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A NEW APPROACH TO THE HOMOGENIZATION OF HETEROGENEOUS MEDIA FOR NEUTRON DIFFUSION CALCULATIONS

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I. INTRODUCTION

We shall develop, in this work, the mathematical formulation of a new method of treating neutron diffusion in certain types of heterogeneous media. The heterogeneity of immediate concern, and toward which this work is directed, is that of a regular array of vacuum channels (such as a square lattice of cylindrical holes) in an otherwise homogeneous medium. With modification of details, the general procedure should be applicable to other types of heterogeneity. However, a requirement which should be imposed is that the heterogeneity results in two characteristic directions. For example, in the case of a regular array of vacuum channels, the two characteristic directions are parallel and perpendicular to the channel axis.

Probably the most important considerations of neutron diffusion in media with holes are those of Behrens (B). However, all previous work, including (B), is devoted to the determination of the effects of heterogeneity on specific parameters relevant to neutron diffusion, rather than to a formulation of a general method from which various descriptions of the neutron distribution can be determined. The general problem may be stated along these lines: In media with heterogeneity, such as the type considered here, it is clear that a "simple homogenization" of the medium for neutron diffusion calculations is not a valid representation. We define a simple homogenization as the process of reducing the medium cross sections merely by the ratio of material volume to material-plus-vacuum volume. The streaming of neutrons in the vacuum channels leads to a spreading of the neutrons in the longitudinal direction (parallel to channel axis) which is larger than that in the transverse direction (perpendicular to channel axis). Thus, we find that not only is the simple homogenization questionable for omnidirectional parameter calculation, but also the anisotropic effects are completely subdued. Of course, the reason that we have for considering homogenization of the medium is the existence of an "arsenal" of possible mathematical attacks for such problems.

The new approach to homogenization, which we shall develop, is based on the reasoning that neutrons traveling with a large component of their velocity in the longitudinal direction probably travel further between collisions, on the average, than those traveling with a large component of velocity in the transverse direction. Let us introduce this effect into the neutron transport equation for

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the homogenized medium by allowing the mean free path to be angular-dependent. Clearly, by so doing, we have embarked on a mathematical fiction since the mean free path, as used in neutron transport calculations, is a local parameter. This concept is certainly no more confusing than the idea of medium homogenization. Let us stress that we are imposing a total cross section which varies according to the direction of neutron travel and not merely the usual scattering directional dependence.

All considerations will be based on the idealization of monoenergetic neutron transport. Although time-dependent equations will be developed, most of the analysis will be devoted to the calculation of stationary states. For the purpose of completeness, we shall develop the mathematical formulation along several lines. Specifically: We shall consider both the neutron flux and collison density as dependent variables. We shall apply the familiar P_N and double-P_N approximations, as well as the moment decomposition. We shall demonstrate that, for the case of isotropic scattering, the normal mode procedure, recently used for the solution of several types of neutron transport problems, is applicable and yields exact, closed-form solutions.

The major part of this work we direct toward the mathematical formulation and physical interpretation of the new theory. We shall, however, devote a section to brief remarks relevant to application. There is meager experimental data available for lattices of the type considered. It will therefore be difficult to evaluate the present theory. These considerations will yield a well-defined, albeit not well-substantiated, route to solution of problems involving neutron diffusion in media pierced by vacuum channels.

II. MATHEMATICAL FORMULATION

It would seem that application of these ideas to finite media dictates the use of non-separable position-angle-dependent mean free paths. The inclusion of position-dependence leads to gross difficulties which we have as yet not resolved. We shall assume that the mean free path in the homogenized medium depends only on the angle between the neutron velocity and the direction of the position variable. In all calculations we assume plane symmetry with the position variable either along the longitudinal or transverse direction.

2.1 The Neutron Flux Equation

The monoenergetic neutron transport equation for homogeneous media with plane symmetry may be written as

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(x,\mu,t) + \mu\frac{\partial}{\partial x}\psi(x,\mu,t) + \sigma(\mu)\,\psi(x,\mu,t)$$
$$= c\int \sigma(\mu')\,f(\underline{\Omega}\cdot\underline{\Omega}')\,\psi(x,\mu',t)\,d\underline{\Omega}' + S(x,\mu,t) \qquad (2.1.1)$$

where $\psi(x,\mu,t)$ is the neutron flux distribution as a function of position, x, direction cosine of neutron travel relative to x-direction, μ, and time, t; v is neutron speed; $\sigma(\mu)$ is the angular-dependent total cross section; c is the mean number of secondary neutrons which emanate from a neutron-nucleus collision; $f(\underline{\Omega}\cdot\underline{\Omega}')$ is the normalized distribution in $\underline{\Omega}$, the neutron post-collision direction of travel, of secondary neutrons produced by collision of a primary neutron with pre-collision direction $\underline{\Omega}'$; and, $S(x,\mu,t)$ is the rate of neutron introduction from sources which are independent of the neutron distribution. Although c and f are not necessarily descriptive of a non-multiplying medium, we shall use the terms scattering probability for c and scattering distribution for f to avoid stilted discourse. We employ an expansion of the scattering distribution in terms of Legendre polynomials $\left\{P_n(\underline{\Omega}\cdot\underline{\Omega}')\right\}$, i.e.,

$$f(\underline{\Omega}\cdot\underline{\Omega}') = \sum_n \frac{2n+1}{4\pi} f_n \; P_n(\underline{\Omega}\cdot\underline{\Omega}') \tag{2.1.2}$$

and the spherical harmonics addition theorem (H, p.143) to eliminate the azimuthal direction dependence appearing in the integral in Eq. (2.1.1). We obtain

$$\begin{aligned}\frac{1}{v}\frac{\partial}{\partial t}\psi(x,\mu,t) + \mu\frac{\partial}{\partial x}\psi(x,\mu,t) + \sigma(\mu)\,\psi(x,\mu,t) \\ = \frac{c}{2}\sum_n (2n+1)\, f_n\, P_n(\mu)\int_{-1}^{+1} P_n(\mu')\,\sigma(\mu')\,\psi(x,\mu',t)\,d\mu' \\ + S(x,\mu,t)\end{aligned} \tag{2.1.3}$$

In order to proceed with a general discussion of the properties of Eq. (2.1.3) we require a more specific representation of $\sigma(\mu)$. We suppose, with little loss in relevant generality, that

$$\sigma(\mu) = \sum_n \sigma_n \, P_n(\mu) \tag{2.1.4}$$

Let us further define the sets $\{\psi_n(x,t)\}$ and $\{S_n(x,t)\}$ by

$$\psi_n(x,t) = \int_{-1}^{+1} P_n(\mu) \, \psi(x,\mu,t) \, d\mu \tag{2.1.5a}$$

$$S_n(x,t) = \int_{-1}^{+1} P_n(\mu) \, S(x,\mu,t) \, d\mu \tag{2.1.5b}$$

Equations (2.1.5a) and (2.1.5b) specify the respective expansion coefficients in Legendre polynomial expansions of the neutron flux and source density, e.g.,

$$\psi(x,\mu,t) = \sum_n \frac{2n+1}{2} \, \psi_n(x,t) \, P_n(\mu) \tag{2.1.6}$$

In these terms, integration of Eq. (2.1.3) over the interval $\mu\epsilon(-1,+1)$ yields the relation

$$\frac{1}{v}\frac{\partial\psi_0}{\partial t} + \frac{\partial\psi_1}{\partial x} + (1 - c) \sum_n \sigma_n \, \psi_n = S_0 \tag{2.1.7}$$

Eq. (2.1.7) is the continuity equation for neutron motion. The only term which appears in an unfamiliar form is that which expresses the total neutron interaction rate. Clearly,

$$\sum_n \sigma_n \, \psi_n(x,t) = \int_{-1}^{+1} \sigma(\mu) \, \psi(x,\mu,t) \, d\mu \tag{2.1.8}$$

A further, familiar, reduction of the transport equation can be made in terms of the sets $\{\psi_n\}$, $\{S_n\}$, and $\{\sigma_n\}$. Using the recurrence relation for Legendre polynomials (H, p. 32),

$$(2n + 1)\ \mu\ P_n(\mu) = n\ P_{n-1}(\mu) + (n + 1)\ P_{n+1}(\mu) \tag{2.1.9}$$

yields the set of coupled differential equations,

$$\frac{1}{v}\frac{\partial}{\partial t}\psi_n(x,t) + \frac{n}{2n+1}\frac{\partial}{\partial x}\psi_{n-1}(x,t) + \frac{n+1}{2n+1}\frac{\partial}{\partial x}\psi_{n+1}(x,t)$$

$$+ \sum_{\ell,m} (2\ell + 1)\ \sigma_m\ A_{\ell mn}\ (1 - cf_n)\ \psi_\ell(x,t)$$

$$= S_n(x,t) \tag{2.1.10}$$

The set $\left\{A_{\ell mn}\right\}$ is defined by

$$A_{\ell mn} = \frac{1}{2}\int_{-1}^{+1} P_\ell(\mu)\ P_m(\mu)\ P_n(\mu)\ d\mu \tag{2.1.11}$$

and has the following properties (H, p. 87):

(i) The order of the indices is unimportant.

(ii) $A_{\ell mn} = 0$ if the sum of any two of the indices is less than the third.

(iii) $A_{\ell mn} = 0$ if $\ell + m + n$ is odd.

(iv) When $A_{\ell mn} \neq 0$, i.e., avoiding (ii) and (iii), we have

$$A_{\ell mn} = \left[\frac{1}{\ell + m + n + 1}\right]\left[\frac{(1)(3)\cdots(\ell + m - n - 1)}{(2)(4)\cdots(\ell + m - n)}\right]$$

$$\left[\frac{(1)(3)\cdots(\ell + n - m - 1)}{(2)(4)\cdots(\ell + n - m)}\right]\left[\frac{(1)(3)\cdots(n + m - \ell - 1)}{(2)(4)\cdots(n + m - \ell)}\right]$$

$$\left[\frac{(2)(4)\cdots(\ell + m + n)}{(1)(3)\cdots(\ell + m + n - 1)}\right] \tag{2.1.12}$$

It is from Eq. (2.1.10) that the various approximations to the transport equation are derived. Before we proceed with detailed examination of some approximations, let us reexamine Eq. (2.1.3). We obtain a simpler integral term if the dependent variable is changed to the neutron collision density,

$$F(x,\mu,t) = \sigma(\mu)\,\psi(x,\mu,t) \quad .$$

2.2 The Collision Density Equation

We reformulate Eq. (2.1.3) in terms of the collision density, $F(x,\mu,t)$, and the mean free path, $\lambda(\mu) = 1/\sigma(\mu)$, and obtain

$$\frac{\lambda(\mu)}{v}\frac{\partial}{\partial t} F(x,\mu,t) + \mu\,\lambda(\mu)\frac{\partial}{\partial x} F(x,\mu,t) + F(x,\mu,t)$$

$$= \frac{c}{2}\sum_n (2n + 1)\, f_n\, P_n(\mu)\int_{-1}^{+1} P_n(\mu')\, F(x,\mu',t)\, d\mu'$$

$$+ S(x,\mu,t) \tag{2.2.1}$$

With the sets $\{F_n(x,t)\}$ and $\{\lambda_n\}$ defined by

$$F(x,\mu,t) = \sum_n \frac{2n+1}{2} F_n(x,t)\; P_n(\mu) \tag{2.2.2a}$$

$$\lambda(\mu) = \sum_n \lambda_n\; P_n(\mu) \tag{2.2.2b}$$

we obtain the set of coupled differential equations (cf., Eq.(2.1.10)),

$$\begin{aligned}\sum_{\ell,m} (2\ell+1)\, A_{\ell mn} \frac{\lambda_m}{v}\frac{\partial}{\partial t} F_\ell(x,t) &+ \frac{n}{2n+1} \sum_{\ell,m} (2\ell+1) A_{\ell m,n-1} \lambda_m \frac{\partial}{\partial x} F_\ell(x,t) \\ &+ \frac{n+1}{2n+1} \sum_{\ell,m} (2\ell+1) A_{\ell m,n+1} \lambda_m \frac{\partial}{\partial x} F_\ell(x,t) \\ &+ (1 - c\, f_n)\, F_n(x,t) \\ &= S_n(x,t)\end{aligned} \tag{2.2.3}$$

We note that if $\lambda_n = 0$ for $n > 0$, i.e., the familiar case of an angle-independent mean free path, then Eq. (2.2.3), with the aid of the properties of $\{A_{\ell mn}\}$, reduces to

$$\begin{aligned}\frac{\lambda_0}{v}\frac{\partial F_n}{\partial t} + \frac{n\lambda_0}{2n+1}\frac{\partial F_{n-1}}{\partial t} + \frac{(n+1)\,\lambda_0}{2n+1}\frac{\partial F_{n+1}}{\partial t} &+ (1 - cf_n)\, F_n \\ &= S_n\end{aligned} \tag{2.2.4}$$

which is the expected result. We note further that the $n = 0$ member of Eq. (2.2.3) yields the continuity equation

$$\sum_m \lambda_m \frac{\partial F_m}{\partial t} + \sum_{\ell,m} (2\ell+1)\, \lambda_m\, A_{\ell m1} \frac{\partial F_\ell}{\partial x} + (1-c)\, F_0 = S_0 \tag{2.2.5}$$

In recognizing Eq. (2.2.5) as the continuity equation, we have used the easily derived relations

$$\psi_0(x,t) = \sum_n \lambda_n F_n(x,t) \tag{2.2.6a}$$

$$\psi_1(x,t) = \sum_{m,n} (2n+1) \lambda_m A_{nm1} F_n(x,t) \tag{2.2.6b}$$

In the remainder of this work we shall assume the case of a stationary state. In most cases this assumption merely leads to simpler algebra and notation and is actually not a requirement for the determination of a solution. We shall discuss the P_N-approximation and double-P_N-approximation as applied to both flux and collision density expansions. We shall consider the moment decomposition for both flux and collision density. And, finally, we shall consider, in greater depth, the case of isotropic scattering using the collision density as dependent variable.

2.3 The P_N-Approximation

We define the P_N-approximation based on a flux expansion, or collision density expansion, by the requirements that in Eq. (2.1.10), or Eq. (2.2.3), $\psi_n(x) = 0$ for $n > N$, or $F_n(x) = 0$ for $n > N$, and the equations labelled by $n > N$ are discarded. Thus, in a P_N-approximation we obtain $N + 1$ coupled differential equations with the $N + 1$ dependent variables $\psi_n(x)$, $n = 0, 1, \cdots, N$, or

$F_n(x)$, $n = 0, 1, \cdots, N$. For example, the P_o-approximation based on a flux expansion gives the single relation

$$\sigma_0 (1 - c)\, \psi_0(x) = 0 \tag{2.3.1}$$

which has the implication $c = 1$ for non-trivial ψ_0. This is a familiar result. The P_o-approximation based on a collision density expansion gives the unusual relation

$$\frac{\lambda_1}{3} \frac{dF_0}{dx} + (1 - c) F_o(x) = 0 \tag{2.3.2}$$

After this present brief comment we shall restrict our consideration to the case of $\lambda(\mu)$ a symmetric function of μ on the interval $\mu \epsilon (-1, +1)$, and, in that case $\lambda_1 = 0$. For the moment, let us suppose that $\lambda_1 \neq 0$. To be specific, we suppose that $\lambda_1 > 0$. According to Eq. (2.3.2), the implication is that $dF_o/dx < 0$ which indicates a neutron flow in the + x - direction. The question which we must pose is: Can F be μ-independent (consistent with P_o-approximation) and yet have $\psi(x,\mu)$ represent a neutron flow in + x - direction (i.e., ψ increase with μ)? Clearly the answer is in the affirmative since, if $\lambda(\mu)$ increases with μ (implied by $\lambda_1 > 0$), then the ratio ψ/λ can be μ-independent if ψ also increases with μ. In all following considerations we assume that $\lambda(\mu)$, and thus $\sigma(\mu)$, is a symmetric function of μ. Thus we shall always require $\lambda_n = 0$, and $\sigma_n = 0$, for n odd.

The P_1-approximation (i.e., diffusion theory) based on a flux expansion gives the two equations

$$\frac{d\psi_1}{dx} + (1 - c)\ \sigma_0\ \psi_0(x) = S_0(x) \tag{2.3.3a}$$

$$\frac{1}{3}\frac{d\psi_0}{dx} + (1 - cf_1)(\sigma_0 + \frac{2}{5}\sigma_2)\ \psi_1(x) = S_1(x) \tag{2.3.3b}$$

If it is further assumed that all neutron sources are isotropic such that $S_n(x) = 0$ for $n > 0$, then Eqs. (2.3.3a) and (2.3.3b) combine to give the usual diffusion theory relations

$$-D\frac{d^2\psi_0}{dx^2} + \sigma'\ \psi_0(x) = S_0(x) \tag{2.3.4a}$$

$$\psi_1(x) = -D\frac{d\psi_0}{dx} \tag{2.3.4b}$$

with diffusion coefficient, D, and "absorption" cross section, σ', given by

$$D = \frac{1}{3(1 - cf_1)(\sigma_0 + 2\sigma_2/5)} \tag{2.3.5a}$$

$$\sigma' = (1 - c)\ \sigma_0 \tag{2.3.5b}$$

It should be noted that only σ_0 and σ_2 enter into the diffusion theory parameters. The $\sigma(\mu)$ expansion was not truncated at a quadratic in order to obtain Eqs. (2.3.5a) and (2.3.5b). In passing, it is also worth noting that, had we included time-dependence, the

P_1-approximation would have given the "telegraphist's equation." The added assumption that $\partial\psi_1/v\,\partial t << \partial\psi_0/\partial x$ results in the form of the familiar time-dependent diffusion theory with D and σ' again given by Eqs. (2.3.5a) and (2.3.5b).

The P_1-approximation based on a collision density expansion gives the two equations

$$(\lambda_0 + \frac{2}{5}\lambda_2)\frac{dF_1}{dx} + (1 - c)\,F_0(x) = S_0(x) \tag{2.3.6a}$$

$$\frac{1}{3}(\lambda_0 + \frac{2}{5}\lambda_2)\frac{dF_0}{dx} + (1 - cf_1)\,F_1(x) = S_1(x) \tag{2.3.6b}$$

In the case of isotropic sources, Eqs. (2.3.6a) and (2.3.6b) combine to give

$$-D'\frac{d^2F_0}{dx^2} + (1 - c)\,F_0(x) = S_0(x) \tag{2.3.7a}$$

$$F_1(x) = -\frac{D'}{(\lambda_0 + 2\lambda_2/5)}\frac{dF_0}{dx} \tag{2.3.7b}$$

where the "diffusion coefficient" is

$$D' = \frac{\left(\lambda_0 + 2\lambda_2/5\right)^2}{3(1 - cf_1)} \tag{2.3.8}$$

As is the case in the flux based diffusion theory, a quadratic truncation of $\lambda(\mu)$ is not necessary to obtain the results given in Eqs. (2.3.7) and (2.3.8).

The usual result of exponential spatial decline away from sources in infinite media is found for flux and collision density. The "period" of the exponential, the diffusion length L, is given by

$$L_\psi^2 = \frac{1}{3\sigma_0(1 - c)(1 - cf_1)(\sigma_0 + 2\sigma_2/5)} \tag{2.3.9}$$

based on a flux expansion, and

$$L_F^2 = \frac{(\lambda_0 + 2\lambda_2/5)^2}{3(1 - c)(1 - cf_1)} \tag{2.3.10}$$

based on a collision density expansion. We have pointed out the dissimilar results obtained from a P_0-approximation based on flux and collision density expansions. We can illustrate the divergence of results for the P_1-approximation by considering the ratio L_ψ/L_F for a given problem. Let us suppose that $\lambda(\mu)$ is actually an even quadratic, i.e., $\lambda_n = 0$ for $n = 1$ and $n > 2$. The corresponding $\sigma(\mu)$ is not a quadratic, but we need compute only σ_0 and σ_2 since these coefficients determine L_ψ. In Figure 1 we display the results for the range $\lambda_2/\lambda_0 \epsilon (0,2)$. Clearly, the flux and collision density based expansions can lead to significantly different results.

It should be expected that the two diffusion theories give different results. We note that whereas the ψ-approximation yields an accurate representation for neutron current, $\psi_1(x)$, but an inaccurate (truncated) representation for collision density, the F-approximation results in an accurate representation for total interaction rate,

$F_0(x)$, but an inaccurate representation for current. For several reasons it will seem that we favor the collision density expansion in this work. However, these reasons are mainly of an algebraic character and it should not be construed that the F-formulation is, in all cases, superior.

Approximations of higher order than P_1 are accomplished following the usual general prescriptions. The added complications due to the angular-dependence of the mean free path place no restrictions on the formalism. Higher order approximations lessen the differences exhibited by the ψ and F-formulations.

2.4 The Double-P_N-Approximation

The double-P_N-approximation is derived from Yvon's method whereby the angular-dependence of the neutron flux, or collision density, is decomposed into contributions from + x - directed and from - x - directed neutrons. In order to simplify notation, let us consider the case of a stationary state in a medium characterized by isotropic scattering. The plane symmetry, monoenergetic neutron transport equation, Eq. (2.1.3), is then

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \sigma(\mu)\, \psi(x,\mu) = \frac{c}{2}\int_{-1}^{+1} \sigma(\mu')\, \psi(x,\mu')\, d\mu' + S(x,\mu) \tag{2.4.1}$$

We use the half-angle-range expansions

$$\psi(x,\mu) = \sum_n (2n + 1)\, \psi_n^+(x)\, P_n(2\mu - 1), \qquad \mu > 0 \tag{2.4.2a}$$

$$= \sum_n (2n + 1)\, \psi_n^-(x)\, P_n(2\mu + 1), \qquad \mu < 0 \tag{2.4.2b}$$

$$S(x,\mu) = \sum_n (2n + 1)\, S_n^+(x)\, P_n(2\mu - 1), \qquad \mu > 0 \tag{2.4.2c}$$

$$= \sum_n (2n + 1)\, S_n^-(x)\, P_n(2\mu + 1), \qquad \mu < 0 \tag{2.4.2d}$$

$$\sigma(\mu) = \sum_n \sigma_n^+\, P_n(2\mu - 1), \qquad \mu > 0 \tag{2.4.2e}$$

$$= \sum_n \sigma_n^-\, P_n(2\mu + 1), \qquad \mu < 0 \tag{2.4.2f}$$

to obtain the set of coupled differential equations

$$\frac{n}{2n+1}\frac{d\psi_{n-1}^{\pm}}{dx} + \frac{n+1}{2n+1}\frac{d\psi_{n+1}^{\pm}}{dx} \pm \frac{d\psi_n^{\pm}}{dx} + 2\sum_{\ell,m}(2\ell + 1)\, A_{\ell mn}\, \sigma_m^{\pm}\, \psi_\ell^{\pm}$$

$$= c\, \delta_{no} \sum_\ell (\sigma_\ell^- \psi_\ell^- + \sigma_\ell^+ \psi_\ell^+) + 2S_n^{\pm} \tag{2.4.3}$$

The double-P_N-approximation is defined by the requirement $\psi_n^{\pm}(x) = 0$ for $n > N$ and the equations labelled by $n > N$ are discarded.

The same analysis is followed for the collision density based approximation. Thus, with the definitions

$$F(x,\mu) = \sum_n (2n+1)\, F_n^+(x)\, P_n(2\mu - 1), \qquad \mu > 0 \tag{2.4.4a}$$

$$= \sum_n (2n+1)\, F_n^-(x)\, P_n(2\mu + 1), \qquad \mu < 0 \tag{2.4.4b}$$

$$\lambda(\mu) = \sum_n \lambda_n^+ P_n(2\mu - 1), \qquad \mu > 0 \tag{2.4.4c}$$

$$= \sum_n \lambda_n^- P_n(2\mu + 1), \qquad \mu < 0 \tag{2.4.4d}$$

we obtain the transport equation in the form

$$\frac{n}{2n+1} \sum_{\ell,m} (2\ell + 1)\, \lambda_m^\pm A_{\ell m,n-1} \frac{dF_\ell^\pm}{dx}$$

$$+ \frac{n+1}{2n+1} \sum_{\ell,m} (2i + 1)\, \lambda_m^\pm A_{m,n+1} \frac{dF_\ell^\pm}{dx}$$

$$\pm \sum_{\ell,m} (2\ell + 1)\, \lambda_m^\pm A_{\ell mn} \frac{dF_\ell^\pm}{dx} + 2F_n^\pm$$

$$= c\, \delta_{n0} (F_0^+ + F_0^-) + 2S_n^\pm \tag{2.4.5}$$

The collision density double-P_N-approximation is defined as in the ψ-formulation.

Besides the usual usefulness of Yvon's method in the solution of problems where accurate representation of source or free boundaries are required, we find an added flexibility for the angular-dependence of the mean free path. Thus, $\lambda(\mu)$ may have one form for $\mu > 0$ determined by the set $\left\{\lambda_n^+\right\}$, or $\left\{\sigma_n^+\right\}$, and another form for $\mu < 0$ following the set $\left\{\lambda_n^-\right\}$, or $\left\{\sigma_n^-\right\}$. For example, using these methods we can express a symmetric $\lambda(\mu)$ which varies linearly with μ for $\mu > 0$ and $\mu < 0$ by using only two terms in each $\lambda(\mu)$ expansion, i.e., set $\lambda_0^+ = \lambda_0^-$, $\lambda_1^+ = -\lambda_1^-$, and $\lambda_n^{\pm} = 0$ for $n > 1$.

2.5 A Moment Decomposition

In the usual theory of neutron transport through homogeneous media, it is well-known that any space-angle moment of the neutron distribution can be found even though the distribution itself is unknown. In fact, an important method of determining the neutron distribution is the construction of a likely flux shape from a finite set of moments. Let us now consider a moment decomposition for the case of an angular-dependent mean free path where, as in our previous considerations, two formulations will be examined, i.e., with ψ and F as dependent variable.

We define the neutron flux and source moments by

$$\psi_n^j = \int_{-\infty}^{+\infty} x^j \, \psi_n(x) \, dx \tag{2.5.1a}$$

$$S_n^j = \int_{-\infty}^{+\infty} x^j \, S_n(x) \, dx \tag{2.5.1b}$$

We assume that the medium is of infinite extent. Multiplication of the stationary state form of Eq. (2.1.10) by x^j and integration over $x \in (-\infty, +\infty)$ yields the set of algebraic moment relations

$$\sum_{\ell,m} (2\ell + 1)\, \sigma_m\, A_{\ell mn}\, (1 - cf_n)\, \psi_\ell^j = S_n^j + \frac{j}{2n + 1}\left[n\psi_{n-1}^{j-1} + (n + 1)\psi_{n+1}^{j-1}\right] \tag{2.5.2}$$

With the collision density moments similiarily defined, i.e.,

$$F_n^j = \int_{-\infty}^{+\infty} x^j\, F_n(x)\, dx \tag{2.5.3}$$

and performing similar operations on the stationary state form of Eq. (2.2.3), we find a set of algebraic equations relating the collision density moments, i.e.,

$$\begin{aligned}(1 - cf_n)\, F_n^j = S_n^j &+ \frac{jn}{2n + 1} \sum_{\ell,m} (2\ell + 1)\, \lambda_m\, A_{\ell m,n-1} F_\ell^{j-1} \\ &+ \frac{j(n + 1)}{2n + 1} \sum_{\ell,m} (2\ell + 1)\, \lambda_m\, A_{\ell m,n+1} F_\ell^{j-1}\end{aligned} \tag{2.5.4}$$

The moments of the neutron distribution resulting from a unit, plane, isotropic source (at $x = 0$) are easily interpretable in terms of important macroscopic parameters. For this source, $S_n^j = \delta_{n0}\, \delta_{j0}$. Let us first consider calculation of flux moments and then contrast this result with the method applied to collision density moments. For the sake of definiteness, let us assume that $\sigma(\mu)$ is an even M degree polynomial in μ. With $\sigma(\mu)$ an even function, it is clear that

$\psi_n^j = 0$ for $j + n$ odd. An examination of Eq. (2.5.2), using the properties of the set $\{A_{\ell mn}\}$, indicates that with $M \neq 0$ and $n + j$ even, ψ_n^j depends on ψ_{n-1}^{j-1}, ψ_{n+1}^{j-1}, $\psi_{|n-M|}^j$, $\psi_{|n-M|+2}^j$, ..., ψ_{n+M-2}^j, ψ_{n+M}^j.

Thus, we find that we cannot determine ψ_n^j without employing a truncation on the set $\{\psi_n(x)\}$. This is, of course, the simplification used in a P_N-approximation and would not yield exact moments. The familiar case of $M = 0$ poses no such difficulties and one can readily find the exact moments.

In considering the collision density moments, let us assume that $\lambda(\mu)$ is an even M degree polynomial in μ. It follows that $F_n^j = 0$ for $n + j$ odd. It is also possible to show, from Eq. (2.5.4) and the properties of $\{A_{\ell mn}\}$, that $F_n^j = 0$ for $n > j(M + 1)$ and therefore, with finite M, we can calculate any collision density moment exactly. For example, we find for the case of $M = 2$,

$$\langle x^2 \rangle = \frac{F_0^2}{F_0^0} = 2L_F^2 + \frac{18}{175} \frac{\lambda_2^2}{(1 - c)(1 - cf_3)} \tag{2.5.5}$$

where $\langle x^2 \rangle$ is the mean value of x^2 and L_F is the diffusion length as calculated by a collision density based P_1-approximation. We note that the result of Eq. (2.5.5) is unlike that found in the angle-independent mean free path case. In that case $M = 0$ and we find that the second spatial moment is given correctly by diffusion theory, i.e., $\langle x^2 \rangle = 2L^2$ both in the exact calculation and in diffusion theory.

In passing, we note that the set $\left\{\psi_n^j\right\}$ can be found from the set $\left\{F_n^j\right\}$ via the easily derived relation

$$\psi_n^j = \sum_{\ell,m} (2\ell + 1)\, \lambda_m\, A_{\ell mn}\, F_\ell^j \tag{2.5.6}$$

The result of Eq. (2.5.6) does not contradict our earlier assertion regarding the problem of finding the flux moments. When the determination of the $\left\{\psi_n^j\right\}$ is approached directly by Eq. (2.5.2), it is the total cross section which is considered to be an M degree polynomial in μ. When $\left\{\psi_n^j\right\}$ is found using $\left\{F_n^j\right\}$ as in Eq. (2.5.6), the mean free path is assumed to be an M degree polynomial in μ.

2.6 The Case of Isotropic Scattering

The stationary state form of Eq. (2.2.1) with the additional assumption of isotropic scattering gives the collision density equation

$$\mu\, \lambda(\mu)\, \frac{\partial}{\partial x} F(x,\mu) + F(x,\mu) = \frac{c}{2} \int_{-1}^{+1} F(x,\mu')\, d\mu' + S(x,\mu) \tag{2.6.1}$$

In this case of isotropic scattering an angle variable change is suggested. Specifically, let us define the angle variable $u = \mu\lambda(\mu)/\lambda(1)$, measure x in units of $\lambda(1)$, and change dependent variable to $F(x,u) = g(u)\, F(x,\mu)$ with $g(u) = |d\mu/du|$. With the source density change $S(x,u) = g(u)\, S(x,\mu)$, Eq. (2.6.1) takes the form

$$u \frac{\partial}{\partial x} F(x,u) + F(x,u) = \frac{c}{2}\, g(u) \int_{-1}^{+1} F(x,u')\, du' + S(x,u) \tag{2.6.2}$$

We shall consider various aspects of the solution of Eq. (2.6.2).

Legendre Polynomial Expansion

Following the procedure used in the derivation of Eq. (2.1.10), we obtain the coupled differential equation form of the transport equation

$$\frac{n}{2n+1}\frac{dF_{n-1}}{dx} + \frac{n+1}{2n+1}\frac{dF_{n+1}}{dx} + F_n(x) = \frac{c\, g_n}{2n+1} F_0(x) + S_n(x) \qquad (2.6.3)$$

where we have used the Legendre polynomial expansions

$$F(x,u) = \sum_n \frac{2n+1}{2} F_n(x)\ P_n(u) \qquad (2.6.4a)$$

$$S(x,u) = \sum_n \frac{2n+1}{2} S_n(x)\ P_n(u) \qquad (2.6.4b)$$

$$g(u) = \sum_n g_n\ P_n(u) \qquad (2.6.4c)$$

It should be noted that the requirement of a symmetric $\lambda(\mu)$ imposes the condition that $g_n = 0$ for n odd.

The idea of a P_N-approximation is equally well-applied here. For example, the P_1-approximation (diffusion theory) takes the usual form (cf., Eq. (2.3.7))

$$-\frac{1}{3}\frac{d^2F_0}{dx^2} + (1 - cg_0)\, F_0 = S_0 \tag{2.6.5a}$$

$$F_1 = -\frac{1}{3}\frac{dF_0}{dx} \tag{2.6.5b}$$

where only isotropic sources are allowed.

Moment Decomposition

With the collision density and source moments defined in the usual manner (i.e., by Eq. (2.5.1)), Eq. (2.6.3) can be transformed to the algebraic set

$$F_n^j = S_n^j + \frac{c\ g_n}{2n + 1} F_0^j$$
$$+ \frac{j}{2n + 1}\left[n\ F_{n-1}^{j-1} + (n + 1)\ F_{n+1}^{j-1}\right] \tag{2.6.6}$$

We have assumed that $\lambda(\mu)$ is symmetric which implies that $g(u)$ is symmetric. We also find that $F_n^j = 0$ for $j + n$ odd, and, for odd n, F_n^j depends only on F_{n-1}^{j-1} and F_{n+1}^{j-1}. Furthermore, we have the interesting property that the spatial moment F_0^j depends only on the set of moments $\left\{F_n^i\ ,\ n + i \leq j\right\}$. Therefore, the calculations of a low-order spatial moment requires the specification of a small number of the g_n and the prior determination of a small number of other moments.

As an example, let us calculate, by these methods, the second spatial moment of the neutron distribution resulting from a unit, plane, isotropic source (at x = 0). In this case $S_n^j = \delta_{n0}\,\delta_{j0}$. The moments F_0^0, F_2^0, and F_1^1 are easily determined and are the only values required in the calculation of F_0^2. We find, for the normalized second spatial moment (cf., Eq. (2.5.5)),

$$\langle x^2 \rangle = \frac{F_0^2}{F_0^0} = \frac{2}{3}\;\frac{1 + 2c\; g_2/5}{1 - c\; g_0} \tag{2.6.7}$$

Normal Mode Expansion

We shall apply the recently developed normal mode technique (C) to the problem of determining the exact and asymptotic solution of Eq. (2.6.2). In so doing we shall arrive at an interesting mathematical problem the details of which we consider in Appendix B. We consider the homogeneous form of Eq. (2.6.2), i.e., S = 0. Translational invariance suggests the "ansatz"

$$F(x,u) = \phi\,(\nu,u)\,\exp(-x/\nu) \tag{2.6.8}$$

where we allow the separation variable, ν, to be complex. We obtain the integral equation

$$(\nu - u)\;\phi(\nu,u) = \frac{c}{2}\,\nu\;g(u)\int_{-1}^{+1}\phi(\nu,u')\,du' \tag{2.6.9}$$

We adopt the normalization

$$\int_{-1}^{+1}\phi(\nu,u)\,du = 1 \tag{2.6.10}$$

If we allow solutions of Eq. (2.6.9) to be distributions (in the sense of L. Schwartz (S)), then we have

$$\phi(\nu,u) = \frac{c}{2}\,\frac{\nu\, g(u)}{\nu - u} + \Lambda(\nu)\,\delta(\nu - u) \tag{2.6.11}$$

We assume that g(u) satisfies a Holder condition (M, p. 11) on the interval of the real line $u \epsilon (-1, +1)$. Any singular integrals which might appear are then of the Cauchy type and we define their evaluation as the Cauchy principal value (M, p. 26).

The normalization required of $\phi(\nu,u)$, i.e., Eq. (2.6.10), leads to a specification of allowed discrete values of ν in the region of the ν-complex-plane excluding the line (-1, +1), and to a specification of the function $\Lambda(\nu)$ for $\nu \epsilon (-1, +1)$. To aid in the analysis of these results, let us define the Cauchy integral, $G(\nu)$, by

$$G(\nu) = \frac{1}{2\pi\, i}\int_{-1}^{+1} \frac{g(u)}{u - \nu}\, du \tag{2.6.12}$$

With $\nu \notin (-1, +1)$, we find that Eq. (2.6.10) gives

$$1 + i\,\pi\, c\,\nu\, G(\nu) = 0 \tag{2.6.13}$$

which has a set of roots which are distinct. With $\nu \in (-1, +1)$, Eq. (2.6.10) yields an explicit formula for the function $\Lambda(\nu)$ and no restrictions are placed on allowed values of ν. We find

$$\Lambda(\nu) = 1 + i\,\pi\, c\,\nu\, G(\nu) \tag{2.6.14}$$

Thus, if we extend the definition of $\Lambda(\nu)$, as expressed in Eq. (2.6.14), to the entire ν-plane, we find that the zeroes of $\Lambda(\nu)$ determine the set of allowed distinct values of ν. Since $g(u)$ is symmetric, $G(-\nu) = -G(\nu)$, whence, $\Lambda(\nu)$ is an even function of ν. The zeroes of $\Lambda(\nu)$, therfore, appear in pairs which we label $\pm\nu_j$.

We have found a set of functions of the angle variable u indexed by ν, $\{\phi(\nu,u)\}$. There is a discrete indexed set with $\nu \notin (-1, +1)$ and members characterized by

$$\phi(\pm\nu_j,u) = \frac{c}{2}\,\frac{\nu_j\, g(u)}{\nu_j \mp u} \tag{2.6.15}$$

and, a continuous indexed set with $\nu \in (-1, +1)$ and of form given by Eq. (2.6.11). The function $\Lambda(\nu)$, which appears in $\phi(\nu,u)$ for $\nu \in (-1,+1)$, is given by Eq. (2.6.14). Furthermore, the zeroes of $\Lambda(\nu)$ for $\nu \notin (-1, +1)$ establish the set of discrete indices $\{\pm\nu_j\}$.

If we assume that $g(u) \neq 0$ for $u \in (-1, +1)$, we may write Eq. (2.6.9), with the normalization of Eq. (2.6.10), in the form

$$\left[1 - \frac{u}{\nu}\right] \frac{\phi(\nu,u)}{g(u)} = \frac{c}{2} \tag{2.6.16}$$

Let us multiply Eq. (2.6.16) with index ν by $\phi(\nu',u)$ and subtract the result from Eq. (2.6.16) with index ν' multiplied by $\phi(\nu,u)$. Employing Eq. (2.6.10) and integrating over $u \in (-1, +1)$ we obtain

$$\left[\frac{1}{\nu} - \frac{1}{\nu'}\right] \int_{-1}^{+1} \frac{u}{g(u)}\, \phi(\nu,u)\, \phi(\nu',u)\, du = 0 \tag{2.6.17}$$

There is clearly no degeneracy and thus Eq. (2.6.17) may be rewritten as the orthogonality relation

$$\int_{-1}^{+1} \frac{u}{g(u)} \phi(\nu,u)\ \phi(\nu',u)\ du = 0 \text{ for } \nu \neq \nu' \tag{2.6.18}$$

The nature of the orthogonality relation including the case $\nu = \nu'$ depends on whether ν is a member of the discrete index set or belongs to the continuum. If ν is a discrete index, then

$$\int_{-1}^{+1} \frac{u}{g(u)} \phi(\pm\nu_j,u)\ \phi(\pm\nu_i,u)\ du = I(\pm\nu_j)\ \delta_{ji} \tag{2.6.19a}$$

$$I(\pm\nu_j) = \frac{c^2}{4}\ \nu_j^2 \int_{-1}^{+1} \frac{u\ g(u)}{(\nu_j \mp u)^2}\ du \tag{2.6.19b}$$

If ν belongs to the continuum, then

$$\int_{-1}^{+1} \frac{u}{g(u)} \phi(\nu,u)\ \phi(\nu',u)\ du = \frac{\nu\ \Lambda^2(\nu)}{g(\nu)}\ \delta(\nu - \nu') \tag{2.6.20}$$

We have found that the set of normal modes, $\{\phi(\nu,u)\}$, is orthogonal, with weight function $u/g(u)$, on the interval $u \in (-1, +1)$. For the remainder of this section we shall assume that the normal modes are also complete in the space of functions which satisfy a Holder condition on the interval $u \in (-1, +1)$. In Appendix B we shall, in measure, substantiate this hypothesis by demonstrating the existence of the modal expansion coefficients. In so doing, we shall generalize the interval of completeness to all physically relevant cases.

Assuming that the normal modes form a complete set on the interval $u \in (-1, +1)$, we have the general solution of Eq. (2.6.2) in the form

$$F(x,u) = \sum_{\nu} a(\nu)\ \phi(\nu,u)\ \exp(-x/\nu) \tag{2.6.21}$$

where the summation indicates integration over continuous spectra when applicable. In many problems we find boundary conditions which can be formulated as

$$F(o,u) = \phi(u) = \sum_{\nu} a(\nu)\ \phi(\nu,u) \text{ for } u \in (-1, +1) \tag{2.6.22}$$

and we can use the orthogonality relations to determine the expansion coefficients, $a(\nu)$. In detail, Eq. (2.6.22) is rewritten as

$$\begin{aligned}\phi(u) = &\sum_{j} a(+\nu_j)\ \phi(+\nu_j,u) + \sum_{j} a(-\nu_j)\ \phi(-\nu_j,u)\\ &+ \int_{-1}^{+1} a(\nu)\ \phi(\nu,u)\ d\nu \text{ for } u \in (-1, +1)\end{aligned} \tag{2.6.23}$$

Direct application of the discrete index orthogonality relation, Eq. (2.6.19), yields the discrete indexed expansion coefficients,

$$a(\pm\nu_j) = \frac{1}{I(\pm\nu_j)} \int_{-1}^{+1} \frac{u}{g(u)}\ \phi(u)\ \phi(\pm\nu_j,u)\ du \tag{2.6.24}$$

Using Eq. (2.6.18) we obtain, from Eq. (2.6.23),

$$\int_{-1}^{+1} \frac{u}{g(u)}\ \phi(u)\ \phi(\nu,u)\ du = \int_{-1}^{+1} du\ \frac{u}{g(u)}\ \phi(\nu,u) \int_{-1}^{+1} a(\nu')\phi(\nu',u)d\nu' \tag{2.6.25}$$

There appears a doubly Cauchy singular integral and thus the order of integration in Eq. (2.6.25) may not be reversed without due caution. The doubly singular term appears as

$$\frac{c^2}{4}\nu\int_{-1}^{+1} du\, \frac{u\, g(u)}{\nu - u}\int_{-1}^{+1} \frac{\nu'\, a(\nu')}{\nu' - u}\, d\nu'$$

We assume that $a(\nu)$ satisfies a Holder condition for $\nu \in (-1, +1)$ and follow the dictates of the Poincaré-Bertrand formula for inverting the integration order (M, p. 57). We find

$$\frac{c^2}{4}\nu\int_{-1}^{+1} du\, \frac{u\, g(u)}{\nu - u}\int_{-1}^{+1} \frac{\nu'\, a(\nu')}{\nu' - u}\, d\nu' = \pi^2\, \frac{c^2}{4}\, \nu^3\, g(\nu)\, a(\nu)$$

$$+ \frac{c^2}{4}\nu\int_{-1}^{+1} d\nu'\, \frac{\nu'\, a(\nu')}{\nu' - u}\int_{-1}^{+1} \frac{u\, g(u)}{\nu - u}\, du \qquad (2.6.26)$$

Using Eqs. (2.6.20) and (2.6.26) we obtain the more useful "orthogonality relation,"

$$\int_{-1}^{+1} du\, \frac{u}{g(u)}\, \phi(\nu,u)\int_{-1}^{+1} a(\nu')\, \phi(\nu',u)\, d\nu' = I(\nu)\, a(\nu) \text{ for } \nu \in (-1, +1) \qquad (2.6.27a)$$

$$I(\nu) = \nu\, g(\nu)\left[\left(\frac{\Lambda(\nu)}{g(\nu)}\right)^2 + \left(\frac{\pi\, c\, \nu}{2}\right)^2\right] \qquad (2.6.27b)$$

Now, applying Eq. (2.6.27) to the problem of finding the continuum expansion coefficients in Eq. (2.6.23), we have

$$a(\nu) = \frac{1}{I(\nu)}\int_{-1}^{+1} \frac{u}{g(u)}\, \phi(u)\, \phi(\nu,u)\, du$$

$$\text{for } \nu \in (-1, +1) \qquad (2.6.28)$$

2.7 The Green's Function for the Case of Isotropic Scattering.

As a specific example of the use of the relations just developed, let us consider the problem of finding the infinite medium Green's function for isotropic, plane sources. In this case, the source density, $S(x,u)$, of Eq. (2.6.2) represents a unit, plane, isotropic emission of neutrons at a position which we choose to designate $x = 0$, i.e., $S(x,u) = g(u)\, \delta(x)/2$. Integration of Eq. (2.6.2) over a vanishing interval about $x = 0$ yields the boundary condition

$$u\left[F(0^+,u) - F(0^-,u)\right] = g(u)/2 \text{ for } u \in (-1, +1) \tag{2.7.1}$$

We impose the additional condition that as $|x| \to \infty$, $F(x,u) \to 0$ and express the solution in the form

$$F(x,u) = \sum_j a(+v_j)\, \phi(+v_j,u) \exp(-x/v_j) + \int_0^{+1} a(v)\, \phi(v,u) \exp(-x/v)\, dv \text{ for } x > 0 \tag{2.7.2a}$$

$$F(x,u) = -\sum_j a(-v_j)\, \phi(-v_j,u) \exp(x/v_j) - \int_{-1}^{0} a(v)\, \phi(v,u) \exp(-x/v)\, dv \text{ for } x < 0 \tag{2.7.2b}$$

The source condition, Eq. (2.7.1), then takes the form of the general boundary condition, Eq. (2.6.23). Specifically, we have

$$\frac{u}{g(u)} \sum_j a(+v_j)\phi(+v_j,u) + \frac{u}{g(u)} \sum_j a(-v_j)\, \phi(-v_j,u) + \frac{u}{g(u)} \int_{-1}^{+1} a(v)\phi(v,u)dv = \frac{1}{2} \tag{2.7.3}$$

Whence, employing the normalization expressed by Eq. (2.6.10),

$$a(\pm\nu_j) = \frac{1}{2I(\pm\nu_j)} \tag{2.7.4a}$$

$$a(\nu) = \frac{1}{2I(\nu)} \quad \text{for } \nu \in (-1, +1) \tag{2.7.4b}$$

and we have completed the solution of the Green's function.

Let us examine some aspects of the Green's function. For simplicity, we assume that there is only one pair of zeroes of $\Lambda(\nu)$, $\pm\nu_0$. We shall develop a sufficient condition for this property in Appendix A. The Green's function is then given by

$$F(x,u) = \frac{\phi(+\nu_0,u)}{2I(+\nu_0)} \exp(-x/\nu_0) + \int_0^{+1} \frac{\phi(\nu,u)}{2I(\nu)} \exp(-x/\nu)\, d\nu \quad \text{for } x > 0 \tag{2.7.5a}$$

$$F(x,u) = -\frac{\phi(-\nu_0,u)}{2I(-\nu_0)} \exp(x/\nu_0) - \int_{-1}^{0} \frac{\phi(\nu,u)}{2I(\nu)} \exp(-x/\nu)\, d\nu \quad \text{for } x < 0 \tag{2.7.5b}$$

With the definition

$$\phi_n(\nu) = \int_{-1}^{+1} \phi(\nu,u)\, P_n(u)\, du \tag{2.7.6}$$

and the easily derived symmetry properties; $I(-\nu_0) = -I(+\nu_0)$ and $I(-\nu) = -I(\nu)$, we find the collision density moments for the neutron distribution from a unit, plane, isotropic source,

$$F_n^j = \frac{j!}{2}\left[\phi_n(+\nu_0) + (-1)^j\, \phi_n(-\nu_0)\right] \frac{\nu_0^{j+1}}{I(+\nu_0)} + \frac{j!}{2}\int_0^{+1} \left[\phi_n(\nu) + (-1)^j\, \phi_n(-\nu)\right] \frac{\nu^{j+1}}{I(\nu)}\, d\nu \tag{2.7.7}$$

Using Eqs. (2.1.9), (2.6.4c), (2.6.9) and (2.6.10), we obtain a recurrence relation for the set $\left\{\phi_n(\nu)\right\}$,

$$(2n + 1)\, \nu\, \phi_n(\nu) = n\, \phi_{n-1}(\nu) + (n + 1)\, \phi_{n+1}(\nu) + c\, \nu\, g_n \tag{2.7.8}$$

and the normalization $\phi_0(\nu) = 1$. We note that Eq. (2.7.8) implies that $\phi_n(\nu)$ is an even, or odd, polynomial in ν of degree n. Therefore, we have $\phi_n(-\nu) = (-1)^n\, \phi_n(+\nu)$, and Eq. (2.7.7) reduces to

$$F_n^j = j! \left[\frac{\nu_0^{j+1}}{I(+\nu_0)}\, \phi_n(+\nu_0) + \int_0^{+1} \frac{\nu^{j+1}}{I(\nu)}\, \phi_n(\nu)\, d\nu \right] \qquad \text{if } j + n \text{ is even} \tag{2.7.9a}$$

$$F_n^j = 0 \qquad \text{if } j + n \text{ is odd} \tag{2.7.9b}$$

We have already considered the moments set $\left\{F_n^j\right\}$. For this particular case the $\left\{F_n^j\right\}$ is determined from Eq. (2.6.6) and the source condition $S_n^j = \delta_{n0}\, \delta_{j0}$. In passing, we note that the consistency of Eq. (2.6.6) and (2.7.9) is easily demonstrated via the recurrence relation Eq. (2.7.8). Moreover, equating the F_0^0 and F_0^2 moments as derived by the two relations, we find

$$\frac{\nu_0}{I(+\nu_0)} + \int_0^{+1} \frac{\nu}{I(\nu)}\, d\nu = \frac{1}{1 - c\, g_0} \tag{2.7.10a}$$

$$\frac{\nu_0^3}{I(+\nu_0)} + \int_0^{+1} \frac{\nu^3}{I(\nu)}\, d\nu = \frac{1}{3}\, \frac{1 + 2c\, g_2/5}{(1 - c\, g_0)^2} \tag{2.7.10b}$$

From Eqs. (2.7.10a) and (2.7.10b) we obtain an explicit expression for the discrete index, ν_0, i.e.,

$$\nu_0^2 = \frac{\frac{1}{3}\,\frac{(1 + 2c\ g_2/5)}{(1 - c\ g_0)^2} - \int_0^{+1} \frac{\nu^3}{I(\nu)}\, d\nu}{\frac{1}{1 - c\ g_0} - \int_0^{+1} \frac{\nu}{I(\nu)}\, d\nu} \tag{2.7.11}$$

For $c < 1$, ν_0 is real and is interpreted as the exact asymptotic diffusion length (here, measured in units of $\lambda(1)$). It should be noted that the integral terms in Eq. (2.6.39) depend on c and $\{g_n\}$ via the dependence of $I(\nu)$ on these parameters (cf., Eq. (2.6.27b)).

III. REMARKS REGARDING APPLICATION OF THE THEORY

We shall develop a limited number of considerations relevant to the application of the theory presented in Section II. These remarks are intended as a brief illustration of possible methods of application of the present theory to physical problems. Many interesting calculations are possible, and with the accomplishment of experimental measurements of neutron distributions in the types of media under discussion, many comparisons of theoretical and experimental results would be profitable.

We certainly require methods of determining the proper variation of mean free path if these mathematical formulations are to be applied to physical problems. In this section we shall discuss the general types of heterogeneity toward which the current theory applies. We shall detail a simple method, using known diffusion lengths, to specify the angular dependence of the mean free path for a particular type of heterogeneity.

3.1 Types of Heterogeneity

As mentioned earlier, the motivation of the present effort is the establishment of a method of homogenization of regular arrays of vacuum channels for the purpose of neutron diffusion calculations. We also imposed the necessary restriction that, in general, the type of heterogeneity considered should yield two characteristic orthogonal directions. As an example of the caution which must be exercised in application of the theory, let us consider a type of heterogeneity which, at first approach, appears to satisfy the necessary requirements, but which actually is unsuitable for these methods. Specifically, we examine the case of a periodic slab array of scatterer and vacuum. This heterogeneity exhibits two characteristic orthogonal directions; perpendicular to slab, and the directions in the plane of the slab. Moreover, the direction perpendicular to the slabs (transverse to slab "channels") yields considerations which are algebraically easily accomplished. If $\lambda(\mu,x)$ represents the mean distance traveled to a collision by a neutron located at a position x to the left of the right-hand-face of a slab of scatterer, traveling with direction cosine μ relative to the slab perpendicular direction, we find

$$\lambda(\mu,x) = \lambda_s + \frac{\frac{T_v}{\mu} \exp(-x/\lambda_s\mu)}{1-\exp(-T_s/\lambda_s\mu)} \quad \text{for } x \leq T_s \text{ , } \mu > 0 \tag{3.1.1}$$

In Eq. (3.1.1), λ_s is the mean free path in the scatterer material which has slab thickness T_s, and T_v is the vacuum slab thickness. For the homogenized medium we require a function $\lambda(\mu)$ which, it would seem, should be a "suitable" average of $\lambda(\mu,x)$. For the case of isotropic scattering, the average

$$\lambda(\mu) = \frac{\int_0^{T_s} \lambda(\mu,x)\, \psi_0(x)\, dx}{\int_0^{T_s} \psi_0(x)\, dx} \tag{3.1.2}$$

is clearly indicated. In Eq. (3.1.2), $\psi_0(x)$ represents the actual angular integrated neutron flux. We note that, in the present case, $\psi_0(x)$ can be found. Far from neutron sources we have $\psi_0(x) \rightarrow \exp(x/L_s)$ where L_s is the "asymptotic" diffusion length in the scatterer material.

A note in passing: If $T_s/L_s \ll 1$, then $\psi_o(x)$ is approximately constant and Eq. (3.1.2) gives the result

$$\lambda(\mu) = \lambda_s \; (1 + T_v/T_s) \tag{3.1.3}$$

which is the "simply homogenized" parameter. Of course, the condition $T_s/L_s \ll 1$ should yield the homogeneous limit.

If x now represents the direction perpendicular to the slabs we have the asymptotic result $\psi_o(x) \longrightarrow \exp(-x/L_s)$ when the position x falls in a scatterer slab and $\psi_o(x)$ is a constant when x falls in a vacuum slab. The "best fit" to this flux, for the homogenized medium, is $\psi_o(x) \longrightarrow \exp(-x/L)$ where L is the simply homogenized diffusion length, i.e., $L = L_s \; (1 + T_v/T_s)$. This result would be obtained if Eq. (3.1.3) were used. In this particular case, we have the situation that a calculation based on an angular-dependent mean free path yields results that are less representative than the simply homogenized calculation. It is expected that, in the orthogonal characteristic direction (i.e., in the plane of the slab), use of an angular-dependent mean free path is indicated.

The case of a calculation in the slab perpendicular direction for a periodic slab array is certainly excluded from the present considerations. Moreover, one should feel no motivation toward developing a theory for that case since it is easily treated by a standard method, i.e., change of position variable to "optical thickness."

We shall now consider the details of a macroscopic-parameter-based calculation for a heterogeneity for which the present methods were clearly intended, i.e., a regular array of cylindrical vacuum channels.

3.2 Cylindrical Channels in a Regular Array

Let us consider a regular array of vacuum channels of cylindrical cross section. With every vacuum channel of cross sectional area A_v we associate a cross sectional area of scatterer material A_s such that $V = A_v/A_s$ is the ratio of vacuum volume to scatterer volume characteristic of the medium. We shall label the axial, or longitudinal, direction with x and direction cosine μ, and the radial, or transverse direction with y and direction cosine η. Due to streaming along channels we expect different diffusion properties in the x and y-directions and both of these cases to be different than the simply homogenized diffusion. The simply homogenized mean free path is given by

$$\lambda_h = \lambda_s \, (1 + V) \tag{3.2.1}$$

Before presenting a specific method for obtaining a representative $\lambda(\mu)$, let us make some general comments regarding the features of such a calculation. It is clear that we have chosen to consider only two representative directions; axial, or x-direction, and transverse, or y direction. It is also clear that in the present lattice the actual description of a straight line path in the transverse direction starting from a point in the scattering material depends upon both the azimuthal angle about the x-direction and the particular position in the scattering material relative to, say, the center of a vacuum channel. A "suitable" averaging technique must be employed. Furthermore, we encounter the same difficulty when considering a description of an axially-oriented path. Let $\lambda_x(\mu)$ and $\lambda_y(\eta)$ represent the angular-dependent mean free paths with respect to x-direction diffusion and y-direction diffusion. The following constraints on the "suitable" averaging technique seem intuitively reasonable:

(i) The average mean free path based on axial and transverse directions should be equal, i.e.,

$$\int_{-1}^{+1} \lambda_x(\mu)\, d\mu = \int_{-1}^{+1} \lambda_y(\eta)\, d\eta \tag{3.2.2}$$

(ii) The axial mean free path in the transverse direction (i.e., at $\mu = 0$) should be equal to the transverse mean free path in the transverse direction (i.e., at $\eta = \pm 1$), i.e.,

$$\lambda_x(0) = \lambda_y(\pm 1) \tag{3.2.3}$$

(iii) Both axial and transverse mean free paths should be symmetric, i.e.,

$$\lambda_x(\mu) = \lambda_x(-\mu) \tag{3.2.4a}$$

$$\lambda_y(\eta) = \lambda_\eta(-\eta) \tag{3.2.4b}$$

A possible method of obtaining $\lambda(\mu)$ is to find, as a function of starting position in the scatterer, the mean free path length traveled in all directions. Then, upon "suitably" weighting this quantity (according to whether $\lambda_x(\mu)$ or $\lambda_y(\eta)$ is desired) an average yields the angular-dependent mean free path. In even the simplest lattice this is a geometric task of considerable magnitude. Here, for the sake of brevity, we shall take an alternate, albeit certainly less self-contained, route. We shall assume that we have given certain macroscopic diffusion parameters, such as diffusion length, and use the general constraints of Eqs. (3.2.2), (3.2.3) and (3.2.4) to obtain a representation of the mean free path which yields the given parameters. To be specific, let us assume that $\lambda_x(\mu)$ and $\lambda_y(\eta)$ are even quadratics of the respective variables. Thus, in terms of the Legendre polynomial expansion

$$\lambda_x(\mu) = \lambda_{x0} + \lambda_{x2} \; P_2(\mu) \tag{3.2.5a}$$

$$\lambda_y(\eta) = \lambda_{y0} + \lambda_{y2} \; P_2(\mu) \tag{3.2.5b}$$

From Eq. (3.2.2), we obtain $\lambda_{x0} = \lambda_{y0}$, and this result used in Eq. (3.2.3) yields $\lambda_{y2} = -\lambda_{x2}/2$. Therefore, in terms of the two unknowns, λ_0 and λ_2 , Eq. (3.2.5) may be reformulated as

$$\lambda_x(\mu) = \lambda_0 + \lambda_2 \; P_2(\mu) \tag{3.2.6a}$$

$$\lambda_y(\eta) = \lambda_0 - \frac{1}{2} \lambda_2 \; P_2(\eta) \tag{3.2.6b}$$

The arguments used here with respect to the mean free path also apply to the determination of the total cross section. Thus, we expect the general constraints:

(i) $$\int_{-1}^{+1} \sigma_x(\mu)\, d\mu = \int_{-1}^{+1} \sigma_y(\eta)\, d\eta$$

(ii) $$\sigma_x(0) = \sigma_y(\pm 1)$$

(iii) $$\sigma_x(\mu) = \sigma_x(-\mu) \text{ and } \sigma_y(\eta) = \sigma_y(-\eta)$$

If the total cross section is assumed to be an even quadratic, then in terms of the two unknowns, σ_0 and σ_2, we have

$$\sigma_x(\mu) = \sigma_0 + \sigma_2\, P_2(\mu)$$

$$\sigma_y(\eta) = \sigma_0 - \frac{1}{2}\sigma_2\, P_2(\eta)$$

For the remainder of this discussion we shall assume that the neutron collision density is used as dependent variable and thus the mean free path is the relevant parameter. One can equally well apply these considerations to the neutron flux and total cross section.

From Eqs. (2.3.10) and (3.2.6) we have the results

$$L_x^2 = \frac{(\lambda_0 + 2\lambda_2/5)^2}{3(1 - c)(1 - cf_1)} \tag{3.2.7a}$$

$$L_y^2 = \frac{(\lambda_0 - \lambda_2/5)^2}{3(1 - c)(1 - cf_1)} \tag{3.2.7b}$$

We also have

$$L_s^2 = \frac{\lambda_s^2}{3(1 - c)(1 - cf_1)} \tag{3.2.8}$$

From Eqs. (3.1.10) and (3.1.11) we obtain

$$\frac{\lambda_0}{\lambda_s} = \frac{\langle L \rangle}{L_s} \tag{3.2.9a}$$

$$\langle L \rangle = L_x/3 + 2L_y/3 \tag{3.2.9b}$$

$$\frac{\lambda_2}{\lambda_s} = \frac{5}{3}\left[\frac{L_x}{L_s} - \frac{L_y}{L_s}\right] \tag{3.2.9c}$$

We can use measured values of L_x and L_y, or other theoretical treatments, to find L_x and L_y in order to determine λ_0/λ_s and λ_2/λ_s . For example, if we use Behren's theoretical formulation (B),

$$\left(\frac{L_x}{L_s}\right)^2 = 1 + 2V + \frac{2RV}{\exp(2R/V) - 1} + 2RV \tag{3.2.10a}$$

$$\left(\frac{L_y}{L_s}\right)^2 = 1 + 2V + \frac{2RV}{\exp(2R/V) - 1} + RV \tag{3.2.10b}$$

where R is the ratio of the vacuum channel radius to λ_s. In Fig. 3 we present L_x/L_s and L_y/L_s as a function of R, $R \in (0,5)$, for the cases V = 0.5, 1.0 and 2.0 as determined by Eq. (3.2.10). Then, in Fig. 4 we have, for the same values of R and V, the results for λ_0/λ_s and λ_2/λ_s based on the curves in Fig. 3.

It should be noted that what we refer to as L_x^2 and L_y^2 is Eq. (3.2.10) are actually calculated by Behrens (B) as $\langle x^2 \rangle /2$ and $\langle y^2 \rangle /2$ and, via Eq. (2.5.5), we have

$$\langle x^2 \rangle = 2L_x^2 + \frac{18}{175} \frac{\lambda_2^2}{(1 - c)(1 - cf_3)} \tag{3.2.11a}$$

$$\langle y^2 \rangle = 2L_y^2 + \frac{18}{175} \frac{\lambda_2^2/4}{(1 - c)(1 - cf_3)} \tag{3.2.11b}$$

If $\lambda_2^2 > 1$ the validity of Fig. 4 as a relevant representation for $\lambda(\mu)$ is questionable. However, truncation of $\lambda(\mu)$ at a quadratic would, in that case, also be of questionable usefullness.

IV. SUMMARY

We have developed the mathematical formulation of a new approach to the homogenization of certain types of heterogeneous media (such as a regular array of vacuum channels) for the purpose of neutron diffusion calculations. The new method is based on the inclusion of an angular-dependent mean free path in the theory of neutron transport. In the present effort, calculations are restricted to media with plane symmetry and monoenergetic neutron theory is employed. Extension to energy-dependent theory and to other symmetries would probably follow the general lines for the familiar, angular-independent case without significant additional complication. However, it seems clear that the requirement of the existence of two orthogonal characteristic directions in the development of the angular dependence of the mean free path must be imposed.

We have found that a neutron flux based theory and a collision density based theory can lead to significantly different results when low-order approximations, such as diffusion theory, are employed in the solution of the transport equation. For the case of isotropic scattering, the normal mode technique is applicable, and exact, closed-form solutions can be determined.

Evaluating the results implied by the present theory with respect to measurements is impossible. There is a current lack of pertinent experimental results for neutron distribution description in the relevant type of media.

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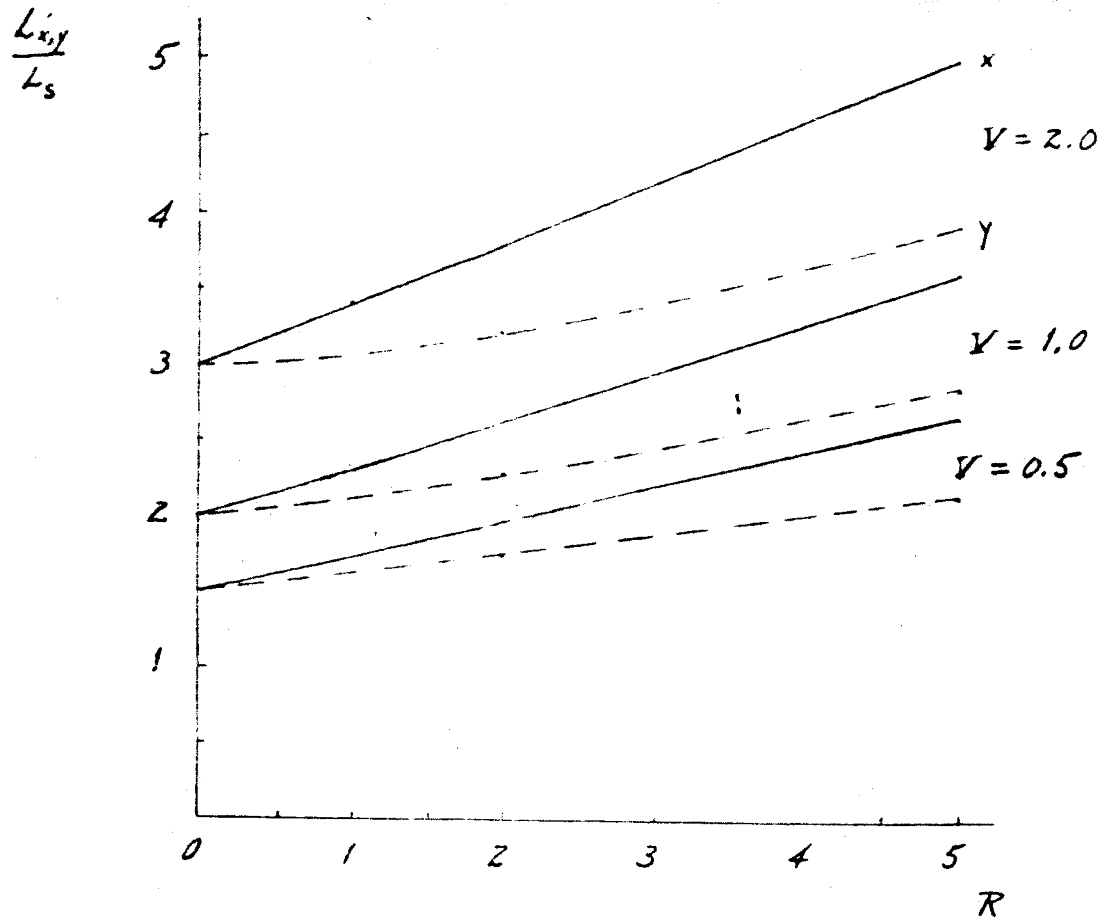

Figure 3. Neutron diffusion lengths as found by Behren's theoretical formulation.

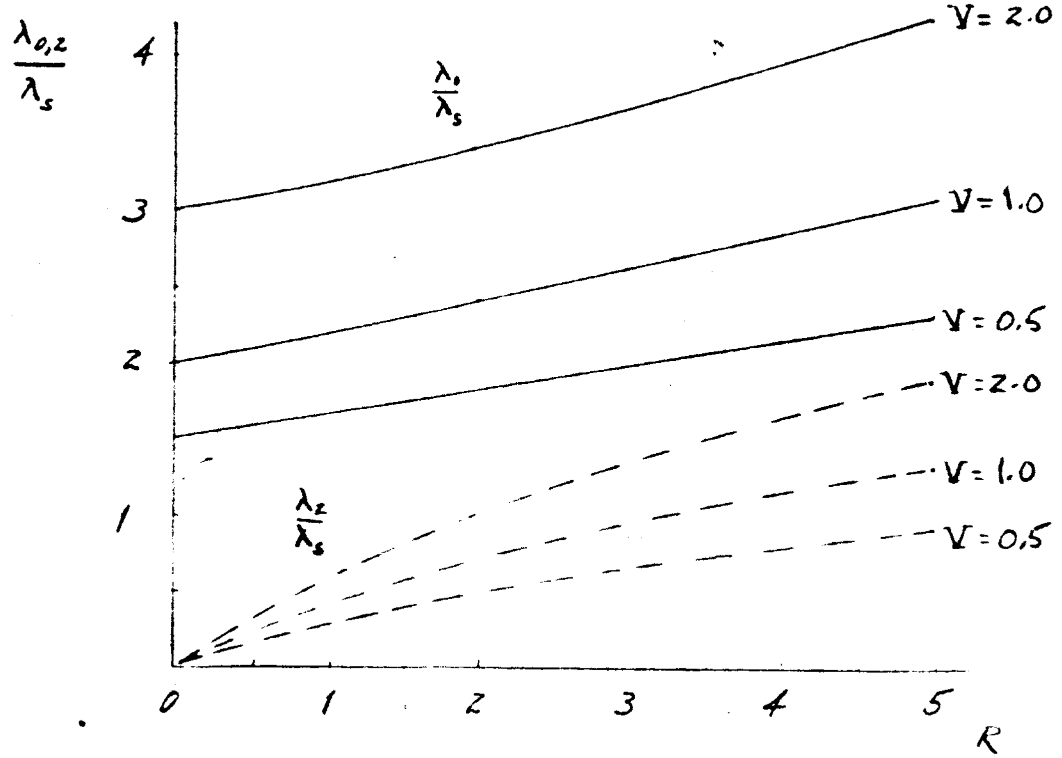

Figure 4. Expansion coefficients for an even quadratic representation of the mean free path angular dependence.

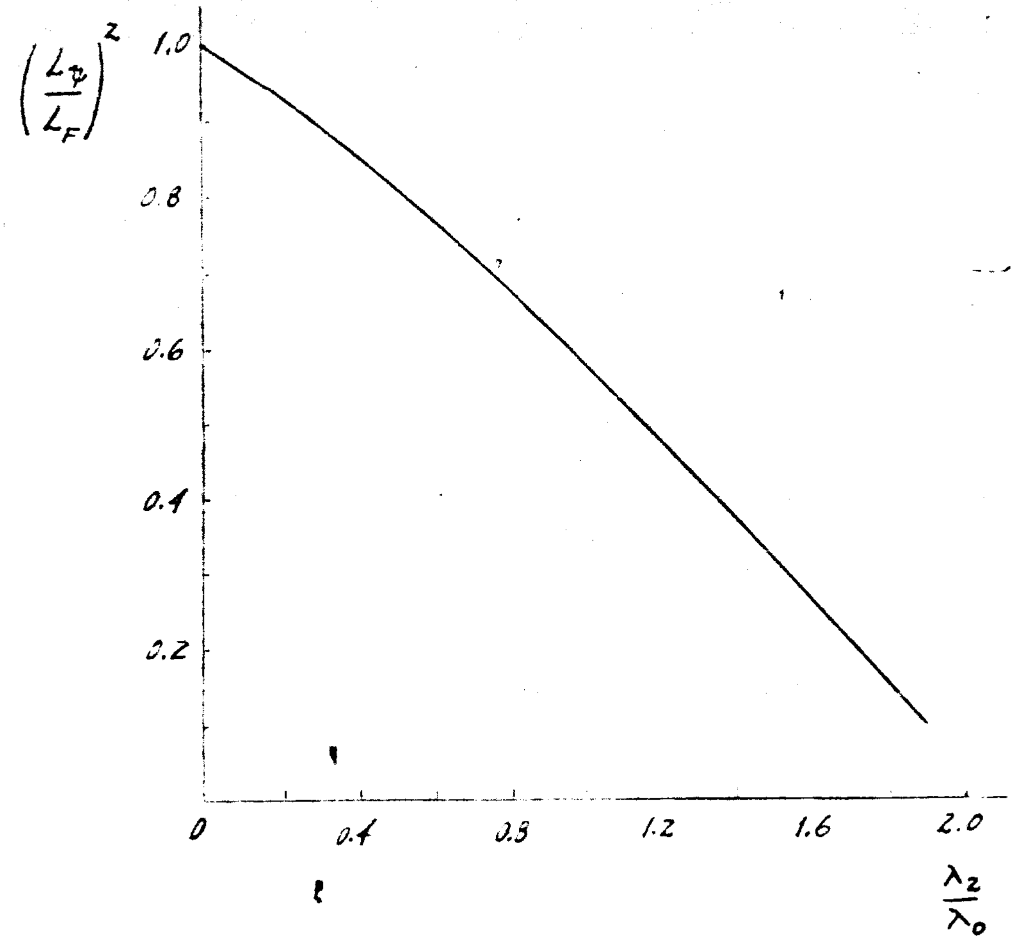

Figure 1. The ratio of diffusion lengths as calculated by a neutron flux based and collision density based P_1-approximation for the case of an even quadratic mean free path, $\lambda(\mu) = \lambda_0 + \lambda_2 P_2(\mu)$.

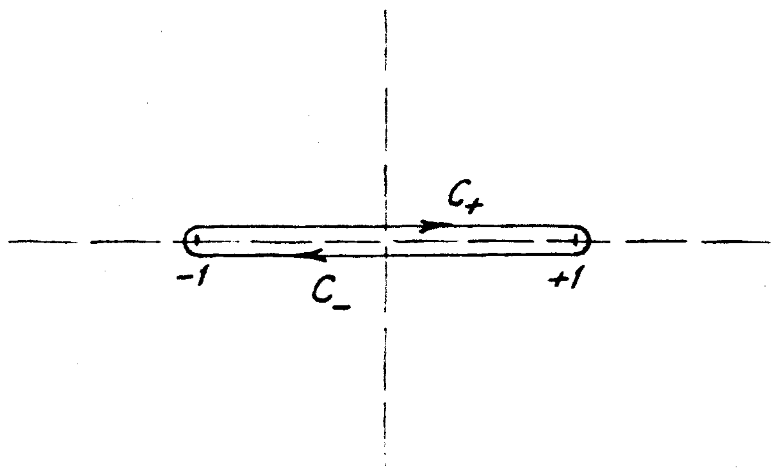

Figure 2. Contour in ν-plane used in determination of the number of zeroes of the function $\Lambda(\nu)$.

APPENDIX A

THE FUNCTION $\Lambda(\nu)$

We have found previously that the zeroes of $\Lambda(\nu)$ for $\nu \notin (-1, +1)$ are the discrete set of normal mode indices and that they appear in pairs, $\pm\nu_j$. Let us now discuss the number of these allowed discrete indices. To this end, and for relations which are useful in Section 2.7, we turn to a brief study of the general properties of the function $\Lambda(\nu)$ as defined by Eqs. (2.6.12) and (2.6.14), i.e.,

$$\Lambda(\nu) = 1 + i\,\pi\, c\, \nu\, G(\nu)$$

$$G(\nu) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{g(u)}{u - \nu}\, du$$

In terms of the set $\{g_n\}$, as defined in Eq. (2.6.4c), $\Lambda(\nu)$ may be rewritten

$$\Lambda(\nu) = 1 - c\,\nu \sum_n g_n\, Q_n(\nu) \tag{A.1}$$

where $Q_n(\nu)$ is a Legendre function of the second kind defined for the entire ν-plane by an extension of the Neumann formula (H, p. 51) to include $\nu \in (-1, +1)$, i.e.,

$$Q_n(\nu) = \frac{1}{2}\int_{-1}^{+1} \frac{P_n(u)}{\nu - u}\, du \tag{A.2}$$

with singular integrals evaluated as the Cauchy principal value. For large ν, $\nu\, Q_n(\nu)$ varies as ν^{-n}. Thus, $\Lambda(\nu)$ is bounded for large ν. Furthermore, the $Q_n(\nu)$ are analytic in the ν-plane excluding $\nu \in (-1, +1)$ and, therefore, $\Lambda(\nu)$ is analytic in this same region. We use the contour illustrated in Fig. 2 and the argument theorem (T, p. 116) to establish the number of zeroes of $\Lambda(\nu)$ in the region $\nu \notin (-1, +1)$. Since the zeroes of $\Lambda(\nu)$ appear in pairs, we denote the number of zeroes by 2J. The argument theorem applied here yields

$$4\pi J = \text{change in arg } \Lambda^+(u) \text{ on } C_+ + \text{change in arg } \Lambda^-(u) \text{ on } C_- \tag{A.3}$$

We have assumed that $g(u)$ satisfies a Holder condition on $u \in (-1, +1)$ and therefore $G(\nu)$ is a Cauchy integral. We apply the Plemelj formulae (M, p. 43) to find the limit values $G^{\pm}(u)$. We find

$$G^{\pm}(u) = G(u) \pm \frac{1}{2}\, g(u) \tag{A.4}$$

where $G^+(u)$ and $G^-(u)$ refer to the limit values of $G(\nu)$ as ν approaches u from above and below the real line respectively. From Eq. (A.4) we obtain the limit values

$$\Lambda^{\pm}(u) = \Lambda(u) \pm \frac{i\,\pi}{2}\, c\; u\, g(u) \tag{A.5}$$

Now, $\Lambda(u)$ with $u \in (-1, +1)$ is a real function (with singularities at $u = \pm 1$), and, we have $\Lambda(o) = 1$ and $g(u)$ is a symmetric function. Whence, we obtain the relations

$$\arg \Lambda^{+}(u) = -\arg \Lambda^{-}(u) \tag{A.6a}$$

$$\arg \Lambda^{+}(o) = \arg \Lambda^{-}(o) = 0 \tag{A.6b}$$

$$\Lambda^{+}(u) = \Lambda^{-}(-u) \tag{A.6c}$$

These results used in Eq. (A.3) yield the number of pairs of zeroes, J, in terms of the single angle $\arg \Lambda^{+}(+1)$, i.e.,

$$J = \frac{1}{\pi} \arg \Lambda^{+}(+1) \tag{A.7}$$

It should be noted that Eq. (A.3) contains the implicit requirement that $\Lambda^{+}(u) = 0$ for $u \in (-1, +1)$. This assumption is not completely necessary, however, it probably applies to most cases of physical interest and its application greatly simplifies these considerations.

We shall develop a sufficient condition for $J = 1$ in the case that $g(u)$ is an N degree polynomial in u, i.e.,

$$g(u) = \sum_{n=o}^{N} g_n \, P_n(u) \tag{A.8}$$

We note that the Legendre functions, $Q_n(\nu)$, can be expressed as

$$Q_n(\nu) = P_n(\nu)\, Q_o(\nu) - W_{n-1}(\nu) \tag{A.9a}$$

$$Q_o(\nu) = \text{arc tanh } \nu, \qquad \nu \in (-1, +1)$$

$$= \text{arc tanh } \frac{1}{\nu}, \qquad \nu \notin (-1, +1) \tag{A.9b}$$

where $W_{n-1}(\nu)$ is an even, or odd, polynomial in ν of degree $n-1$ (H, p. 51). In these terms $\Lambda(\nu)$ is rewritten as

$$\Lambda(\nu) = 1 - c\,\nu\, Q_0(\nu) \sum_{n=0}^{N} g_n\, P_n(\nu) + \sum_{n=0}^{N} g_n\, W_{n-1}(\nu) \tag{A.10}$$

We also have as $u \to +1$, $Q_0(u) \to +\infty$ and $P_n(+1) = 1$. Clearly, $W_n(+1)$ is bounded, and thus, if

$$\sum_{n=0}^{N} g_{2n} > 0 \tag{A.11}$$

then as $u \to +1$, $\Lambda(u) \to -\infty$. From Eq. (A.5), in the present case, we have

$$\Lambda^+(u) = \Lambda(u) + \frac{i\,\pi}{2}\, c\, u \sum_{n=0}^{N} g_n\, P_n(u) \tag{A.12}$$

Therefore, we may conclude the following: If Eq. (A.11) holds and, in the range $u \in (0, +1)$,

$$\sum_{n=0}^{N} g_n\, P_n(u) > 0 \tag{A.13}$$

then $\arg \Lambda^+(+1) = \pi$ and we have the desired result, $J = 1$. Let us stress that Eqs. (A.11) and (A.13) give a sufficient, not necessary, condition for the number of pairs of discrete indexed normal modes to be unity.

APPENDIX B

A RELEVANT HILBERT PROBLEM

In Section 2.6 the existence of the modal expansion coefficients $\left\{a(\pm\nu_j),\ j = 1,\ 2, \cdots,\ J;\ a(\nu),\ \nu \in (-1,\ +1)\right\}$ was assumed. Moreover, the orthogonality relations are based on the whole angle range $u \in (-1,\ +1)$ and thus only provide a means of determining expansion coefficients for the case of a boundary condition given over all angles. By reducing the problem of finding expansion coefficients to the solution of an inhomogeneous Hilbert problem, we find that one can demonstrate the existence of expansion coefficients, and, prescribe a method for determining the value of the coefficients for problems involving all physically relevant boundary conditions. We follow closely the techniques elegantly described by Muskhelishvili (M).

<u>Reduction of Transport Problem to an Inhomogeneous Hilbert Problem</u>

We shall, in general, encounter transport problem boundary conditions of the form

$$\phi(u) = \sum_{j=1}^{J} a(+\nu_j)\ \phi(+\nu_j,u) + \sum_{j=1}^{J} a(-\nu_j,u)\ \phi(-\nu_j,u)$$

$$+ \int_{\alpha}^{\beta} a(\nu)\ \phi(\nu,u)\ d\nu \quad \text{for } u \in (\alpha,\beta) \tag{B.1}$$

where $-1 \leq \alpha < \beta \leq +1$. Let us suppose that by some method we are able to determine the set of discrete indexed coefficients

$\left\{a(\pm\nu_j),\ j = 1, 2, \cdots, J\right\}$, and define

$$\phi'(u) = \phi(u) - \sum_{j=1}^{J} a(+\nu_j)\ \phi(+\nu_j, u) - \sum_{j=1}^{J} a(-\nu_j)\ \phi(-\nu_j, u) \tag{B.2}$$

We then have an integral equation for $a(\nu)$, $\nu \in (\alpha, \beta)$, i.e.,

$$\int_{\alpha}^{\beta} a(\nu)\ \phi(\nu, u)\ d\nu = \phi'(u) \text{ for } u \in (\alpha, \beta) \tag{B.3}$$

Using the derived form of $\phi(\nu, u)$ Eq. (2.6.11), we obtain

$$\Lambda(u)\ a(u) + \frac{c}{2}\ g(u) \int_{\alpha}^{\beta} \frac{\nu\ a(\nu)}{\nu - u}\ d\nu = \phi'(u) \text{ for } u \in (\alpha, \beta) \tag{B.4}$$

From Eq. (A.5)

$$\frac{c}{2}\ u\ g(u) = \frac{1}{2\pi i}\left[\Lambda^{+}(u) - \Lambda^{-}(u)\right] \tag{B.5a}$$

$$\Lambda(u) = \frac{1}{2}\left[\Lambda^{+}(u) + \Lambda^{-}(u)\right] \tag{B.5b}$$

and therefore Eq. (B.4) may be rewritten as

$$\frac{1}{2}\left[\Lambda^{+}(u) + \Lambda^{-}(u)\right] ua(u) + \left[\Lambda^{+}(u) - \Lambda^{-}(u)\right] A(u)$$

$$= u\ \phi'(u) \tag{B.6a}$$

$$A(u) = \frac{1}{2\pi i} \int_{\alpha}^{\beta} \frac{\nu\ a(\nu)}{\nu - u}\ d\nu$$

$$\text{for } u \in (\alpha, \beta) \tag{B.6b}$$

We have assurance that $A(u)$, as defined in Eq. (B.6b), exists if $a(u)$ satisfies a Holder condition on $u \in (\alpha, \beta)$. For the moment, let us assume that this condition is fulfilled and define the Cauchy integral, $A(\nu)$, over the entire ν-plane,

$$A(\nu) = \frac{1}{2\pi i} \int_{\alpha}^{\beta} \frac{u\, a(u)}{u - \nu}\, du \tag{B.7}$$

The Plemelj formulae yield the limit relations on the line $u \in (\alpha, \beta)$,

$$A^{+}(u) - A^{-}(u) = u\, a(u) \tag{B.8a}$$

$$A^{+}(u) + A^{-}(u) = 2A(u) \tag{B.8b}$$

The results of Eqs. (B.5) and (B.8) applied to Eq. (B.6) give the alternate form

$$\Lambda^{+}(u)\, A^{+}(u) - \Lambda^{-}(u)\, A^{-}(u) = u\, \phi'(u) \text{ for } u \in (\alpha, \beta) \tag{B.9}$$

We have assumed that $\Lambda^{\pm}(u) \neq 0$ for $u \in (\alpha, \beta)$. With this restriction we can easily transform Eq. (B.9) to the form of a boundary condition for an inhomogeneous Hilbert problem on an arc (M, ch. 10). Restating the problem of determining $a(u)$, in these terms: Find the sectionally analytic function, $A(\nu)$, vanishing at infinity, with boundary condition on the line $u \in (\alpha, \beta)$,

$$A^{+}(u) = \frac{\Lambda^{-}(u)}{\Lambda^{+}(u)}\, A^{-}(u) + \frac{u\, \phi'(u)}{\Lambda^{+}(u)} \tag{B.10}$$

We note that the assumptions on $g(u)$ and $\Lambda^{\pm}(u)$ imply that $\Lambda^{-}(u)/\Lambda^{+}(u)$ is a function satisfying a Holder condition and not vanishing on $u \in (\alpha, \beta)$, and, if we assume that the angle boundary condition, $\phi(u)$, satisfies a Holder condition and $a(\pm\nu_j)$ exist, then $u\, \phi'(u)/\Lambda^{+}(u)$ satisfies a Holder condition on $u \in (\alpha, \beta)$.

Let us help clarify our procedure by summarizing. If we assume (what we wish to prove) that $a(u)$ satisfies a Holder condition on $u \in (\alpha,\beta)$, then the integral $A(\nu)$, defined by Eq. (B.7), is of the Cauchy type. Now, Cauchy integrals are sectionally analytic functions with boundary the line of integration. Specifically, if (α,β) is the line of integration:

(i) $A(\nu)$ is analytic in ν-plane excluding (α, β).

(ii) $A(\nu)$ approaches well-defined limits as $u \in (\alpha, \beta)$ is approached from either side of (α, β) with possible exception of the end points, α and/or β.

(iii) Near the end points, $A(\nu)$ satisfies the conditions

$$|A(\nu)| \leq \frac{A}{|\nu - \alpha|^{a}} \qquad \text{as } \nu \longrightarrow \alpha$$

$$|A(\nu)| \leq \frac{B}{|\nu - \beta|^{b}} \qquad \text{as } \nu \longrightarrow \beta$$

where a, b, A and B are real constants, and $a < 1$ and $b < 1$.

Moreover, $A(\nu)$ vanishes as $|\nu| \to \infty$. We have transformed the integral equation for $a(u)$ into the boundary condition Eq. (B.10) which is the form of an inhomogeneous Hilbert problem boundary condition. Thus, we have reduced the original transport problem to an inhomogeneous Hilbert problem. If we can find a solution, $A(\nu)$, which introduces no physical ambiguity, then our assumption of the existence of $a(u)$, $u \in (\alpha, \beta)$, will be substantiated.

Solution of the Hilbert Problem

In terms of $\theta(u) = \arg \Lambda^+(u)$, we have $\Lambda^-(u)/\Lambda^+(u) = \exp(-2i\ \theta(u))$ and the Hilbert problem boundary condition (cf., Eq. (B.10))

$$A^+(u) = \exp(-2i\ \theta(u)\ )\ A^-(u) + \frac{u\ \phi'(u)}{\Lambda^+(u)} \quad \text{for } u \in (\alpha, \beta) \tag{B.11}$$

Since $A(\nu)$ must also vanish as $|\nu| \to \infty$, the solution is (M)

$$A(\nu) = \frac{H(\nu)}{2\pi i} \int_\alpha^\beta \frac{u\ \phi'(u)}{(u - \nu)\ \Lambda^+(u)\ H^+(u)}\ du \tag{B.12}$$

where $H(\nu)$ is the fundamental solution of the associated homogeneous Hilbert problem and is given by

$$H(\nu) = (\alpha - \nu)^{-\theta(\alpha)/\pi}\ (\beta - \nu)^{\theta(\beta)/\pi}\ e^{\Theta(\nu)} \tag{B.13}$$

The Cauchy integral $\Theta(\nu)$ is defined by

$$\Theta(\nu) = -\frac{1}{\pi} \int_\alpha^\beta \frac{\theta(u)}{u - \nu}\ du \tag{B.14}$$

Providing $\kappa = \Theta(\beta)/\pi - \Theta(\alpha)/\pi$ is a positive integer, we have the κ additional requirements

$$\int_{\alpha}^{\beta} \frac{u^{n+1}\ \phi'(u)}{\Lambda^{+}(u)\ H^{+}(u)}\, du = 0 \quad \text{for } n = 0, 1, \cdots, \kappa - 1 \tag{B.15}$$

These additional requirements are a necessary feature of the solution. It should be recalled that the function $\phi'(u)$, $u \in (\alpha, \beta)$, is not completely specified, i.e., the discrete indexed expansion coefficients, $a(\pm v_j)$, in Eq. (B.2) are, as yet, unknown. For the general problems considered later, it will be demonstrated that, in each case, the κ requirements are necessary and sufficient for the complete specification of all discrete and continuum expansion coefficients.

Application of the Hilbert Problem Solution

Plane symmetry transport problems fall into two general categories:

(i) Infinite media problems with full-angle-range boundary conditions (such as the Green's function solved in Section 2.7).

(ii) Half-space media problems with half-angle-range boundary conditions (such as albedo or Milne type problems).

Combinations of the solutions of these type problems lead to the solution of cases with finite media (slabs). For full-range boundary conditions, the orthogonality of the normal modes provides a direct method for determining expansion coefficients. The solution of the Hilbert problem in these cases demonstrates the existence of the coefficients and thus partially supports the completeness hypothesis. For half-range boundary problems, there are no apparent orthogonality properties of the normal modes. In these cases, the solution of the Hilbert problem not only provides proof of existence, but also gives a well-defined prescription for the determination of expansion coefficients. We shall now outline the application of the Hilbert problem solution to the categories of full-range and half-range boundary conditions.

In the case of an infinite medium, full-range boundary condition problem, a source condition is usually given at some position, which we choose to designate $x = 0$. For $c < 1$, it follows that $F(x,u)$ should vanish as $|x| \to \infty$. Thus, the general form of solution is that given in Eq. (2.7.2). The source condition can be formulated as

$$\phi(u) = \sum_{j=1}^{J} a(+\nu_j)\ \phi(+\nu_j,u) + \sum_{j=1}^{J} a(-\nu_j)\ \phi(-\nu_j,u) + \int_{-1}^{+1} a(\nu)\ \phi(\nu,u)\ d\nu \quad \text{for } u \in (-1, +1) \tag{B.16}$$

Instead of using the obviously indicated orthogonality properties, let us consider the coefficient evaluation by the route prescribed in the Hilbert problem solution. Note that $\alpha = -1$ and $\beta = +1$. From

Eqs. (A.6) and (A.7), we have the results $\theta(-1) = -J\pi$ and $\theta(+1) = J\pi$. Therefore, in this case, we find that $\kappa = 2J$ and there are 2J requirements of the form of Eq. (2.7.15). Specifically,

$$\int_{-1}^{+1} \frac{u^{n+1} \ \phi'(u)}{\Lambda^+(u) \ H^+(u)} du = 0 \quad \text{for } n = 0, 1, \cdots, 2J - 1 \tag{B.17}$$

Eq. (B.17) provides a sufficient number of equations to find the discrete indexed expansion coefficients, $a(\pm\nu_j)$, $j = 1, 2, \cdots, J$. The fundamental solution, $H(\nu)$, is given by (cf., Eq. (B.13)

$$H(\nu) = (-1 - \nu)^J \ (1 - \nu)^J \ e^{\Theta(\nu)} \tag{B.18a}$$

$$\Theta(\nu) = -\frac{1}{\pi} \int_{-1}^{+1} \frac{\theta(u)}{u - \nu} du \tag{B.18b}$$

Thus, $A(\nu)$ is determined (by Eq. (B.12)) and we can find $a(u)$ for $u \in (-1, +1)$ from the limit relation (cf., Eq. (B.8a)

$$u \ a(u) = A^+(u) - A^-(u) \quad \text{for } u \in (-1, +1) \tag{B.19}$$

Since the problem has been completely and unambiguously solved, it is clear that the supposition that $a(u)$ satisfy a Holder condition is substantiated and we have demonstrated the existence of the expansion coefficients.

For half-space media we consider two types of problems. An "albedo problem" is described by a boundary condition at the medium surface ($x = 0$ with medium occupying $x > 0$) specified for $u \in (0, +1)$ and the condition that $F(x,u)$ vanish as $x \to \infty$. A "Milne problem" is described by a similar boundary condition at $x = 0$, but with $F(x,u) \longrightarrow \phi(-\nu,u)\exp(x/\nu)$ with $\nu = \nu_j$, $j = 1, 2, \cdots, J$, or $\nu \in (0, +1)$, as $x \to \infty$. We have specified these problems as boundary conditions on the half-range $u \in (0, +1)$. With obvious modifications, the procedure is easily applied to half-space media occupying $x < 0$ and boundary conditions on $u \in (-1, 0)$. With the half-space occupying $x > 0$, the general solution of an albedo problem is

$$F(x,u) = \sum_{j=1}^{J} a(+\nu_j)\ \phi(+\nu_j,u)\ \exp(-x/\nu_j) + \int_0^{+1} a(\nu)\ \phi(\nu,u)\ \exp(-x/\nu)\ d\nu \text{ for } x > 0 \tag{B.20}$$

and for a Milne problem,

$$F(x,u) = A\ \phi(-\nu,u)\ \exp(x/\nu) + \sum_{j=1}^{J} a(+\nu_j)\ \phi(+\nu_j,u)\exp(-x/\nu_j) + \int_0^{+1} a(\nu)\ \phi(\nu,u)\ \exp(-x/\nu)\ d\nu \text{ for } x > 0 . \tag{B.21}$$

In both cases, the boundary condition at $x = 0$ can be expressed in the form of Eq. (B.1), i.e.,

$$\phi(u) = \sum_{j=1}^{J} a(+\nu_j)\ \phi(+\nu_j,u) + \int_0^{+1} a(\nu)\ \phi(\nu,u)\ d\nu \quad \text{for } u \in (0, +1) \tag{B.22}$$

Now, $\alpha = 0$ and $\beta = +1$ and, from Eqs. (A.6) and (A.7), $\theta(0) = 0$ and $\theta(+1) = J\pi$. Thus, $\kappa = J$ and we have the J requirements

$$\int_0^{+1} \frac{u^{n+1} \quad \phi'(u)}{\Lambda^+(u) \; H^+(u)} du = 0 \quad \text{for } n = 0, 1, \cdots, J - 1 \qquad \text{(B.23)}$$

These are sufficient to determine the discrete indexed coefficients, $a(+\nu_j)$, $j = 1, 2, \cdots, J$. The fundamental solution takes the form

$$H(\nu) = (1 - \nu)^J \exp(\Theta(\nu)) \qquad \text{(B.24a)}$$

$$\Theta(\nu) = -\frac{1}{\pi} \int_0^{+1} \frac{\theta(u)}{u - \nu} du \qquad \text{(B.24b)}$$

The Hilbert problem solution, $A(\nu)$, $\nu \in (0, +1)$, and the continuum expansion coefficients, $a(u)$, $u \in (0, +1)$, are found as in the case of a full-range boundary problem. Again, we find substantiation for the supposition of the existence of the relevant members of $\{a(\nu)\}$. Moreover, we find a prescription for calculating the expansion coefficients when the use of orthogonality conditions is impossible.