{\partial^2 H}{\partial u_m \partial x_1} \right] (\Delta x_1) + \left[\frac{\partial^2 H}{\partial u_m \partial \lambda_1} \right] (\Delta \lambda_1) + \left[\frac{\partial^2 H}{\partial u_m \partial u_k} \right] (\Delta u_k) \\ m &= 1, 2, \dots, r \end{aligned}$$

$$\begin{aligned}
\Delta x_1 &= x_1 - x_1^e & l &= 1, 2, \dots, n \\
\Delta \lambda_1 &= \lambda_1 - \lambda_1^e & k &= 1, 2, \dots, r \\
\Delta u_k &= u_k - u_k^e
\end{aligned}$$

The last equation in the preceding expansions can be solved for Δu_k and the results substituted into the first two equations. Then the first two equations can be arranged as a linear system of differential equations with time varying matrix coefficients:

$$\begin{aligned}
\dot{x}_i &= [A_1(t)] x_i + [B_1(t)] \lambda_i + C_{i1}(t) \\
\dot{\lambda}_i &= [A_2(t)] x_i + [B_2(t)] \lambda_i + C_{i2}(t) .
\end{aligned}$$

The coefficients A, B, and C are generated along the trial trajectory using the arbitrary guesses for x_{i0} , λ_{i0} , t_0 , t_f , and the arbitrary control functions $u_k(t)$.

To obtain a general solution to the above linear system of differential equations one particular solution (x_i^P and λ_i^P) of the entire system must be generated by numerical integration and $2n$ particular solutions to the homogeneous part (i.e., with the C_{i1} and C_{i2} terms left off) must be generated. The initial conditions for the particular solution to the nonhomogeneous system are chosen to be the same as the initial conditions for the trial trajectory. The initial conditions for the $2n$ homogeneous solutions for convenience are chosen as follows and written in matrix notation:

$$x_{iq}(t_0) = [I] \quad \lambda_{iq}(t_0) = [0] \quad q = 1, 2, \dots, n .$$

Also

$$x_{is}(t_0) = [0] \quad \lambda_{is}(t_0) = [I] \quad s = 1, 2, \dots, n .$$

Each of the columns in the previous set of matrices is considered to be a set of initial conditions to yield the $2n$ particular solutions to the homogeneous system. All of the numerical integrations can be performed at the same time, and the general solution is then written:

$$x_i(t) = x_i^P(t) + \left[x_{iq}(t) \right] K_q + \left[x_{is}(t) \right] K_s .$$

$$\lambda_i(t) = \lambda_i^P(t) + \left[\lambda_{iq}(t) \right] K_q + \left[\lambda_{is}(t) \right] K_s .$$

The constants K_q and K_s in the previous relations are determined so that the general solution will satisfy the D_j and E_j boundary conditions. To do this the boundary conditions are also expanded in a Taylor Series with only the first-order terms retained. This yields the following expressions:

$$0 = D_j^e + \left[\frac{\partial D_j^e}{\partial x_i} \right]_{t_0} \Delta x_i(t_0) + \left[\frac{\partial D_j^e}{\partial \lambda_i} \right]_{t_0} \Delta \lambda_i(t_0) + \left[\frac{\partial D_j^e}{\partial u_k} \right]_{t_0} \Delta u_k(t_0) + \dot{D}_j^e(t_0) \Delta t_0$$

$$0 = E_j^e + \left[\frac{\partial E_j^e}{\partial x_i} \right]_{t_f} \Delta x_i(t_f) + \left[\frac{\partial E_j^e}{\partial \lambda_i} \right]_{t_f} \Delta \lambda_i(t_f) + \left[\frac{\partial E_j^e}{\partial u_k} \right]_{t_f} \Delta u_k(t_f) + \dot{E}_j^e(t_f) \Delta t_f$$

$$\Delta x_i(t_f) = x_i(t_f) - x_i^e(t_f)$$

$$\Delta \lambda_i(t_f) = \lambda_i(t_f) - \lambda_i^e(t_f) .$$

Again the third equation in the system of equations is used to obtain Δu_k in terms of Δx_i and $\Delta \lambda_i$. Then the general solutions for $x_i(t)$ and $\lambda_i(t)$ evaluated at t_0 and t_f are substituted into the Δx_i and $\Delta \lambda_i$ expressions, and the boundary conditions become $2n + 2$ linear equations in the unknowns K_q , K_s , Δt_0 , and Δt_f , which are easily solved.

Using the newly computed values for K_q , K_s , Δt_0 , and Δt_f , the original system of equation and the linear system with time varying coefficients are re-integrated. The initial conditions for the original system of equations remain the same, but the initial conditions for the linear system are given by evaluating the general solutions for $x_i(t)$ and $\lambda_i(t)$ at t_0 . This yields

$$\begin{aligned}
x_i(t_0)_{\text{new}} &= x_i^e(t_0) + K_q \\
\lambda_i(t_0)_{\text{new}} &= \lambda_i^e(t_0) + K_s \\
t_0_{\text{new}} &= t_0^e + \Delta t_0 \\
t_f_{\text{new}} &= t_f^e + \Delta t_f .
\end{aligned}$$

During the reintegration process the third equation of the system of equations is again used to compute Δu_k , which is added to the old control function to produce the new control function for the next iteration cycle. This procedure does not converge until both the boundary conditions are satisfied and $\frac{\partial H}{\partial u_k} = 0$ over the interval $t_0 \leq t \leq t_f$.

To increase the chances of convergence the process can be made to creep toward a solution by not requiring that D_j , E_j , and $\frac{\partial H}{\partial u_k}$ be zero on a particular trial, but only that it be smaller than the previous trial (i.e., zero in the Taylor

Series expansions for D_j , E_j , and $\frac{\partial H}{\partial u_k}$ is replaced by $K \begin{bmatrix} D_j^e \\ E_j^e \end{bmatrix}$ and $k \begin{bmatrix} \frac{\partial H}{\partial u_k} \end{bmatrix}$

where $0 \leq K < 1$ and $0 \leq k < 1$). This is the same idea that was suggested for the first and second methods of satisfying the boundary conditions.

Computational Considerations

All three of the preceding methods require that the units of length, mass, and time be adjusted or scaled such that each of the variables has the same order of magnitude. This scaling is needed mainly to retain good numerical precision when the matrix inverse in Newton's formula is taken or when the system of linear equations is solved in the third method. For trajectory optimization problems this is easily accomplished as is shown in the Computational Procedure Section of Appendix II.

None of the methods is sensitive to the choice of the quantities that must be estimated to begin the iteration cycle, except the first method. If the Δx values estimated for the numerical determination of the partial derivatives are not chosen properly, this method may not converge. The second method has been found to converge for almost any values of the quantities that must be estimated, except all zeroes. For example a heliocentric transfer from the Earth's orbit to the orbit of Mars converged in only 10 iterations with very crude guesses for the initial conditions. In Reference 3, this same problem with the same crude guesses took 13 iterations and the procedure used (which is similar to method three) is slightly more complicated to program on a computer. Because the second method has proven to be so effective, no effort has been made by the author to program the third procedure. The explanation of the third procedure is included as a generalization and extension of the ideas presented in Reference 3, and also to point out this method's similarity to the steepest descent idea.

Inequality constraints on the control variables can be handled very easily by all three methods. When the control variables are on the constraint boundary, H is not considered to be a function of the control u ; therefore, all of the terms in the equations that contain first and higher order partial derivatives of H with respect to u are considered to be zero. To handle discontinuities or inequality constraints on the state variables or functions of the state variables and the control variables, modification of the necessary conditions is needed. The modifications have been developed in References 4, 5, and 6. All of the modifications to the necessary conditions usually require that additional boundary conditions be satisfied at other than the first and last points. Because the three methods discussed for satisfying boundary conditions have been formulated for two points (the first and last), the extension to three or more points should be obvious. Also, because the iterations on the boundary conditions are performed at both ends of the trajectory, all three of the methods can be integrated backward or forward. The backward integration is helpful when the final conditions are extremely sensitive to changes in the initial conditions, but not "vice versa."

CONCLUSIONS

Practical methods for solving boundary value problems associated with the optimization of trajectories have been discussed. Actual experience with the construction of computer programs and the numerical results of computer programs has indicated that the second method described is usually the most effective for solving boundary value problems. Efficient use of the new IBM FORMAC computer language, which enables the computer to obtain functional

forms for the partial derivatives of functions with respect to the variables that appear in the function, will also assure that the second method becomes an easily programmed and economical computer program. More about the IBM FORMAC computer language can be found in Reference 7.

The detailed application of the second method to a simple trajectory optimization problem is outlined in Appendix II. Because the theoretical formulation for more difficult problems is also available, subsequent efforts will be directed toward the application of the second method to these problems to generate additional computer programs.

APPENDIX I
DERIVATION OF THE NECESSARY CONDITIONS
FOR THE OPTIMIZATION OF S

The problem is to determine a $\bar{u}(t)$ that satisfies the system of differential equations

$$\begin{aligned} \dot{x}_i &= f_i(x, u, t) & i &= 1, 2, \dots, n \\ x &= x_1, x_2, \dots, x_n \\ u &= u_1, u_2, \dots, u_r \end{aligned}$$

with boundary conditions

$$\begin{aligned} F_\alpha(x_0, t_0) &= 0 & \alpha &= 1, 2, \dots, l \leq n+1 \\ G_\beta(x_f, t_f) &= 0 & \beta &= 1, 2, \dots, m \leq n+1 \end{aligned}$$

and that maximizes or minimizes the quantity

$$S = \sum_{i=1}^n c_i x_i(t_f) \quad .$$

Because the x_i 's are assumed to be continuous, S can be rewritten in integral form.

$$S = \int_{t_0}^{t_f} \left[\sum_{i=1}^n c_i \dot{x}_i \right] dt + \sum_{i=1}^n c_i x_i(t_0)$$

To examine the effects of the constraints on the maximization of S, a new function S' is defined.

$$\begin{aligned} S' &= \int_{t_0}^{t_f} \left\{ \sum_{i=1}^n c_i \dot{x}_i + \sum_{i=1}^n \lambda_i [\dot{x}_i - f_i(x, u, t)] \right\} dt \\ &+ \sum_{i=1}^n c_i x_i(t_0) + \sum_{\alpha=1}^l p_\alpha F_\alpha(x_0, t_0) + \sum_{\beta=1}^m \rho_\beta G_\beta(x_f, t_f) \end{aligned}$$

A minimization of S' is equivalent to a minimization of S, if the boundary conditions and the system of differential equations are also satisfied. Assuming a solution to the problem statement exists, the optimum S' can be written as follows:

$$\begin{aligned} \bar{S}' = & \int_{\bar{t}_0}^{\bar{t}_f} \left\{ \sum_{i=1}^n c_i \bar{x}_i + \sum_{i=1}^n \bar{\lambda}_i [\bar{x}_i - f_i(\bar{x}, \bar{u}, \bar{t})] \right\} dt \\ & + \sum_{i=1}^n c_i \bar{x}_i(\bar{t}_0) + \sum_{\alpha=1}^l \bar{p}_\alpha F_\alpha(\bar{x}_0, \bar{t}_0) + \sum_{\beta=1}^m \bar{\rho}_\beta G_\beta(\bar{x}_f, \bar{t}_f) \quad . \end{aligned}$$

The bar over all of the variables means that they have their optimum value.

Now a variation in all of the variables is written as follows:

$$\begin{aligned} S' = \bar{S}' + \Delta S' = & \int_{\bar{t}_0 + \Delta t_0}^{\bar{t}_f + \Delta t_f} \left\{ \sum_{i=1}^n c_i (\bar{x}_i + \Delta \dot{x}_i) \right. \\ & + \left. \sum_{i=1}^n (\bar{\lambda}_i + \Delta \lambda_i) [(\bar{x}_i + \Delta \dot{x}_i) - f_i(\bar{x} + \Delta x, \bar{u} + \Delta u, t)] \right\} dt \\ & + \sum_{i=1}^n c_i [\bar{x}_i(\bar{t}_0) + \Delta x_i(\bar{t}_0)] + \sum_{\alpha=1}^l (\bar{p}_\alpha + \Delta p_\alpha) F_\alpha[\bar{x}_0(\bar{t}_0) + \Delta x(\bar{t}_0), \bar{t}_0 + \Delta t_0] \\ & + \sum_{\beta=1}^m (\bar{\rho}_\beta + \Delta \rho_\beta) G_\beta[\bar{x}_f(\bar{t}_f) + \Delta x(\bar{t}_f), \bar{t}_f + \Delta t_f] \quad . \end{aligned}$$

In the preceding expression the variations for times greater than \bar{t}_f and less than \bar{t}_0 are taken from the values at \bar{t}_0 and \bar{t}_f . For example

$$\Delta x(t) = x(t) - \bar{x}(t_0) \quad t \leq \bar{t}_0$$

$$\Delta x(t) = x(t) - \bar{x}(t_f) \quad t \geq \bar{t}_f \quad , \text{ etc.}$$

Subtracting \bar{S}' from $\bar{S}' + \Delta S'$ yields $\Delta S'$.

$$\begin{aligned} \Delta S' = & \int_{\bar{t}_0 + \Delta t_0}^{\bar{t}_0} \left\{ \sum_{i=1}^n c_i (\bar{x}_i + \Delta \dot{x}_i) + \sum_{i=1}^n (\bar{\lambda}_i + \Delta \lambda_i) [(\bar{x}_i + \Delta \dot{x}_i) - f_i(\bar{x} + \Delta x, \bar{u} + \Delta u, t)] \right\} dt \\ & + \int_{\bar{t}_0}^{\bar{t}_f} \left\{ \sum_{i=1}^n c_i \Delta \dot{x}_i + \sum_{i=1}^n \bar{\lambda}_i [\Delta \dot{x}_i + f_i(\bar{x}, \bar{u}, t) - f_i(\bar{x} + \Delta x, \bar{u} + \Delta u, t)] \right. \\ & + \left. \sum_{i=1}^n \Delta \lambda_i [(\bar{x}_i + \Delta \dot{x}_i) - f_i(\bar{x} + \Delta x, \bar{u} + \Delta u, t)] \right\} dt \end{aligned}$$

$$\begin{aligned}
& + \int_{\bar{t}_f}^{\bar{t}_f + \Delta t_f} \left\{ \sum_{i=1}^n c_i (\bar{x}_i + \Delta \dot{x}_i) + \sum_{i=1}^n (\bar{\lambda}_i + \Delta \lambda_i) [(\bar{x}_i + \Delta \dot{x}_i) - f_i(\bar{x} + \Delta x, \bar{u} + \Delta u, t)] \right\} dt \\
& + \sum_{i=1}^n c_i \Delta x_i(\bar{t}_0) + \sum_{\alpha=1}^1 \bar{p}_\alpha \left\{ F_\alpha[\bar{x}(\bar{t}_0) + \Delta x(\bar{t}_0), \bar{t}_0 + \Delta t_0] - F_\alpha[\bar{x}(\bar{t}_0), \bar{t}_0] \right\} \\
& + \sum_{\alpha=1}^1 \Delta p_\alpha F_\alpha[\bar{x}(\bar{t}_0) + \Delta \bar{x}(\bar{t}_0), \bar{t}_0 + \Delta t_0] \\
& + \sum_{\beta=1}^m \bar{\rho}_\beta \left\{ G_\beta[\bar{x}(\bar{t}_f) + \Delta x(\bar{t}_f), \bar{t}_0 + \Delta t_0] - G_\beta[\bar{x}(\bar{t}_f), \bar{t}_f] \right\} \\
& + \sum_{\beta=1}^m \Delta \rho_\beta \left\{ G_\beta[\bar{x}(\bar{t}_f) + \Delta \bar{x}(\bar{t}_f), \bar{t}_f + \Delta t_f] \right\}
\end{aligned}$$

If $\Delta S'$ is greater than zero for all the variations, S' will be a minimum and "vice versa." To see how $\Delta S'$ can be made greater than or less than zero the following steps are taken. The expression for $\Delta S'$ is simplified by assuming that all of the variations considered are small. This allows Taylor Series expansions to be used, and all second-order terms can then be neglected. The following expansions are made with only linear terms of the Taylor Series retained. The symbol δ means a smaller variation (meaning first-order terms are sufficient) than the symbol Δ .

$$\begin{aligned}
f_i(\bar{x} + \delta x, \bar{u} + \Delta u, t) &= f_i(\bar{x}, \bar{u} + \Delta u, t) + \sum_{l=1}^n \left[\frac{\partial f_i(\bar{x}, \bar{u} + \Delta u, t)}{\partial x_l} \right] \delta x_l \\
F_\alpha[\bar{x}(\bar{t}_0) + \delta x(\bar{t}_0), \bar{t}_0 + \delta t_0] &= F_\alpha[\bar{x}(\bar{t}_0), \bar{t}_0] + \sum_{i=1}^n \left\{ \frac{\partial F_\alpha[\bar{x}(\bar{t}_0), \bar{t}_0]}{\partial x_i} \right\} \delta x_i(\bar{t}_0) \\
&+ \left\{ \sum_{i=1}^n \left[\frac{\partial F_\alpha[\bar{x}(\bar{t}_0), \bar{t}_0]}{\partial x_i} \right] \bar{x}_i(\bar{t}_0) \right\} \delta t_0 + \left\{ \frac{\partial F_\alpha[\bar{x}(\bar{t}_0), \bar{t}_0]}{\partial t} \right\} \delta t_0 \\
G_\beta[\bar{x}(\bar{t}_f) + \delta x(\bar{t}_f), \bar{t}_f + \delta t_f] &= G_\beta[\bar{x}(\bar{t}_f), \bar{t}_f] + \sum_{i=1}^n \left\{ \frac{\partial G_\beta[\bar{x}(\bar{t}_f), \bar{t}_f]}{\partial x_i} \right\} \delta x_i(\bar{t}_f) \\
&+ \left\{ \sum_{i=1}^n \left[\frac{\partial G_\beta[\bar{x}(\bar{t}_f), \bar{t}_f]}{\partial x_i} \right] \bar{x}_i(\bar{t}_f) \right\} \delta t_f + \left\{ \frac{\partial G_\beta[\bar{x}(\bar{t}_f), \bar{t}_f]}{\partial t} \right\} \delta t_f
\end{aligned}$$

Substitution into the expression for $\Delta S'$ and elimination of the second-order terms caused by the smaller variations (denoted by a δ) yield:

$$\begin{aligned}
\delta S' = & \left\{ \sum_{i=1}^n c_i \bar{x}_i(\bar{t}_0) + \sum_{i=1}^n \bar{\lambda}_i(\bar{t}_0) [\dot{\bar{x}}_i(\bar{t}_0) - f_i(\bar{x}_0, \bar{u}_0 + \Delta u_0, \bar{t}_0)] \right\} \delta t_0 \\
& + \int_{\bar{t}_0}^{\bar{t}_f} \left\{ \sum_{i=1}^n c_i \delta \dot{\bar{x}}_i + \sum_{i=1}^n \bar{\lambda}_i \left[\delta \dot{\bar{x}}_i + f_i(\bar{x}, \bar{u}, t) - f_i(\bar{x}, \bar{u} + \Delta u, t) \right. \right. \\
& \quad \left. \left. - \sum_{l=1}^n \frac{\partial f_i(\bar{x}, \bar{u} + \Delta u, t)}{\partial x_l} (\delta x_l) \right] \right. \\
& \quad \left. + \sum_{i=1}^n \delta \lambda_i [\bar{x}_i - f_i(\bar{x}, \bar{u} + \Delta u, t)] \right\} dt \\
& + \left\{ \sum_{i=1}^n c_i \bar{x}_i(\bar{t}_f) + \sum_{i=1}^n \bar{\lambda}_i(\bar{t}_f) [\bar{x}_i(\bar{t}_f) - f_i(\bar{x}_f, \bar{u}_f + \Delta u, \bar{t}_f)] \right\} \delta t_f \\
& + \sum_{i=1}^n c_i \delta x_i(\bar{t}_0) + \sum_{\alpha=1}^l \bar{p}_\alpha \left\{ \sum_{i=1}^n \left[\frac{\partial F_\alpha(\bar{x}_0, \bar{t}_0)}{\partial x_i} \right] \delta x_i(\bar{t}_0) \right. \\
& \quad \left. + \sum_{i=1}^n \left[\frac{\partial F_\alpha(\bar{x}_0, \bar{t}_0)}{\partial x_i} \right] \bar{x}_i(\bar{t}_0) \delta t_0 + \left[\frac{\partial F_\alpha(\bar{x}_0, \bar{t}_0)}{\partial t} \right] \delta t_0 \right\} \\
& + \sum_{\alpha=1}^l \delta p_\alpha F_\alpha[\bar{x}_0, \bar{t}_0] + \sum_{\beta=1}^m \delta \rho_\beta G_\beta[\bar{x}_f, \bar{t}_f] \\
& + \sum_{\beta=1}^m \bar{\rho}_\beta \left\{ \sum_{i=1}^n \left[\frac{\partial G_\beta(\bar{x}_f, \bar{t}_f)}{\partial x_i} \right] \delta x_i(\bar{t}_f) + \sum_{i=1}^n \left[\frac{\partial G_\beta(\bar{x}_f, \bar{t}_f)}{\partial x_i} \right] \bar{x}_i(\bar{t}_f) \delta t_f \right. \\
& \quad \left. + \left[\frac{\partial G_\beta(\bar{x}_f, \bar{t}_f)}{\partial t} \right] \delta t_f \right\} .
\end{aligned}$$

A combination of certain terms under the integral sign may be integrated by parts as is shown:

$$\int_{\bar{t}_0}^{\bar{t}_f} \left[\sum_{i=1}^n (c_i + \bar{\lambda}_i) \delta \dot{x}_i \right] dt = \left[\sum_{i=1}^n (c_i + \bar{\lambda}_i) \delta x_i \right]_{\bar{t}_0}^{\bar{t}_f} - \int_{\bar{t}_0}^{\bar{t}_f} \left[\sum_{i=1}^n \bar{\lambda}_i \delta x_i \right] dt \quad .$$

Substitution of this expression into $\delta S'$ and further rearrangement yields an expression of the form.

$$\begin{aligned} \delta S' = & \left\{ \sum_{i=1}^n c_i \bar{x}_i(\bar{t}_0) + \sum_{i=1}^n \bar{\lambda}_i(\bar{t}_0) [\dot{x}_i(\bar{t}_0) - f_i(\bar{x}_0, \bar{u}_0 + \Delta u_0, \bar{t}_0)] \right\} \delta t_0 \\ & + \sum_{\alpha=1}^1 \bar{p}_\alpha \left\{ \sum_{i=1}^n \left[\frac{\partial F_\alpha(\bar{x}_0, \bar{t}_0)}{\partial x_i} \right] \bar{x}_i(\bar{t}_0) \delta t_0 + \left[\frac{\partial F_\alpha(\bar{x}_0, \bar{t}_0)}{\partial t} \right] \delta t_0 \right\} \\ & \sum_{i=1}^n \left[-\bar{\lambda}_i(\bar{t}_0) + \sum_{\alpha=1}^1 \bar{p}_\alpha \frac{\partial F_\alpha(\bar{x}_0, \bar{t}_0)}{\partial x_i} \right] \delta x_i(\bar{t}_0) + \sum_{\alpha=1}^1 \delta p_\alpha F_\alpha(\bar{x}_0, \bar{t}_0) \\ & - \int_{\bar{t}_0}^{\bar{t}_f} \left\{ \sum_{i=1}^n \left[\bar{\lambda}_i + \sum_{l=1}^n (\bar{\lambda}_l) \frac{\partial f_l(\bar{x}, \bar{u} + \Delta u, t)}{\partial x_i} \right] \delta x_i \right. \\ & + \sum_{i=1}^n \bar{\lambda}_i [f_i(\bar{x}, \bar{u} + \Delta u, t) - f_i(\bar{x}, \bar{u}, t)] \\ & \left. - \sum_{i=1}^n \delta \lambda_i [\bar{x}_i - f_i(\bar{x}, \bar{u} + \Delta u, t)] \right\} dt \\ & + \left\{ \sum_{i=1}^n c_i \bar{x}_i(\bar{t}_f) + \sum_{i=1}^n \bar{\lambda}_i(\bar{t}_f) [\dot{x}_i(\bar{t}_f) - f_i(\bar{x}_f, \bar{u}_f + \Delta u_f, \bar{t}_f)] \right\} \delta t_f \\ & + \sum_{\beta=1}^m \bar{\rho}_\beta \left\{ \sum_{i=1}^n \left[\frac{\partial G_\beta(\bar{x}_f, \bar{t}_f)}{\partial x_i} \right] \bar{x}_i(\bar{t}_f) \delta t_f + \left[\frac{\partial G_\beta(\bar{x}_f, \bar{t}_f)}{\partial t} \right] \delta t_f \right\} \\ & \sum_{i=1}^n \left[c_i + \bar{\lambda}_i(\bar{t}_f) + \sum_{\beta=1}^m \bar{\rho}_\beta \frac{\partial G_\beta(\bar{x}_f, \bar{t}_f)}{\partial x_i} \right] \delta x_i(\bar{t}_f) + \sum_{\beta=1}^m \delta \rho_\beta G_\beta(\bar{x}_f, \bar{t}_f) \end{aligned}$$

The only way that $\delta S'$ can be made greater than or less than zero for either positive or negative values of the variations (the δ values) is to make the coefficients of all the variations zero. This yields the following necessary conditions:

$$F_{\alpha}(\bar{x}_0, \bar{t}_0) = 0$$

$$\bar{\lambda}_i(\bar{t}_0) = \sum_{\alpha=1}^1 \bar{p}_{\alpha} \frac{\partial F_{\alpha}(\bar{x}_0, \bar{t}_0)}{\partial x_i}$$

$$\sum_{i=1}^n c_i \dot{\bar{x}}_i(\bar{t}_0) + \sum_{i=1}^n \bar{\lambda}_i(\bar{t}_0) [\dot{\bar{x}}_i(\bar{t}_0) - f_i(\bar{x}_0, \bar{u}_0 + \Delta u_0, \bar{t}_0)]$$

$$+ \sum_{\alpha=1}^1 \bar{p}_{\alpha} \left\{ \sum_{i=1}^n \left[\frac{\partial F_{\alpha}(\bar{x}_0, \bar{t}_0)}{\partial x_i} \right] \bar{x}_i(\bar{t}_0) + \frac{\partial F_{\alpha}(\bar{x}_0, \bar{t}_0)}{\partial t} \right\} = 0$$

$$G_{\beta}(\bar{x}_f, \bar{t}_f) = 0$$

$$\bar{\lambda}_i(\bar{t}_f) = -c_i - \sum_{\beta=1}^m \bar{\rho}_{\beta} \frac{\partial G_{\beta}(\bar{x}_f, \bar{t}_f)}{\partial x_i}$$

$$\sum_{i=1}^n c_i \dot{\bar{x}}_i(\bar{t}_f) + \sum_{i=1}^n \bar{\lambda}_i(\bar{t}_f) [\dot{\bar{x}}_i(\bar{t}_f) - f_i(\bar{x}_f, \bar{u}_f + \Delta u_f, \bar{t}_f)]$$

$$+ \sum_{\beta=1}^m \bar{\rho}_{\beta} \left\{ \sum_{i=1}^n \left[\frac{\partial G_{\beta}(\bar{x}_f, \bar{t}_f)}{\partial x_i} \right] \bar{x}_i(\bar{t}_f) + \frac{\partial G_{\beta}(\bar{x}_f, \bar{t}_f)}{\partial t} \right\} = 0$$

$$\bar{\dot{x}}_i = f_i(\bar{x}, \bar{u} + \Delta u, t) \qquad \bar{t}_0 \leq t \leq \bar{t}_f$$

$$\bar{\lambda}_i = - \sum_{l=1}^n \bar{\lambda}_l \left[\frac{\partial f_l(\bar{x}, \bar{u} + \Delta u, t)}{\partial x_i} \right] \qquad \bar{t}_0 \leq t \leq \bar{t}_f$$

Then the expression for $\delta S'$ becomes

$$\delta S' = - \int_{\bar{t}_0}^{\bar{t}_f} \left\{ \sum_{i=1}^n \bar{\lambda}_i [f_i(\bar{x}, \bar{u} + \Delta u, t) - f_i(\bar{x}, \bar{u}, t)] \right\} dt$$

Now $\delta S'$ will be positive if the variations Δu make the variations in the differences of the f_i either zero or negative. This means the variations in the differences of the f_i must be negative for some finite interval in the time period $\bar{t}_0 \leq t \leq \bar{t}_f$. This allows bounds to be placed on the control variables. From the opposite viewpoint, $\delta S'$ will be negative if the variations Δu make the variations in the differences of the f_i either zero or positive. The same considerations hold for bounds on the control variable. If $\Delta S'$ is positive for all small variations in u , then S' is a local minimum with respect to u and "vice versa." Defining a new quantity H and combining and rearranging the previous results allows the final form of the necessary conditions to be written

$$H = \sum_{i=1}^n \lambda_i f_i(x, u, t) \quad .$$

Then,

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}$$

$$\lambda_i \dot{=} - \frac{\partial H}{\partial x_i} \quad .$$

Because H must be a maximum or a minimum with respect to u

$$\frac{\partial H}{\partial u} = 0, \text{ and } H \text{ a maximum for } S \text{ a minimum}$$

$$H \text{ a minimum for } S \text{ a maximum}$$

$$F_\alpha(x_0, t_0) = 0$$

$$\lambda_i(t_0) = \sum_{\alpha=1}^1 p_\alpha \frac{\partial F_\alpha(x_0, t_0)}{\partial x_i}$$

$$\sum_{i=1}^n c_i \dot{x}_i(t_0) + H(t_0) + \sum_{\alpha=1}^1 p_\alpha \frac{\partial F_\alpha(x_0, t_0)}{\partial t} = 0$$

$$G_\beta(x_f, t_f) = 0$$

$$\lambda_i(t_f) = -c_i - \sum_{\beta=1}^m \rho_\beta \frac{\partial G_\beta(x_f, t_f)}{\partial x_i}, \quad H(t_f) - \sum_{\beta=1}^m \rho_\beta \frac{\partial G_\beta(x_f, t_f)}{\partial t} = 0$$

This development and form for the necessary conditions were chosen because they allow the part of the second variation associated with the control functions (i.e., the maximization or minimization of H) to be examined and satisfied. In reality all of the second order terms in the Taylor Series expansions should be retained and examined. Combinations of the second order terms would produce quadratic forms, and all of these would have to be positive or negative definite to assure a minimization or a maximization of S . From a computational standpoint, it is impossible to assure positive or negative definiteness for any but the terms associated with the control functions alone. The other terms can be examined as a test, but there is no freedom to correct the terms if the test is not satisfied. Therefore, a solution is usually obtained satisfying the necessary conditions given, and physical reasoning is used to determine if the trajectory is acceptable without resorting to the extra effort necessary to test for the sufficiency conditions.

APPENDIX II
APPLICATION OF THE NECESSARY CONDITIONS
TO A TRAJECTORY PROBLEM

A simple model for the equations of motion of a rocket-powered vehicle in three dimensions is given by the following system of differential equations:

$$\ddot{x} = \frac{F}{m} \sin \chi_p - \frac{\mu x}{R^3}$$

$$\ddot{y} = \frac{F}{m} \cos \chi_p \cos \chi_y - \frac{\mu y}{R^3}$$

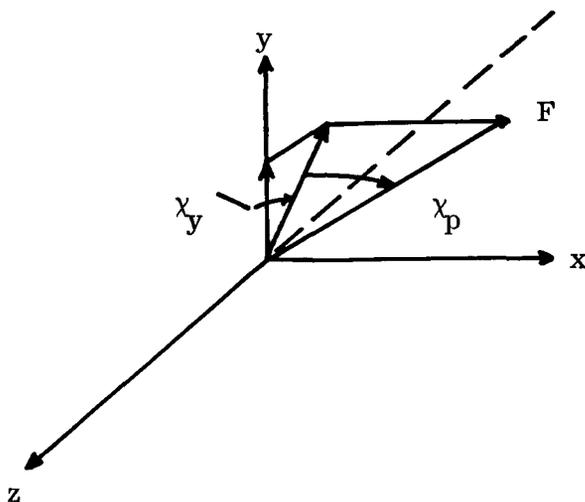
$$\ddot{z} = -\frac{F}{m} \cos \chi_p \sin \chi_y - \frac{\mu z}{R^3}$$

$$R = \sqrt{x^2 + y^2 + z^2}$$

$$m(t) = m_0 - \dot{m}_0(t-t_0)$$

$$F(t) = F_0 \quad .$$

These equations are based on the same assumptions that were used for the two-dimensional model described in the first part of the General Discussion. The control variables are χ_p and χ_y that locate the missile axis or thrust with respect to the coordinate system that is shown in the diagram below.



The following substitutions are made so that the system of differential equations will be in the form required for the application of the necessary conditions:

$$x_1 = \dot{x}$$

$$x_2 = \dot{y}$$

$$x_3 = \dot{z}$$

$$x_4 = x$$

$$x_5 = y$$

$$x_6 = z$$

$$x_7 = m$$

$$x_8 = F$$

$$u_1 = \chi_p$$

$$u_2 = \chi_y$$

Then

$$f_1 = \frac{x_8}{x_7} \sin u_1 - \frac{\mu x_4}{(x_4^2 + x_5^2 + x_6^2)^{3/2}}$$

$$f_2 = \frac{x_8}{x_7} \cos u_1 \cos u_2 - \frac{\mu x_5}{(x_4^2 + x_5^2 + x_6^2)^{3/2}}$$

$$f_3 = \frac{x_8}{x_7} \cos u_1 \sin u_2 - \frac{\mu x_6}{(x_4^2 + x_5^2 + x_6^2)^{3/2}}$$

$$f_4 = x_1$$

$$f_5 = x_2$$

$$f_6 = x_3$$

$$f_7 = -\dot{m}_0$$

$$f_8 = 0$$

Now H can be written:

$$H = \lambda_1 \left[\left(\frac{x_8}{x_7} \right) \sin u_1 - \frac{\mu x_4}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] + \lambda_2 \left[\left(\frac{x_8}{x_7} \right) \cos u_1 \cos u_2 - \frac{\mu x_5}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] \\ + \lambda_3 \left[- \left(\frac{x_8}{x_7} \right) \cos u_1 \sin u_2 - \frac{\mu x_6}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] + \lambda_4 x_1 + \lambda_5 x_2 + \lambda_6 x_3 - \lambda_7 \dot{m}_0 \quad .$$

Then the IBM FORMAC language can be used to obtain fortran expressions for part of the necessary conditions as follows:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \quad i = 1, 2, \dots, 8$$

$$\dot{\lambda}_i = - \frac{\partial H}{\partial x_i} .$$

Expressions for the control variables that appear in the \dot{x}_i and $\dot{\lambda}_i$ equations are obtained by solving the third equation in the system of equations for u_1 and u_2 .

$$\frac{\partial H}{\partial u_1} = 0 = \lambda_1 \left(\frac{x_8}{x_7} \right) \cos u_1 - \lambda_2 \left(\frac{x_8}{x_7} \right) \sin u_1 \cos u_2 + \lambda_3 \left(\frac{x_8}{x_7} \right) \sin u_1 \sin u_2$$

$$\frac{\partial H}{\partial u_2} = 0 = - \lambda_2 \left(\frac{x_8}{x_7} \right) \cos u_1 \sin u_2 - \lambda_3 \left(\frac{x_8}{x_7} \right) \cos u_1 \cos u_2$$

From the second equation

$$\tan u_2 = - \frac{\lambda_3}{\lambda_2} .$$

This is assuming that $\left(\frac{x_8}{x_7} \right)$ and $\cos u_1$ are not zero, but L'Hospital's rule can be used to show that the expression is still true as both approach zero as a limit. From the previous relation

$$\sin u_2 = \frac{-\lambda_3}{\pm \sqrt{\lambda_2^2 + \lambda_3^2}}$$

$$\cos u_2 = \frac{+\lambda_2}{\pm \sqrt{\lambda_2^2 + \lambda_3^2}} .$$

Substitution of these equations into the equation for $\frac{\partial H}{\partial u_1}$ gives:

$$\lambda_1 \cos u_1 - \frac{\lambda_2^2}{\pm \sqrt{\lambda_2^2 + \lambda_3^2}} \sin u_1 - \frac{\lambda_3^2}{\pm \sqrt{\lambda_2^2 + \lambda_3^2}} \sin u_1 = 0$$

$$\lambda_1 \cos u_1 \pm \sqrt{\lambda_2^2 + \lambda_3^2} \sin u_1 = 0$$

$$\tan u_1 = \frac{\lambda_1}{\pm \sqrt{\lambda_2^2 + \lambda_3^2}} .$$

$$\begin{aligned}\therefore \sin u_1 &= \frac{\lambda_1}{\pm\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \\ \cos u_1 &= \frac{\pm\sqrt{\lambda_2^2 + \lambda_3^2}}{\pm\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\end{aligned}$$

The ambiguities on the signs of \sin and \cos of u_1 and u_2 can be resolved by examining $\frac{\partial^2 H}{\partial u_1^2}$ and $\frac{\partial^2 H}{\partial u_2^2}$. Because it is desired to minimize

$$S = \sum_{i=1}^8 c_i x_i(t_f) \quad \text{where } c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_8 = 0 \text{ and } c_7 = -1, \frac{\partial^2 H}{\partial u_1^2} \text{ and}$$

$\frac{\partial^2 H}{\partial u_2^2}$ must be negative. The above statements say that $S = -m(t_f)$ is to be a

minimum. This is equivalent to stating that $m(t_f)$ is to be a maximum. For

$-m(t_f)$ to be a minimum H during the period $t_0 \leq t \leq t_f$ must be a maximum;

therefore $\frac{\partial^2 H}{\partial u_1}$ and $\frac{\partial^2 H}{\partial u_2}$ must be negative at all times in the interval

$$t_0 \leq t \leq t_f .$$

$$\begin{aligned}\frac{\partial^2 H}{\partial u_1^2} &= -\lambda_2 \left(\frac{x_8}{x_7} \right) \cos u_1 \cos u_2 + \lambda_3 \left(\frac{x_8}{x_7} \right) \cos u_1 \sin u_2 \\ &= \left(\frac{x_8}{x_7} \right) \{ -\lambda_1 \sin u_1 + \cos u_1 [-\lambda_2 \cos u_2 + \lambda_3 \sin u_2]\}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 H}{\partial u_2^2} &= -\lambda_2 \left(\frac{x_8}{x_7} \right) \cos u_1 \cos u_2 + \lambda_3 \left(\frac{x_8}{x_7} \right) \cos u_1 \sin u_2 \\ &= \frac{x_8}{x_7} \{ \cos u_1 [-\lambda_2 \cos u_2 + \lambda_3 \sin u_2]\}\end{aligned}$$

From the physics of the problem it is obvious that $\left(\frac{x_8}{x_7} \right)$ will always be positive.

Also, the positive sign is arbitrarily chosen for $\cos u_1$.

$$\cos u_1 = \frac{\sqrt{\lambda_2^2 + \lambda_3^2}}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

Then $\frac{\partial^2 H}{\partial u_2^2}$ will be negative if the following signs are chosen for $\sin u_2$ and $\cos u_2$.

$$\cos u_2 = \frac{\lambda_2}{\sqrt{\lambda_2^2 + \lambda_3^2}}$$

$$\sin u_2 = \frac{-\lambda_3}{\sqrt{\lambda_2^2 + \lambda_3^2}}$$

Also, $\frac{\partial^2 H}{\partial u_1^2}$ will be negative if the following signs are chosen for $\sin u_1$.

$$\sin u_1 = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

The preceding choice of signs make $\frac{\partial^2 H}{\partial u_1 \partial u_2}$ and $\frac{\partial^2 H}{\partial u_2 \partial u_1}$ zero;

therefore H is a maximum because $\frac{\partial^2 H}{\partial u_1^2}$ and $\frac{\partial^2 H}{\partial u_2^2}$ are both negative. With

the expressions for the control angles defined, the boundary conditions can now be discussed.

For this problem the boundary conditions at t_0 are assumed to have the form shown below:

$$F_1 = x_1 - \dot{x}_0 = 0$$

$$F_2 = x_2 - \dot{y}_0 = 0$$

$$F_3 = x_3 - \dot{z}_0 = 0$$

$$F_4 = x_4 - x_0 = 0$$

$$F_5 = x_5 - y_0 = 0$$

$$F_6 = x_6 - z_0 = 0$$

$$F_7 = x_7 - m_0 = 0$$

$$F_8 = x_8 - F_0 = 0$$

$$F_9 = t - t_0 = 0$$

Then the transversality conditions become

$$\lambda_1(t_0) = p_1$$

$$\lambda_5(t_0) = p_5$$

$$\lambda_2(t_0) = p_2$$

$$\lambda_6(t_0) = p_6$$

$$\lambda_3(t_0) = p_3$$

$$\lambda_7(t_0) = p_7$$

$$\lambda_4(t_0) = p_4$$

$$\lambda_8(t_0) = p_8$$

$$\dot{m}_0 + H(t_0) + p_9 = 0$$

The last equation determines p_9 , and the other eight p 's or λ 's must be determined to satisfy the boundary conditions and transversality conditions at t_f . For this problem the boundary conditions are assumed to have the form

$$G_1 = x_4^2 + x_5^2 + x_6^2 - R_{t_f}^2 = 0$$

$$G_2 = x_1^2 + x_2^2 + x_3^2 - v_{t_f}^2 = 0$$

$$G_3 = x_1x_4 + x_2x_5 + x_3x_6 - (Rv \cos \vartheta)_{t_f} = 0$$

The quantities $R_{t_f}^2$, $v_{t_f}^2$, and $(Rv \cos \vartheta)_{t_f}$ are desired constants that characterize a certain orbit where R is the radius, v is the velocity, and $Rv \cos \vartheta$ is the dot product of \bar{R} and \bar{v} . Then the transversality conditions become

$$\lambda_1(t_f) = 2\rho_2x_1 + \rho_3x_4$$

$$\lambda_2(t_f) = 2\rho_2x_2 + \rho_3x_5$$

$$\lambda_3(t_f) = 2\rho_2x_3 + \rho_3x_6$$

$$\lambda_4(t_f) = 2\rho_1x_4 + \rho_3x_1$$

$$\lambda_5(t_f) = 2\rho_1x_5 + \rho_3x_2$$

$$\lambda_6(t_f) = 2\rho_1x_6 + \rho_3x_3$$

$$\lambda_7(t_f) = 1$$

$$\lambda_8(t_f) = 0$$

$$H(t_f) = 0$$

This is nine transversality conditions and three boundary conditions for a total of twelve conditions that must be satisfied by the eight initial λ 's or p 's and the three ρ 's at t_f and t_0 . The three ρ 's at t_f can be eliminated by using three of the transversality conditions to solve for the ρ 's in terms of x 's and λ 's. To do this in an easy manner the following steps are taken:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 2\rho_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \rho_3 \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} = 2\rho_1 \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} + \rho_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

Vector algebra is used on these two vector equations to eliminate the three ρ 's. Crossing the first vector equation with $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ yields

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 2\rho_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \rho_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= \rho_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} .$$

Crossing the second vector equation with $\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$ yields

$$\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \times \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} = 2\rho_1 \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \times \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} + \rho_3 \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \rho_3 \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

Adding the two resulting equations gives

$$\begin{aligned}
& \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \times \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} \\
& = \rho_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} + \rho_3 \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad .
\end{aligned}$$

Performing the cross multiplications yields the three desired independent scalar equations:

$$\begin{aligned}
& \lambda_3 x_2 - \lambda_2 x_3 + \lambda_6 x_5 - \lambda_5 x_6 = 0 \\
& -\lambda_3 x_1 + \lambda_1 x_3 - \lambda_6 x_4 + \lambda_4 x_6 = 0 \\
& \lambda_2 x_1 - \lambda_1 x_2 + \lambda_5 x_4 - \lambda_4 x_5 = 0 \quad .
\end{aligned}$$

Now the eight λ 's or ρ 's at t_0 and t_f are used to satisfy the following nine conditions at t_f .

$$\begin{aligned}
E_1 & = x_4^2 + x_5^2 + x_6^2 - R_{t_f}^2 = 0 \\
E_2 & = x_1^2 + x_2^2 + x_3^2 - v_{t_f}^2 = 0 \\
E_3 & = x_1 x_4 + x_2 x_5 + x_3 x_6 - (Rv \cos \vartheta)_{t_f} = 0 \\
E_4 & = \lambda_3 x_2 - \lambda_2 x_3 + \lambda_6 x_5 - \lambda_5 x_6 = 0 \\
E_5 & = -\lambda_3 x_1 + \lambda_1 x_3 - \lambda_6 x_4 + \lambda_4 x_6 = 0 \\
E_6 & = \lambda_2 x_1 - \lambda_1 x_2 + \lambda_5 x_4 - \lambda_4 x_5 = 0 \\
E_7 & = \lambda_7 - 1 = 0 \\
E_8 & = \lambda_8 \\
E_9 & = H = \lambda_1 \left[\left(\frac{x_8}{x_7} \right) \sin u_1 - \frac{\mu x_4}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] \\
& \quad + \lambda_2 \left[\left(\frac{x_8}{x_7} \right) \cos u_1 \cos u_2 - \frac{\mu x_5}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right]
\end{aligned}$$

$$+ \lambda_3 \left[- \left(\frac{x_8}{x_7} \right) \cos u_1 \sin u_2 - \frac{\mu x_6}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] + \lambda_4 x_1 + \lambda_5 x_2 + \lambda_6 x_3 - \lambda_7 \dot{m}_0 = 0 .$$

The preceding discussion and the discussion in the main part of this report should be sufficient for an understanding of the following computational procedure that was used in constructing the actual computer program.

Computational Procedure

Preload

$x_1^0 = \dot{x}$ (m/sec)	λ_1^0
$x_2^0 = \dot{y}$ (m/sec)	λ_2^0
$x_3^0 = \dot{z}$ (m/sec)	λ_3^0
$x_4^0 = x$ (m)	λ_4^0
$x_5^0 = y$ (m)	λ_5^0
$x_6^0 = z$ (m)	λ_6^0
$x_7^0 = m \left(\frac{\text{kg sec}^2}{\text{m}} \right)$	λ_7^0
$x_8^0 = F$ (kg)	λ_8^0
$\dot{F}_0 =$ (kg/sec)	$R_{t_f}^2$ cutoff altitude squared (m ²)
$\dot{m}_0 = \left(\frac{\text{kg sec}}{\text{m}} \right)$	$v_{t_f}^2$ cutoff velocity squared (m ² /sec ²)
GM (m ³ /sec ²)	$(Rv \cos \vartheta)_{t_f}$ (determines path angle at cutoff)
t_0 (sec)	K
t_f (sec)	
Tolerance = $.5 \times 10^{-6}$	
Δt (sec) for integration	

Preload Computations

All input must be scaled

$$K_1 = 1.5698587 \times 10^{-7} \quad (\text{scales length})$$

$$K_2 = 1.241825 \times 10^{-3} \quad (\text{scales time})$$

$$K_3 = 0.16001332 \times 10^{-4} \quad (\text{scales weight})$$

The preceding scale factors are for near earth trajectories, and they cause the initial radius $(\sqrt{x_4^2 + x_5^2 + x_6^2})$, the initial mass, and μ to be unity for the test case data given at the end of this computational procedure.

$$x_1^0 = x_1^0 \left(\frac{K_1}{K_2} \right)$$

$$x_2^0 = x_2^0 \left(\frac{K_1}{K_2} \right)$$

$$x_3^0 = x_3^0 \left(\frac{K_1}{K_2} \right)$$

$$x_4^0 = x_4^0 (K_1)$$

$$x_5^0 = x_5^0 (K_1)$$

$$x_6^0 = x_6^0 (K_1)$$

$$x_7^0 = x_7^0 \left(\frac{K_3 K_2^2}{K_1} \right)$$

$$x_8^0 = x_8^0 (K_3)$$

$$\dot{F}_0 = \dot{F}_0 \left(\frac{K_3}{K_2} \right)$$

$$\dot{m}_0 = \dot{m}_0 \left(\frac{K_3 K_2}{K_1} \right)$$

$$GM = GM \left(\frac{K_1^3}{K_2^2} \right)$$

$$t_0 = t_0 \quad (K_2)$$

$$t_f = t_f \quad (K_2)$$

$$\Delta t = \Delta t \quad (K_2)$$

$$R_{t_f}^2 = R_{t_f}^2 \quad (K_1^2)$$

$$v_{t_f}^2 = v_{t_f}^2 \left(\frac{K_1^2}{K_2^2} \right)$$

$$(Rv \cos \vartheta)_{t_f} = (Rv \cos \vartheta)_{t_f} \left(\frac{K_1^2}{K_2} \right)$$

Preload Computations for the Isolation Routine

Set up the following matrices:

$$\left[\frac{\partial x_i}{\partial \lambda_j^0} \right]_{t_0} = [0] \quad i \text{ (number of rows)} = 8$$

$$j \text{ (number of columns)} = 8$$

$$\left[\frac{\partial \lambda_i}{\partial \lambda_j^0} \right]_{t_0} = [I]$$

Suggested Order for Computing Line "n" for the Isolation Routine

$x_i, \lambda_i, \left[\frac{\partial x_i}{\partial \lambda_j^0} \right]$, and $\left[\frac{\partial \lambda_i}{\partial \lambda_j^0} \right]$ are brought forward from the previous (t) to a new (t + Δt) by Runge-Kutta integration.

The following equation is to be used with the IBM FORMAC language to obtain functional (fortran) expressions for the partial derivatives needed for the subsequent calculations:

$$H = \lambda_1 \left[\left(\frac{x_8}{x_7} \right) \sin u_1 - \frac{(GM) x_4}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] + \lambda_2 \left[\left(\frac{x_8}{x_7} \right) \cos u_1 \cos u_2 - \frac{(GM) x_5}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] \\ + \lambda_3 \left[-\frac{x_8}{x_7} \cos u_1 \sin u_2 - \frac{(GM) x_6}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] + \lambda_4 x_1 + \lambda_5 x_2 + \lambda_6 x_3 - \lambda_7 \dot{m}_0 + \lambda_8 \dot{F}_0$$

$$(1) \sin u_1 = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

$$(2) \cos u_1 = \frac{\sqrt{\lambda_2^2 + \lambda_3^2}}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

$$(3) \sin u_2 = \frac{-\lambda_3}{\sqrt{\lambda_2^2 + \lambda_3^2}}$$

$$(4) \cos u_2 = \frac{\lambda_2}{\sqrt{\lambda_2^2 + \lambda_3^2}}$$

$$(5) \dot{x}_i = \frac{\partial H}{\partial \lambda_i} \quad i = 1, 2, \dots, 8$$

$$(6) \lambda_i^* = -\frac{\partial H}{\partial x_i} \quad i = 1, 2, \dots, 8$$

Construct the following matrices with the IBM FORMAC language:

$$\left. \begin{array}{l} \left[\frac{\partial^2 H}{\partial u_k \partial x_1} \right] \\ \left[\frac{\partial^3 H}{\partial u_k \partial \lambda_1} \right] \end{array} \right\} \begin{array}{l} k(\text{rows}) = 1, 2 \\ l(\text{columns}) = 1, 2, \dots, 8 \end{array}$$

$$\left. \left[\frac{\partial^2 H}{\partial u_k \partial u_m} \right] \right\} \begin{array}{l} k(\text{rows}) = 1, 2, \\ m(\text{columns}) = 1, 2 \end{array}$$

$$\left. \begin{array}{l} \left[\frac{\partial^2 H}{\partial x_i \partial x_j} \right] \\ \left[\frac{\partial^2 H}{\partial x_i \partial \lambda_j} \right] \end{array} \right\} \begin{array}{l} i(\text{rows}) = 1, 2, \dots, 8 \\ j(\text{columns}) = 1, 2, \dots, 8 \end{array}$$

Then compute:

$$(7) \quad \left[\frac{\partial u_k}{\partial \lambda_j^0} \right] = - \left[\frac{\partial^2 H}{\partial u_k \partial u_m} \right]^{-1} \left\{ \left[\frac{\partial^2 H}{\partial u_k \partial x_l} \right] \left[\frac{\partial x_i}{\partial \lambda_j^0} \right] + \left[\frac{\partial^2 H}{\partial u_k \partial \lambda_l} \right] \left[\frac{\partial \lambda_i}{\partial \lambda_j^0} \right] \right\}$$

$$(8) \quad \left[\frac{\partial \dot{x}_i}{\partial \lambda_j^0} \right] = \left[\frac{\partial^2 H}{\partial x_i \partial \lambda_j} \right]^T \left[\frac{\partial x_i}{\partial \lambda_j^0} \right] + \left[\frac{\partial^2 H}{\partial u_k \partial \lambda_l} \right]^T \left[\frac{\partial u_k}{\partial \lambda_l^0} \right]$$

$$(9) \quad \left[\frac{\partial \dot{\lambda}_i}{\partial \lambda_j^0} \right] = - \left[\frac{\partial^2 H}{\partial x_i \partial x_j} \right] \left[\frac{\partial x_i}{\partial \lambda_j^0} \right] - \left[\frac{\partial^2 H}{\partial u_k \partial x_l} \right]^T \left[\frac{\partial u_k}{\partial \lambda_j^0} \right] - \left[\frac{\partial^2 H}{\partial x_i \partial \lambda_j} \right] \left[\frac{\partial \lambda_i}{\partial \lambda_j^0} \right]$$

Now \dot{x}_i , $\dot{\lambda}_i$, $\left[\frac{\partial \dot{x}_i}{\partial \lambda_j^0} \right]$, and $\left[\frac{\partial \dot{\lambda}_i}{\lambda_j^0} \right]$ are integrated by Runge-Kutta integration to give x_i , λ_i , $\left[\frac{\partial x_i}{\partial \lambda_j^0} \right]$, and $\left[\frac{\partial \lambda_i}{\partial \lambda_j^0} \right]$ for the next line. This integration is stopped when $t = t_f$.

All of the following equations are to be computed at the time $t = t_f$ on a particular trial:

$$(1) \quad E_1 = x_4^2 + x_5^2 + x_6^2 - R_{t_f}^2$$

$$(2) \quad E_2 = x_1^2 + x_2^2 + x_3^2 - v_{t_f}^2$$

$$(3) \quad E_3 = x_1 x_4 + x_2 x_5 + x_3 x_6 - (Rv \cos \vartheta)_{t_f}$$

$$(4) \quad E_4 = \lambda_3 x_2 - \lambda_2 x_3 + \lambda_6 x_5 - \lambda_5 x_6$$

$$(5) \quad E_5 = -\lambda_3 x_1 + \lambda_1 x_3 - \lambda_6 x_4 + \lambda_4 x_6$$

$$(6) E_6 = \lambda_2 x_1 - \lambda_1 x_2 + \lambda_5 x_4 - \lambda_4 x_5$$

$$(7) E_7 = \lambda_7 - 1$$

$$(8) E_8 = \lambda_8$$

$$(9) E_9 = \lambda_1 \left[\left(\frac{x_8}{x_7} \right) \sin u_1 - \frac{(GM) x_4}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] \\ + \lambda_2 \left[\left(\frac{x_8}{x_7} \right) \cos u_1 \cos u_2 - \frac{(GM) x_5}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] \\ + \lambda_3 \left[- \left(\frac{x_8}{x_7} \right) \cos u_1 \sin u_2 - \frac{(GM) x_6}{(x_4^2 + x_5^2 + x_6^2)^{3/2}} \right] \\ + \lambda_4 x_1 + \lambda_5 x_2 + \lambda_6 x_3 - \lambda_7 \dot{m}_0 + \lambda_8 \dot{F}_0$$

$$(10) \text{ Define } E = [E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9]$$

Construct the following matrices or vectors using the automatic partial differentiation routine:

$$\left[\frac{\partial E_R}{\partial x_i} \right], \left[\frac{\partial E_R}{\partial u_k} \right], \left[\frac{\partial E_R}{\partial \lambda_j} \right], \text{ and } \left[\frac{\partial E_R}{\partial t} \right]$$

$$R(\text{rows}) = 1, 2, \dots, 9$$

$$i(\text{columns}) = 1, 2, \dots, 8$$

$$j(\text{columns}) = 1, 2, \dots, 8$$

$$k(\text{columns}) = 1, 2$$

Then compute:

$$(11) \left[\frac{\partial E_R}{\partial \lambda_j^0} \right] = \left[\frac{\partial E_R}{\partial x_i} \right] \left[\frac{\partial x_i}{\partial \lambda_j^0} \right] + \left[\frac{\partial E_R}{\partial u_k} \right] \left[\frac{\partial u_k}{\partial \lambda_j^0} \right] + \left[\frac{\partial E_R}{\partial \lambda_j} \right] \left[\frac{\partial \lambda_j}{\partial \lambda_j^0} \right]$$

$$(12) \left[W_{km} \right] = \begin{bmatrix} (\lambda_2 \sin u_1 \sin u_2 + \lambda_3 \sin u_1 \cos u_2) \\ (\lambda_3 \cos u_1 \sin u_2 - \lambda_1 \sin u_1 - \lambda_2 \cos u_1 \cos u_2) \\ (\lambda_3 \cos u_1 \sin u_2 - \lambda_2 \cos u_1 \cos u_2) \\ (\lambda_2 \sin u_1 \sin u_2 + \lambda_3 \sin u_1 \cos u_2) \end{bmatrix}$$

$$(13) \dot{u}_k = \left[W_{km} \right]^{-1} \begin{bmatrix} \dot{\lambda}_2 \cos u_1 \sin u_2 + \dot{\lambda}_3 \cos u_1 \cos u_2 \\ \dot{\lambda}_2 \sin u_1 \cos u_2 - \dot{\lambda}_1 \cos u_1 - \dot{\lambda}_3 \sin u_1 \sin u_2 \end{bmatrix}$$

$$(14) \dot{E}_R = \left[\frac{\partial E_R}{\partial x_i} \right] \dot{x}_i + \left[\frac{\partial E_R}{\partial u_k} \right] \dot{u}_k + \left[\frac{\partial E_R}{\partial \lambda_j} \right] \dot{\lambda}_j + \frac{\partial E_R}{\partial t}$$

$$(15) \Delta \bar{E} = \bar{E}$$

$$\text{Compute } |\Delta \bar{E}| = \sqrt{E_1^2 + E_2^2 + E_3^2 + E_4^2 + E_5^2 + E_6^2 + E_7^2 + E_8^2 + E_9^2}$$

$$(16) \begin{bmatrix} z_{qs} \end{bmatrix} = \begin{bmatrix} \frac{\partial E_R}{\partial \lambda_j^0} \\ \dot{E}_R \end{bmatrix}$$

If $|\Delta \bar{E}|$ is \leq tolerance go to converged case run saving λ_i^0 (old) and t_f (old). If not continue:

$$(17) \bar{p} = \begin{bmatrix} z_{qs} \end{bmatrix}^{-1} \bar{E} \quad \begin{array}{l} \text{Compute } \begin{bmatrix} z_{qs} \end{bmatrix}^{-1} \text{ in double precision} \\ \text{and test } \begin{bmatrix} z_{qs} \end{bmatrix} \begin{bmatrix} z_{qs} \end{bmatrix}^{-1} = [I] . \end{array}$$

where

$$\bar{p} = [\Delta \lambda_1^0, \Delta \lambda_2^0, \Delta \lambda_3^0, \Delta \lambda_4^0, \Delta \lambda_5^0, \Delta \lambda_6^0, \Delta \lambda_7^0, \Delta \lambda_8^0, \Delta t_f]$$

On the first trial skip the next three "if" statements and go to "now for a new trial" :

$$\text{If } \frac{|\Delta \bar{E}_n|}{|\Delta \bar{E}_{n-1}|} > (1-K + \frac{1}{6} K) \text{ set } K = \frac{K}{1.8} .$$

$$\text{If } \frac{|\Delta \bar{E}_n|}{|\Delta \bar{E}_{n-1}|} \leq (1-K + \frac{1}{6} K) \text{ set } K = 1.8 K .$$

$$\text{If } K > 1 \text{ set } K = 1 .$$

Now for a new trial:

$$\lambda_i^0 \text{ (new)} = \lambda_i^0 \text{ (old)} - K \Delta \lambda_i$$

$$\Delta t_f \text{ (new)} = t_f \text{ (old)} - K \Delta t_f .$$

Converged Case Run

Reintegrate the trajectory with the converged $\lambda_i(t_0)$ and t_f and compute additionally at each Δt .

$$R = \sqrt{x_4^2 + x_5^2 + x_6^2} \quad \left(\frac{1}{K_1} \right)$$

$$v = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad \left(\frac{K_2}{K_1} \right)$$

$$\cos \vartheta = \frac{x_1x_4 + x_2x_5 + x_3x_6}{Rv}$$

$$\sin \vartheta = \sqrt{1 - \cos^2 \vartheta}$$

$$\left[\begin{array}{l} \vartheta = \arctan \left(\frac{\sin \vartheta}{\cos \vartheta} \right) \quad (\text{convert to degrees}) \\ u_1 = \arctan \left(\frac{\sin u_1}{\cos u_1} \right) \quad (\text{convert to degrees}) \\ u_2 = \arctan \left(\frac{\sin u_2}{\cos u_2} \right) \quad (\text{convert to degrees}) \end{array} \right.$$

Print out at each Δt :

- (1) On the first step of each trial print

$$t, x_i, \dot{x}_i, \lambda_i, \dot{\lambda}_i, \left[\frac{\partial u_k}{\partial \lambda_j^0} \right], \left[\frac{\partial x_i}{\partial \lambda_j^0} \right], \left[\frac{\partial \dot{x}_i}{\partial \lambda_j^0} \right], \left[\frac{\partial \lambda_i}{\partial \lambda_j^0} \right], \text{ and } \left[\frac{\partial \dot{\lambda}_i}{\partial \lambda_j^0} \right] .$$

- (2) At $t = t_f$ on each trial print all of (1) plus the following

$$\bar{E}, \Delta \bar{E}, \left[\frac{\partial E_R}{\partial \lambda_j^0} \right], \dot{u}_k, \dot{E}_R, \left[z_{qs} \right], \left[z_{qs} \right]^{-1}, \left[z_{qs} \right] \left[z_{qs} \right]^{-1}, \text{ and } \bar{p} .$$

- (3) At each Δt of the converged case run print

$$t, u_1, u_2, R, v, \vartheta, x_i, \lambda_i, \dot{x}_i, \dot{\lambda}_i .$$

Data for an Example Trajectory

$t_0 = 150.01366$
 $x_1 = 2378.9375$
 $x_2 = 1181.8890$
 $x_3 = -1259.8308$
 $x_4 = 143391.19$
 $x_5 = 6447134.8$
 $x_6 = -40967.611$
 $x_7 = 6361.8535$
 $x_8 = 40599.685$
 $\dot{F}_0 = 0.0$
 $\dot{m}_0 = 9.6776706$
 $GM = .39860160 \times 10^{15}$
 $\lambda_1^0 = 1.0$
 $\lambda_2^0 = 1.0$
 $\lambda_3 = 1.0$
 $\lambda_4 = 1.0$
 $\lambda_5 = 1.0$
 $\lambda_6 = 1.0$
 $\lambda_7 = 1.0$
 $\lambda_8 = 1.0$
 $t_f = 650$
 $\Delta t = 1.0$
 Tolerance = $.5 \times 10^{-6}$
 $R_{t_f}^2 = .44193245 \times 10^{14}$
 $v_{t_f}^2 = .59878362 \times 10^8$
 $(Rv \cos \vartheta)_{t_f} = 0$
 $K = .1$

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BOUNDARY VALUE PROBLEMS ASSOCIATED
WITH OPTIMIZATION THEORY

By Hugo L. Ingram

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