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Neutron Stars II. Neutrino-Cooling and Observability*

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ABSTRACT

Calculations of the rates of the cooling reactions $n + n \rightarrow n + p + e^- + \bar{\nu}_e$ and $n + \pi^- \rightarrow n + e^- + \bar{\nu}_e$ are presented; the rates of the closely related muon-producing reactions and the four inverse processes are also given. Several different arguments are used to obtain estimates of the relevant matrix elements. The nucleons are assumed to form a normal Fermi fluid with a continuous excitation spectrum. The calculated cooling rates indicate that a neutron star containing quasi-free pions would cool within a few days to a temperature so low that the star would be unobservable. The surface of a star that does not contain quasi-free pions would cool to 10^7 °K in a few months and would reach 4×10^6 °K in about 100 years. The calculated cooling rates strongly indicate that the discrete x-ray sources located in the direction of the galactic center are not neutron stars.

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I. INTRODUCTION

Measurements made on recent rocket flights above the earth's atmosphere have demonstrated the existence of several discrete sources of galactic x-rays.[1-4] Several authors[5-7] have suggested that some of the observed sources may be hot neutron stars radiating x-rays from their surfaces, while other authors have suggested that the observed x-rays may be synchrotron radiation from energetic electrons in magnetic fields[8] or bremsstrahlung radiation from hot clouds of electrons and nuclei.[8,9]

The neutron-star hypothesis is the most specific of the suggested x-ray producing mechanisms, and it is thus the easiest hypothesis to disprove observationally. The most obvious property of a neutron star, its small size, has led to observational proof[10] that the principal x-ray source in the Crab nebula is not a neutron star; the results of the recent occultation experiment indicate that the source in the Crab has a diameter of the order of one light year. In the present work, we consider in detail another important property of neutron stars, their fast cooling by neutrino emission, and find that the calculated cooling rates imply important restrictions on the observability of neutron stars.

We calculate the rates at which a star loses energy by emitting neutrinos in the reactions

$$n + n \rightarrow n + p + e^- + \bar{\nu}_e \quad , \tag{1}$$

$$n + n \rightarrow n + p + \mu^- + \bar{\nu}_\mu \quad , \tag{2}$$

$$\pi^- + n \rightarrow n + e^- + \bar{\nu}_e \quad , \tag{3}$$

and

$$\pi^- + n \rightarrow n + \mu^- + \bar{\nu}_\mu \quad , \tag{4}$$

as well as the inverse processes

$$n + p + e^- \rightarrow n + n + \nu_e \quad , \tag{5}$$

$$n + p + \mu^- \rightarrow n + n + \nu_\mu \quad , \tag{6}$$

$$n + e^- \rightarrow n + \pi^- + \nu_e \quad , \tag{7}$$

and

$$n + \mu^- \rightarrow n + \pi^- + \nu_\mu \quad . \tag{8}$$

Reactions (1) and (5) were first discussed by Chiu and Salpeter[7] and the corresponding neutrino luminosities have been calculated by several authors.[11-13] We have previously reported crude estimates[13] of the rates of reactions (4) and (8).

We expect that reactions (1) - (8) should be the dominant means of neutrino production in neutron stars. In the Appendix, we consider the rates of various other neutrino-producing reactions, and conclude that these processes do not contribute importantly to the neutrino luminosity.

In our calculations of the rates of reactions (1) - (8), we have assumed that the spectrum of excited states available to a dense neutron gas is continuous, just as it is for a normal Fermi gas. Ginzburg and Kirzhnits[14] have pointed out that the excitation spectrum of the nucleon gas may not be continuous, but may instead resemble the spectrum of a gas of superconducting electrons. The existence of superfluidity might greatly modify the cooling rates of neutron stars, and we expect to consider the question of superfluidity of the nucleon gas in a future paper. Our present treatment of the neutrino-producing reactions and the related conclusions about the observability of neutron stars are expected to be accurate only if the nucleon gas does not form a superfluid.

Our calculations indicate that reactions (1) and (5) cause a mass M_s of neutron star matter to lose energy at a rate given by

$$L_\nu^{nn-e} = (6 \times 10^{38} \text{ erg sec}^{-1})(M_s/M_\odot)(\rho_{nucl}/\rho)^{1/3} \; T_9^{\,8} \quad , \tag{9}$$

where $M_\odot$ is the mass of the sun, ρ_{nucl} is the density of nuclear matter (3.7×10^{14} gm/cc), ρ is the density of the neutron-star matter, and T_9 is the stellar temperature in units of 10^9 $^\circ$K. The neutrino luminosity due to reactions (2) and (6) is equal to $F \times L_\nu^{nn-e}$, where F is equal to zero when the electron Fermi energy $W_F(e)$ is less than $m_\mu c^2$, and is equal to the ratio of the muon Fermi momentum to the electron Fermi momentum if $W_F(e)$ is greater than $m_\mu c^2$. Thus the net energy loss by reactions (1), (2), (5), and (6) is equal to $(1 + F)\, L_\nu^{nn-e}$.

We find that the rate of energy loss by neutrinos produced by reactions (3), (4), (7), and (8) is given approximately by

$$L_\nu^{\pi n} \approx (10^{46} \text{ erg sec}^{-1})(n_\pi/n_b)(M_s/M_\odot) \; T_9^{\,6} \quad , \tag{10}$$

where n_π/n_b is the ratio of the number density of quasi-free π^- mesons to the number density of baryons. The luminosity $L_\nu^{\pi n}$ is greater than L_ν^{nn-e} if n_π/n_b is greater than about 10^{-7}. As we have shown in Sec. III of the preceding paper, one cannot say with any degree of certainty whether or not quasi-free pions are present in neutron stars. We confine ourselves in the present work to consideration of the consequences of the presence of quasi-free pions in neutron stars, setting aside the much more difficult problem of whether such pions are actually present.

We combine Eqs. (9) and (10) with the results of the neutron-star models of Tsuruta[15] (computed using the equations of stellar structure and various simple laws for the equation of state) to estimate cooling times of hot neutron stars. A neutron star containing quasi-free pions would cool so fast by neutrino emission that its x-ray luminosity would be

negligible within a few days after the formation of the star. Thus our cooling rates indicate that the observed x-ray sources cannot be neutron stars that contain quasi-free pions.

The surface temperature of a neutron star that is cooling by reactions (1) and (5) should reach 10^7 °K within about a year after the star's formation and should reach 4×10^6 °K after about one-hundred years. The flux from a neutron star with a surface temperature of 4×10^6 °K could not be detected above the background flux using current techniques, unless the star were less than one kiloparsec from us. Thus our cooling rates suggest that neutron stars would be observable with present techniques only if they happened to be formed within a small volume (~ 1 cubic kiloparsec) centered at the sun, or if they happened to be observed in the process of formation.

About half of the observed x-ray sources are in the direction of the galactic center; our cooling times indicate that any observed source that is actually located near the galactic center (which is about 8 kiloparsecs away) could be a neutron star only if it was formed less than a week before it was observed, an extremely unlikely possibility. However, it has been suggested[3,16] that the brightest source, the one that appears to be in the constellation Scorpius, may be of the order of 30 parsecs from the sun. If the Scorpius source is in fact only 30 parsecs away, the observed flux from it is consistent with the hypothesis[3] that the source is a neutron star with a surface temperature of about 3×10^6 °K. Our cooling times indicate that such a star could be thousands of years old. However, a black body at 3×10^6 °K would not produce the large numbers of short-wavelength photons recently observed[17] for the Scorpius source.

We begin the detailed discussion of the reaction rates by formulating in Sec. II the general problem of neutrino emission from neutron stars. Then in Sec. III, we use simple heuristic arguments to obtain approximate expressions for the rates of reactions (1) and (3). The problem of neutrino opacity is treated in Sec. IV, where we show that the mean free paths of all neutrinos involved in reactions (1) to (8) are large compared to the radius of a neutron star. Section V contains a detailed calculation of the rate of energy loss by reactions (1), (2), (5), and (6), while Sec. VI contains an analogous treatment of the pion processes, reactions (3), (4), (7), and (8). Finally, in Sec. VII, we use information from neutron-star models[15] to calculate the rate of cooling of the surface of a typical hot neutron star (i.e., the decrease of the x-ray luminosity with time). We then apply our calculated cooling rates to the recent observations of Bowyer et al.[3]

II. GENERAL FORMULATION

In order to compute cooling times, one must consider the excited states of a neutron star. A neutron star is almost completely isothermal, except for an extremely thin atmosphere. For the purposes of calculating the rate of neutrino emission, one can neglect the atmosphere and imagine that the excited states of the star are populated (according to the usual Boltzmann factor) by placing the star in contact with a thermal bath at a finite temperature T. The star then has a definite baryon number and total electric charge but does not have a definite energy. The rate of energy loss (cooling) by neutrino emission is given by an expression of the form:

$$L_\nu = (2\pi/\hbar) \sum_\nu \sum_{\beta < \alpha} |\langle S_\beta;\ \nu\ |H_w|\ S_\alpha\rangle|^2\ E_\nu\ \delta(E_\alpha - E_\beta - E_\nu)\ \exp\ (-\ E_\alpha/kT)\ , \tag{11}$$

where S_α, S_β are states of the entire star, H_w is the weak-interaction Hamiltonian, E_ν is the energy of the emitted neutrino ν, and the summation over β is limited to states for which $E_\beta < E_\alpha$.

In practice, cooling rates must be computed with the help of a model; we adopt an independent-particle model whose general characteristics have been discussed in the preceding paper.[18] We shall in fact use several slightly different versions of the independent-particle model in order to estimate the uncertainties in our results. We also approximate the thermal average (Eq. (11)) over the states of the star by assigning a Fermi-Dirac or Bose-Einstein distribution function to each kind of particle in the star. As discussed in paper I, it is not possible to decide at present whether or not neutron stars contain a significant number of quasi-free pions; hence our calculations have been carried out for both assumptions, pions present and pions not present.

III. HEURISTIC CALCULATIONS

One can estimate the order of magnitude of the energy loss due to processes (1) - (8) by a simple heuristic argument that is not entirely fraudulent. The main feature of this argument is that only Fermions on the edge of their degenerate seas can undergo inelastic scattering. Thus only a small fraction of the order of (kT/E_F) of the Fermions of a given type can participate in the cooling reactions. Since neutrinos escape from a neutron star (see Section IV) this argument does not apply to them. However, the net amount of energy transferred to a neutrino in

any of the cooling reactions must be, by conservation of energy, of the order of kT. As a guess, we replace the dimensionless neutrino phase space, which is proportional to E_ν^2, by $(kT)^2/\left[E_F(n)\ E_F(p)\right]$ for reactions (1) and (2) and similar factors for reactions (3) and (4).

The energy loss from reaction (1) can now be crudely estimated from the familiar arguments of kinetic theory. One writes for the energy loss from a volume Ω by reaction (1):

$$L_\nu^{(1)} \sim \Omega\ n(n)^2\ \langle\sigma v\rangle\ E_\nu \left[\frac{kT}{E_F(n)}\right]^{+6} \left[\frac{kT}{E_F(p)}\right] \tag{12}$$

where $n(n)$ is the neutron number density, the weak-interaction cross section $\sigma \sim 10^{-43}\left[E_F(n)/1\ \text{MeV}\right]^2\ \text{cm}^2$, the relative velocity $v \sim c/3$, the neutrino energy $E_\nu \sim kT/3$, and the various Fermi energies can be estimated from Eqs. (5) of paper I. We have included in Eq. (12) one factor of kT/E_F for each degenerate fermion that occurs in process (1); we have also made use of the fact that $E_F(e)$ is, according to Sec. II of paper I, approximately equal to $E_F(n)$. We consider a mass M_s of neutron-star matter at a uniform density ρ and a uniform temperature T. Using Eq. (5) of paper I in Eq. (12), one finds that the neutrino luminosity due to reaction (1) is given by

$$L_\nu^{(1)} \sim (6 \times 10^{39}\ \text{erg-sec}^{-1})(M_s/M_\odot)(\rho_{nucl}/\rho)^3\ T_9^{\ 8} \quad , \tag{13}$$

where $M_\odot$ is the mass of the sun and T_9 is the temperature in billions of degrees. Equation (13) yields energy losses that are not enormously different from the energy losses computed from our more complicated analysis of Sec. V. Moreover, Eq. (13) gives correctly the crucial

dependence of $L_\nu^{(1)}$ on temperature, although the density dependence cannot be obtained correctly without a more careful kinematical analysis.

A similar crude argument can be used to obtain an estimate of the energy losses from reactions (3). Note that process (3) contains two fewer Fermions than processes (1) and (2); hence the rate of (3) is faster than (1) by a factor of the order of $(E_F(n)/kT)^2$. Thus:

$$L_\nu^{(3)} \sim (4 \times 10^{45}\ \text{erg-sec}^{-1})(n_\pi/n_n)(M_s/M_\odot)(\rho_{nucl}/\rho)^{8/3}\ T_9^{\ 6} \quad . \qquad (14)$$

The heuristic arguments show clearly what quantities must be calculated in a careful analysis, namely, the phase-space integrals (which we have approximated by factors of kT/E_F) and the nuclear matrix elements (which we have approximated by an average weak-interaction cross section).

IV. NEUTRINO OPACITY

Neutrinos produced by the reactions discussed in the previous section have typical energies of the order of kT, with kT less than or of the order of 100 keV. For neutrinos of such energies, the largest contribution to the neutrino-opacity comes from neutrino-electron scattering for ν_e and neutrino-muon scattering for ν_μ. This result can easily be established by examining the possible reactions. We consider first electron neutrinos, ν_e.

The following reactions are forbidden for typical neutron-star conditions by conservation of energy and momentum: $\nu_e + n \rightarrow p + e^-$, $\bar{\nu}_e + p \rightarrow n + e^+$, and $\bar{\nu}_e + p + n \rightarrow n + n + e^+$. The reaction $\nu_e + n + n \rightarrow p + e^- + n'$ and related reactions involving strange particles, e.g., Λ^0's or Σ^-'s, occur rarely because the cross section is of the order of 10^{-42} cm^2 times several factors of (kT/E_F).

Neutrino absorption by heavier elements on the surface of the star is negligible because the cross sections are small and the heavier elements are rare. Thus neutrino-electron scattering is the most important interaction for ν_e.

A similar analysis has been carried out for muon neutrinos and shows that the only interactions allowed by the selection rules and by energy conservation are $\nu_\mu - \mu^-$ and $\bar{\nu}_\mu - \mu^-$ scattering.

The cross section for neutrino-electron scattering in a degenerate gas is,[19] for $E_\nu \ll E_F(e)$:

$$\sigma \sim 2 \times 10^{-44} \left(E_\nu / m_e c^2\right)^2 \quad . \tag{15}$$

Equation (15) should be multiplied by one-third for antineutrino-electron scattering. For $\nu_\mu - \mu^-$ scattering, the cross section is again given by Eq. (15). For $\bar{\nu}_\mu - \mu^-$ scattering, Eq. (15) should be multiplied by one-third.

The mean free path of an electron neutrino is:

$$\lambda_{\nu_e} \equiv (\sigma n_e)^{-1} \quad , \tag{16a}$$

and therefore:

$$\lambda_{\nu_e} \gtrsim 2.5 \times 10^{+2} \left(\rho_{nucl}/\rho\right)^{+2} \text{ Km} \quad , \quad \rho \lesssim 4\, \rho_{nucl} \quad . \tag{16b}$$

In obtaining Eq. (16b), we have used Eq. (5b) of I. The mean free path of a muon neutrino is larger than λ_{ν_e} since muons are less numerous than electrons.

Note that the values of the mean free path given by Eqs. (16) are large compared to the radius of a neutron star (~ 10 Km). Thus the opacity of a neutrino star to low-energy neutrinos is entirely negligible.

V. NUCLEON-NUCLEON COOLING

A. General Expressions

We now make explicit use of the independent-particle model to calculate the rate of reaction (1). We describe the state of the entire star in terms of the states of its individual particles, introducing corrections to account for the interactions among the various particles. Following the work of Gomes et al.,[20] we label each single-particle state by its momentum $\underset{\sim}{p}$; as in paper I, the energy assigned to a state of particle species i with momentum $\underset{\sim}{p}$ is given by

$$E_i(p) = \sqrt{m_i^2 c^4 + p^2 c^2} + U_i(p) - m_i c^2 \quad . \tag{17a}$$

The Fermi energy $E_F(i)$ is defined by

$$E_F(i) = \sqrt{m_i^2 c^4 + \left[P_F(i)\right]^2 c^2} - m_i c^2 \quad , \tag{17b}$$

where $P_F(i)$ is the Fermi momentum for a particle of species i. The zero point of $U_i(p)$ is defined such that $U_i\left[P_F(i)\right]$ is equal to the binding energy $B(i)$ as defined in Sec. III-B of paper I. Thus, $E_i(p)$ is the energy required to take a particle of type i from infinity and place it in the neutron star in a state with momentum $\underset{\sim}{p}$ (gravitational interactions not considered). The quantities $W_i(p)$ and $W_F(i)$ are defined to be equal, respectively, to $\left[E_i(p) + m_i c^2\right]$ and $\left[E_F(i) + m_i c^2\right]$.

The neutrino luminosity $L_\nu^{(1)}$ arising from reaction (1) $(n + n \rightarrow n + p + e^- + \bar{\nu}_e)$ in a volume Ω is:

$$L_\nu^{(1)} = \pi \hbar^{-1} \sum_{\text{spins}} \int d^3\underset{\sim}{n}_1 \, d^3\underset{\sim}{n}_2 \, d^3\underset{\sim}{n}_1{}' \, d^3\underset{\sim}{n}_p \, d^3\underset{\sim}{n}_e \, d^3\underset{\sim}{n}_{\bar{\nu}} \; S \, \delta(E_i - E_f) \, E_{\bar{\nu}}$$

$$\times \; |\langle n,p,e,\bar{\nu}_e \, |H_W| \, n,n\rangle|^2 \qquad , \tag{18a}$$

where the subscripts 1, 2, 1', p, e, and $\bar{\nu}$ denote the two initial neutrons, the final neutron, the proton, the electron, and the antineutrino, respectively. We have included a factor of one-half which arises from the identity of the two initial neutrons. The density of individual-particle states can be expressed in terms of the particle momenta as follows:

$$d^3\underset{\sim}{n}_i = (2\pi\hbar)^{-3} \; \Omega \; p_i^{\,2} \, dp_i \, d\Omega_i \qquad . \tag{18b}$$

The quantity S is a product of Fermi-Dirac distribution functions for each particle appearing in reaction (1), except the neutrino; S corrects the density-of-state factors for the effect of the exclusion principle in the final state and gives the appropriate occupation numbers in the initial state. More explicitly,

$$S = \prod_{i=1}^{5} S(i) \qquad , \tag{18c}$$

where for the two initial neutrons,

$$S(n) = \left\{1 + \exp\left[E_n - E_F(n)\right]/kT\right\}^{-1} \qquad , \tag{18d}$$

and for the proton, neutron, and electron appearing in the final state,

$$S(i) = \left\{1 + \exp\left[E_F(i) - E_i\right]/kT\right\}^{-1} \qquad . \tag{18e}$$

The weak Hamiltonian is:

$$H_w = 2^{-1/2}\, G \int d^3\underset{\sim}{x} \left[\bar{\psi}_p(x)\, \gamma_\alpha\, (C_V - C_A\, \gamma_5)\, \psi_n(x)\right] \left[\bar{\psi}_e\, \gamma_\alpha\, (1 + \gamma_5)\, \psi_\nu\right] + \text{h.c.} \tag{18f}$$

In order to separate out the center-of-mass motion of the nucleons in the matrix element $\langle n,p,e,\bar{\nu}_e\, |H_w|\, n,n\rangle$, we introduce the following center-of-mass and relative coordinates:

$$\underset{\sim}{K} = (\underset{\sim}{k}_1 + \underset{\sim}{k}_2) \quad , \tag{19a}$$

$$\underset{\sim}{k} = 2^{-1}\, (\underset{\sim}{k}_1 - \underset{\sim}{k}_2) \quad , \tag{19b}$$

$$\underset{\sim}{R} = 2^{-1}\, (\underset{\sim}{r}_1 + \underset{\sim}{r}_2) \quad , \tag{19c}$$

and

$$\underset{\sim}{r} = (\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \quad , \tag{19d}$$

where $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{r}_1$, and $\underset{\sim}{r}_2$ are, respectively, the wave numbers and positions of the two nucleons in the initial state. Primed variables will be used for the analogous final-state quantities. The nucleonic wave functions in the initial and final states are of the form:

$$\Psi_{nn} = \Omega^{-1} \exp\, (i\, \underset{\sim}{K} \cdot \underset{\sim}{R})\, \psi_{nn}^{S}\, (\underset{\sim}{k};\, \underset{\sim}{r})\, \chi(S,\, M_S) \tag{20a}$$

and

$$\Psi_{np} = \Omega^{-1} \exp\, (i\, \underset{\sim}{K}' \cdot \underset{\sim}{R})\, \psi_{np}^{S'}\, (\underset{\sim}{k}';\, \underset{\sim}{r})\, \chi(S',\, M_S') \quad . \tag{20b}$$

In Eqs. (20), the functions ψ describe the relative motion of the pairs of nucleons; the incoming part of the asymptotic form of $\psi_{np}^{S'}\, (\underset{\sim}{k}';\, \underset{\sim}{r})$ is the same as the incoming part of a plane wave with wave vector $\underset{\sim}{k}'$. The function $\chi(S',\, M_{S'})$ describes a two-particle spin-state with total spin S' and z-component $M_{S'}$.

The nucleon matrix element that appears in Eq. (18a) can now be expressed as an integral over the relative wave functions ψ_{nn}^{S} and $\psi_{np}^{S'}$. Before writing down an explicit formula for the matrix element, we make two simplifications: (1) we assume that the nucleon-nucleon potential acts only in even-parity states; and (2) we neglect all terms involving the lepton momenta. The first assumption has frequently been used in nuclear-matter calculations and does not appear to give rise to any large errors. The second simplification can be shown to introduce errors of the order of 15% if the first approximation is valid. One may reasonably expect the errors in the calculated neutrino luminosity arising from these approximations to be small compared to the uncertainties that arise from our lack of a fundamental theory of strong interactions from which one would hope to calculate the scattering of nucleons in a neutron star.

Making the simplifications described above, we square the matrix element and sum over all spins, obtaining

$$\sum_{\text{spins}} |\langle n,p,e,\bar{\nu}_e \,|H_w|\, n,n\rangle|^2$$

$$= 8\, G^2 \left[C_V^{\,2}\, |M_V|^2 + 3\, C_A^{\,2}\, |M_A|^2\right] (2\pi)^3\, \lambda\!\!\!^{-}_{\pi}{}^{6}\, \Omega^{-5}\, \delta^{(3)}\, (\underset{\sim}{K}' - \underset{\sim}{K}) \quad , \tag{21a}$$

where $\lambda\!\!\!^{-}_{\pi}$ is the Compton wavelength of the pion, and the dimensionless matrix elements are defined by

$$M_V = \lambda\!\!\!^{-}_{\pi}{}^{-3} \int d^3r \;\; \psi_{np}^{*0}(\underset{\sim}{k}'; \underset{\sim}{r}) \;\; \psi_{nn}^{0}(\underset{\sim}{k}; \underset{\sim}{r}) \tag{21b}$$

and

$$M_A = \lambda\!\!\!^{-}_{\pi}{}^{-3} \int d^3\underset{\sim}{r} \;\; \psi_{np}^{*1}(\underset{\sim}{k}'; \underset{\sim}{r}) \;\; \psi_{nn}^{0}(\underset{\sim}{k}; \underset{\sim}{r}) \quad . \tag{21c}$$

Substituting Eq. (21) in Eq. (18), we find:

$$L_\nu^{(1)} = 64\,\pi^4\,\Omega\,G^2\,\hbar^{-1}\,\lambda\!\!\!^{-}_\pi{}^{-9}\left[C_V^{\,2}\,|M_V|^2 + 3\,C_A^{\,2}\,|M_A|^2\right]\,P \quad , \tag{22}$$

where the dimensionless phase-space factor P is given by the following equation:

$$P = \Omega^{-6}\,\lambda\!\!\!^{-}_\pi{}^{15}\int \prod_{i=1}^{6} d^3\underset{\sim}{n}_i\;S\,E_{\bar{\nu}}\;\delta^{(3)}(\underset{\sim}{K}' - \underset{\sim}{K})\;\delta(E_f - E_i) \quad . \tag{23}$$

Since each factor $d^3\underset{\sim}{n}_i$ is proportional to the volume Ω, the phase-space integral P is actually independent of Ω. Thus, $L_\nu^{(1)}$ is proportional to Ω.

Inserting the appropriate numerical values in the expression for $L_\nu^{(1)}$, one finds:

$$\Omega^{-1}\,L_\nu^{(1)} = (5.2 \times 10^{48}\ \text{erg-cm}^{-3}\text{-sec}^{-1})\;P\;(|M_V|^2 + 4.3\;|M_A|^2) \quad . \tag{24}$$

As was apparent from our earlier heuristic discussion, two types of quantities must be calculated, the nuclear matrix elements M_A and M_V and the phase-space factor P. Equation (24) has been derived only for the case of reaction (1); we shall consider in Sec. V-D the modifications necessary to account for reactions (2), (5), and (6).

B. The Phase-Space Factor

1. General Discussion

Chemical equilibrium among the different types of particles present in a neutron star is ensured by various weak-interaction processes, particularly reactions (1) and (2). The concentrations of the various particles can be brought to their equilibrium values in typical weak-interaction times of the order of 10^{-6} to 10^{-8} sec. However, the exclusion principle greatly inhibits all these reactions when the stellar matter is

near chemical equilibrium at low temperature. For example, the lifetime of a neutron in a neutron star at equilibrium at 10^9 ^oK is of the order of 10^{12} sec, which is 10^{+18} to 10^{+20} times longer than the time required to establish chemical equilibrium.

This enormous reduction in the reaction rates near equilibrium results from a decrease in the number of available initial and final states. Equation (13) of paper I states that, in a neutron star at equilibrium at 0^oK, two neutrons at the top of their Fermi distribution have just enough energy to produce a neutron, a proton, and an electron at the top of their respective Fermi seas, plus a zero-energy neutrino. At temperatures greater than zero but still small compared to the relevant Fermi energies, neutrons with energies near $E_F(n)$ have sufficient energy to produce a neutron, proton, and electron in unoccupied states near the tops of their respective Fermi seas, plus a neutrino with an energy of the order of kT. Thus the neutrons destroyed in reaction (1) all come from a narrow band of states with energies within a few kT of $E_F(n)$, and the neutrons, protons, and electrons produced in reaction (1) must have energies within a few kT of their respective Fermi energies. The relatively slow rate of reactions (1) and (2) at equilibrium is due to the fact that only a small fraction of the total number of particle states can actually be involved in the reactions. The phase-space factor, P, of Eq. (23), which we evaluate in the following paragraphs, contains a quantitative description of the inhibition of the reaction rate due to the small number of available states. The phase-space factors for the allowed reactions (1) and (2) are the principal quantities that determine their absolute rates, just as the ordinary phase-space factor (usually denoted by f) primarily determines the laboratory decay rates of superallowed nuclear beta decays.

2. Initial Approximations

The integrations involved in the phase-space factor P can all be performed analytically; the approximations required for carrying out the integrations give rise to errors of only a few percent. One can evaluate the integrals relatively accurately because of the simplifications that result from the fact that kT is, for the problems of interest, much less than the relevant Fermi energies. For example, the energy kT is 0.086 MeV at 10^9 °K, whereas $E_F(n)$, $E_F(e)$, and $E_F(p)$ are, respectively, of the order of 70 MeV, 70 MeV, and 3 MeV at nuclear density.

The integrand of P is negligible except in the restricted "important" region of phase space where all the particle energies are within a few kT of their Fermi energies. It is convenient to neglect contributions to the integral from certain regions that are far from the "important" region. In particular, we consider only those parts of the region of integration that satisfy the inequalities

$$p_s + |p_1 - p_2| < p_1' < p_1 + p_2 - p_s \tag{25}$$

and

$$p_1 > p_s \quad , \tag{26a}$$

where

$$p_s = p_p + p_e + p_{\bar{\nu}} \quad . \tag{26b}$$

The largest error made in restricting ourselves to the domain described by relations (25) and (26) is of the order of $e^{-E_F(p)/kT}$, which is less than 10^{-3} for the temperatures and densities of interest.

3. The Angular Integral

We begin the evaluation of the phase-space factor P given in Eq. (23) by performing the integrations over the solid angles. Let the angular integral A be defined by the relation

$$A = \int d\Omega_1 \int d\Omega_p \int d\Omega_e \int d\Omega_{\bar{\nu}} \int d\Omega_2 \int d\Omega_1' \; \delta^3(\underset{\sim}{K}' - \underset{\sim}{K}) \quad . \tag{27}$$

We can rewrite $\delta^3(\underset{\sim}{K}' - \underset{\sim}{K})$ as follows:

$$\delta^3(\underset{\sim}{K}' - \underset{\sim}{K}) = \hbar^3 \, (p_1')^{-2} \, \delta^2(\Omega_1' - \Omega_{\underset{\sim}{q}}) \, \delta(p_1' - q) \tag{28}$$

where $\underset{\sim}{q} = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 - \underset{\sim}{p}_p - \underset{\sim}{p}_e - \underset{\sim}{p}_{\bar{\nu}}$. The angular delta function $\delta^2(\Omega_1' - \Omega_{\underset{\sim}{q}})$, which requires that the directions of $\underset{\sim}{p}_1'$ and $\underset{\sim}{q}$ be the same, allows one to perform the integration on Ω_1' immediately. We note that

$$\delta(p_1' - q) = \delta\left[p_1' - (s^2 + p_2{}^2 + 2s \, p_2\mu)^{1/2}\right] \quad , \tag{29}$$

where $\underset{\sim}{s} = \underset{\sim}{p}_1 - \underset{\sim}{p}_p - \underset{\sim}{p}_e - \underset{\sim}{p}_{\bar{\nu}}$ and μ is the cosine of the angle between $\underset{\sim}{p}_2$ and $\underset{\sim}{s}$. Inequality (25) requires that the quantity $p_1' - q$ be equal to zero for some value of μ between -1 and +1. Hence the integral over Ω_2 can be carried out immediately, with the result that

$$A = 2\pi\hbar^3 \, (p_2 \, p_1')^{-1} \int d\Omega_1 \int d\Omega_p \int d\Omega_e \int d\Omega_{\bar{\nu}} \; s^{-1} \quad . \tag{30}$$

Repeated use of inequality (26) allows one to perform the remaining integrations directly. One finds that

$$A = (4\pi)^5 \, \hbar^3 \, (2 \, p_1 \, p_2 \, p_1')^{-1} \quad . \tag{31}$$

4. The Radial Integral

We now perform the integrations on the lengths of the momentum vectors in Eq. (23). Substituting Eq. (31) into Eq. (23) and using Eq. (18b), we obtain:

$$P = B \int \prod_{i=1}^{6} p_i^2 \, dp_i \; S \, E_{\bar{\nu}} \; \delta(E_f - E_i) \; (p_1 \, p_2 \, p_1')^{-1} \quad , \tag{32a}$$

where

$$B = 2^{-9} \, \pi^{-13} \, (m_\pi c)^{-15} \quad . \tag{32b}$$

The integration over the neutrino momentum $p_{\bar{\nu}}$, which one can perform immediately using the energy delta function, contributes a factor $(E_1 + E_2 - E_1' - W_e - E_p)^3 \, c^{-3}$. Defining the nucleon effective masses as in paper I, we find that

$$p_p \, dp_p = m_p^* \, (p_p) \, dE_p \tag{33a}$$

and

$$p_n \, dp_n = m_n^* \, (p_n) \, dE_n \quad . \tag{33b}$$

The electron energy, W_e, is nearly equal to $p_e c$, since the electrons are highly relativistic.

It is convenient to express P in terms of the following dimensionless variables:

$$x_1 = \beta \left[E_1 - E_F(n) - B(n) \right] \quad , \tag{34a}$$

$$x_2 = -\beta \left[E_1' - E_F(n) - B(n) \right] \quad , \tag{34b}$$

$$x_3 = -\beta \left[W_e - W_F(e) \right] \quad , \tag{34c}$$

$$x_4 = \beta \left[E_2 - E_F(n) - B(n) \right] \quad , \tag{34d}$$

and

$$x_5 = -\beta \left[E_p - E_F(p) - B(p) - m_p + m_n \right] \quad , \tag{34e}$$

where, as usual, $\beta = (kT)^{-1}$.

Substituting Eqs. (33) and (34) into the expression for P, and using the equilibrium condition, Eq. (13) of paper I, we obtain

$$P = B\, c^{-4} (kT)^{8} \int_{-y_n}^{\infty} dx_1 \int_{-\infty}^{y_n} dx_2 \int_{-\infty}^{y_e} dx_3 \int_{-y_n}^{\infty} dx_4 \int_{-(x_1+x_2+x_3+x_4)}^{y_p} dx_5 \; J \;, \tag{35a}$$

where

$$y_n = \beta \left[E_F(n) + B(n) - E_n(\underset{\sim}{p} = 0) \right] \;, \tag{35b}$$

$$y_e = \beta \left[W_F(e) - m_e\, c^2 \right] \;, \tag{35c}$$

$$y_p = \beta \left[E_F(p) + B(p) + m_p - m_n - E_p(\underset{\sim}{p} = 0) \right] \;, \tag{35d}$$

$$J = \left(\sum_{i=1}^{5} x_i \right)^3 \theta \left(\sum_{i=1}^{5} x_i \right) \prod_{i=1}^{5} \left(1 + e^{x_1} \right)^{-1} \times \left[\prod_{i=1}^{3} m_n^{*}(E_i) \right] \cdot m_p^{*}(E_p)\; p_p(E_p)\; p_e^{2}(W_e) \;. \tag{35e}$$

The function $\theta(y)$ is defined to be equal to unity when y is positive and zero when y is negative.

The factor $\theta(\Sigma_i x_i)\, \Pi_i\, (1 + e^{x_i})^{-1}$ is always less than e^{-x_5}. Hence, replacing the limit y_p by infinity increases the integral in Eq. (35) by a term proportional to e^{-y_p}. The quantity y_p is approximately equal to $\beta\, E_F(p)$, which is greater than ten for the temperatures of interest here. Hence, terms proportional to e^{-y_p} can be neglected, and the limit y_p can be replaced by infinity. The limits y_n and y_e can also be replaced by infinity since they are at least ten times larger than y_p.

The effective masses and momenta contained in J can be expanded in a power series. For example, one can express $m_n^*(E_1)$ in the form

$$m_n^*(E_1) = m_n^* \left[E_F(n)\right] + \sum_{n=1}^{\infty} x_1^n \left[kT/E_F(n)\right]^n a_n \quad .$$

Thus if we neglect small terms of order e^{-y_p}, the integral P can be expressed as a power series in kT. Since $kT/E_F(p)$ is less than one-tenth for neutron-star temperatures and densities of interest, we can obtain an adequate approximation for P by considering just the first term in the power series expansion of

$$\left[\prod_{i=1}^{3} m_n^*(E_i)\right] m_p^*(E_p)\, p_p(E_p)\, p_e^2(W_e) \quad .$$

We then obtain

$$P = B\,(kT)^8\, c^{-4}\, (m_n^*)^3\, m_p^*\, P_F(p)\, P_F(e)^2\, I \quad , \tag{36a}$$

where

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} dx_4 \int_{-(x_1+x_2+x_3+x_4)}^{\infty} dx_5 \left(\sum_{i=1}^{5} x_i\right)^3 \prod_{i=1}^{5} \left(1 + e^{x_1}\right)^{-1} \tag{36b}$$

$$= \frac{11{,}513\,\pi^8}{120{,}960} \quad ; \tag{36c}$$

and

$$m_n^* = m_n^* \left[E_F(n)\right] \quad , \tag{36d}$$

$$m_p^* = m_p^* \left[E_F(p)\right] \quad . \tag{36e}$$

Setting $P_F(p)$ and $P_F(e)$ equal to $c^{-1} W_F(e)$, we can write the phase-space factor in the convenient form

$$P \approx 2.6 \times 10^{-29} \left(\frac{m_n^*}{m_n}\right)^3 \left(\frac{m_p^*}{m_p}\right) \left[\frac{W_F(e)}{m_\pi c^2}\right]^3 T_9^{\,8} \quad . \tag{37}$$

The phase-space factor is, as expected from the heuristic argument given in Sec. III, proportional to T^8; it is also proportional to the product of the effective masses of the four nucleons involved, because the number of single-nucleon states per unit energy is proportional to the nucleon effective mass.

Although the integrations involved in P are accurate to within a few per cent, the numerical value of P is difficult to estimate to much better than a factor of two because of the uncertainties in the effective masses and the electron Fermi energy. Using Eqs. (29) - (33) of paper I, we estimate that the product $(m_n^*/m_n)^3 (m_p^*/m_p)$ is equal to 0.6 ± 0.3. The electron Fermi energy depends on $B(n) - B(p)$, the difference between the binding energies of the neutron and proton. This difference might easily be as large as 20 MeV at nuclear density, but unfortunately no reliable theoretical estimates of $B(n) - B(p)$ are yet available. We shall assume that $B(n) - B(p)$ is much smaller than 70 MeV and use the free-particle relation, Eq. (5c) of paper I, for the electron Fermi energy. We then obtain a simple but approximate expression for P,

$$P \approx 1.9 \times 10^{-30} (\rho/\rho_{nucl})^2 \; T_9^{\,8} \quad . \tag{38}$$

C. Estimates of the Matrix Elements

1. Definitions

The dimensionless matrix elements of Eq. (24) can be written as follows:

$$M_V = \lambda\!\!\!^{-}_{\pi}{}^{-3} \int d^3\underset{\sim}{r} \left[\cos \underset{\sim}{k}' \cdot \underset{\sim}{r} + \Delta^0_{np}(\underset{\sim}{r})\right]^* \left[\cos \underset{\sim}{k} \cdot \underset{\sim}{r} + \Delta^0_{nn}(\underset{\sim}{r})\right] \quad ; \tag{39}$$

$$M_A = \lambda\!\!\!^{-}_{\pi}{}^{-3} \int d^3\underset{\sim}{r} \left[\cos \underset{\sim}{k}' \cdot \underset{\sim}{r} + \Delta^1_{np}(\underset{\sim}{r})\right]^* \left[\cos \underset{\sim}{k} \cdot \underset{\sim}{r} + \Delta^0_{nn}(\underset{\sim}{r})\right] \quad . \tag{40}$$

The initial-state wave function $\left[\cos \underset{\sim}{k} \cdot \underset{\sim}{r} + \Delta^0_{nn}(\underset{\sim}{r})\right]$ describes the relative motion of two neutrons with total spin zero. The functions $\left[\cos \underset{\sim}{k}' \cdot \underset{\sim}{r} + \Delta^0_{np}(\underset{\sim}{r})\right]$ and $\left[\cos \underset{\sim}{k}' \cdot \underset{\sim}{r} + \Delta^1_{np}(\underset{\sim}{r})\right]$ correspond to neutron-proton pairs in states with spin zero and spin one, respectively. We consider only states that are even under exchange of the positions of the two nucleons because we have neglected nucleon-nucleon scattering in odd-parity states.

Our lack of detailed knowledge of the effects of strong interactions makes accurate calculation of M_A and M_V difficult. In the following subsection we use a dimensional argument to guess the order of magnitude and density dependence of the matrix elements. We then use two specific models for the nucleon-nucleon collisions to obtain more detailed estimates of M_A and M_V.

2. Dimensional Estimate

The integrals over $\underset{\sim}{r}$ in Eqs. (39) and (40) must yield a quantity proportional to the cube of a length. Thus we can estimate M_V and M_A by considering the physical lengths that are involved. There are two lengths associated with the nucleon-nucleon potential: the attractive potential

has a range of about $\lambda\!\!\!^{-}_{\pi}$ and the core radius is about $0.4\ \lambda\!\!\!^{-}_{\pi}$. The relevant wave numbers K, k, and k' are all large fractions of $P_F(n)\ \hbar^{-1}$, and $\hbar\ P_F(n)^{-1} \cong 0.4\ \lambda\!\!\!^{-}_{\pi}\ (\rho/\rho_{nucl})^{-1/3}$.

Since all the lengths involved are nearly equal at nuclear density, we expect $|M_A|^2$ and $|M_V|^2$ to be of the order of unity at nuclear density. Furthermore, the effective range of Δ is probably determined primarily by k, k', or $P_F(n)\ \hbar^{-1}$. Thus we might expect M_A and M_V to be proportional to $P_F(n)^{-3}$, i.e., to decrease as ρ^{-1}. In any event, we expect M_A and M_V to decrease slowly with increasing density, for moderate densities.

3. Scattering Model

In this model, we assume that the function Δ in Eqs. (39) and (40) is an outgoing scattered wave; that is, we assume that

$$\Delta \sim \sum_{\ell} e^{i\delta_\ell} \sin \delta_\ell\ P_\ell(\Omega)\ e^{ikr}/kr \tag{41}$$

for $kr \gg 1$. Equation (41) does not describe the wave function for the region $kr < 1$, a region which contributes a large part ($\gtrsim$ one-half) of the integrals M_V and M_A. In order to estimate the wave function for small radii, we must assume a specific form for the interaction potential. We adopt the separable potential suggested by Yamaguchi.[21] The corresponding s-wave scattering wave function is given by

$$\cos \underset{\sim}{k} \cdot \underset{\sim}{r} + e^{i\delta} \sin \delta\ \left(e^{ikr} - e^{-\beta r}\right) (kr)^{-1} \quad , \tag{42}$$

where

$$e^{i\delta} \sin \delta = \left\{-i + \frac{\beta}{k}\left[-\frac{1}{2} + \frac{1}{2}\left(\frac{k}{\beta}\right)^2 + (2\pi^2\lambda\beta)^{-1}\left(\beta^2 + k^2\right)^2\right]\right\}^{-1} \quad . \tag{43}$$

The parameters λ and β, which represent, respectively, the coupling strength and range of the separable potential, can be determined from the singlet and triplet scattering data. The effective Hamiltonians acting on the space parts of the singlet and triplet wave functions are different. But the two singlet wave functions contained in M_V are eigenfunctions of the same Hamiltonian; since the two eigenfunctions correspond to different nucleon energies, they are orthogonal. Thus the free-scattering model implies that M_V equals zero.

We have computed M_A using values of β and λ that reproduce the experimental phase shifts between 25 and 100 MeV. The resulting expression for M_A is complicated, but, for $\rho \lesssim \rho_{nucl}$ it can be accurately approximated as follows:

$$|M_A|^2 \approx 0.3\,(\rho_{nucl}/\rho)^{7/3} \quad . \tag{44}$$

Note that the model described above neglects all correlations between the colliding nucleons and the other nucleons that are present.

4. Nuclear Matter Calculation

In using the scattering model discussed above, we have neglected the fact that the exclusion principle prohibits scattering into occupied states. Nearly all the states that are energetically accessible to two colliding nucleons are, in fact, occupied in a neutron star; hence there is almost no free scattering. The wave function describing the relative motion of two nucleons in a neutron star or in nuclear matter is a symmetrized plane wave, except for some distortion for small internucleon separations. This distortion is described by the functions Δ in Eqs. (39) and (40). One can describe the collision between two particles most

simply by using a two-particle Schrödinger equation. The effect of the interactions between the two colliding particles and the other nucleons can be represented approximately by replacing the free-particle masses by the effective masses. However, the Schrödinger equation must also be modified to take account of the fact that the states below the relevant Fermi levels are largely occupied; the appropriate modified form of the Schrödinger equation is the Bethe-Goldstone equation, which is often used in nuclear-matter calculations.[20] In the Bethe-Goldstone equation, the usual potential-energy term $V(\underset{\sim}{r})\ \psi(\underset{\sim}{r})$ is replaced by $qV(\underset{\sim}{r})\ \psi(\underset{\sim}{r})$, where q is a projection operator that eliminates those Fourier components of $V(\underset{\sim}{r})\ \psi(\underset{\sim}{r})$ that correspond to occupied states. Since the operator $qV(\underset{\sim}{r})$ is not hermitian, the solutions to the Bethe-Goldstone equation for different energies are not necessarily orthogonal. Thus M_V need not be zero as it was in the scattering model of subsection 3.

We follow Gomes <u>et al.</u>[20] in assuming spin-independent forces, which implies that M_A and M_V are equal. However, Δ_{nn} and Δ_{np} are not equal, since the exclusion principle differentiates between neutrons and protons. Using the fact that $|\underset{\sim}{k}|$ is different from $|\underset{\sim}{k}'|$ to show that

$$\int d^3r \cos \underset{\sim}{k}' \cdot \underset{\sim}{r} \cos \underset{\sim}{k} \cdot \underset{\sim}{r} = 0 \quad ,$$

we can rewrite Eqs. (39) and (40) in the form

$$M_A = M_V \tag{45a}$$

$$= \lambda\!\!\!^{-}_{\pi}{}^{-3} \int d^3r \left[\cos \underset{\sim}{k}' \cdot \underset{\sim}{r}\ \Delta_{nn}(\underset{\sim}{r}) + \cos \underset{\sim}{k} \cdot \underset{\sim}{r}\ \Delta_{np}(\underset{\sim}{r}) + \Delta_{np}(\underset{\sim}{r})\ \Delta_{nn}(\underset{\sim}{r})\right] \ . \tag{45b}$$

The function $\Delta_{nn}(\underset{\sim}{r})$ has no Fourier components corresponding to the scattering of either neutron into an occupied state, i.e., $\Delta_{nn}(\underset{\sim}{r})$ has

no components with wave number $\underset{\sim}{p}$ for which $|\frac{1}{2}\underset{\sim}{K} + \underset{\sim}{p}| < P_F(n)\,\hbar^{-1}$. Since $\underset{\sim}{k}'$ is approximately one-half $\underset{\sim}{K}$, $\Delta_{nn}(\underset{\sim}{r})$ has no Fourier component with wave number $\pm\,\underset{\sim}{k}'$, and

$$\int d^3r \cos \underset{\sim}{k}' \cdot \underset{\sim}{r}\, \Delta_{nn}(\underset{\sim}{r}) = 0 \qquad .$$

We follow Gomes et al. in assuming that the nucleon-nucleon potential consists of an attractive square well and a hard core. The long-range attractive well has little effect on the wave function for densities comparable to ρ_{nucl}; the distortion functions Δ are due almost entirely to the hard core. We consider the case where the core radius, a, is much less than $\hbar\left[P_F(n)\right]^{-1}$. The resulting low-density approximation should be reasonably accurate up to densities about equal to nuclear density. In the low-density limit, one can make the following simplifications: first, we need only consider s-waves; second, we can neglect the last term in Eq. (45b) because the product $\Delta_{np}(\underset{\sim}{r})\,\Delta_{nn}(\underset{\sim}{r})$ is of second order in $P_F(n)\,a$; third, in computing $\Delta_{np}(\underset{\sim}{r})$ we can neglect the leakage of the wave function inside the core as well as the changes in the wave function's normalization caused by the distortion terms Δ. One can then use the Bethe-Goldstone equation to find the Fourier component of $\Delta_{np}(\underset{\sim}{r})$ that corresponds to the momentum $\underset{\sim}{k}$. In this way, one finds that

$$\begin{aligned} |M_A|^2 &= |M_V|^2 \\ &= \left[(4\pi a)\,(k'^2 - k^2)^{-1}\,\lambda\!\!\!^{-}_{\pi}{}^{-3}\right]^2 \qquad . \end{aligned} \tag{46}$$

The values of k and k' are determined by kinematics and the exclusion principle. We found in Sec. V-B that the particles involved in

reactions (1) and (5) must be in a narrow band of states at the top of their respective Fermi seas. Thus the momentum of each particle involved in a reaction must be nearly equal to the Fermi momentum for that particle. The neutron Fermi momentum is large compared to the proton and electron Fermi momenta; the neutrino momentum, which is of the order of kT/c, is completely negligible. Hence the momentum $\underset{\sim}{p}_1'$ of the final neutron must be approximately equal to the momentum in the initial state, $\underset{\sim}{p}_1 + \underset{\sim}{p}_2$. If we neglect the momenta of all particles except the neutrons, we find that the three neutron momenta form an equilateral triangle with sides of length $P_F(n)$. It follows that k is equal to $3^{1/2}\,(2\hbar)^{-1}\,P_F(n)$ and k' is equal to $(2\hbar)^{-1}\,P_F(n)$. Substituting these values of k and k' in Eq. (46), using Eq. (5e) of paper I, and choosing the core radius a to be 4×10^{-14} cm, we find that

$$|M_A|^2 = |M_V|^2$$

$$\simeq 1.0\,(\rho_{nucl}/\rho)^{4/3} \quad . \tag{47}$$

5. Summary

The scattering model and the model based on the usual picture of nuclear matter both predict that $|M_A|^2$ is of the order of unity near nuclear density and that $|M_A|^2$ decreases with increasing density. The relatively small difference between Eqs. (44) and (47), and the agreement of both equations with a dimensional analysis, indicates that the value of the total matrix element is not critically sensitive to the uncertainty in our knowledge of the strong internucleon force.

D. Related Reactions

1. The Inverse Reaction

We have calculated so far only the rate of neutrino energy loss via reaction (1). At the temperatures and densities for which reactions (1) and (5) are the dominant means of ensuring chemical equilibrium in the n-e-p system, the rates of reactions (1) and (5) must be equal in order to preserve the equilibrium. We shall now show that the rates of neutrino energy loss by the two reactions are in fact equal within the approximations we have used in calculating the rate of reaction (1).

For reaction (1), Eq. (21) provides an evaluation of $\Sigma_{spins} |\langle f |H| i\rangle|^2$; this equation is accurate if the lepton momenta are small compared to the neutron momenta. The expression for $\Sigma_{spins} |\langle f |H| i\rangle|^2$ for reaction (5) is identical to Eq. (21) if the lepton momenta are again neglected. The nucleon matrix elements M_A and M_V for reaction (5) are the complex conjugates of M_A and M_V for reaction (1). Furthermore one can easily show that Eq. (35) for the phase-space factor P holds equally well for reactions (1) and (5). Thus, Eq. (22), which gives the neutrino luminosity in terms of M_V, M_A, and P, predicts the same rates of energy loss for the direct and inverse reactions.

2. Muon Production

Muons are present in a neutron star if the electron Fermi energy is greater than the muon rest energy $m_\mu c^2$;[18] muon neutrinos are then produced by reactions (2) and (6). The rate of reactions (2) and (6) can be computed by the method used for reactions (1) and (5). The only difference in the rates of production of muon and electron neutrinos results from the fact that the density of muon states at the top of the muon Fermi sea differs from the density of electron states at the top of the electron

Fermi sea by a factor F, where, for $W_F(e)$ greater than $m_\mu c^2$,

$$F = P_F(\mu)/P_F(e) \quad . \tag{48}$$

Using the equilibrium relations (Eqs. (10) and (11) of paper I), we obtain

$$F = \left(1 - \left[m_\mu c^2/W_F(e)\right]^2\right)^{1/2} \quad . \tag{49}$$

The ratio F is of course zero when $E_F(e)$ is less than $m_\mu c^2$. Using Eq. (5c) of paper I to estimate $W_F(e)$, we find that

$$F = \left[1 - 2.25\,(\rho_{nucl}/\rho)^{4/3}\right]^{1/2} \qquad \text{for} \quad \rho > 1.8\,\rho_{nucl} \tag{50a}$$

and

$$F = 0 \qquad \text{for} \quad \rho < 1.8\,\rho_{nucl} \quad . \tag{50b}$$

E. Numerical Expressions

We now combine the results of the last four subsections to obtain convenient numerical expressions for the rate of energy loss by neutrino emission. Substituting Eqs. (38) and (47) into Eq. (24), and multiplying by $2(1 + F)$ to take account of reactions (2), (5), and (6), we find that the rate of neutrino energy loss by the two-nucleon reactions is given by

$$L_\nu^{nn} = (10^{20}\ \text{erg cm}^{-3}\ \text{sec}^{-1})\,(\rho/\rho_{nucl})^{2/3}\ T_9^{\,8}\,(1 + F) \quad , \tag{51}$$

where F is given in Eq. (50).

The luminosity of a mass M_s of neutron-star matter with a uniform density ρ is given by the expression

$$L_\nu^{nn} \approx (6 \times 10^{38}\ \text{erg sec}^{-1})\,(M_s/M_\odot)\,(\rho_{nucl}/\rho)^{1/3}\ T_9^{\,8}\,(1 + F) \quad , \tag{52}$$

where $M_\odot$ is the mass of the sun.[22]

Equations (51) and (52) give estimates of the neutrino luminosity from reactions involving two nucleons. Two-nucleon reactions are expected to dominate the neutrino production as long as there are no quasi-free pions present.

F. Comparison with Previous Work

Chiu and Salpeter[7] first suggested that reactions (1) and (5) might contribute importantly to the cooling of neutron stars. They used a dimensional analysis to obtain the expression

$$L_\nu^{CS} = (2 \times 10^{36} \text{ erg/sec})\, T_9^{\,8} \left[E_F(n)/50 \text{ MeV}\right]^{-2.25} (M_s/M_\odot)$$

for the rate of energy loss by neutrinos produced in reactions (2) and (5). The result given by Chiu and Salpeter has the correct temperature dependence, but it is typically two or three orders of magnitude smaller than our best estimate (as given in Eq. (52)).

Finzi[11] has performed a detailed calculation of the rate of reaction (2) at a density of $1.6\ \rho_{nucl}$. Although he did not explicitly calculate the rate of energy loss by reaction (5), he correctly assumed it to be equal to the neutrino luminosity arising from reaction (1). His treatment of the matrix element differs from ours in several ways. First, he neglected the effects of the exclusion principle on the relative motion of two colliding nucleons. Second, he treated the strong nucleon-nucleon interaction as a first-order perturbation; the nucleon scattering matrix element was assumed to be equal to a constant, which was determined by the requirement that the same first-order perturbation treatment yield a value of 3×10^{-26} cm^2 for the scattering cross section for free nucleons.

Third, he treated the nucleons and leptons as scalar particles (instead of Fermions) in calculating the amplitude associated with the weak vertex. Finzi's treatment of the phase-space factor P differs from ours in two ways: first, a minor error in his integrations results in an extra factor that is approximately equal to 2/3; second, he uses the free masses m_n and m_p instead of effective masses m_n^* and m_p^* to describe the density of single-particle states. Finzi gave the following expression for the luminosity of $0.6\ M_\odot$ of neutron star matter at $1.6\ \rho_{nucl}$:

$$L_\nu^F = (8.83 \times 10^{37}\ \mathrm{erg/sec})\ T_9^8 \quad .$$

This result differs from the luminosity predicted by Eq. (52) for the same mass and density by about a factor of one-fifth (if we set F equal to zero). The disagreement between the two answers is small compared to the obvious uncertainties in either approach. The closeness of the two results for the rate of energy loss arises partly from the fact that the matrix element is, as we mentioned in Sec. V-C, relatively insensitive to the details of the model used to calculate it.

Ellis[12] has recently reported a similar calculation of the rate of energy loss by reactions (1) and (5). Following Finzi, he employed second-order perturbation theory to estimate the transition amplitude, using the known nucleon-nucleon scattering data to determine the coupling at the strong vertex; he also neglected the effects of the surrounding neutrons on the relative motion of the colliding nucleons. Unlike Finzi, Ellis treated the nucleons and leptons as fermions, and he performed the calculation for a range of densities. Although he did treat the nucleons relativistically, he did not consider the protons to be degenerate, despite the fact that $E_F(p)/kT$ is of the order of 50 for most temperatures and

densities expected in neutron stars. Ellis performed part of the integration over phase space by a Monte Carlo technique; he gave the following formula, which accurately represents his numerical results:

$$L_\nu^E = (6 \times 10^{38}\ \mathrm{erg/sec}) \left[E_F(n)/50\ \mathrm{MeV}\right]^{-1.9} (M_s/M_\odot)\ T_9^{8.7} \quad .$$

The peculiar temperature dependence is due primarily to the fact that he assumed that the protons were non-degenerate. The above relation does not differ from that obtained by Finzi or by us by more than a factor of ten in the most interesting domains of temperature and density.

We have extended and refined the work of previous authors in several respects. First, the rate of energy loss by muon neutrinos and the luminosity due to the inverse processes (reactions (5) and (6)) have been explicitly calculated in the present work. Second, we have attempted to modify the single-particle picture to take account of strong interactions. In particular, we have used the methods developed for nuclear-matter calculations to estimate the density of single-particle levels (as expressed by the effective masses) and to treat the nucleon-nucleon scattering in a manner consistent with the exclusion principle. We have also been able to calculate the phase-space factor more accurately by expressing it in a form that permits accurate analytic evaluation.

VI. PION COOLING

A. General Discussion

In this section we calculate the rates of several neutrino-producing reactions that will occur if quasi-free pions are present in neutron matter. We showed in Sec. V of paper I that quasi-free pions, if they are present

at all in a neutron star, must be highly degenerate; that is, nearly all the pions must be in the lowest-energy single-particle state. The momentum $\underset{\sim}{p}_\pi$ and energy ω_π of this lowest single-particle state are not known. The reaction rate fortunately does not depend sensitively on $\underset{\sim}{p}_\pi$, and we can assume that $\underset{\sim}{p}_\pi$ is zero without making a serious error. The energy ω_π can be written

$$\omega_\pi = B(\pi^-) + m_\pi c^2 \quad , \tag{53}$$

where $B(\pi^-)$, the pion binding energy, was defined in Sec. III of paper I.

The most important neutrino-producing processes that involve pions are reactions (3), (4), (7), and (8). We shall first derive an expression for the rate of energy loss by reaction (3), and then modify the formula to take account of other reactions.

The rate of energy loss per pion by reaction (3) is given by

$$L_\nu^{(3)} = 2\pi\hbar^{-1} \sum_{\text{spins}} \int d^3\underset{\sim}{n}_1\, d^3\underset{\sim}{n}_1{}'\, d^3\underset{\sim}{n}_e\, d^3\underset{\sim}{n}_{\bar{\nu}}\; S\, \delta(E_f - E_i)$$

$$E_{\bar{\nu}}\; |\langle (n,e^-,\bar{\nu})\, |H_w|\, (n,\pi^-)\, +\rangle|^2 \quad . \tag{54}$$

The notation used in Eq. (54) is similar to that used in Eq. (18): the differentials $d^3\underset{\sim}{n}_1$, $d^3\underset{\sim}{n}_1{}'$, $d^3\underset{\sim}{n}_e$, and $d^3\underset{\sim}{n}_{\bar{\nu}}$ refer to the initial neutron, the final neutron, the electron, and the antineutrino, respectively. The statistical factor S is identical to that defined in Eq. (18c), except that it only includes factors for the two neutrons and the electron (all pions are assumed to be in the lowest energy state). The initial state vector $|(n,\pi^-)\, +\rangle$ is an eigenstate of the strong Hamiltonian; the incoming part of $|(n,\pi^-)\, +\rangle$ corresponds to a neutron with momentum $\underset{\sim}{p}_1$ and a pion with

momentum $\underset{\sim}{p}_\pi$. The final state vector $|(n,e^-,\bar{\nu})\rangle$ is a product of momentum eigenstates representing a neutron (with momentum $\underset{\sim}{p}_1$), an electron (with momentum $\underset{\sim}{p}_e$), and a neutrino (with momentum $\underset{\sim}{p}_{\bar{\nu}}$).

We again find it convenient to separate the neutrino luminosity into a dimensionless phase-space factor, a dimensionless matrix element, and a constant factor. The matrix element is nearly constant over those regions of phase space where the statistical factor S is non-negligible. Thus we can remove the matrix element from the integral and write the neutrino luminosity in the form

$$L_\nu^{(3)} = P\, M^2 \left[G^2 (2\pi)^4 \hbar^{-1} \lambda\!\!\!^{-}_\pi{}^{-6} \right] \quad , \tag{55a}$$

where

$$P = (2\pi)^{-12} (m_\pi c)^{-9} \int d^3\underset{\sim}{p}_1\, d^3\underset{\sim}{p}_1{}'\, d^3\underset{\sim}{p}_e\, d^3\underset{\sim}{p}_{\bar{\nu}}\; \delta(E_f - E_i)\, \delta^3(\underset{\sim}{P}_f - \underset{\sim}{P}_i)\, S\, E_{\bar{\nu}} \quad , \tag{55b}$$

$$M^2\, \delta^3(\underset{\sim}{P}_f - \underset{\sim}{P}_i) = G^{-2}\, \Omega^4\, \lambda\!\!\!^{-}_\pi{}^{-3} (2\pi\hbar)^{-3} \times \sum_{\text{spins}} \left| \langle (n,e^-,\bar{\nu})\, |H_w|\, (n,\pi^-)\, + \rangle \right|^2 \quad , \tag{55c}$$

and $\underset{\sim}{P}_i$ and $\underset{\sim}{P}_f$ are the initial and final momenta, respectively.

In the following sections, we estimate the values of P and M^2, employing arguments that are analogous to those we have previously used to calculate the nucleon-nucleon cooling rate. We shall see, however, that our knowledge of the relevant matrix elements is much less accurate for pionic cooling than it is for nucleon-nucleon cooling.

B. The Phase-Space Factor

As in the case of nucleon-nucleon cooling, we describe the density of available initial and final states by the phase-space factor P, which, for reaction (3), is defined in Eq. (55b). The integrand in Eq. (55b) is concentrated in the small "important region" of phase space where the energy of each particle is within a few kT of its Fermi energy. Just as in Sec. V-B, we neglect the contribution to the integral P from certain regions far from the "important region"; in particular, we consider only the parts of phase space satisfying the following inequalities:

$$p_1 + p_{\bar{\nu}} + p_\pi - p_e < p_1' < p_1 - p_{\bar{\nu}} - p_\pi + p_e \quad ; \tag{56a}$$

$$p_1 > p_{\bar{\nu}} + p_\pi \quad . \tag{56b}$$

The phase-space factor for reaction (3) can be evaluated by the methods used to calculate P for reaction (1). We use inequalities (56) to evaluate the angular integral A, where

$$A = \int d\Omega_1 \int d\Omega_{\bar{\nu}} \int d\Omega_e \int d\Omega_1' \; \delta^3(\underset{\sim}{p}_1 + \underset{\sim}{p}_\pi - \underset{\sim}{p}_1' - \underset{\sim}{p}_e - \underset{\sim}{p}_{\bar{\nu}}) \quad . \tag{57}$$

The result is

$$A = 32\pi^3 \, (p_1' \, p_e \, p_1)^{-1} \quad . \tag{58}$$

Substituting Eq. (58) into Eq. (55b) and using the energy delta function to integrate over the neutrino momentum yields

$$-P = C \int_0^\infty p_1 \, dp_1 \int_0^\infty p_1' \, dp_1' \int_{p_m}^\infty p_e \, dp_e \, (E_o)^3 \, S \quad , \tag{59a}$$

where

$$C = 2^{-7} \, \pi^{-9} \, m_\pi^{-9} \, c^{-12} \quad , \tag{59b}$$

$$E_o = E_1 + \omega_\pi - E_1' - W_e \quad , \tag{59c}$$

and p_m is defined such that E_o is equal to zero when p_e equals p_m.

As in Sec. V-B, it is convenient to change variables: we define $m_n^*(E_n)$, x_1, x_2, and x_3 by Eqs. (33b), (34a), (34b), and (34c) respectively. We then use the conditions of chemical equilibrium (Eqs. (11b) and (15) of paper I) to obtain

$$P = C\,(kT)^6 c^{-2} \int_{-y_n}^{\infty} m_n^*(E_1)\,dx_1 \int_{-\infty}^{y_n} m_n^*(E_1')\,dx_2 \int_{-(x_1+x_2)}^{y_e} dx_3$$

$$\times \left[\omega_\pi - kT\,x_3\right] (x_1 + x_2 + x_3)^3 \; S \quad , \tag{60}$$

where y_n and y_e were defined in Eqs. (35b) and (35c). To lowest order in $\left[kT/E_F(n)\right]$ we can set the effective masses equal to their value at the neutron Fermi energy and neglect $x_3\,kT$ relative to ω_π. Replacing the limits y_n and y_e by infinity causes errors of the order of $e^{-\beta E_F(n)}$ and $e^{-\beta E_F(n)}$, respectively; these errors can be neglected since $\beta E_F(n)$ and $\beta E_F(e)$ are both much greater than 100. We then obtain the following expression for the phase-space factor:

$$P = 2^{-7}\,\pi^{-9}\,(\omega_\pi/m_\pi c^2)\,(kT/m_\pi c^2)^6\,(m_n^*/m_\pi)^2\;I \quad , \tag{61a}$$

where

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-(x_1+x_2)}^{\infty} dx_3\,(x_1+x_2+x_3)^3 \prod_{i=1}^{3} \left(1 + e^{x_i}\right)^{-1} \quad , \tag{61b}$$

$$= (457/5040)\,\pi^6 \quad . \tag{61c}$$

The phase-space factor is proportional to T^6, as expected from the heuristic argument in Sec. II. The factor P for reaction (3) depends on the density only through the effective mass m_n^* and the pion ground-state energy ω_π. Referring to the results of Sec. IV-B of paper I, we assume that the neutron effective mass is 1.0 m_n. We also assume that the pion binding energy $B(\pi^-)$ is small compared to $m_\pi c^2$. Then the pion phase-space factor can be conveniently expressed in the form

$$P = 5.6 \times 10^{-23} \; T_9^{\,6} \quad . \tag{62}$$

C. Matrix Element

We present several arguments that can, in the absence of a detailed theory of strong interactions, be used to obtain crude estimates of the matrix elements for reaction (3).

1. Dimensional Argument

The physical lengths involved in the matrix element M are the following: $\hbar c\, \omega_\pi^{-1}$, $\hbar(m_e c)^{-1}$, $\hbar(m_n c)^{-1}$, $\hbar\left[P_F(e)\right]^{-1}$, $\hbar c\left[E_{\bar{\nu}}\right]^{-1}$, $\hbar\left[P_F(n)\right]^{-1}$, and the range of the pion-nucleon potential. The range of the pion-nucleon potential is of the order of the scale length $\lambda\!\!\!^{-}_\pi$. We assume that the pion binding energy $B(\pi^-)$ is not large compared to $m_\pi c^2$; then $\hbar c\, \omega_\pi^{-1}$ and $\hbar\left[P_F(e)\right]^{-1}$ are also of the order of $\lambda\!\!\!^{-}_\pi$.

There remain four relevant lengths that are not approximately equal to $\lambda\!\!\!^{-}_\pi$: $\hbar c\, E_{\bar{\nu}}^{-1}$, $\hbar(m_e c)^{-1}$, $\hbar(m_n c)^{-1}$, and $\hbar\left[P_F(n)\right]^{-1}$. The neutrino energy $E_{\bar{\nu}}$ enters the matrix element only through the combinations $E_e \pm E_{\bar{\nu}}$; since $E_{\bar{\nu}}$ is much smaller than E_e, it follows that M is essentially independent of $\hbar c\left[E_{\bar{\nu}}\right]^{-1}$. The momentum and energy transferred to the leptons do not depend strongly on m_e, m_n, or $P_F(n)$, because of the

equilibrium relations that obtain (cf. paper I, Eq. (13)). Hence these three quantities do not contribute strongly to the energy denominators corresponding to the important virtual states (the virtual states involved in Fig. 1 for example). The amplitudes at the vertices are not strongly dependent on $P_F(n)$; consequently, the entire matrix element M is approximately independent of $\hbar\left[P_F(n)\right]^{-1}$. We shall see later that the amplitude at the weak vertex can, for some diagrams, be proportional to the electron mass, and the contributions from these diagrams are consequently inhibited by a factor of (m_e/m_π). The contributions from the dominant diagrams, however, are essentially independent of $\hbar(m_e c)^{-1}$. The effect of the nucleon mass on the matrix element is more subtle; the masses of the hadrons and the coupling constants characterizing their interactions are connected in a complicated way. The ratio $(\lambda\!\!\!^{-}_\pi/\lambda\!\!\!^{-}_n)$ or m_n/m_π, is typical of the dimensionless quantities arising in strong interaction calculations. Our dimensional reasoning can only suggest that M should be of the order of unity, within perhaps a couple of factors of m_n/m_π.

2. Pion Decay

We first estimate the rate of reaction (3) by considering the diagram shown in Fig. 1a; the pion is assumed to beta decay during a collision with a neutron. The diagram suggests factoring the matrix element of H_w as follows:

$$\langle (n,e^-,\bar{\nu})\ |H_w|\ (n,\pi^-)\ +\rangle = \langle e\ (p_e)\ \bar{\nu}\ (p_{\bar{\nu}})\ |H_w|\ \pi\ (p_e + p_{\bar{\nu}})\rangle$$

$$\times\ \langle n\ (p_1')\ \pi\ (p_e + p_{\bar{\nu}})\ |\ (n,\pi^-)\ +\rangle \quad , \qquad (63)$$

where $e(p_e)$, for example, represents an electron state with four-momentum p_e. This factorization can be justified formally by writing an explicit expression for $\langle e,n,\bar{\nu}\,|H_w|\,(n,\pi^-)+\rangle$ in terms of a double integral over the neutron and pion coordinates.

The weak Hamiltonian is the product of a leptonic weak current $G^{1/2}\,2^{-1/4}\,\bar{\psi}_e\,\gamma_\alpha\,(1+\gamma_5)\,\psi_\nu$ and a pionic weak current Q_α. The current Q_α is proportional to the four momentum $(p_e + p_{\bar{\nu}})_\alpha$ of the decaying pion, since the four-momentum is the only vector associated with the spinless particle. Including all the necessary normalization factors, we can write the required matrix element as follows:

$$\langle e(p_e)\,\bar{\nu}(p_{\bar{\nu}})\,|H_w|\,\pi(p_e + p_{\bar{\nu}})\rangle = \frac{1}{4}\,G\,K\left[(p_e + p_{\bar{\nu}})^2\right]\,\hbar^{-3/2}\,m_\pi c$$

$$\times\left[\omega_e\,\omega_{\bar{\nu}}\,(\omega_e + \omega_{\bar{\nu}})\,\Omega\right]^{-1/2}\,(p_e + p_{\bar{\nu}})_\alpha\,\bar{u}_e(p_e)\,\gamma_\alpha\,(1+\gamma_5)\,v_{\bar{\nu}}(-p_{\bar{\nu}}) \quad , \tag{64}$$

where $\omega_e c$ and $\omega_{\bar{\nu}} c$ are the muon and neutrino energies and $K\left[(p_e + p_{\bar{\nu}})^2\right]$ is a dimensionless scalar that indicates the relative strength of the pionic part of the weak current. The value of K for a pion on the mass shell can be determined empirically from the known half life $\tau_\pi(\frac{1}{2})$ of the pion at rest:

$$|K(-m_\pi^2)|^2 = 8\pi(\ln 2)\,\hbar^7\,c^{-4}\,G^{-2}\,m_\pi^{-3}\,m_\mu^{-2}\,(1 - m_\mu^2/m_\pi^2)^{-2}\left[\tau_\pi(\tfrac{1}{2})\right]^{-1} \tag{65}$$

$$\approx 0.88 \quad . \tag{66}$$

We shall assume for simplicity that

$$|K[(p_e + p_\nu)^2]|^2 \approx |K(-m_e^2)|^2 \tag{67a}$$

$$\approx |K(-m_\pi^2)|^2 \tag{67b}$$

$$= |K|^2 \quad . \tag{67c}$$

It is easy to establish the dimensional forms of the overlap matrix element $\langle n(p_1') \pi(p_e + p_{\bar{\nu}}) | (n,\pi^-) + \rangle$, which we abbreviate by $\langle \phi | \psi \rangle$. The matrix element involves two integrals, one over the center-of-mass coordinates and one over the relative position of the nucleon and pion. The integration over the center-of-mass coordinate yields a momentum delta function; the integration over the relative position yields an effective overlap volume, which must be of the order of $\lambda\!\!\!^{-}_\pi{}^3$. Since the wave functions are normalized in a volume Ω, we find that

$$|\langle \phi | \psi \rangle|^2 = B^2 \lambda\!\!\!^{-}_\pi{}^6 \, \Omega^{-3} \, (2\pi\hbar)^3 \, \delta^3(\underset{\sim}{P}_f - \underset{\sim}{P}_i) \quad , \tag{68}$$

where B is expected to be of the order of unity. Substituting Eqs. (64), (67), and (68) in Eq. (55c), one obtains, after summing over nucleon and leptons spins,

$$M^2 = - B^2 \, |K|^2 \, m_e^2 \, c^2 \, (\omega_\pi \, m_\pi)^{-1} \, p_e \cdot p_{\bar{\nu}} \, (\omega_e \, \omega_{\bar{\nu}})^{-1} \quad . \tag{69}$$

The integrand of the phase space factor is independent of $\underset{\sim}{p}_{\bar{\nu}}$ to first order in kT; hence we can average over the direction of $\underset{\sim}{p}_{\bar{\nu}}$ in Eq. (68). Substituting the value of K from Eq. (66) in Eq. (69), we then find that

$$- \quad M^2 = 0.9 \, B^2 \, m_e^2 \, c^2 \, (\omega_\pi \, m_\pi)^{-1} \quad . \tag{70}$$

The factor B is a dimensionless number characterizing the strong pion-nucleon interaction. It should be equal to unity within perhaps a couple of powers of (m_n/m_π).

3. Born Approximation for Pion Decay

We now use a specific model to treat the pion-nucleon interaction. We assume an interaction Hamiltonian given by[23]

$$H_s = i\, g\, \bar{\psi}_N \underset{\sim}{\tau}\, \gamma_5\, \psi_N \cdot \underset{\sim}{\phi} \quad , \tag{71a}$$

where

$$g \approx 14 \tag{71b}$$

and $\underset{\sim}{\tau}$ and $\underset{\sim}{\phi}$ are vectors in the isotopic spin spaces of the nucleon and pion, respectively. We treat the strong interaction as if it were a small perturbation and consider just the diagram shown in Fig. 1b. The assumption that the strong interaction is a small perturbation is of course not valid because of the large value of g, but we use the first-order treatment in the hope that it will provide some insight into the relevant physical quantities that enter the problem and perhaps serve as a guide in the estimation of the factor B in Eq. (70).

The Feynman rules permit one to calculate easily the amplitude corresponding to the diagram of Fig. 1b. We use the free-particle propagators for the pion and nucleon, make the non-relativistic approximation for the nucleons, neglect m_π compared to m_n, and average over the directions of the neutrino momentum. The result is

$$M^2 \approx \frac{1}{2}\, g^4\, |K|^2\, (m_e/m_n)^2\, (1 - m_e{}^2/m_\pi{}^2)^{-2} \quad , \tag{72a}$$

or

$$M^2 \approx 380\, (m_e/m_\pi)^2 \quad , \tag{72b}$$

which corresponds to

$$B \approx 21 \quad . \tag{72c}$$

from Fig. 1c. For example, the contribution from Fig. 1b would have been large compared to that from Fig. 1c, had it not been for the factor $(m_e/m_\pi)^2$, which resulted from the form of the weak Hamiltonian. The constant in Eq. (72b) is large because it contains four factors of g, while the constant in Eq. (75) contains only two.

The estimates of the matrix element M^2 given in sub-sections (1) to (4) can be summarized by stating that M^2 is expected to be of the order of ten but is uncertain by one or two powers of ten.

D. Related Reactions

Muons are expected to be present in neutron stars that contain pions if ω_π is greater than $m_\mu c^2$ (cf. Eq. (11b) and (15) of paper I). When muons are present, reaction (4) contributes to the rate of neutrino production. The phase space factor for reaction (4) is the same as for reaction (3) if, as expected, $\omega_\pi - m_\mu c^2$ is much larger than kT. The matrix element M is, on the other hand, not the same for decays producing muons and electrons. In Sec. VI-C we found that diagrams such as Fig. 1c that involve the decay of a neutron into a proton, electron, and anti-neutrino were much more important than diagrams such as Figs. 1a and 1b that involve the decay of a virtual π^- into an e and a $\bar{\nu}_e$. However, the pion-decay processes that are inhibited by a factor of $(m_e/m_\pi)^2$ in the case of decay into e and $\bar{\nu}_e$ are only inhibited by a factor of $(m_\mu/m_\pi)^2$ in the case of decay into a μ and $\bar{\nu}_\mu$. Thus diagrams such as Figs. 1a and 1b may contribute importantly to the rate of production of muon neutrinos. The rates of production of electron and muon neutrinos may nevertheless be of the same order of magnitude, and, lacking an accurate estimate of either rate, we shall assume that the rates of energy loss by muon and electron neutrinos are equal.

As in the case of the nucleon-nucleon reactions, the rate of energy loss by the inverse processes (reactions (7) and (8)) can be proved equal to the rate of energy loss by the forward processes (reactions (3) and (4)).

E. Numerical Expressions

The rate of energy loss by neutrinos produced in pion reactions can be obtained by substituting values of M^2 and P in Eq. (55a). In particular, we use Eq. (62) for the phase space factor and set M^2 equal to ten. Multiplying by four to account for the muonic decay (reaction (4)) and the inverse processes (reaction (7) and (8)), we find the expression

$$L_\nu^{\pi n} \approx 10^{-11} \text{ erg/sec} \tag{76}$$

for the rate of energy loss per pion. The neutrino luminosity of a mass M_s of stellar matter is then given by

$$L_\nu^{\pi n} \approx (10^{46} \text{ erg/sec})\, T_9^{\,6}\, (n_\pi/n_b)\, (M_s/M_\odot) \quad , \tag{77}$$

where n_π/n_b is the ratio of the number density of quasi-free pions to the number density of baryons. Equations (76) and (77) are probably accurate to within a factor of something like one-hundred. The result given in Eq. (77) is about twice the rate indicated by the heuristic discussion in Sec. II, and is almost identical to the result of our previous calculation.[24] We note that the energy loss by the pionic process is of the order of $10^7\, T_9^{\,-2}$ times the energy loss by the nucleon-nucleon processes if a significant number of quasi-free pions are present.

VII. COOLING TIMES AND OBSERVABILITY

A. Temperature Distribution

The interior of a neutron star is nearly isothermal because of the high conductivity of the degenerate electrons. The effective surface temperature T_e is, on the basis of the models of Tsuruta,[15] of the order of 10^{-2} times the central temperature T, but the temperature drop from T to T_e occurs almost entirely in a thin surface layer of non-degenerate and partially degenerate matter. The energy loss by neutrino emission depends on the interior temperature T, but the rate at which photons are emitted from the surface is governed by the effective surface temperature T_e.

The thermal energy U of a neutron star is approximately equal to the energy of the thermal excitations in the neutron gas if there are no gaps in the energy spectrum of the gas. We thus find that

$$U \approx (5 \times 10^{47} \text{ erg})\, T_9^{\,2}\, (\rho/\rho_{nucl})^{-2/3}\, (M_s/M_\odot) \quad , \tag{78}$$

where T_9 is the interior temperature of the star in units of 10^9 °K.

B. Cooling Rates

We assume that the star radiates photons from its surface like a black body; the detailed atmospheric calculations of Orszag[25] indicate that the black body assumption is a fairly accurate overall approximation. The photon luminosity is then given by

$$L_\gamma = (7 \times 10^{36} \text{ erg sec}^{-1})\, T_{e7}^{\,4}\, R_{10}^{\,2} \quad , \tag{79}$$

where T_{e7} is the effective surface temperature in units of 10^7 °K, and R_{10} is the radius of the star in units of 10 km. For convenience, we rewrite Eqs. (52) and (77) for the energy loss by the nucleon-nucleon and pion-nucleon processes:

The factors of $(m_e/m_\pi)^2$ in Eqs. (69), (70), and (72) result from our assumption that the leptons are produced in the decay of a π^-. A similar factor occurs, for the same reason, in the well-known and experimentally verified prediction of the V-A theory for the decay of a free pion. In the next section we consider the production of an electron and a neutrino by the decay of a neutron (Fig. 1c) and find that the corresponding matrix element is not inhibited by factors of m_e/m_π.

4. Neutron Decay

The diagram shown in Fig. 1c is the simplest one in which reaction (3) takes place by neutron decay. At the strong vertex, we use the Hamiltonian given in Eq. (71), but the weak vertex now involves the nucleon current. We assume a pure V-A form

$$H_w = 2^{-1/2}\, G\, \bar{\psi}_p\, \gamma_\alpha\, (1+\gamma_5)\, \psi_n\, \bar{\psi}_e\, \gamma_\alpha\, (1+\gamma_5)\, \psi_\nu \tag{73}$$

for the weak Hamiltonian. The coefficient of the axial vector part of the nucleon current has, for simplicity, been set equal to unity.

The dimensionless factor M^2 can be calculated using Feynman rules. Making the same approximations as in sub-section (3), we find that

$$M^2 \approx 2\, g^2\, (m_\pi/m_n)^2 \tag{74}$$

$$\approx 9 \quad . \tag{75}$$

We note that the contribution to M from the diagram of Fig. 1c contains no factors of m_e/m_π and is consequently larger than the contributions from the diagrams involving pion decay. The value of M^2 given in Eq. (75) is not reliable, however, because of our use of perturbation theory. Terms of higher order in g^2 may be larger than the contribution

$$L_\nu^{nn} \approx (6 \times 10^{38} \text{ erg sec}^{-1}) (M_s/M_\odot) (\rho_{nucl}/\rho)^{1/3} \; T_9^{\,8} (1 + F) \quad , \tag{52}$$

$$L_\nu^{\pi n} \approx (10^{46} \text{ erg sec}^{-1}) (M_s/M_\odot) (n_\pi/n_b) \; T_9^{\,6} \quad , \tag{77}$$

where n_π and n_b are the number densities of pions and baryons, respectively; the factor F, which represents the contribution from muonic decays, was defined in Eq. (49).

The rate of change of the interior temperature can easily be computed (if the ratio of interior to surface temperature is known) using the relation

$$\frac{dU}{dt} = - L_\gamma - L_\nu^{nn} - L_\nu^{\pi n} \tag{80}$$

and Eq. (78). If quasi-free pions are present in significant numbers, the pion-nucleon cooling reactions are dominant, and the time required for the interior to cool from an initial temperature $T(i)$ to a final temperature $T(f)$ is

$$\Delta t \text{ (pions)} = (8 \times 10^{-7} \text{ yr}) (n_b/n_\pi) (\rho/\rho_{nucl})^{-2/3} \left[T_9(f)^{-4} - T_9(i)^{-4}\right] \; . \tag{81}$$

The luminosity $L^{\pi n}$ is zero if no quasi-free pions are present. Then we can solve Eqs. (78), (79), (52), and (80), finding that the time required for a star's interior to cool from $T(i)$ to $T(f)$ is given by

$$\Delta t \approx a\, \alpha^6 \left\{ \left[\alpha\, T_9(f)\right]^{-2} - \left[\alpha\, T_9(i)\right]^{-2} + b \left[\tan^{-1} x_f - \tan^{-1} x_i\right] \right\} \quad , \tag{82a}$$

where

$$a \approx (1900 \text{ yr}) (M_s/M_\odot)^{1/3} \quad , \tag{82b}$$

$$b \approx 8.5 \, (\rho/\rho_{nucl})^{1/6} (M_s/M_\odot)^{1/6} \quad , \tag{82c}$$

$$x_i = b\left[\alpha T_9(i)\right]^2 \quad , \tag{82d}$$

and

$$x_f = b\left[\alpha T_9(f)\right]^2 \quad . \tag{82e}$$

We have assumed that the temperature parameter α, defined by

$$\alpha(T) = 10^{-2}\, T/T_e \tag{83a}$$

or

$$\alpha(T) = T_9/T_{e7} \quad , \tag{83b}$$

is approximately constant for T between T(i) and T(f).

It is clear from Eq. (82) that the cooling rate depends strongly on the parameter α, which must be determined from theoretical models of neutron stars. We wish to stress that α is, in fact, the only quantity derived from neutron-star models that enters at all sensitively into the theoretical predictions of the cooling rates. It is primarily through α that the models affect the question of the observability of neutron stars, and future models calculations should therefore attempt to establish the uncertainty in $\alpha(T)$ due to, for example, uncertainties in the equation of state.

We have computed cooling times for a typical neutron star, with the results shown in Fig. 2. The curves represent cooling by the pion-nucleon reaction (Eq. (81)), by the nucleon-nucleon processes, by photons radiating from the surface, and by the nucleon-nucleon process and photon cooling operating together (Eq. (82)). We considered a star with average density ρ_{nucl} and mass $M_\odot$. The quantity $\alpha(T)$ is a slowly varying function of temperature; we chose values of $\alpha(T)$ in agreement with a neutron-star model constructed by Tsuruta[15] (see Table I).

C. Observability of Neutron Stars

The probability of ever observing a neutron star depends strongly on the rates at which such stars cool. A star containing quasi-free pions would emit detectable x-rays for no more than a few days, and the probability of observing it would be small. A star that cools only by the nucleon-nucleon and photon processes would be detectable for a longer time (cf. Fig. 2).

We have previously pointed out[26] that the rate of decrease of the x-ray intensity from a neutron star could be used as an observational test of theories of neutron-star cooling; we have given a convenient formula for making the appropriate observational comparisons, should a neutron star ever be discovered.

We now consider the flux of photons that would be produced at a distance r by a neutron star with effective temperature T_e. The flux Φ of photons with wavelengths less than λ_m is given approximately by

$$\Phi \approx (0.4\ \mathrm{cm}^{-2}\ \mathrm{sec}^{-1})\ R_{10}^{\ 2}\ r_{kpc}^{\ -2}\ (T_e/3 \times 10^6\ {}^\circ\mathrm{K})^3 \quad (\tfrac{1}{2}x^2 + x + 1)\ e^{-x} \quad , \tag{84a}$$

where R_{10} is the stellar radius in units 10 km, r_{kpc} is the distance to the star in kiloparsecs (1 kpc $\approx 3.08 \times 10^{21}$ cm), and x is defined as follows:

$$x = 4.8\ (10\ \mathring{A}/\lambda_m)\ (3 \times 10^6\ {}^\circ\mathrm{K}/T_e) \quad . \tag{84b}$$

Approximately ten x-ray sources have been identified by Giacconi *et al.*,[1] Bowyer *et al.*,[2,3] and Clark *et al.*[4] These sources are concentrated near the galactic plane, and about half of them are located in the

direction of the center of the galaxy. The weakest source detected by Bowyer et al. produced a measured flux of $0.7\ cm^{-2}\ sec^{-1}$, and, because of absorption in the earth's atmosphere and in the counter itself, the observed x-rays must have been concentrated in the wavelength range from 1.5 Å to 8 Å; since the sun is approximately 8 kiloparsecs from the galactic center, we conclude from Eq. (84) that the effective temperature of an observed source located at the galactic center must be greater than $2 \times 10^7\ {}^{o}K$, if the source is no larger than a neutron star. Comparison with Fig. 2 indicates that a neutron star with a temperature of $2 \times 10^7\ {}^{o}K$ would have to be less than a day old. The x-ray sources located in the direction of the galactic center have been observed several times in the last few years,[1,3,17] and the flux from these sources has not changed, within the observational uncertainties (about a factor of two or three). Hence we conclude that the sources in the direction of the galactic center are almost certainly not neutron stars.

The strongest x-ray source appears to be in the direction of Scorpius. We have used Eq. (84) to calculate the distance at which a neutron star with a given surface temperature could produce the flux observed from the Scorpius source; this distance is calculated for various surface temperatures. The corresponding cooling times computed from Eq. (82) are shown in the third column of Table II. In computing the second column of Table II, we assumed that all the observed photons had wavelengths less than 8 Å; we also assumed that the neutron star had a radius of 10 km. It has been suggested that the Scorpius source may be only of the order of 30 parsecs from the sun. According to Table II, a distance of 30 parsecs corresponds to a surface

temperature of about 3×10^6 °K and to a reasonable cooling time of approximately 10^3 yr. However, a black-body at 3×10^6 °K would not produce nearly enough radiation with wavelength less than 2 Å to be consistent with the spectral measurements recently performed on the Scorpius source by Giacconi et al.[17]

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APPENDIX

Sections V and VI are devoted to the calculation of the rates of neutrino loss by reactions (1) - (8); in this appendix, we explain why we expect reactions (1) - (8) to dominate the neutrino production if the nucleon gas has a continuous excitation spectrum. In the following paragraphs, we consider various types of reactions and show that their contributions to the neutrino production is small compared to the contributions from reactions (1) - (8).

We first consider reactions that do not involve either electromagnetic interactions or quasi-free pions. The rate of ordinary neutron decay

$$n \rightarrow p + e^- + \bar{\nu} \tag{A1}$$

is negligible compared to the rate of reaction (1). As explained in Sec. V-B, the condition of chemical equilibrium and conservation of energy imply that the rates of processes involving only neutrons, protons, electrons, and neutrinos are dominated by reactions in which the neutrons, protons, and electrons concerned have energies near their respective Fermi energies, and the neutrinos produced have energies of the order of kT. But momentum cannot be conserved in reaction (A1) if p_n is near $P_F(n)$, p_p is near $P_F(p)$, p_e is near $P_F(e)$, and $p_{\bar{\nu}}$ is of the order of kTc^{-1}, because

$$P_F(n) - P_F(p) - P_F(e) \gg kTc^{-1} \quad .$$

Consequently, reaction (A1) must involve the emission of electrons and protons with momenta small compared to their Fermi momenta, and the probability of finding such low-energy states unoccupied is of the order

of $\exp\left[-E_F(n)/kT\right]$, which is extremely small. Conservation of momentum is easily satisfied if the decaying neutron is allowed to collide with another particle, as in reaction (1).

Reactions that involve large numbers of particles are slow because only a small fraction (of the order of $kT/E_F(n)$) of the particles of a given species are near enough to their Fermi level to scatter into unoccupied states. For example, the reaction

$$n + n + n \rightarrow n + n + p + e^- + \bar{\nu} \tag{A2}$$

is slower than reaction (1) by a factor of the order of $\left[kT/E_F(n)\right]^2$.

A reaction that, like

$$e^+ + n + n \rightarrow n + p + \bar{\nu} \quad , \tag{A3}$$

involves an incident positron, produces few neutrinos because the concentration of positrons is proportional to $\exp\left[-E_F(e)/kT\right]$. Positron-producing reactions like

$$n + p \rightarrow n + n + e^+ + \bar{\nu} \tag{A4}$$

are slowed by the same factor of $\exp\left[-E_F(e)/kT\right]$, because the number of neutron-proton pairs with enough energy to produce two neutrons in unoccupied states is proportional to $\exp\left[-E_F(e)/kT\right]$.

Applying the arguments of the last few paragraphs to all of the obvious neutrino-producing processes that do not involve either quasi-free pions or electromagnetic interactions, we find that none of these processes are faster than reactions (1), (2), (5), and (6).

We now consider reactions that do not involve pions, but do involve electromagnetic interactions. Photons propagating through a neutron star interact with the charged particles in the stellar medium. Creation of

one of these quasi-free photons (usually called "plasmons") requires an energy greater than $\hbar\omega_o$, where ω_o is the plasma frequency in the medium. Consequently, the rate of a reaction such as

$$\gamma \rightarrow \nu + \bar{\nu} \quad , \tag{A5}$$

which involves one external plasmon is proportional to $e^{-\hbar\omega_o/kT}$. Rates of such reactions are small for temperatures less than 10^9 °K because $\hbar\omega_o$ is of the order of 5 MeV at neutron-star densities.[27]

Reactions involving more than one neutrino are generally slow because of the small amount of phase space available to such processes. The amount of phase space available to a neutrino with energy less than kT is proportional to $(kT)^3$. Consequently, the rate of the reaction

$$p + \mu^- \rightarrow p + e^- + \bar{\nu}_e + \nu_\mu \quad , \tag{A6}$$

for example, is smaller than the rate of reaction (1) by a factor of the order of $\left[kT/E_F(n)\right]^2$.

More detailed work on processes involving electromagnetic interactions is now in progress,[28,29] but we have not yet found any such processes that are more important than reaction (1).

Turning to reactions involving quasi-free pions, we can use the arguments presented in the last few paragraphs to show that the following types of pion reactions are slower than reactions (3), (4), (7), and (8): the free decay of the pion $(\pi^- \rightarrow \mu^- + \bar{\nu}_\mu)$, reactions involving large numbers of fermions, positron processes, and pionic reactions involving more than one neutrino. The reaction

$$\pi^- + \pi^- \rightarrow \pi^- + \mu^- + \bar{\nu}_\mu \quad , \tag{A7}$$

however, might be faster than reactions (3), (4), (7), and (8) if the lowest quasi-free pion state has a momentum greater than about $\frac{1}{3} P_F(\mu)$; the energy and momentum of the lowest pion state are completely unknown. However, the question of whether reaction (A7) proceeds faster than reactions (3), (4), (7), and (8) is not particularly important, because reactions (3), (4), (7), and (8) alone would be sufficient to cause a neutron star containing quasi-free pions to cool too fast to allow it to be observed.

TABLE I.

Temperature parameter α ($= 10^{-2}\ T/T_e$). The values of α were obtained by interpolation of a table given by Tsuruta.[a]

T_{e7}	α
2.0	1.92
1.0	1.65
0.8	1.61
0.6	1.59
0.4	1.53
0.3	1.48
0.2	1.39
0.1	1.10

[a] S. Tsuruta, Ph.D. thesis, Columbia University (1964) (unpublished), p. 322.

TABLE II.

Possible distances and cooling times for the Scorpius source.

T_{e7}	r_{kpc}	$t_{cooling}$
2.0	2.3	5 hr
1.0	0.7	60 da
0.8	0.5	1 yr
0.6	0.3	8 yr
0.4	0.09	100 yr
0.3	0.03	800 yr
0.2	6×10^{-3}	10^{4} yr
0.1	4×10^{-5}	3×10^{5} yr

FIGURE CAPTIONS

Fig. 1: Several Feynman diagrams for the reaction $n + \pi^- \rightarrow n' + e^- + \bar{\nu}_e$.

Fig. 2: Cooling times calculated for a typical neutron star. The curves marked πn and nn were calculated assuming neutrino loss by the pion-nucleon and nucleon-nucleon reactions, respectively. The curve γ represents a star cooling by radiation from the surface only, and the curve $nn + \gamma$ gives the cooling time of a star emitting neutrinos from its interior by the nucleon-nucleon processes and radiating photons from its surface.

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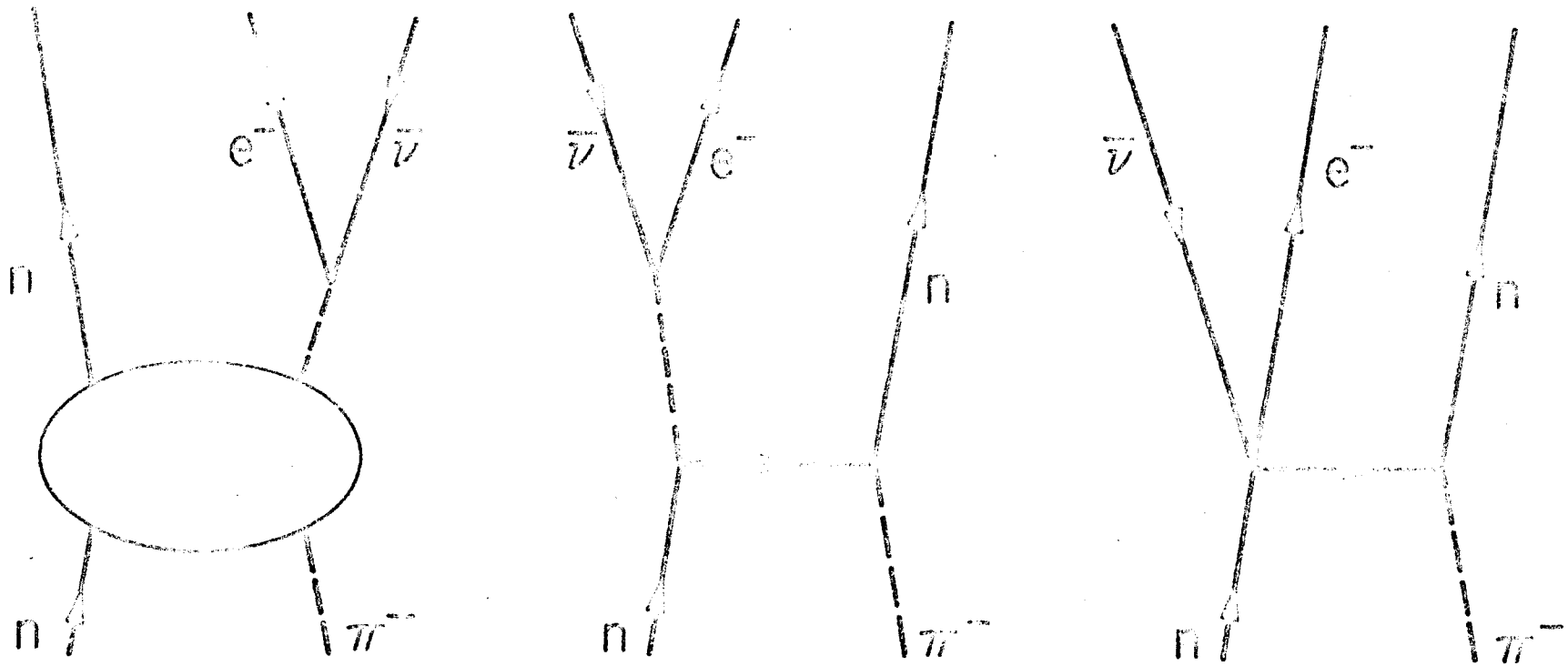
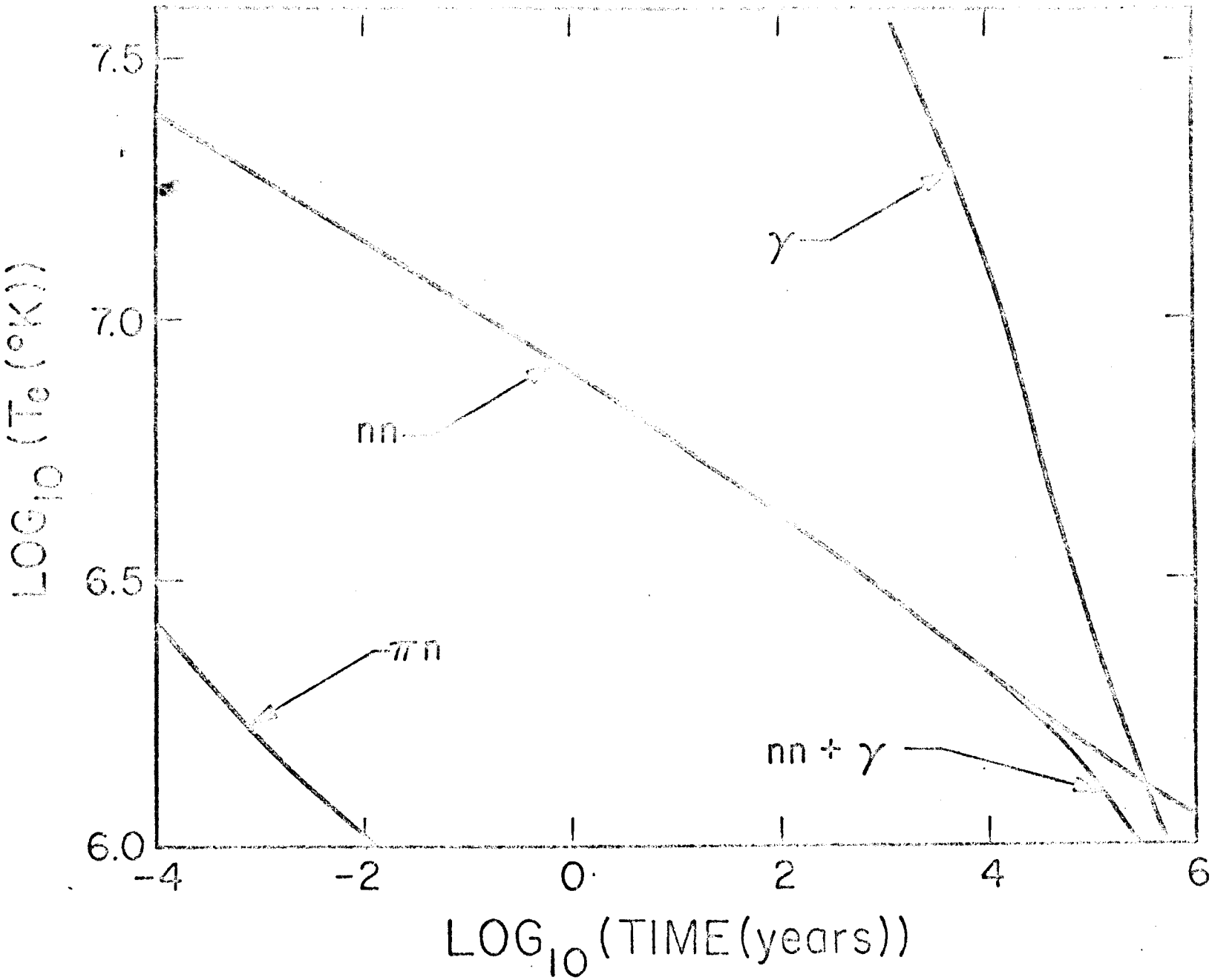


(a) (b) (c)


7.5
7.0
6.5
6.0
-4
-2
0
2
4
6
LOG_{10} (T_e (°K))
LOG_{10} (TIME (years))
γ
nn
π n
nn + γ