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ORBITAL PREDICTION AND DIFFERENTIAL CORRECTION USING VINTI'S SPHEROIDAL THEORY FOR ARTIFICIAL SATELLITES

BY

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ABSTRACT

The spheroidal theory developed by Vinti for determining the orbit of an artificial satellite of an oblate planet is presented in algorithmic form, in which empirically derived initial conditions are used to obtain the co-ordinate and velocity components of an unretarded satellite at any time. A differential orbit improvement method utilizing observational data is described. This method produces a mean set of orbital elements by an iterated least-squares fitting of the equations of condition. The results of preliminary applications of the orbit generator and differential correction to two artificial satellites of the Earth, through use of a high-speed digital electronic computer, is shown in tabular and graphical form.

Author.

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INTRODUCTION

The spheroidal method for satellite orbits provides a procedure for calculating the orbit of any satellite of an oblate planet, when all forces except those of the primary's gravitational field are neglected. Determining the effect of the oblateness of a planet on the orbit of a satellite sufficiently near to the planet so that the forces of other bodies may be neglected is one of the central problems of satellite astronomy.

Vinti, in a series of research papers (listed in the references at the conclusion of this report), has found a gravitational potential for the exterior of an axially symmetric oblate planet which is able to produce an "intermediary reference orbit" accounting for more than 99.5 percent of the deviation of the Earth's potential from spherical symmetry. The Vinti potential is a very accurate approximation for the Earth's gravitational potential, which both satisfies Laplace's equation and leads to separability of the Hamilton-Jacobi equation in oblate spheroidal co-ordinates, the most appropriate system for an oblate planet. Use of this form for the potential reduces the problem of satellite motion to the analytic solution of quartic polynomials and avoids the use of perturbation theory entirely in deriving an accurate intermediary orbit. The Vinti potential is actually much closer to the empirically accepted one for the Earth than any previously used as the starting point of a calculation. In the case of the Earth, the resulting intermediary orbit reproduces the even zonal harmonics exactly through the second and approximately through the fourth. The secular solution can be obtained to arbitrarily high order in the second harmonic oblateness parameter, and, by means of rapidly converging infinite series, the periodic solution can easily be obtained through second order. The solution holds for all angles of inclination (in the case of equatorial or near-equatorial orbits, certain simplifications can be made in the equations) and contains no critical inclination or long-periodic terms. For such a reference orbit, error can never accumulate because of the exactness of the secular terms.

This method of solution for unretarded satellite orbits has been adapted for computational purposes on a high-speed digital electronic computer primarily by means of the FORTRAN programming language. The function of the present paper is to provide the computational procedure for determining and correcting an orbit in algorithmic form, adopting algebraic symbols consistent with those in Vinti's papers. A summary of preliminary results utilizing observational data from artificial satellites is included.

INPUT PARAMETERS

The fundamental physical units employed are those of the canonical Vanguard system. In this system, the fundamental unit of length is the Earth's equatorial radius (taken to be 6378.388 km), and the fundamental unit of mass is the terrestrial mass (taken to be 5.983×10^{24} kgm). The fundamental unit of time is adjusted so that the Newtonian gravitational constant G is set equal to unity; this process yields a value for the Vanguard unit of time of 806.832 seconds. To obtain a physical significance for this time, consider a satellite "orbiting" the Earth at its surface. This time unit is then seen to be the time required for such a satellite to traverse one radian.

The inertial co-ordinate system takes the Earth's polar axis as the Z-axis (which is also the planetary axis of symmetry and the axis of rotation). The X-Y plane is the equatorial plane, with the X-axis pointing toward the vernal equinox (the first point of Aries), the Y-axis orthogonally to the east to form a right-handed system, and the Earth's center of mass at the origin.

The following constants are required in the computations:

$\mu \equiv$ GM, where G is the Newtonian gravitational constant and M is the Earth's mass. From the preceding remarks, it is seen that $\mu = 1$ in the Vanguard system.

$c \equiv r_e \sqrt{J_2}$, where r_e is the equatorial radius of the Earth (unity in the Vanguard system) and J_2 is the coefficient of the second zonal harmonic in the infinite series expansion of the Earth's potential. The value of J_2 is approximately 1.0823×10^{-3}.

t_i, the initial time.

t_f, the final time.

Δt, the time increment used in generating position and velocity components for equal time intervals following t_i and preceding t_f.

X_i, Y_i, Z_i, the initial conditions of position.

$\dot{X}_i$, $\dot{Y}_i$, $\dot{Z}_i$, the initial conditions of velocity. Note that the set $X_i, Y_i, Z_i, \dot{X}_i, \dot{Y}_i, \dot{Z}_i$ of initial conditions is also referred to as the set of injection conditions if t_i, the initial time, is taken to be the time of injection of the satellite into orbit.

CO-ORDINATE CONVERSION

We now compute the following quantities. The square of the magnitude of the position vector:

$$r_i^2 = X_i^2 + Y_i^2 + Z_i^2.$$

The dot product of the position and velocity vectors:

$$r_i \dot{r}_i = X_i \dot{X}_i + Y_i \dot{Y}_i + Z_i \dot{Z}_i.$$

The oblate spheroidal co-ordinates (ρ, η, ϕ) and their time derivatives:

$$\rho_i^2 = \frac{1}{2}\left[(r_i^2 - c^2) + \sqrt{(r_i^2 - c^2)^2 + 4c^2 Z_i^2}\right]$$

$$\eta_i^2 = \frac{1}{2c^2}\left[-(r_i^2 - c^2) + \sqrt{(r_i^2 - c^2)^2 + 4c^2 Z_i^2}\right].$$

Then ρ_i and η_i are found by extracting square-roots, with the condition that the sign of η_i is the same as the sign of Z_i.

$$\dot{\rho}_i = \frac{1}{2\rho_i}\left[r_i \dot{r}_i + \frac{r_i \dot{r}_i (r_i^2 - c^2) + 2c^2 Z_i \dot{Z}_i}{\sqrt{(r_i^2 - c^2)^2 + 4c^2 Z_i^2}}\right]$$

$$\dot{\eta}_i = \frac{1}{2c^2\eta_i}\left[-r_i \dot{r}_i + \frac{r_i \dot{r}_i (r_i^2 - c^2) + 2c^2 Z_i \dot{Z}_i}{\sqrt{(r_i^2 - c^2)^2 + 4c^2 Z_i^2}}\right]$$

In the above, $\rho_i \neq 0$, but if $\eta_i = 0$, then $\dot{\eta}_i = \dot{Z}_i / \rho_i$.

$$\sin \phi_i = \frac{Y_i}{\sqrt{(\rho_i^2 + c^2)(1 - \eta_i^2)}}$$

$$\cos \phi_i = \frac{X_i}{\sqrt{(\rho_i^2 + c^2)(1 - \eta_i^2)}}$$

From the above trigonometric relations, we obtain ϕ_i within the limits $0 \leq \phi_i < 2\pi$.

THE JACOBI CONSTANTS OF GENERALIZED MOMENTA

Compute:

$$\alpha_1 = \frac{1}{2}(\dot{X}_i^2 + \dot{Y}_i^2 + \dot{Z}_i^2) - \mu \rho_i (\rho_i^2 + c^2 \eta_i^2)^{-1}$$

$$\alpha_3 = X_i \dot{Y}_i - Y_i \dot{X}_i$$

$$\alpha_2 = \left[(\rho_i^2 + c^2 \eta_i^2)^2 \dot{\eta}_i^2 + \alpha_3^2 - 2\alpha_1 c^2 \eta_i^2 (1 - \eta_i^2)\right]^{1/2} (1 - \eta_i^2)^{-1/2}$$

FACTORING THE QUARTICS: PRIME CONSTANTS

Compute:

$$x_D^2 = -2\alpha_1 \alpha_2^2 \mu^{-2}$$

$$p_0 = \alpha_2^2 \mu^{-1}$$

$$y_D^2 = \left(\frac{\alpha_3}{\alpha_2}\right)^2$$

$$k_0 = c^2 p_0^{-2}$$

$$A = -2 k_0 p_0 y_D^2 \left[1 - k_0 (3 x_D^2 y_D^2 - 2 x_D^2 - 8 y_D^2 + 4)\right]$$

$$B = k_0 p_0^2 (1 - y_D^2) \left[1 + k_0 y_D^2 (4 - x_D^2)\right]$$

$$b_1 = -\frac{1}{2} A$$

$$b_2 = \sqrt{B}$$

$$a = p_0 x_D^{-2} \left[1 - k_0 x_D^2 y_D^2 + k_0^2 x_D^2 y_D^2 (3 x_D^2 y_D^2 - 2 x_D^2 - 8 y_D^2 + 4)\right]$$

$$p = a^{-1} p_0^2 x_D^{-2} \left[1 - k_0 y_D^2 (4 - x_D^2) - 16 k_0^2 y_D^2 (2 y_D^2 - 1)\right.$$
$$\left. - k_0^2 x_D^2 y_D^2 (x_D^2 y_D^2 - x_D^2 - 20 y_D^2 + 12)\right]$$

$$g = k_0 y_D^2 (3x_D^2 - 4) - 16 k_0^2 y_D^2 (2 y_D^2 - 1)$$

$$- k_0^2 x_D^2 y_D^2 (2 x_D^2 y_D^2 - 5 x_D^2 - 28 y_D^2 + 20)$$

$$e = \left[1 - x_D^2 (1 + g)\right]^{1/2}$$

If $(\alpha_2^2 - \alpha_3^2) = 0$, then $\eta_0 = 0$ and $\eta_2^{-2} = k_0 x_D^2 (1 - k_0 x_D^2)$.

If $(\alpha_2^2 - \alpha_3^2) \neq 0$, then calculate η_0 from

$$\eta_0^{-2} = \frac{\alpha_2^2 - 2\alpha_1 c^2}{2(\alpha_2^2 - \alpha_3^2)} \left\{1 + \sqrt{1 + \frac{8\alpha_1 c^2 (\alpha_2^2 - \alpha_3^2)}{(\alpha_2^2 - 2\alpha_1 c^2)^2}}\right\}.$$

Also,

$$\eta_2^{-2} = \frac{\alpha_2^2 - 2\alpha_1 c^2}{2(\alpha_2^2 - \alpha_3^2)} \left\{1 - \sqrt{1 + \frac{8\alpha_1 c^2 (\alpha_2^2 - \alpha_3^2)}{(\alpha_2^2 - 2\alpha_1 c^2)^2}}\right\}.$$

MUTUAL CONSTANTS

$$q = \eta_0 \eta_2^{-1}$$

If $\eta_0 \neq 0$, then compute:

$$A_1 = (1 - e^2)^{1/2}\, p \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n \left(\frac{b_1}{b_2}\right) R_{n-2} \left[(1 - e^2)^{1/2}\right]$$

where $P_n(x)$ is the Legendre polynomial with argument x of degree n, and where $R_n(x) \equiv x^n P_n(1/x)$. The infinite series above (and those that follow) is computed by an iterative method, with computation of terms ceasing when the absolute value of the ratio of successive terms minus unity is less than or equal to some pre-selected tolerance, i.e., computation ceases when

$$\left| \frac{(A_1)_{i-1}}{(A_1)_i} - 1 \right| \leq \epsilon$$

where ϵ might be 10^{-7}. Convergence should be attained by consideration of the first several terms, in most cases. To increase computational speed, the first term (for $n = 2$) of the above series may be given explicitly by

$$(1 - e^2)^{1/2} \left(\frac{b_2^2}{p}\right) P_2 \left(\frac{b_1}{b_2}\right).$$

If $\eta_0 = 0$ (corresponding to an orbit in the equatorial plane), compute instead:

$$A_1 = (1 - e^2)^{1/2}\, p \sum_{n=2}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{b_1}{2p}\right)^n R_{n-2} \left[(1 - e^2)^{1/2}\right]$$

where the first term (for $n = 2$) is given explicitly by

$$\frac{3}{2} (1 - e^2)^{1/2} \left(\frac{b_1^2}{p}\right).$$

If $\eta_0 \neq 0$, compute:

$$A_2 = (1 - e^2)^{1/2} \, p^{-1} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_n \left[(1 - e^2)^{1/2}\right]$$

where P_n and R_n are defined as above, and where the first term (for $n = 0$) of the above series is given explicitly by

$$(1 - e^2)^{1/2} \, p^{-1} .$$

If $\eta_0 = 0$, compute instead:

$$A_2 = (1 - e^2)^{1/2} \, p^{-1} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{b_1}{2p}\right)^n R_n \left[(1 - e^2)^{1/2}\right]$$

where the first term (for $n = 0$) is given explicitly by

$$(1 - e^2)^{1/2} \, p^{-1} .$$

Then compute:

$$A_3 = (1 - e^2)^{1/2} \, p^{-3} \sum_{n=0}^{\infty} D_n R_{n+2} \left[(1 - e^2)^{1/2}\right]$$

where, for $\eta_0 \neq 0$, D_n is computed as follows:

$$D_n = D_{2i} = \sum_{m=0}^{i} (-1)^{i-m} \left(\frac{c}{p}\right)^{2i-2m} \left(\frac{b_2}{p}\right)^{2m} P_{2m} \left(\frac{b_1}{b_2}\right)$$

(n an even integer)

$$D_n = D_{2i+1} = \sum_{m=0}^{i} (-1)^{i-m} \left(\frac{c}{p}\right)^{2i-2m} \left(\frac{b_2}{p}\right)^{2m+1} P_{2m+1} \left(\frac{b_1}{b_2}\right)$$

(n an odd integer).

If $\eta_0 = 0$, use instead:

$$D_n = D_{2i} = \sum_{m=0}^{i} (-1)^{i-m} \left(\frac{c}{p}\right)^{2i-2m} \frac{(4m)!}{\left[(2m)!\right]^2} \left(\frac{b_1}{2p}\right)^{2m}$$

(n an even integer)

$$D_n = D_{2i+1} = \sum_{m=0}^{i} (-1)^{i-m} \left(\frac{c}{p}\right)^{2i-2m} \frac{(4m+2)!}{\left[(2m+1)!\right]^2} \left(\frac{b_1}{2p}\right)^{2m+1}$$

(n an odd integer).

Then compute:

$$B_1 = \frac{1}{2} + \frac{3}{16} q^2 + \frac{15}{128} q^4$$

$$B_2 = 1 + \frac{1}{4} q^2 + \frac{9}{64} q^4$$

$$B_3 = 1 - (1 - \eta_2^{-2})^{-1/2} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m}$$

where

$$\gamma_m = \frac{(2m)!}{2^{2m} (m!)^2} \sum_{n=1}^{m-1} \frac{(2n)!}{2^{2n} (n!)^2} \eta_0^{2n}$$

$$A_{11} = \frac{3}{4} (1 - e^2)^{1/2} p^{-3} e (- 2 b_1 b_2^2 p + b_2^4)$$

$$A_{12} = \frac{3}{32} (1 - e^2)^{1/2} p^{-3} b_2^4 e^2$$

If $(b_1/b_2) < 1$, then compute:

$$A_{21} = (1 - e^2)^{1/2} p^{-1} e \left[b_1 p^{-1} + (3 b_1^2 - b_2^2) p^{-2} - \frac{9}{2} b_1 b_2^2 \left(1 + \frac{1}{4} e^2\right) p^{-3} + \frac{3}{8} b_2^4 p^{-4} (4 + 3e^2) \right]$$

$$A_{22} = (1 - e^2)^{1/2} p^{-1} \left[\frac{1}{8} e^2 (3 b_1^2 - b_2^2) p^{-2} - \frac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3}{32} b_2^4 p^{-4} (6e^2 + e^4) \right]$$

$$A_{23} = \frac{1}{8} (1 - e^2)^{1/2} p^{-1} e^3 (- b_1 b_2^2 p^{-3} + b_2^4 p^{-4})$$

$$A_{24} = \frac{3}{256} (1 - e^2)^{1/2} p^{-5} b_2^4 e^4$$

$$A_{31} = (1 - e^2)^{1/2}\, p^{-3}\, e \left[2 + b_1 p^{-1} \left(3 + \frac{3}{4} e^2\right) - p^{-2} \left(\frac{1}{2} b_2^2 + c^2\right) (4 + 3e^2)\right]$$

$$A_{32} = (1 - e^2)^{1/2}\, p^{-3} \left[\frac{1}{4} e^2 + \frac{3}{4} b_1 p^{-1} e^2 - p^{-2} \left(\frac{1}{2} b_2^2 + c^2\right)\left(\frac{3}{2} e^2 + \frac{1}{4} e^4\right)\right]$$

$$A_{33} = (1 - e^2)^{1/2}\, p^{-3}\, e^3 \left[\frac{1}{12}\, p^{-1}\, b_1 - \frac{1}{3} p^{-2} \left(\frac{1}{2} b_2^2 + c^2\right)\right]$$

$$A_{34} = -\frac{1}{32}\, (1 - e^2)^{1/2}\, e^4\, p^{-5} \left(\frac{1}{2}\, b_2^2 + c^2\right)$$

If $(b_1/b_2) \geq 1$ (corresponding to a near-equatorial orbit or an equatorial orbit), then compute instead:

$$A_{21} = (1 - e^2)^{1/2}\, p^{-1}\, e \left[b_1 p^{-1} + (3 - b_2^2\, b_1^{-2})\, c^4\, p^{-4}\, (1 - \eta_0^2)^2\right]$$

$$A_{22} = \frac{1}{8} (1 - e^2)^{1/2}\, p^{-1}\, e^2\, (3 - b_2^2 b_1^{-2})\, c^4\, p^{-4}\, (1 - \eta_0^2)^2$$

A_{23} and A_{24} as given above.

$$A_{31} = (1 - e^2)^{1/2}\, p^{-3} e \left[2 + \left(3 + \frac{3}{4} e^2\right) c^2\, p^{-2}\, (1 - \eta_0^2) - (4 + 3\,e^2)\, c^2 p^{-2}\right]$$

$$A_{32} = (1 - e^2)^{1/2}\, p^{-3}\, e^2 \left[\frac{1}{4} + \frac{3}{4} c^2\, p^{-2}\, (1 - \eta_0^2) - \left(\frac{1}{4}\, e^2 + \frac{3}{2}\right) c^2\, p^{-2}\right]$$

$$A_{33} = \frac{1}{3} (1 + e^2)^{1/2}\, p^{-5}\, e^3\, c^2 \left[\frac{1}{4}\, (1 - \eta_0^2) - 1\right]$$

$$A_{34} = -\frac{1}{32}\, (1 - e^2)^{1/2}\, p^{-5}\, e^4\, c^2$$

Now compute:

$$2\pi\nu_1 = (-2\alpha_1)^{1/2}\, (a + b_1 + A_1 + c^2\, \eta_0^2\, A_2\, B_1\, B_2^{-1})^{-1}$$

If $(b_1/b_2) < 1$, compute:

$$2\pi\nu_2 = (\alpha_2^2 - \alpha_3^2)^{1/2}\, \eta_0^{-1}\, A_2\, B_2^{-1}\, (a + b_1 + A_1 + c^2\, \eta_0^2\, A_2\, B_1\, B_2^{-1})^{-1}$$

If $(b_1/b_2) \geq 1$, compute instead:

$$2\pi\nu_2 = \alpha_2 [1 - 2\alpha_1 \alpha_2^{-2} c^2 (1 - \eta_0^2)]^{1/2} A_2 B_2^{-1} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} .$$

Then compute:

$$e' = ae(a + b_1)^{-1} .$$

Note that the condition $e' \leq e$ must be fulfilled.

THE JACOBI CONSTANTS OF GENERALIZED CO-ORDINATES

If $e \neq 0$, then compute:

$$\sin E_i = \frac{\dot{\rho}_i (\rho_i^2 + \eta_i^2 c^2)}{ae \sqrt{(-2\alpha_1)(\rho_i^2 + A\rho_i + B)}}$$

$$\cos E_i = (1 - \rho_i a^{-1}) e^{-1} .$$

From the above, we obtain E_i within the limits $0 \leq E_i < 2\pi$.

$$\sin v_i = \frac{(1 - e^2)^{1/2} \sin E_i}{1 - e \cos E_i}$$

$$\cos v_i = \frac{\cos E_i - e}{1 - e \cos E_i} .$$

From the above, we obtain v_i within the limits $0 \leq v_i < 2\pi$.

If $e = 0$, then:

$$v_i = 0$$

and

$$E_i = 0 .$$

If $\eta_0 \neq 0$, then compute:

$$\cos \psi_i = \frac{\dot{\eta}_i (\rho_i^2 + \eta_i^2 c^2)}{c\eta_0 \sqrt{(-2\alpha_1)(\eta_2^2 - \eta_i^2)}}$$

$$\sin \psi_i = \frac{\eta_i}{\eta_0} .$$

From the above, we obtain ψ_i within the limits $0 \leq \psi_i < 2\pi$.

$$\sin \chi_i = \frac{(1-\eta_0^2)^{1/2} \sin \psi_i}{\sqrt{1-\eta_0^2 \sin^2 \psi_i}}$$

$$\cos \chi_i = \frac{\cos \psi_i}{\sqrt{1-\eta_0^2 \sin^2 \psi_i}}.$$

From the above, we obtain χ_i within the limits $0 \leq \chi_i < 2\pi$.

If $\eta_0 = 0$, then: $\psi_i = \phi_i$ and $\chi_i = \phi_i$.

Now compute:

$$\sin n v_i \quad \text{for} \quad n = 2, 3, 4.$$

$$\sin n\psi_i \quad \text{for} \quad n = 2, 4.$$

If $(b_1/b_2) < 1$, then compute:

$$\begin{aligned}\beta_1 = {} & (-2\alpha_1)^{-1/2} [b_1 E_i + a(E_i - e \sin E_i) + A_1 v_i \\ & + A_{11} \sin v_i + A_{12} \sin 2v_i] + c^2(\alpha_2^2 - \alpha_3^2)^{-1/2} \eta_0^3 \Big[B_1 \psi_i \\ & - \frac{1}{8}(2 + q^2) \sin 2\psi_i + \frac{1}{64} q^2 \sin 4\psi_i\Big] - t_i\end{aligned}$$

$$\begin{aligned}\beta_2 = {} & -\alpha_2(-2\alpha_1)^{-1/2} [A_2 v_i + A_{21} \sin v_i + A_{22} \sin 2v_i \\ & + A_{23} \sin 3v_i + A_{24} \sin 4v_i] + (\alpha_2^2 - \alpha_3^2)^{-1/2} \eta_0 \alpha_2 \Big[B_2 \psi_i \\ & - \frac{1}{32} q^2(4 + 3q^2) \sin 2\psi_i + \frac{3}{256} q^4 \sin 4\psi_i\Big].\end{aligned}$$

If $(b_1/b_2) \geq 1$, then compute instead:

$$\begin{aligned}\beta_1 = {} & (-2\alpha_1)^{-1/2} [b_1 E_i + a(E_i - e \sin E_i) + A_1 v_i \\ & + A_{11} \sin v_i + A_{12} \sin 2v_i] + c^2 \eta_0^2 \alpha_2^{-1} [1 - 2\alpha_1 \alpha_2^{-2} c^2 (1 - \eta_0^2)]^{-1/2} \Big[B_1 \psi_i \\ & - \frac{1}{8}(2 + q^2) \sin 2\psi_i + \frac{1}{64} q^2 \sin 4\psi_i\Big] - t_i\end{aligned}$$

$$\beta_2 = -\alpha_2(-2\alpha_1)^{-1/2}[A_2 v_i + A_{21}\sin v_i + A_{22}\sin 2v_i$$

$$+ A_{23}\sin 3v_i + A_{24}\sin 4v_i] + [1 - 2\alpha_1\alpha_2^{-2}c^2(1-\eta_0^2)]^{-1/2}\Big[B_2\psi_i$$

$$-\frac{1}{32}q^2(4+3q^2)\sin 2\psi_i + \frac{3}{256}q^4\sin 4\psi_i\Big].$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) < 1$, then compute:

$$\beta_3 = \phi_i - \alpha_3(\alpha_2^2 - \alpha_3^2)^{-1/2}\eta_0\Big[(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\chi_i$$

$$+ B_3\psi_i + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi_i\Big] + c^2\alpha_3(-2\alpha_1)^{-1/2}[A_3 v_i$$

$$+ A_{31}\sin v_i + A_{32}\sin 2v_i + A_{33}\sin 3v_i + A_{34}\sin 4v_i].$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) \geq 1$, compute instead:

$$\beta_3 = \phi_i - \alpha_3\alpha_2^{-1}[1 - 2\alpha_1\alpha_2^{-2}c^2(1-\eta_0^2)]^{-1/2}\ [(1-\eta_0^2)^{-1/2}\ (1-\eta_2^{-2})^{-1/2}\chi_i$$

$$+ B_3\psi_i] + c^2\alpha_3(-2\alpha_1)^{-1/2}[A_3 v_i + A_{31}\sin v_i + A_{32}\sin 2v_i$$

$$+ A_{33}\sin 3v_i + A_{34}\sin 4v_i]$$

If $\eta_0 = 0$, compute instead:

$$\beta_3 = \phi_i - \beta_2(\operatorname{sgn}\alpha_3) - \alpha_3(-2\alpha_1)^{-1/2}[A_2 v_i$$

$$+ A_{21}\sin v_i + A_{22}\sin 2v_i] + c^2\alpha_3(-2\alpha_1)^{-1/2}[A_3 v_i$$

$$+ A_{31}\sin v_i + A_{32}\sin 2v_i + A_{33}\sin 3v_i + A_{34}\sin 4v_i]$$

where

$$\operatorname{sgn}\alpha_3 \equiv \frac{\alpha_3}{|\alpha_3|}.$$

THE ORBIT GENERATOR OF POSITION AND VELOCITY COMPONENTS

In this section, parameters arise which are time-dependent. Initially, the value for time t is equal to t_i, but on subsequent iterations $t = t_i + n(\Delta t)$, $n = 1, 2, 3, \ldots$. Here Δt is the time increment input parameter used in generating position and velocity components for equal time intervals following t_i and preceding or coincident with the final time t_f.

If $\eta_0 \neq 0$ and if $(b_1/b_2) < 1$, then compute:

$$M_s = 2\pi\nu_1(t + \beta_1 - c^2\beta_2\alpha_2^{-1}\eta_0^2 B_1 B_2^{-1})$$

$$\psi_s = 2\pi\nu_2[t + \beta_1 + \beta_2\alpha_2^{-1}A_2^{-1}(a + b_1 + A_1)]$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) \geq 1$, then compute instead:

$$M_s = (-2\alpha_1)^{1/2} \left[\frac{B_2(t+\beta_1) - c^2\beta_2\alpha_2^{-1}\eta_0^2 B_1}{(a+b_1+A_1)B_2 + c^2\eta_0^2 A_2B_1}\right]$$

ψ_s as given above.

If $\eta_0 = 0$, then compute instead:

$$M_s = (-2\alpha_1)^{1/2}\,(t+\beta_1)(a+b_1+A_1)^{-1}$$

$$\psi_s = (1-2\alpha_1\alpha_2^{-2}c^2)^{1/2}\,[\beta_2 + \alpha_2 A_2 (t+\beta_1)(a+b_1+A_1)^{-1}]$$

We now solve the following equation for $(M_s + E_0)$:

$$M_s + E_0 - e' \sin (M_s + E_0) = M_s .$$

If we let $\mathcal{E} = M_s + E_0$, then we can solve this equation (known as Kepler's equation) by use of the iterative Newton-Raphson method.

$$\mathcal{E}_{n+1} = \mathcal{E}_n - \frac{(\mathcal{E}_n - e' \sin \mathcal{E}_n - M_s)}{(1 - e' \cos \mathcal{E}_n)}$$

$$- \frac{(\mathcal{E}_n - e' \sin \mathcal{E}_n - M_s)^2 (e' \sin \mathcal{E}_n)}{2(1 - e' \cos \mathcal{E}_n)^3}$$

For the initial value, $(\mathcal{E}_n)_{n=0} = M_s$. Iteration ceases when

$$\left| \frac{\mathcal{E}_n}{\mathcal{E}_{n+1}} - 1 \right| \leq \epsilon$$

where ϵ is a pre-selected tolerance (e.g., 10^{-7}). Convergence should be attained with several iterations.

Now use the anomaly connections:

$$\cos v' = (\cos \mathcal{E} - e)(1 - e \cos \mathcal{E})^{-1}$$

$$\sin v' = (1 - e^2)^{1/2}\,(1 - e \cos \mathcal{E})^{-1} \sin \mathcal{E} .$$

From the above, we obtain v' within the limits $0 \leq v' < 2\pi$. The angle v' is then placed within the same circle of revolution as the angle $\mathcal{E}$ (which is not taken modulo 2π).

Then compute: $$v_0 = v' - M_s .$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) < 1$, then compute:

$$\psi_0 = (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} A_2 B_2^{-1} v_0$$

$$M_1 = (a + b_1)^{-1} \Big[-(A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}) v_0$$

$$+ \frac{1}{4} c^2 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \eta_0^3 \sin (2\psi_s + 2\psi_0) \Big]$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) \geq 1$, then compute instead:

$$\psi_0 = (-2\alpha_1)^{-1/2} \left[1 - 2\alpha_1 \alpha_2^{-2} c^2 (1 - \eta_0^2) \right]^{1/2} \alpha_2 A_2 B_2^{-1} v_0$$

$$M_1 = (a + b_1)^{-1} \Big[-(A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}) v_0$$

$$+ \frac{1}{4} c^2 (-2\alpha_1)^{1/2} \left[1 - 2\alpha_1 \alpha_2^{-2} c^2 (1 - \eta_0^2) \right]^{-1/2} \eta_0^2 \alpha_2^{-1} \sin (2\psi_s + 2\psi_0) \Big]$$

If $\eta_0 = 0$, then compute instead:

$$\psi_0 = (-2\alpha_1)^{-1/2} (1 - 2\alpha_1 \alpha_2^{-2} c^2)^{1/2} \alpha_2 A_2 v_0$$

$$M_1 = -(a + b_1)^{-1} A_1 v_0$$

Continuing, if $\eta_0 \neq 0$ and $(b_1/b_2) < 1$, <u>or</u> if $\eta_0 = 0$, then compute:

$$E_1 = (1 - e' \cos \mathcal{E})^{-1} M_1 - \frac{1}{2} e' (1 - e' \cos \mathcal{E})^{-3} M_1^2 \sin \mathcal{E}$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) \geq 1$, then compute instead:

$$E_1 = (1 - e' \cos \mathcal{E})^{-1} M_1$$

Now use the anomaly connections again:

$$\cos v'' = [\cos (\mathcal{E} + E_1) - e] [1 - e \cos (\mathcal{E} + E_1)]^{-1}$$

$$\sin v'' = (1 - e^2)^{1/2} [1 - e \cos (\mathcal{E} + E_1)]^{-1} \sin (\mathcal{E} + E_1)$$

From the above, find the angle v'' and place it within the same circle of revolution as the angle $(\mathcal{E} + E_1)$.

Then compute: $$v_1 = v'' - v'$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) < 1$, then compute:

$$\psi_1 = (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} B_2^{-1} (A_2 v_1 + A_{21} \sin v'$$

$$+ A_{22} \sin 2v') + \frac{1}{8} q^2 B_2^{-1} \sin (2\psi_s + 2\psi_0)$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) \geq 1$, then compute instead:

$$\psi_1 = (-2\alpha_1)^{-1/2} [1 - 2\alpha_1 \alpha_2^{-2} c^2 (1 - \eta_0^2)]^{1/2} \alpha_2 B_2^{-1} (A_2 v_1$$

$$+ A_{21} \sin v' + A_{22} \sin 2v') + \frac{1}{8} q^2 B_2^{-1} \sin (2\psi_s + 2\psi_0)$$

If $\eta_0 = 0$, then compute instead:

$$\psi_1 = (-2\alpha_1)^{-1/2} (1 - 2\alpha_1 \alpha_2^{-2} c^2)^{1/2} \alpha_2 (A_2 v_1 + A_{21} \sin v' + A_{22} \sin 2v')$$

Now if $(b_1/b_2) < 1$, we continue this procedure one step further to obtain terms of second order, as follows. Compute:

$$M_2 = -(a + b_1)^{-1} \Big\{ A_1 v_1 + A_{11} \sin v' + A_{12} \sin 2v'$$

$$+ c^2 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \eta_0^3 \Big[B_1 \psi_1 - \frac{1}{2} \psi_1 \cos (2\psi_s + 2\psi_0)$$

$$- \frac{1}{8} q^2 \sin (2\psi_s + 2\psi_0) + \frac{1}{64} q^2 \sin (4\psi_s + 4\psi_0) \Big] \Big\}$$

$$E_2 = [1 - e' \cos (\mathcal{E} + E_1)]^{-1} M_2$$

Let

$$E = \mathcal{E} + E_1 + E_2$$

and use the anomaly connections once again:

$$\cos v''' = (\cos E - e)(1 - e \cos E)^{-1}$$

$$\sin v''' = (1 - e^2)^{1/2} (1 - e \cos E)^{-1} \sin E$$

From the above, find the angle v'' and place it within the same circle of revolution as the angle E.

Then compute:

$$v_2 = v''' - v''$$

$$\psi_2 = (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} B_2^{-1} (A_2 v_2 + A_{21} v_1 \cos v'$$

$$+ 2A_{22} v_1 \cos 2v' + A_{23} \sin 3v' + A_{24} \sin 4v')$$

$$+ \frac{1}{4} q^2 B_2^{-1} \Big[\psi_1 \cos (2\psi_s + 2\psi_0) + \frac{3}{8} q^2 \sin (2\psi_s + 2\psi_0) - \frac{3}{64} q^2 \sin (4\psi_s + 4\psi_0) \Big]$$

Finally, let:

$$v = M_s + v_0 + v_1 + v_2$$

$$\psi = \psi_s + \psi_0 + \psi_1 + \psi_2$$

Now if $(b_1/b_2) \geq 1$, we omit computation of M_2, E_2, v_2, and ψ_2. In such case, these terms become of the third order and hence negligible.

Instead, we let:

$$E = \mathcal{E} + E_1$$

$$v = M_s + v_0 + v_1$$

$$\psi = \psi_s + \psi_0 + \psi_1 \; .$$

Continuing, compute:

$$\sin \chi = (1 - \eta_0^2)^{1/2} (1 - \eta_0^2 \sin^2 \psi)^{-1/2} \sin \psi$$

$$\cos \chi = (1 - \eta_0^2 \sin^2 \psi)^{-1/2} \cos \psi$$

From the above, find the angle χ and place it within the same circle of revolution as the angle ψ.

If $(b_1/b_2) < 1$, then compute:

$$\rho = (1 + e \cos v)^{-1} p$$

If $(b_1/b_2) \geq 1$, then compute instead:

$$\rho = a (1 - e \cos E)$$

Then, if $\eta_0 \neq 0$ and if $(b_1/b_2) < 1$, compute:

$$\eta = \eta_0 \sin \psi$$

$$\phi = \beta_3 + \alpha_3 (\alpha_2^2 - \alpha_3^2)^{-1/2} \eta_0 \Big[(1 - \eta_0^2)^{-1/2} (1 - \eta_2^{-2})^{-1/2} \chi$$

$$+ B_3 \psi + \frac{3}{32} \eta_0^2 \eta_2^{-4} \sin 2\psi\Big] - c^2 \alpha_3 (-2\alpha_1)^{-1/2} (A_3 v$$

$$+ A_{31} \sin v + A_{32} \sin 2v + A_{33} \sin 3v + A_{34} \sin 4v)$$

If $\eta_0 \neq 0$ and if $(b_1/b_2) \geq 1$, compute instead:

η as given above.

$$\phi = \beta_3 + \alpha_3 \alpha_2^{-1} \left[1 - 2\alpha_1 \alpha_2^{-2} c^2 (1 - \eta_0^2)\right]^{-1/2} \Big[(1 - \eta_0^2)^{-1/2} (1 - \eta_2^{-2})^{-1/2} \chi$$

$$+ B_3 \psi\Big] - c^2 \alpha_3 (-2\alpha_1)^{-1/2} (A_3 v + A_{31} \sin v$$

$$+ A_{32} \sin 2v + A_{33} \sin 3v + A_{34} \sin 4v)$$

If $\eta_0 = 0$, compute instead

$$\eta = 0$$

$$\phi = \beta_3 + \beta_2 (\mathrm{sgn}\ \alpha_3) + \alpha_3 (-2\alpha_1)^{-1/2} (A_2 v + A_{21} \sin v + A_{22} \sin 2v) - c^2 \alpha_3 (-2\alpha_1)^{-1/2} (A_3 v + A_{31} \sin v + A_{32} \sin 2v + A_{33} \sin 3v + A_{34} \sin 4v)$$

where

$$\mathrm{sgn}\ \alpha_3 \equiv \frac{\alpha_3}{|\alpha_3|}.$$

The oblate spheroidal co-ordinates must satisfy the following conditions:

$$\rho \geq 0$$

$$-1 \leq \eta \leq +1$$

Now the co-ordinates and velocities may be found as follows:

$$h_\phi^2 = (\rho^2 + c^2)(1 - \eta^2)$$

$$\dot{\rho} = a\, e\, (-2\alpha_1)^{1/2} (\rho^2 + A\rho + B)^{1/2} (\rho^2 + \eta^2 c^2)^{-1} \sin E$$

$$\dot{\eta} = c\, \eta_0 (-2\alpha_1)^{1/2} (\eta_2^2 - \eta^2)^{1/2} (\rho^2 + \eta^2 c^2)^{-1} \cos \psi$$

$$X = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \cos \phi$$

$$Y = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \sin \phi$$

$$Z = \rho\, \eta$$

$$\dot{X} = X \left[\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right] - Y \left(\frac{\alpha_3}{h_\phi^2}\right)$$

$$\dot{Y} = Y \left[\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right] + X \left(\frac{\alpha_3}{h_\phi^2}\right)$$

$$\dot{Z} = \rho\, \dot{\eta} + \eta\, \dot{\rho}$$

This completes the algorithm for predicting orthogonal position and velocity components of the satellite based upon a set of initial conditions.

COMPUTATION OF DIRECTION COSINES

Often the set of initial conditions (position and velocity components) provided is only approximate at best, and thus the orbit that is predicted based on these initial conditions will similarly contain inaccuracies. In order to remove these inaccuracies and to account for the effects of forces not considered by the analytical theory (e.g., aerodynamic drag, solar radiation, meteoric bombardment, etc.) the orbital parameters are corrected by comparison with those found by direct observation. The orbit improvement method produces a mean set of orbital elements through an iterated least-squares fitting of the differential solution to numerous observational values.

To perform the differential correction process, the following data must be available in addition to the constants listed in the section on "Input Parameters" above:

$f \equiv 1 - r_p/r_e$, the flattening coefficient of the Earth (where r_p is the polar radius of the Earth, and r_e is the equatorial radius), taken to be approximately $1/298.3 = 3.3523299 \times 10^{-3}$.

ω, the rotational rate of the Earth in radians per mean solar hour (taken to be 0.26251614).

$(\lambda_E)_i$, $i = 1, 2, \ldots, s$, the geodetic (or geographic) longitude of the terrestrial tracking stations in radians, as measured eastward from Greenwich (a negative sign must be prefixed if measured westward from Greenwich). We assume that there are s tracking stations reporting observational data used for comparison purposes.

$(\theta_D)_i$, $i = 1, 2, \ldots, s$, the geodetic (or geographic) latitude of the stations in radians, measured as positive north of the equator and as negative south of the equator $(-\pi/2 \le \theta_D \le +\pi/2)$.

$(H)_i$, $i = 1, 2, \ldots, s$, the altitude of the stations in feet, measured as positive above sea level and as negative below sea level.

$(\lambda_0)_d$, $d = 1, 2, \ldots$, the angle in radians, measured eastward from the vernal equinox (the first point of Aries) to the Greenwich meridian at midnight Greenwich mean time for each day d during the period that observations are provided. The apparent sidereal time (the hour angle of the first point of Aries) at midnight Greenwich mean time for each day throughout the year is tabulated in "The American Ephemeris and Nautical Almanac."

t_0, a reference time preceding or coinciding with the time of the first observation provided, which is used as the zero point in determining t, the relative observation time. It may be the time of injection if the tracking data includes observations made during the first several orbits of the satellite. The purpose of determining a relative observation time t is to eliminate any reference to the calendar.

We now describe the observation data cards, which are effectively input for the differential correction scheme. There are several methods of recording satellite tracking data; we present here one of the most common methods, referred to as Minitrack observation data. (Refer to the appendix of this report for discussion of another method.) A set of observation data of this type includes the following parameters for each recorded spacecraft observation:

t', the date and time of observation. As given, t' is a calendar time. We remove any dependence on the calendar by determining $t = t' - t_0$, where t is the relative observation time and t_0 is a reference calendar time. Then t becomes a time interval, measured in Vanguard units of time from the zero point t_0. As mentioned above, t_0 is chosen so that for all observations $t \ge 0$. It is convenient to have the choice of t_0 coincide with that corresponding to the initial position and velocity conditions $X_i, Y_i, Z_i, \dot{X}_i, \dot{Y}_i, \dot{Z}_i$. When this choice is made, then t_0 is known as an initial or epoch time.

k, the code number for the tracking station reporting the observation. Generally, the range of k is the set of integers 1, 2, 3, . . . , s.

L_0, the observed direction cosine in the X-direction.

M_0, the observed direction cosine in the Y-direction.

w_L and w_M, the weighting factors corresponding to observations L_0 and M_0, respectively. This information is optional; if not provided, then it is assumed that w_L and w_M are each unity.

The co-ordinate system employed for the observation data is centered at the tracking station on the Earth's surface, with the X-Y plane tangent to the surface. It is a right-handed, orthogonal system with the X-axis extending in an easterly direction along the line of latitude, the Y-axis extending in a northerly direction along the line of longitude, and the Z-axis normal to the surface and pointing toward the geodetic zenith.

We first compute the so-called "auxiliary functions" S and C (refer to the "Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac") from the relations:

$$C = [\cos^2 \theta_D + (1 - f)^2 \sin^2 \theta_D]^{-1/2}$$

$$S = (1 - f)^2 C$$

Here and in the following, we eliminate use of the subscript "i" referring to an individual one of the s tracking stations, and assume that the computations given are performed for each respective station. The value of H is then converted from its input units of feet to units of the Earth's equatorial radius (the conversion factor is 4.77865×10^{-8}) so as to conform to the canonical Vanguard system of units used throughout (see above under "Input Parameters"). Then the geocentric latitude is given by:

$$\theta_G = \arctan \left[\left(\frac{S + H}{C + H}\right) \tan \theta_D\right].$$

Now the geocentric distance of the observation point (i.e., tracking station), in units of the Earth's equatorial radius, is found:

$$\hat{\rho} = [(S + H)^2 \sin^2 \theta_D + (C + H)^2 \cos^2 \theta_D]^{1/2}.$$

The angle δ, between the vernal equinox and the observation meridian plane, is computed in radian measure by the following expression:

$$\delta = (\lambda_0)_d + \omega\,(\Delta T) + \lambda_E.$$

Here, $(\lambda_0)_d$ is as defined above with the value chosen (designated by the subscript "d") corresponding to the midnight immediately preceding observation time. Also, ΔT is the computed time, in hours, between observation time and midnight preceding observation time. Thus, the second term in the expression for δ accounts for the fact that the Greenwich meridian rotates while the vernal equinox remains fixed in inertial space.

The inertial geocentric co-ordinates of the observation point are now converted from a spherical to a Cartesian system by means of the following equations:

$$X_T = \hat{\rho} \cos \theta_G \cos \delta$$

$$Y_T = \hat{\rho} \cos \theta_G \sin \delta$$

$$Z_T = \hat{\rho} \sin \theta_G$$

The angle ψ_x, measured in the observation latitude plane between the vernal equinox and the tracking station's X-co-ordinate axis, is given in radian measure by:

$$\psi_x = \delta + \frac{\pi}{2}$$

The local co-ordinates of the satellite (X_M, Y_M, Z_M) can now be determined from the inertial position vector (X, Y, Z) computed by the orbit generator. Here "local" refers to co-ordinates measured at the tracking station. The orbit generator will produce the position components (X, Y, Z) at the observation time. Then the local co-ordinates are given by the matrix relation:

$$\begin{bmatrix} X_M \\ Y_M \\ Z_M \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \theta_D & \cos \theta_D \\ 0 & -\cos \theta_D & \sin \theta_D \end{bmatrix} \begin{bmatrix} \cos \psi_x & \sin \psi_x & 0 \\ -\sin \psi_x & \cos \psi_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} X_T \\ Y_T \\ Z_T \end{bmatrix} \right\}$$

Here, the difference of column matrices on the extreme right represents a translation from the Earth's center to the tracking station position; the center matrix on the right represents a rotation in the latitude plane about the polar axis through an angle of ψ_x to bring the inertial X-axis into coincidence with the station's X_M-axis; the left matrix represents a rotation in the longitude plane about the X_M-axis through an angle of $(\pi/2 - \theta_D)$ to bring the inertial Z-axis into coincidence with the station's Z_M-axis. This matrix equation is equivalent to:

$$\begin{bmatrix} X_M \\ Y_M \\ Z_M \end{bmatrix} = \begin{bmatrix} \cos \psi_x & \sin \psi_x & 0 \\ -\sin \psi_x \sin \theta_D & \cos \psi_x \sin \theta_D & \cos \theta_D \\ \sin \psi_x \cos \theta_D & -\cos \psi_x \cos \theta_D & \sin \theta_D \end{bmatrix} \begin{bmatrix} X - X_T \\ Y - Y_T \\ Z - Z_T \end{bmatrix}$$

or, explicitly stated:

$$X_M = (X - X_T) \cos \psi_x + (Y - Y_T) \sin \psi_x$$

$$Y_M = -(X - X_T) \sin \psi_x \sin \theta_D + (Y - Y_T) \cos \psi_x \sin \theta_D + (Z - Z_T) \cos \theta_D$$

$$Z_M = (X - X_T) \sin \psi_x \cos \theta_D - (Y - Y_T) \cos \psi_x \cos \theta_D + (Z - Z_T) \sin \theta_D$$

We now find the computed values of the direction cosines L_c (in the X-direction) and M_c (in the Y-direction) in terms of the local co-ordinates.

$$L_c = \frac{X_M}{(X_M^2 + Y_M^2 + Z_M^2)^{1/2}}$$

$$M_c = \frac{Y_M}{(X_M^2 + Y_M^2 + Z_M^2)^{1/2}}$$

Of course, the computed value of the third direction cosine N_c (in the Z-direction) is predetermined by L_c and M_c through the relation:

$$N_c = (1 - L_c^2 - M_c^2)^{1/2}$$

THE STANDARD DEVIATION OF FIT

The differences between the observed and computed values of the direction cosines can now be found:

$$\Delta L = L_0 - L_c$$

$$\Delta M = M_0 - M_c$$

These differences are sometimes referred to as "residuals", although this term is also used in a different sense in the method of fitting by least squares. We compute these differences for each observation in the set of observation data. The number of observations in the set is variable, and it may be determined by an input parameter n.

The average residual is given by:

$$\bar{R} = \frac{1}{2n} \sum_{i=1}^{n} (\Delta L_i + \Delta M_i),$$

where the subscript "i" ranges over individual observations.

The standard deviation of the residuals from their mean value is found from:

$$\sigma = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \left[(\Delta L_i - \bar{R})^2 + (\Delta M_i - \bar{R})^2\right]}$$

The standard deviation of the residuals (from zero) is called the standard deviation of fit, and is given by:

$$\sigma_f = \sqrt{\frac{1}{2n-6} \sum_{i=1}^{n} \left[(\Delta L_i)^2 + (\Delta M_i)^2\right]}$$

As is customary, the larger multiplicative factor $(2n - 6)^{-1}$ is used to indicate the excess of simultaneous equations of condition over the number of independent coefficients (see below under section titled, "Fitting by Method of Least Squares").

We may also determine an acceptable range of values for the residuals, bounded by an lower limit r_1 and an upper limit r_2, based upon the standard deviation. If a residual falls outside this range, it may be rejected, with statistical validity, from inclusion in the fitting process. For example, we may choose:

$$r_1 = \bar{R} - j\sigma$$

$$r_2 = \bar{R} + j\sigma$$

For normal (Gaussian) distributions, 68.27 percent of the cases are included within one standard deviation on either side of the mean ($j = 1$ above), 95.45 percent of the cases are included within two standard deviations ($j = 2$ above), and 99.73 percent of the cases are included within three standard deviations ($j = 3$ above). For moderately skewed distributions, the above percentages may hold approximately. If certain of the residuals are rejected on this statistical basis, the standard deviation of the accepted residuals only may be computed as a "working" standard deviation of fit. Its value is computed exactly as is σ_f above, with certain terms omitted in the summation, and should be substantially smaller in magnitude than σ_f.

ANALYTICAL PROCEDURE OF DIFFERENTIAL CORRECTION

The first-order Taylor series expansion of the equations of condition may be written:

$$\Delta L = L_0 - L_c = \sum_{i=1}^{6} \frac{\partial L_c}{\partial q_i} \Delta q_i$$

$$\Delta M = M_0 - M_c = \sum_{i=1}^{6} \frac{\partial M_c}{\partial q_i} \Delta q_i$$

where q_i ($i = 1, 2, \ldots, 6$) are the mean or Izsak elements given below.

$q_1 = a$, the semi-major axis.

$q_2 = e$, the eccentricity.

$q_3 = \eta_0 \equiv \sin I$, where I is the inclination of the orbital plane to the equator.

$q_4 = \beta_1$, corresponds to the negative of the time of passage through perigee in Keplerian motion.

$q_5 = \beta_2$, corresponds to the argument of perigee in Keplerian motion.

$q_6 = \beta_3$, corresponds to the right ascension of the ascending node in Keplerian motion.

We may expand the above partial derivatives by the chain rule as follows:

$$\frac{\partial L_c}{\partial q_i} = \frac{\partial L_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial L_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial L_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i}$$

$$\frac{\partial M_c}{\partial q_i} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i}$$

From the equations for L_c and M_c in terms of the local co-ordinates given earlier (refer to the section titled, "Computation of Direction Cosines"), we find directly:

$$\frac{\partial L_c}{\partial X_M} = (X_M^2 + Y_M^2 + Z_M^2)^{-1/2} - X_M^2 (X_M^2 + Y_M^2 + Z_M^2)^{-3/2}$$

$$\frac{\partial L_c}{\partial Y_M} = - X_M Y_M (X_M^2 + Y_M^2 + Z_M^2)^{-3/2}$$

$$\frac{\partial L_c}{\partial Z_M} = - X_M Z_M (X_M^2 + Y_M^2 + Z_M^2)^{-3/2}$$

$$\frac{\partial M_c}{\partial X_M} = - X_M Y_M (X_M^2 + Y_M^2 + Z_M^2)^{-3/2} = \frac{\partial L_c}{\partial Y_M}$$

$$\frac{\partial M_c}{\partial Y_M} = (X_M^2 + Y_M^2 + Z_M^2)^{-1/2} - Y_M^2 (X_M^2 + Y_M^2 + Z_M^2)^{-3/2}$$

$$\frac{\partial M_c}{\partial Z_M} = - Y_M Z_M (X_M^2 + Y_M^2 + Z_M^2)^{-3/2}$$

Since the co-ordinates X_T, Y_T, Z_T and the angles ψ_x and θ_D are independent of orbital parameters (and merely geodesic functions), we have the matrix relation:

$$\begin{bmatrix} \frac{\partial X_M}{\partial q_i} \\ \frac{\partial Y_M}{\partial q_i} \\ \frac{\partial Z_M}{\partial q_i} \end{bmatrix} = \begin{bmatrix} \cos\psi_x & \sin\psi_x & 0 \\ -\sin\psi_x \sin\theta_D & \cos\psi_x \sin\theta_D & \cos\theta_D \\ \sin\psi_x \cos\theta_D & -\cos\psi_x \cos\theta_D & \sin\theta_D \end{bmatrix} \begin{bmatrix} \frac{\partial X}{\partial q_i} \\ \frac{\partial Y}{\partial q_i} \\ \frac{\partial Z}{\partial q_i} \end{bmatrix}$$

We find $\frac{\partial X}{\partial q_i}, \frac{\partial Y}{\partial q_i}, \frac{\partial Z}{\partial q_i}$ by substituting:

$$\rho = a(1 - e\cos E) \quad \text{and} \quad \eta = \eta_0 \sin\psi$$

in the relations:

$$X = \sqrt{(\rho^2 + c^2)(1 - \eta^2)}\cos\phi$$

$$Y = \sqrt{(\rho^2 + c^2)(1 - \eta^2)}\sin\phi$$

$$Z = \rho\eta$$

and then determining $\partial E/\partial q_i$, $\partial\psi/\partial q_i$, and $\partial\phi/\partial q_i$. Here E and ψ are uniformizing variables, analogous respectively to the eccentric anomaly and the argument of latitude in elliptic motion. The parameter ϕ is the third oblate spheroidal co-ordinate, the geocentric right ascension. The procedure for determining the eighteen partial derivatives $\partial E/\partial q_i$, $\partial\psi/\partial q_i$, and $\partial\phi/\partial q_i$ is a rather lengthy one which is initiated in the next section.

Before embarking upon this procedure, the following comments are in order. The analytical partial derivatives in the differential correction given in this report correspond to the case where $(b_1/b_2) < 1$ only. This excludes equatorial and near-equatorial orbits. More specifically, orbits where the inclination I is such that

$$0 \leq I \leq I_c \quad \text{or} \quad 180° - I_c \leq I \leq 180°,$$

where I_c might be as large as 1°54' for an orbit sufficiently close to the Earth, are excluded. The analytical partial derivatives for the case in which $(b_1/b_2) \geq 1$ are, in fact, simpler in form, and they are derived in an analogous manner from the equations for this special case presented earlier in this report.

The differential correction process may be carried through to terms of second order, or it may be simplified to omit terms of purely second order. This is not quite the same as carrying the process through to terms of first order, since some second-order effects are included even when terms of purely second order are dropped. Generally, the speed of computation in the differential correction will be increased considerably by neglecting purely second-order terms, without risking any great loss in precision of the final differential coefficients for the conditional equations. It should be remembered, however, that even a slight loss in the precision of the differential coefficients may cause an additional iteration in the least-squares fitting to be necessary. An option to choose the method desired in the differential correction may be provided by inclusion of an input variable assuming either one of two values as appropriate for the choice. The terms that are to be omitted in the simplified version will be indicated as such in the sections that follow.

PRIME CONSTANTS II

The following parameters are utilized extensively throughout the differential correction process and must therefore be evaluated beforehand. In many cases, the parameters are those that were computed previously by approximation methods, and are here re-determined by more accurate expressions.

$$p = a(1 - e^2)$$

$$D = (ap - c^2)(ap - c^2 \eta_0^2) + 4a^2 c^2 \eta_0^2$$

$$D' = D + 4a^2 c^2 (1 - \eta_0^2)$$

$$A = -2ac^2 D^{-1} (ap - c^2 \eta_0^2)(1 - \eta_0^2)$$

$$B = c^2 \eta_0^2 D^{-1} D'$$

$$b_1 = -\frac{1}{2} A$$

$$b_2 = \sqrt{B}$$

$$\alpha_1 = -\frac{1}{2} \mu (a + b_1)^{-1}$$

$$\alpha_2 = [\mu (a + b_1)^{-1}]^{1/2} \, [ap D' D^{-1} - c^2 (1 - \eta_0^2)]^{1/2}$$

$$\alpha_3 = \alpha_2 \left\{1 - c^2 \eta_0^2 \, [ap D' D^{-1} - c^2(1 - \eta_0^2)]^{-1}\right\}^{1/2} (1 - \eta_0^2)^{1/2}$$

$$\eta_2^{-2} = c^2 D \, (ap D')^{-1}$$

$$q = \eta_0 \eta_2^{-1}$$

$$k = c^2 p^{-2}$$

DIFFERENTIAL CORRECTION: TIME-INDEPENDENT PARTIAL DERIVATIVES

Compute the following in the order indicated:

$$\frac{\partial p}{\partial a} = 1 - e^2$$

$$\frac{\partial p}{\partial e} = -2 a e$$

$$\frac{\partial p^{-1}}{\partial a} = -(1 - e^2) \, p^{-2}$$

$$\frac{\partial p^{-1}}{\partial e} = 2 a^{-1} e \, (1 - e^2)^{-2}$$

$$\frac{\partial D}{\partial a} = 8 a c^2 \eta_0^2 + 2 p \, [2 ap - c^2 (1 + \eta_0^2)]$$

$$\frac{\partial D}{\partial e} = [2ap - c^2 (1 + \eta_0^2)] \, a \frac{\partial p}{\partial e}$$

$$\frac{\partial D}{\partial \eta_0} = 8 a^2 c^2 \eta_0 - 2 (a p - c^2) c^2 \eta_0$$

$$\frac{\partial D'}{\partial a} = \frac{\partial D}{\partial a} + 8 a c^2 (1 - \eta_0^2)$$

$$\frac{\partial D'}{\partial e} = \frac{\partial D}{\partial e}$$

$$\frac{\partial D'}{\partial \eta_0} = \frac{\partial D}{\partial \eta_0} - 8 a^2 c^2 \eta_0$$

$$\frac{\partial D^{-1}}{\partial a} = -D^{-2}\,\frac{\partial D}{\partial a}$$

$$\frac{\partial D^{-1}}{\partial e} = -D^{-2}\,\frac{\partial D}{\partial e}$$

$$\frac{\partial D^{-1}}{\partial \eta_0} = -D^{-2}\,\frac{\partial D}{\partial \eta_0}$$

$$\frac{\partial b_1}{\partial a} = b_1 \left[\frac{2p}{a\,p - c^2\,\eta_0^2} + D\,\frac{\partial D^{-1}}{\partial a} + a^{-1}\right]$$

$$\frac{\partial b_1}{\partial e} = a\,c^2\,(1 - \eta_0^2)\left[a\,D^{-1}\,\frac{\partial p}{\partial e} + (a\,p - c^2\,\eta_0^2)\,\frac{\partial D^{-1}}{\partial e}\right]$$

$$\frac{\partial b_1}{\partial \eta_0} = a c^2 \left\{\left[\frac{\partial D^{-1}}{\partial \eta_0}\,(a\,p - c^2\,\eta_0^2) - 2\,c^2\,\eta_0\,D^{-1}\right](1 - \eta_0^2) - 2\eta_0\,D^{-1}\,(a\,p - c^2\,\eta_0^2)\right\}$$

$$\frac{\partial b_2}{\partial a} = \frac{1}{2}\,c\,\eta_0\,(D'\,D^{-1})^{-1/2}\left(D^{-1}\,\frac{\partial D'}{\partial a} + D'\,\frac{\partial D^{-1}}{\partial a}\right)$$

$$\frac{\partial b_2}{\partial e} = \frac{1}{2}\,c\,\eta_0\,(D'\,D^{-1})^{-1/2}\left(D^{-1}\,\frac{\partial D'}{\partial e} + D'\,\frac{\partial D^{-1}}{\partial e}\right)$$

$$\frac{\partial b_2}{\partial \eta_0} = \frac{1}{2}\,c\,\eta_0\,(D'\,D^{-1})^{-1/2}\left(D^{-1}\,\frac{\partial D'}{\partial \eta_0} + D'\,\frac{\partial D^{-1}}{\partial \eta_0}\right) + c\,(D'\,D^{-1})^{1/2}$$

$$\frac{\partial \alpha_1}{\partial a} = \frac{1}{2}\,\mu\,(a + b_1)^{-2}\left(1 + \frac{\partial b_1}{\partial a}\right)$$

$$\frac{\partial \alpha_1}{\partial e} = \frac{1}{2}\,\mu\,(a + b_1)^{-2}\,\frac{\partial b_1}{\partial e}$$

$$\frac{\partial \alpha_1}{\partial \eta_0} = \frac{1}{2}\,\mu\,(a + b_1)^{-2}\,\frac{\partial b_1}{\partial \eta_0}$$

$$\frac{\partial \alpha_2}{\partial a} = \frac{1}{2}\,[\mu\,(a + b_1)^{-1}]^{1/2}\left\{\left[2pD'\,D^{-1} + ap\left(D^{-1}\,\frac{\partial D'}{\partial a} + D'\,\frac{\partial D^{-1}}{\partial a}\right)\right][ap\,D'D^{-1} - c^2\,(1 - \eta_0^2)]^{-1/2}\right.$$

$$\left. - (a + b_1)^{-1}\,[ap\,D'D^{-1} - c^2\,(1 - \eta_0^2)]^{1/2}\left(1 + \frac{\partial b_1}{\partial a}\right)\right\}$$

$$\frac{\partial \alpha_2}{\partial e} = \frac{1}{2}\left[\mu\,(a + b_1)^{-1}\right]^{1/2} \left\{\left[aD'D^{-1}\,\frac{\partial p}{\partial e} + ap\left(D^{-1}\,\frac{\partial D'}{\partial e} + D'\,\frac{\partial D^{-1}}{\partial e}\right)\right]\left[ap\,D'D^{-1} - c^2\,(1 - \eta_0^2)\right]^{-1/2}\right.$$

$$\left. - (a + b_1)^{-1}\left[apD'D^{-1} - c^2\,(1 - \eta_0^2)\right]^{1/2}\,\frac{\partial b_1}{\partial e}\right\}$$

$$\frac{\partial \alpha_2}{\partial \eta_0} = \frac{1}{2}\left[\mu\,(a + b_1)^{-1}\right]^{1/2} \left\{\left[ap\left(D^{-1}\,\frac{\partial D'}{\partial \eta_0} + D'\,\frac{\partial D^{-1}}{\partial \eta_0}\right) + 2c^2\,\eta_0\right]\left[ap\,D'D^{-1} - c^2\,(1 - \eta_0^2)\right]^{-1/2}\right.$$

$$\left. - (a + b_1)^{-1}\left[ap\,D'D^{-1} - c^2\,(1 - \eta_0^2)\right]^{1/2}\,\frac{\partial b_1}{\partial \eta_0}\right\}$$

$$\frac{\partial \alpha_2^{-1}}{\partial a} = -\,\alpha_2^{-2}\,\frac{\partial \alpha_2}{\partial a}$$

$$\frac{\partial \alpha_2^{-1}}{\partial e} = -\,\alpha_2^{-2}\,\frac{\partial \alpha_2}{\partial e}$$

$$\frac{\partial \alpha_2^{-1}}{\partial \eta_0} = -\,\alpha_2^{-2}\,\frac{\partial \alpha_2}{\partial \eta_0}$$

$$\frac{\partial \alpha_3}{\partial a} = (1 - \eta_0^2)^{1/2}\left\{\frac{\partial \alpha_2}{\partial a}\left[1 - \frac{c^2\,\eta_0^2}{ap\,D'D^{-1} - c^2\,(1 - \eta_0^2)}\right]^{1/2}\right.$$

$$\left. + \frac{c^2\eta_0^2\,\alpha_2}{2\left[ap\,D'\,D^{-1} - c^2(1 - \eta_0^2)\right]^2}\left[1 - \frac{c^2\,\eta_0^2}{ap\,D'\,D^{-1} - c^2(1 - \eta_0^2)}\right]^{-1/2}\left[2p\,D'\,D^{-1} + ap\left(D^{-1}\,\frac{\partial D'}{\partial a} + D'\,\frac{\partial D^{-1}}{\partial a}\right)\right]\right\}$$

$$\frac{\partial \alpha_3}{\partial e} = (1 - \eta_0^2)^{1/2}\left\{\frac{\partial \alpha_2}{\partial e}\left[1 - \frac{c^2\,\eta_0^2}{ap\,D'D^{-1} - c^2\,(1 - \eta_0^2)}\right]^{1/2}\right.$$

$$\left. + \frac{c^2\eta_0^2\,\alpha_2}{2\left[apD'\,D^{-1} - c^2(1 - \eta_0^2)\right]^2}\left[1 - \frac{c^2\,\eta_0^2}{ap\,D'\,D^{-1} - c^2\,(1 - \eta_0^2)}\right]^{-1/2}\left[a\,D'\,D^{-1}\,\frac{\partial p}{\partial e} + ap\left(D^{-1}\,\frac{\partial D'}{\partial e} + D'\,\frac{\partial D^{-1}}{\partial e}\right)\right]\right\}$$

$$\frac{\partial \alpha_3}{\partial \eta_0} = (1-\eta_0^2)^{1/2}\left\{\frac{\partial \alpha_2}{\partial \eta_0}\left[1 - \frac{c^2\eta_0^2}{ap\,D'D^{-1} - c^2\,(1-\eta_0^2)}\right]^{1/2}\right.$$

$$\left. + \frac{c^2\eta_0\alpha_2}{2[apD'D^{-1} - c^2(1-\eta_0^2)]^2}\left[1 - \frac{c^2\eta_0^2}{ap\,D'\,D^{-1} - c^2\,(1-\eta_0^2)}\right]^{-1/2}\left[2(c^2 - ap\,D'\,D^{-1}) + ap\,\eta_0\left(D^{-1}\frac{\partial D'}{\partial \eta_0} + D'\frac{\partial D^{-1}}{\partial \eta_0}\right)\right]\right\}$$

$$-\eta_0\,\alpha_3\,(1-\eta_0^2)^{-1}$$

$$\frac{\partial \eta_2}{\partial a} = \frac{1}{2}pc^{-2}\,\eta_2^{-1}\left[D^{-1}\left(2\,D' + a\,\frac{\partial D'}{\partial a}\right) + a\,D'\,\frac{\partial D^{-1}}{\partial a}\right]$$

$$\frac{\partial \eta_2}{\partial e} = \frac{1}{2}\,a\,c^{-2}\,\eta_2^{-1}\left[D^{-1}\left(D'\,\frac{\partial p}{\partial e} + p\,\frac{\partial D'}{\partial e}\right) + p\,D'\,\frac{\partial D^{-1}}{\partial e}\right]$$

$$\frac{\partial \eta_2}{\partial \eta_0} = \frac{1}{2}a\,p\,c^{-2}\,\eta_2^{-1}\left(D^{-1}\frac{\partial D'}{\partial \eta_0} + D'\,\frac{\partial D^{-1}}{\partial \eta_0}\right)$$

$$\frac{\partial q}{\partial a} = -\,\eta_0\,\eta_2^{-2}\,\frac{\partial \eta_2}{\partial a}$$

$$\frac{\partial q}{\partial e} = -\,\eta_0\,\eta_2^{-2}\,\frac{\partial \eta_2}{\partial e}$$

$$\frac{\partial q}{\partial \eta_0} = \eta_2^{-1}\left(1 - q\,\frac{\partial \eta_2}{\partial \eta_0}\right)$$

$$\frac{\partial e'}{\partial a} = (a + b_1)^{-2}\,e\left(b_1 - a\,\frac{\partial b_1}{\partial a}\right)$$

$$\frac{\partial e'}{\partial e} = (a + b_1)^{-2}\,a\left(a + b_1 - e\,\frac{\partial b_1}{\partial e}\right)$$

$$\frac{\partial e'}{\partial \eta_0} = -\,(a + b_1)^{-2}\,ae\,\frac{\partial b_1}{\partial \eta_0}$$

$$\frac{\partial B_1}{\partial a} = \left(\frac{3}{8}\,q + \frac{15}{32}\,q^3\right)\frac{\partial q}{\partial a}$$

$$\frac{\partial B_1}{\partial e} = \left(\frac{3}{8}\,q + \frac{15}{32}\,q^3\right)\frac{\partial q}{\partial e}$$

$$\frac{\partial B_1}{\partial \eta_0} = \left(\frac{3}{8} q + \frac{15}{32} q^3\right) \frac{\partial q}{\partial \eta_0}$$

$$\frac{\partial B_2}{\partial a} = \left(\frac{1}{2} q + \frac{9}{16} q^3\right) \frac{\partial q}{\partial a}$$

$$\frac{\partial B_2}{\partial e} = \left(\frac{1}{2} q + \frac{9}{16} q^3\right) \frac{\partial q}{\partial e}$$

$$\frac{\partial B_2}{\partial \eta_0} = \left(\frac{1}{2} q + \frac{9}{16} q^3\right) \frac{\partial q}{\partial \eta_0}$$

$$\frac{\partial B_2^{-1}}{\partial a} = - B_2^{-2} \frac{\partial B_2}{\partial a}$$

$$\frac{\partial B_2^{-1}}{\partial e} = - B_2^{-2} \frac{\partial B_2}{\partial e}$$

$$\frac{\partial B_2^{-1}}{\partial \eta_0} = - B_2^{-2} \frac{\partial B_2}{\partial \eta_0}$$

$$\frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right) = b_2^{-1} \frac{\partial b_1}{\partial a} - b_1 b_2^{-2} \frac{\partial b_2}{\partial a}$$

$$\frac{\partial}{\partial e}\left(\frac{b_1}{b_2}\right) = b_2^{-1} \frac{\partial b_1}{\partial e} - b_1 b_2^{-2} \frac{\partial b_2}{\partial e}$$

$$\frac{\partial}{\partial \eta_0}\left(\frac{b_1}{b_2}\right) = b_2^{-1} \frac{\partial b_1}{\partial \eta_0} - b_1 b_2^{-2} \frac{\partial b_2}{\partial \eta_0}$$

$$\frac{\partial}{\partial a}\left(\frac{b_2}{p}\right) = p^{-1} \frac{\partial b_2}{\partial a} - b_2 p^{-2} \frac{\partial p}{\partial a}$$

$$\frac{\partial}{\partial e}\left(\frac{b_2}{p}\right) = p^{-1} \frac{\partial b_2}{\partial e} - b_2 p^{-2} \frac{\partial p}{\partial e}$$

$$\frac{\partial}{\partial \eta_0}\left(\frac{b_2}{p}\right) = p^{-1} \frac{\partial b_2}{\partial \eta_0}$$

$$\frac{\partial A_1}{\partial a} = A_1 p^{-1} \frac{\partial p}{\partial a}$$

$$+p\,(1-e^2)^{1/2}\left\{\frac{\partial}{\partial a}\left(\frac{b_2}{p}\right)\sum_{n=2}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1} P_n\left(\frac{b_1}{b_2}\right)R_{n-2}[(1-e^2)^{1/2}]\right.$$

$$\left.+\frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right)\sum_{n=2}^{\infty}\left(\frac{b_2}{p}\right)^{n} P_n'\left(\frac{b_1}{b_2}\right)R_{n-2}\,[(1-e^2)^{1/2}]\right\}$$

where $P_n(x)$ is the Legendre polynomial with argument x of degree n, and $P_n'(x)$ is the derivative of the Legendre polynomial with respect to the argument. The definition of R is as given previously, viz., $R_n(x) \equiv x^n P_n(1/x)$. All infinite series are computed by an iterative method, with computation of terms ceasing when the absolute value of the ratio of successive terms minus unity is less than or equal to some pre-selected tolerance.

$$\frac{\partial A_1}{\partial e} = A_1 p^{-1} \frac{\partial p}{\partial e} - A_1\, e\,(1-e^2)^{-1}$$

$$+p\,(1-e^2)^{1/2}\left\{\frac{\partial}{\partial e}\left(\frac{b_2}{p}\right)\sum_{n=2}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1} P_n\left(\frac{b_1}{b_2}\right)R_{n-2}\,[(1-e^2)^{1/2}]\right.$$

$$\left.+\frac{\partial}{\partial e}\left(\frac{b_1}{b_2}\right)\sum_{n=2}^{\infty}\left(\frac{b_2}{p}\right)^{n} P_n'\left(\frac{b_1}{b_2}\right)R_{n-2}\,[(1-e^2)^{1/2}]\right\}$$

$$+\,pe\left\{(1-e^2)^{-1}\sum_{n=3}^{\infty}\left(\frac{b_2}{p}\right)^{n}[(1-e^2)^{1/2}]^{n-2}\,P_n\left(\frac{b_1}{b_2}\right)P_{n-2}'\,[(1-e^2)^{-1/2}]\right.$$

$$\left.-\sum_{n=3}^{\infty}(n-2)\left(\frac{b_2}{p}\right)^{n}[(1-e^2)^{1/2}]^{n-3}\,P_n\left(\frac{b_1}{b_2}\right)P_{n-2}\,[(1-e^2)^{-1/2}]\right\}$$

$$\frac{\partial A_1}{\partial \eta_0} = (1-e^2)^{1/2}\left\{\frac{\partial b_2}{\partial \eta_0}\sum_{n=2}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1} P_n\left(\frac{b_1}{b_2}\right)R_{n-2}\,[(1-e^2)^{1/2}]\right.$$

$$\left.+\,p\frac{\partial}{\partial \eta_0}\left(\frac{b_1}{b_2}\right)\sum_{n=2}^{\infty}\left(\frac{b_2}{p}\right)^{n} P_n'\left(\frac{b_1}{b_2}\right)R_{n-2}\,[(1-e^2)^{1/2}]\right\}$$

$$\frac{\partial A_2}{\partial a} = A_2\, p\, \frac{\partial p^{-1}}{\partial a}$$

$$+ (1 - e^2)^{1/2}\, p^{-1} \left\{ \frac{\partial}{\partial a}\left(\frac{b_2}{p}\right) \sum_{n=1}^{\infty} n \left(\frac{b_2}{p}\right)^{n-1} P_n \left(\frac{b_1}{b_2}\right) R_n\, [(1 - e^2)^{1/2}]\right.$$

$$\left. + \frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right) \sum_{n=1}^{\infty} \left(\frac{b_2}{p}\right)^{n} P_n' \left(\frac{b_1}{b_2}\right) R_n\, [(1 - e^2)^{1/2}]\right\}$$

$$\frac{\partial A_2}{\partial e} = A_2\, p\, \frac{\partial p^{-1}}{\partial e} - A_2\, e\, (1 - e^2)^{-1}$$

$$+ (1 - e^2)^{1/2}\, p^{-1} \left\{ \frac{\partial}{\partial e}\left(\frac{b_2}{p}\right) \sum_{n=1}^{\infty} n \left(\frac{b_2}{p}\right)^{n-1} P_n \left(\frac{b_1}{b_2}\right) R_n\, [(1 - e^2)^{1/2}]\right.$$

$$\left. + \frac{\partial}{\partial e}\left(\frac{b_1}{b_2}\right) \sum_{n=1}^{\infty} \left(\frac{b_2}{p}\right)^{n} P_n' \left(\frac{b_1}{b_2}\right) R_n\, [(1 - e^2)^{1/2}]\right\}$$

$$+ p^{-1}\, e \left\{ (1 - e^2)^{-1} \sum_{n=1}^{\infty} \left(\frac{b_2}{p}\right)^{n} [(1 - e^2)^{1/2}]^{n}\, P_n \left(\frac{b_1}{b_2}\right) P_n'\, [(1 - e^2)^{-1/2}]\right.$$

$$\left. - \sum_{n=1}^{\infty} n \left(\frac{b_2}{p}\right)^{n} [(1 - e^2)^{1/2}]^{n-1}\, P_n \left(\frac{b_1}{b_2}\right) P_n\, [(1 - e^2)^{-1/2}]\right\}$$

$$\frac{\partial A_2}{\partial \eta_0} = (1 - e^2)^{1/2}\, p^{-1} \left\{ \frac{\partial}{\partial \eta_0}\left(\frac{b_2}{p}\right) \sum_{n=1}^{\infty} n \left(\frac{b_2}{p}\right)^{n-1} P_n \left(\frac{b_1}{b_2}\right) R_n\, [(1 - e^2)^{1/2}]\right.$$

$$\left. + \frac{\partial}{\partial \eta_0}\left(\frac{b_1}{b_2}\right) \sum_{n=1}^{\infty} \left(\frac{b_2}{p}\right)^{n} P_n' \left(\frac{b_1}{b_2}\right) R_n\, [(1 - e^2)^{1/2}]\right\}$$

$$\frac{\partial A_2^{-1}}{\partial a} = - A_2^{-2}\, \frac{\partial A_2}{\partial a}$$

$$\frac{\partial A_2^{-1}}{\partial e} = - A_2^{-2}\, \frac{\partial A_2}{\partial e}$$

$$\frac{\partial A_2^{-1}}{\partial \eta_0} = - A_2^{-2}\, \frac{\partial A_2}{\partial \eta_0}$$

$$\frac{\partial A_{21}}{\partial a} = - A_{21}\, p^{-1}\, \frac{\partial p}{\partial a} + (1 - e^2)^{1/2} p^{-2} e \Bigg\{ - b_1\, p^{-1}\, \frac{\partial p}{\partial a}$$

$$+ \frac{\partial b_1}{\partial a} - 2\, p^{-2} (3 b_1^2 - b_2^2) \frac{\partial p}{\partial a} + 2\, p^{-1} \left(3\, b_1\, \frac{\partial b_1}{\partial a} - b_2\, \frac{\partial b_2}{\partial a} \right)$$

$$+ \frac{9}{2}\, b_2\, p^{-2} \left(1 + \frac{1}{4}\, e^2\right) \left(3 b_1\, b_2\, p^{-1}\, \frac{\partial p}{\partial a} - b_2\, \frac{\partial b_1}{\partial a} - 2 b_1\, \frac{\partial b_2}{\partial a} \right)$$

$$+ \frac{3}{2}\, b_2^3\, p^{-3}\, (4 + 3\, e^2) \left(\frac{\partial b_2}{\partial a} - b_2\, p^{-1}\, \frac{\partial p}{\partial a} \right) \Bigg\}$$

$$\frac{\partial A_{21}}{\partial e} = A_{21} \left[-e\, (1 - e^2)^{-1} + e^{-1} - p^{-1}\, \frac{\partial p}{\partial e} \right]$$

$$+ (1 - e^2)^{1/2}\, p^{-2}\, e \Bigg\{ - b_1\, p^{-1}\, \frac{\partial p}{\partial e} + \frac{\partial b_1}{\partial e} - 2\, p^{-2}\, (3\, b_1^2 - b_2^2)\, \frac{\partial p}{\partial e}$$

$$+ 2\, p^{-1} \left(3\, b_1\, \frac{\partial b_1}{\partial e} - b_2\, \frac{\partial b_2}{\partial e} \right) + \frac{9}{2}\, b_2\, p^{-2} \left(1 + \frac{1}{4}\, e^2 \right) \left(3\, b_1\, b_2\, p^{-1}\, \frac{\partial p}{\partial e} - b_2\, \frac{\partial b_1}{\partial e} - 2\, b_1\, \frac{\partial b_2}{\partial e} \right)$$

$$+ \frac{3}{2}\, b_2^3\, p^{-3}\, (4 + 3\, e^2) \left(\frac{\partial b_2}{\partial e} - b_2\, p^{-1}\, \frac{\partial p}{\partial e} \right) + \frac{9}{4}\, b_2^2\, p^{-2}\, e\, (b_2^2\, p^{-1} - b_1) \Bigg\}$$

$$\frac{\partial A_{21}}{\partial \eta_0} = (1 - e^2)^{1/2}\, p^{-2}\, e \Bigg\{ \frac{\partial b_1}{\partial \eta_0} + 2 p^{-1} \left(3\, b_1\, \frac{\partial b_1}{\partial \eta_0} - b_2\, \frac{\partial b_2}{\partial \eta_0} \right)$$

$$- \frac{9}{2}\, b_2\, p^{-2} \left(1 + \frac{1}{4}\, e^2 \right) \left(b_2\, \frac{\partial b_1}{\partial \eta_0} + 2 b_1\, \frac{\partial b_2}{\partial \eta_0} \right) + \frac{3}{2}\, b_2^3\, p^{-3}\, (4 + 3\, e^2)\, \frac{\partial b_2}{\partial \eta_0} \Bigg\}$$

$$\frac{\partial A_{22}}{\partial a} = - A_{22}\, p^{-1}\, \frac{\partial p}{\partial a} + \frac{1}{4}\, p^{-3}\, e^2\, (1 - e^2)^{1/2} \Bigg\{ \Bigg[- p^{-1}\, (3 b_1^2 - b_2^2)$$

$$+ \frac{27}{2}\, p^{-2}\, b_1\, b_2^2 - \frac{3}{2}\, p^{-3}\, b_2^4\, (6 + e^2) \Bigg] \frac{\partial p}{\partial a}$$

$$+ (3 b_1 - \frac{9}{2}\, p^{-1}\, b_2^2)\, \frac{\partial b_1}{\partial a} + \left[- 9\, p^{-1}\, b_1\, b_2 + \frac{3}{2}\, p^{-2}\, b_2^3\, (6 + e^2) - b_2 \right] \frac{\partial b_2}{\partial a} \Bigg\}$$

$$\frac{\partial A_{22}}{\partial e} = - A_{22}\left[p^{-1}\frac{\partial p}{\partial e} + e\,(1-e^2)^{-1}\right]$$

$$+\frac{1}{4}p^{-3}\,e^2\,(1-e^2)^{1/2}\left\{\left[-p^{-1}\,(3b_1^2 - b_2^2) + \frac{27}{2}p^{-2}\,b_1\,b_2^2\right.\right.$$

$$\left.-\frac{3}{2}p^{-3}\,b_2^4\,(6+e^2)\right]\frac{\partial p}{\partial e} + \left(3b_1 - \frac{9}{2}p^{-1}\,b_2^2\right)\frac{\partial b_1}{\partial e}$$

$$\left.+\left[-9\,p^{-1}\,b_1\,b_2 + \frac{3}{2}p^{-2}\,b_2^3\,(6+e^2) - b_2\right]\frac{\partial b_2}{\partial e}\right\}$$

$$+\frac{1}{4}p^{-3}\,e\,(1-e^2)^{1/2}\left[3b_1^2 - b_2^2 - 9\,p^{-1}\,b_1\,b_2^2 + \frac{3}{2}p^{-2}\,b_2^4\,(3+e^2)\right]$$

$$\frac{\partial A_{22}}{\partial \eta_0} = \frac{1}{4}p^{-3}\,e^2\,(1-e^2)^{1/2}\left\{\left(3b_1 - \frac{9}{2}p^{-1}\,b_2^2\right)\frac{\partial b_1}{\partial \eta_0}\right.$$

$$\left.+\left[-9\,p^{-1}\,b_1\,b_2 + \frac{3}{2}p^{-2}\,b_2^3\,(6+e^2) - b_2\right]\frac{\partial b_2}{\partial \eta_0}\right\}$$

$$\frac{\partial B_3}{\partial a} = \frac{\partial \eta_2}{\partial a}\left[(1-\eta_2^{-2})^{-3/2}\,\eta_2^{-3} + 2\sum_{m=2}^{\infty} m\,\gamma_m\,\eta_2^{-2m-1}\right]$$

where γ_m has been given above in the section titled "Mutual Constants."

$$\frac{\partial B_3}{\partial e} = \frac{\partial \eta_2}{\partial e}\left[(1-\eta_2^{-2})^{-3/2}\,\eta_2^{-3} + 2\sum_{m=2}^{\infty} m\,\gamma_m\,\eta_2^{-2m-1}\right]$$

$$\frac{\partial B_3}{\partial \eta_0} = \frac{\partial \eta_2}{\partial \eta_0}\left[(1-\eta_2^{-2})^{-3/2}\,\eta_2^{-3} + 2\sum_{m=2}^{\infty} m\,\gamma_m\,\eta_2^{-2m-1}\right] - \sum_{m=2}^{\infty}\left(\frac{\partial \gamma_m}{\partial \eta_0}\right)\eta_2^{-2m}$$

where

$$\frac{\partial \gamma_m}{\partial \eta_0} = \frac{(2m)!}{2^{2m}\,(m!)^2}\sum_{n=1}^{m-1}\frac{(2n)!\,(2n)}{2^{2n}\,(n!)^2}\,\eta_0^{2n-1}$$

$$\frac{\partial A_3}{\partial a} = -3\,A_3\,p^{-1}\frac{\partial p}{\partial a} + (1-e^2)^{1/2}\,p^{-3}\sum_{n=1}^{\infty} R_{n+2}\,[(1-e^2)^{1/2}]\,\frac{\partial D_n}{\partial a}$$

where $\partial D_n/\partial a$ is computed as follows:

If n is an even integer, then

$$\frac{\partial D_n}{\partial a} = \frac{\partial D_{2i}}{\partial a} = -2cp^{-2}\frac{\partial p}{\partial a}\sum_{m=0}^{i}(-1)^{i-m}(i-m)\left(\frac{c}{p}\right)^{2(i-m)-1}\left(\frac{b_2}{p}\right)^{2m}P_{2m}\left(\frac{b_1}{b_2}\right)$$

$$+2\frac{\partial}{\partial a}\left(\frac{b_2}{p}\right)\sum_{m=0}^{i}(-1)^{i-m}m\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m-1}P_{2m}\left(\frac{b_1}{b_2}\right)$$

$$+\frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{i}(-1)^{i-m}\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m}P'_{2m}\left(\frac{b_1}{b_2}\right)$$

If n is an odd integer, then

$$\frac{\partial D_n}{\partial a} = \frac{\partial D_{2i+1}}{\partial a} = -2cp^{-2}\frac{\partial p}{\partial a}\sum_{m=0}^{i}(-1)^{i-m}(i-m)\left(\frac{c}{p}\right)^{2(i-m)-1}\left(\frac{b_2}{p}\right)^{2m+1}P_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$+\frac{\partial}{\partial a}\left(\frac{b_2}{p}\right)\sum_{m=0}^{i}(-1)^{i-m}(2m+1)\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m}P_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$+\frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{i}(-1)^{i-m}\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m+1}P'_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$\frac{\partial A_3}{\partial e} = -A_3\left[3p^{-1}\frac{\partial p}{\partial e}+e(1-e^2)^{-1}\right]+(1-e^2)^{1/2}p^{-3}\sum_{n=1}^{\infty}R_{n+2}\left[(1-e^2)^{1/2}\right]\frac{\partial D_n}{\partial e}$$

$$+ep^{-3}\left\{(1-e^2)^{-1}\sum_{n=0}^{\infty}\left[(1-e^2)^{1/2}\right]^{n+2}D_nP'_{n+2}\left[(1-e^2)^{-1/2}\right]-\sum_{n=0}^{\infty}(n+2)\left[(1-e^2)^{1/2}\right]^{n+1}D_nP_{n+2}\left[(1-e^2)^{-1/2}\right]\right\}$$

where D_n has been given above in the section titled "Mutual Constants," and where $\partial D_n/\partial e$ is computed as follows:

If n is an even integer, then

$$\frac{\partial D_n}{\partial e} = \frac{\partial D_{2i}}{\partial e} = -2cp^{-2}\frac{\partial p}{\partial e}\sum_{m=0}^{i}(-1)^{i-m}(i-m)\left(\frac{c}{p}\right)^{2(i-m)-1}\left(\frac{b_2}{p}\right)^{2m}P_{2m}\left(\frac{b_1}{b_2}\right)$$

$$+2\frac{\partial}{\partial e}\left(\frac{b_2}{p}\right)\sum_{m=0}^{i}(-1)^{i-m}m\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m-1}P_{2m}\left(\frac{b_1}{b_2}\right)+\frac{\partial}{\partial e}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{i}(-1)^{i-m}\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m}P'_{2m}\left(\frac{b_1}{b_2}\right)$$

If n is an odd integer, then

$$\frac{\partial D_n}{\partial e} = \frac{\partial D_{2i+1}}{\partial e} = -2\,cp^{-2}\frac{\partial p}{\partial e}\sum_{m=0}^{i}(-1)^{i-m}(i-m)\left(\frac{c}{p}\right)^{2(i-m)-1}\left(\frac{b_2}{p}\right)^{2m+1}P_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$+\frac{\partial}{\partial e}\left(\frac{b_2}{p}\right)\sum_{m=0}^{i}(-1)^{i-m}(2m+1)\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m}P_{2m+1}\left(\frac{b_1}{b_2}\right)+\frac{\partial}{\partial e}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{i}(-1)^{i-m}\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m+1}P'_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$\frac{\partial A_3}{\partial \eta_0} = (1-e^2)^{1/2}p^{-3}\sum_{n=1}^{\infty}R_{n+2}\left[(1-e^2)^{1/2}\right]\frac{\partial D_n}{\partial \eta_0}$$

where $\partial D_n/\partial \eta_0$ is computed as follows:

If n is an even integer, then

$$\frac{\partial D_n}{\partial \eta_0} = \frac{\partial D_{2i}}{\partial \eta_0} = 2\,p^{-1}\frac{\partial b_2}{\partial \eta_0}\sum_{m=0}^{i}(-1)^{i-m}m\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m-1}P_{2m}\left(\frac{b_1}{b_2}\right)$$

$$+\frac{\partial}{\partial \eta_0}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{i}(-1)^{i-m}\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m}P'_{2m}\left(\frac{b_1}{b_2}\right)$$

If n is an odd integer, then

$$\frac{\partial D_n}{\partial \eta_0} = \frac{\partial D_{2i+1}}{\partial \eta_0} = p^{-1}\frac{\partial b_2}{\partial \eta_0}\sum_{m=0}^{i}(-1)^{i-m}(2m+1)\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m}P_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$+\frac{\partial}{\partial \eta_0}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{i}(-1)^{i-m}\left(\frac{c}{p}\right)^{2(i-m)}\left(\frac{b_2}{p}\right)^{2m+1}P'_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$\frac{\partial A_{31}}{\partial a} = -3A_{31}p^{-1}\frac{\partial p}{\partial a}+(1-e^2)^{1/2}p^{-4}e\left\{\frac{\partial p}{\partial a}\left[-b_1p^{-1}\left(3+\frac{3}{4}e^2\right)\right.\right.$$

$$\left.\left.+2p^{-2}(4+3e^2)\left(\frac{1}{2}b_2^2+c^2\right)\right]+\frac{\partial b_1}{\partial a}\left(3+\frac{3}{4}e^2\right)-\frac{\partial b_2}{\partial a}b_2p^{-1}(4+3e^2)\right\}$$

$$\frac{\partial A_{31}}{\partial e} = -3A_{31}p^{-1}\frac{\partial p}{\partial e}-p^{-3}\left[(1-e^2)^{-1/2}e^2-(1-e^2)^{1/2}\right]\left[2+b_1p^{-1}\left(3+\frac{3}{4}e^2\right)\right.$$

$$\left.-p^{-2}\left(\frac{1}{2}b_2^2+c^2\right)(4+3e^2)\right]+(1-e^2)^{1/2}p^{-4}e\left\{\frac{\partial p}{\partial e}\left[-b_1p^{-1}\left(3+\frac{3}{4}e^2\right)\right.\right.$$

$$+ 2p^{-2}(4 + 3e^2)\left(\frac{1}{2}b_2^2 + c^2\right)\Bigg] + \frac{\partial b_1}{\partial e}\left(3 + \frac{3}{4}e^2\right) - \frac{\partial b_2}{\partial e}b_2 p^{-1}(4 + 3e^2)$$

$$+ \frac{3}{2}b_1 e - 6p^{-1}e\left(\frac{1}{2}b_2^2 + c^2\right)\Bigg\}$$

$$\frac{\partial A_{31}}{\partial \eta_0} = (1 - e^2)^{1/2} p^{-4} e \left[\left(3 + \frac{3}{4}e^2\right)\frac{\partial b_1}{\partial \eta_0} - b_2 p^{-1}(4 + 3e^2)\frac{\partial b_2}{\partial \eta_0}\right]$$

$$\frac{\partial A_{32}}{\partial a} = -3A_{32}p^{-1}\frac{\partial p}{\partial a} + (1 - e^2)^{1/2}p^{-4}e^2\left\{\frac{\partial p}{\partial a}\left[p^{-2}\left(\frac{1}{2}b_2^2 + c^2\right)\left(3 + \frac{1}{2}e^2\right) - \frac{3}{4}p^{-1}b_1\right]\right.$$

$$\left. + \frac{3}{4}\frac{\partial b_1}{\partial a} - \frac{1}{2}p^{-1}b_2\left(3 + \frac{1}{2}e^2\right)\frac{\partial b_2}{\partial a}\right\}$$

$$\frac{\partial A_{32}}{\partial e} = -3A_{32}p^{-1}\frac{\partial p}{\partial e} - A_{32}e(1 - e^2)^{-1} + (1 - e^2)^{1/2}p^{-4}e^2\left\{\frac{\partial p}{\partial e}\left[p^{-2}\left(\frac{1}{2}b_2^2 + c^2\right)\left(3 + \frac{1}{2}e^2\right)\right.\right.$$

$$\left.\left. - \frac{3}{4}p^{-1}b_1\right] + \frac{3}{4}\frac{\partial b_1}{\partial e} - \frac{1}{2}p^{-1}b_2\left(3 + \frac{1}{2}e^2\right)\frac{\partial b_2}{\partial e}\right\}$$

$$+ (1 - e^2)^{1/2}p^{-3}e\left\{\frac{1}{2}(1 + 3b_1 p^{-1}) - p^{-2}\left(\frac{1}{2}b_2^2 + c^2\right)(3 + e^2)\right\}$$

$$\frac{\partial A_{32}}{\partial \eta_0} = (1 - e^2)^{1/2}p^{-4}e^2\left[\frac{3}{4}\frac{\partial b_1}{\partial \eta_0} - \frac{1}{2}p^{-1}b_2\left(3 + \frac{1}{2}e^2\right)\frac{\partial b_2}{\partial \eta_0}\right]$$

$$\frac{\partial A_{33}}{\partial a} = -3A_{33}p^{-1}\frac{\partial p}{\partial a} + (1 - e^2)^{1/2}p^{-4}e^3\left\{\frac{\partial p}{\partial a}\left[\frac{2}{3}p^{-2}\left(\frac{1}{2}b_2^2 + c^2\right) - \frac{1}{12}p^{-1}b_1\right]\right.$$

$$\left. + \frac{1}{12}\frac{\partial b_1}{\partial a} - \frac{1}{3}p^{-1}b_2\frac{\partial b_2}{\partial a}\right\}$$

$$\frac{\partial A_{33}}{\partial e} = -3A_{33}p^{-1}\frac{\partial p}{\partial e} + p^{-4}e^2\left[3(1 - e^2)^{1/2} - (1 - e^2)^{-1/2}e^2\right]\left[\frac{1}{12}b_1 - \frac{1}{3}p^{-1}\left(\frac{1}{2}b_2^2 + c^2\right)\right]$$

$$+ (1 - e^2)^{1/2}p^{-4}e^3\left\{\frac{\partial p}{\partial e}\left[\frac{2}{3}p^{-2}\left(\frac{1}{2}b_2^2 + c^2\right) - \frac{1}{12}p^{-1}b_1\right] + \frac{1}{12}\frac{\partial b_1}{\partial e} - \frac{1}{3}p^{-1}b_2\frac{\partial b_2}{\partial e}\right\}$$

$$\frac{\partial A_{33}}{\partial \eta_0} = (1-e^2)^{1/2} p^{-4} e^3 \left[\frac{1}{12}\frac{\partial b_1}{\partial \eta_0} - \frac{1}{3} p^{-1} b_2 \frac{\partial b_2}{\partial \eta_0}\right]$$

$$\frac{\partial A_{34}}{\partial a} = -5A_{34} p^{-1} \frac{\partial p}{\partial a} - \frac{1}{32}(1-e^2)^{1/2} p^{-5} e^4 b_2 \frac{\partial b_2}{\partial a}$$

$$\frac{\partial A_{34}}{\partial e} = -5A_{34} p^{-1} \frac{\partial p}{\partial e} + \frac{1}{8} e^3 p^{-5} \left[\frac{1}{4}(1-e^2)^{-1/2} e^2 - (1-e^2)^{1/2}\right] \left(\frac{1}{2} b_2^2 + c^2\right) - \frac{1}{32}(1-e^2)^{1/2} p^{-5} e^4 b_2 \frac{\partial b_2}{\partial e}$$

$$\frac{\partial A_{34}}{\partial \eta_0} = -\frac{1}{32}(1-e^2)^{1/2} p^{-5} e^4 b_2 \frac{\partial b_2}{\partial \eta_0}$$

$$\frac{\partial \nu_1}{\partial a} = -\frac{1}{2\pi}(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \left\{(-2\alpha_1)^{-1/2} \frac{\partial \alpha_1}{\partial a}\right.$$

$$+ \left[1 + \frac{\partial b_1}{\partial a} + \frac{\partial A_1}{\partial a} + c^2 \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial a} + c^2 \eta_0^2 A_2 \left(B_2^{-1} \frac{\partial B_1}{\partial a}\right.\right.$$

$$\left.\left.\left. + B_1 \frac{\partial B_2^{-1}}{\partial a}\right)\right] (-2\alpha_1)^{1/2} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1}\right\}$$

$$\frac{\partial \nu_1}{\partial e} = -\frac{1}{2\pi}(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \left\{(-2\alpha_1)^{-1/2} \frac{\partial \alpha_1}{\partial e}\right.$$

$$+ \left[\frac{\partial b_1}{\partial e} + \frac{\partial A_1}{\partial e} + c^2 \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial e} + c^2 \eta_0^2 A_2 \left(B_2^{-1} \frac{\partial B_1}{\partial e}\right.\right.$$

$$\left.\left.\left. + B_1 \frac{\partial B_2^{-1}}{\partial e}\right)\right] (-2\alpha_1)^{1/2} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1}\right\}$$

$$\frac{\partial \nu_1}{\partial \eta_0} = -\frac{1}{2\pi}(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \left\{(-2\alpha_1)^{-1/2} \frac{\partial \alpha_1}{\partial \eta_0}\right.$$

$$+ \left[\frac{\partial b_1}{\partial \eta_0} + \frac{\partial A_1}{\partial \eta_0} + c^2 \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial \eta_0} + c^2 \eta_0^2 A_2 \left(B_2^{-1} \frac{\partial B_1}{\partial \eta_0}\right.\right.$$

$$\left.\left.\left. + B_1 \frac{\partial B_2^{-1}}{\partial \eta_0}\right) + 2c^2 \eta_0 A_2 B_1 B_2^{-1}\right] (-2\alpha_1)^{1/2} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1}\right\}$$

$$\frac{\partial \nu_2}{\partial a} = \frac{1}{2\pi} \eta_0^{-1} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \Bigg\{ (\alpha_2^2 - \alpha_3^2)^{1/2} B_2^{-1} \frac{\partial A_2}{\partial a}$$

$$+ (\alpha_2^2 - \alpha_3^2)^{-1/2} A_2 B_2^{-1} \left(\alpha_2 \frac{\partial \alpha_2}{\partial a} - \alpha_3 \frac{\partial \alpha_3}{\partial a} \right)$$

$$+ (\alpha_2^2 - \alpha_3^2)^{1/2} A_2 \frac{\partial B_2^{-1}}{\partial a} - (\alpha_2^2 - \alpha_3^2)^{1/2} A_2 B_2^{-1} \Bigg[1 + \frac{\partial b_1}{\partial a} + \frac{\partial A_1}{\partial a}$$

$$+ c^2 \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial a} + c^2 \eta_0^2 A_2 \left(B_2^{-1} \frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_2^{-1}}{\partial a} \right) \Bigg] (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \Bigg\}$$

$$\frac{\partial \nu_2}{\partial e} = \frac{1}{2\pi} \eta_0^{-1} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \Bigg\{ (\alpha_2^2 - \alpha_3^2)^{1/2} B_2^{-1} \frac{\partial A_2}{\partial e}$$

$$+ (\alpha_2^2 - \alpha_3^2)^{-1/2} A_2 B_2^{-1} \left(\alpha_2 \frac{\partial \alpha_2}{\partial e} - \alpha_3 \frac{\partial \alpha_3}{\partial e} \right) + (\alpha_2^2 - \alpha_3^2)^{1/2} A_2 \frac{\partial B_2^{-1}}{\partial e}$$

$$- (\alpha_2^2 - \alpha_3^2)^{1/2} A_2 B_2^{-1} \Bigg[\frac{\partial b_1}{\partial e} + \frac{\partial A_1}{\partial e} + c^2 \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial e}$$

$$+ c^2 \eta_0^2 A_2 \left(B_2^{-1} \frac{\partial B_1}{\partial e} + B_1 \frac{\partial B_2^{-1}}{\partial e} \right) \Bigg] (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \Bigg\}$$

$$\frac{\partial \nu_2}{\partial \eta_0} = \frac{1}{2\pi} \eta_0^{-1} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \Bigg\{ (\alpha_2^2 - \alpha_3^2)^{1/2} B_2^{-1} \frac{\partial A_2}{\partial \eta_0}$$

$$+ (\alpha_2^2 - \alpha_3^2)^{-1/2} A_2 B_2^{-1} \left(\alpha_2 \frac{\partial \alpha_2}{\partial \eta_0} - \alpha_3 \frac{\partial \alpha_3}{\partial \eta_0} \right) + (\alpha_2^2 - \alpha_3^2)^{1/2} A_2 \frac{\partial B_2^{-1}}{\partial \eta_0}$$

$$- (\alpha_2^2 - \alpha_3^2)^{1/2} A_2 B_2^{-1} \Bigg[\frac{\partial b_1}{\partial \eta_0} + \frac{\partial A_1}{\partial \eta_0} + c^2 \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial \eta_0}$$

$$+ c^2 \eta_0^2 A_2 \left(B_2^{-1} \frac{\partial B_1}{\partial \eta_0} + B_1 \frac{\partial B_2^{-1}}{\partial \eta_0} \right) + 2 c^2 \eta_0 A_2 B_1 B_2^{-1} \Bigg] (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1}$$

$$- (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} A_2 B_2^{-1} \Bigg\}$$

The following time-independent partial derivatives are computed only if the differential correction is carried through terms of second order:

$$\frac{\partial A_{11}}{\partial a} = -3 A_{11} p^{-1} \frac{\partial p}{\partial a} - \frac{3}{2} (1 - e^2)^{1/2} p^{-3} e b_2 \left[b_1 b_2 \frac{\partial p}{\partial a} + b_2 p \frac{\partial b_1}{\partial a} + 2 (b_1 p - b_2^2) \frac{\partial b_2}{\partial a} \right]$$

$$\frac{\partial A_{11}}{\partial e} = \frac{3}{4} p^{-3} b_2 (2 b_1 b_2 p - b_2^3) \left[(1-e^2)^{-1/2} e^2 + 3 (1-e^2)^{1/2} p^{-1} e \frac{\partial p}{\partial e} - (1 - e^2)^{1/2} \right]$$

$$- \frac{3}{2} (1 - e^2)^{1/2} p^{-3} e b_2 \left[b_1 b_2 \frac{\partial p}{\partial e} + b_2 p \frac{\partial b_1}{\partial e} + 2 (b_1 p - b_2^2) \frac{\partial b_2}{\partial e} \right]$$

$$\frac{\partial A_{11}}{\partial \eta_0} = - \frac{3}{2} (1 - e^2)^{1/2} p^{-3} e b_2 \left[b_2 p \frac{\partial b_1}{\partial \eta_0} + 2 (b_1 p - b_2^2) \frac{\partial b_2}{\partial \eta_0} \right]$$

$$\frac{\partial A_{12}}{\partial a} = \frac{3}{8} (1 - e^2)^{1/2} p^{-3} e^2 b_2^3 \left(\frac{\partial b_2}{\partial a} - \frac{3}{4} p^{-1} b_2 \frac{\partial p}{\partial a} \right)$$

$$\frac{\partial A_{12}}{\partial e} = \frac{3}{8} (1 - e^2)^{1/2} p^{-3} e^2 b_2^3 \left(\frac{\partial b_2}{\partial e} - \frac{3}{4} p^{-1} b_2 \frac{\partial p}{\partial e} \right) + \frac{3}{16} p^{-3} e b_2^4 \left[(1 - e^2)^{1/2} - \frac{1}{2} (1 - e^2)^{-1/2} e^2 \right]$$

$$\frac{\partial A_{12}}{\partial \eta_0} = \frac{3}{8} (1 - e^2)^{1/2} p^{-3} e^2 b_2^3 \frac{\partial b_2}{\partial \eta_0}$$

$$\frac{\partial A_{23}}{\partial a} = - A_{23} p^{-1} \frac{\partial p}{\partial a} + \frac{1}{8} (1 - e^2)^{1/2} p^{-4} e^3 b_2 \left[(3 b_1 b_2 p^{-1} - 4 b_2^3 p^{-2}) \frac{\partial p}{\partial a} \right.$$

$$\left. - b_2 \frac{\partial b_1}{\partial a} + 2 (2 b_2^2 p^{-1} - b_1) \frac{\partial b_2}{\partial a} \right]$$

$$\frac{\partial A_{23}}{\partial e} = - A_{23} p^{-1} \frac{\partial p}{\partial e} + \frac{1}{8} (1 - e^2)^{1/2} p^{-4} e^3 b_2 \left[(3 b_1 b_2 p^{-1} - 4 b_2^3 p^{-2}) \frac{\partial p}{\partial e} - b_2 \frac{\partial b_1}{\partial e} + 2 (2 b_2^2 p^{-1} - b_1) \frac{\partial b_2}{\partial e} \right]$$

$$+ \frac{1}{8} p^{-4} e^2 b_2^2 (b_2^2 p^{-1} - b_1) \left[3 (1 - e^2)^{1/2} - (1 - e^2)^{-1/2} e^2 \right]$$

$$\frac{\partial A_{23}}{\partial \eta_0} = \frac{1}{8} (1 - e^2)^{1/2} p^{-4} e^3 b_2 \left[- b_2 \frac{\partial b_1}{\partial \eta_0} + 2 (2 b_2^2 p^{-1} - b_1) \frac{\partial b_2}{\partial \eta_0} \right]$$

$$\frac{\partial A_{24}}{\partial a} = A_{24}\left(4\, b_2^{-1}\frac{\partial b_2}{\partial a} - 5\, p^{-1}\frac{\partial p}{\partial a}\right)$$

$$\frac{\partial A_{24}}{\partial e} = A_{24}\left(4\, b_2^{-1}\frac{\partial b_2}{\partial e} - 5\, p^{-1}\frac{\partial p}{\partial e}\right) + \frac{3}{256}\, p^{-5}\, b_2^4\, e^3\left[4\,(1-e^2)^{1/2} - (1-e^2)^{-1/2}\, e^2\right]$$

$$\frac{\partial A_{24}}{\partial \eta_0} = 4\, A_{24}\, b_2^{-1}\frac{\partial b_2}{\partial \eta_0}$$

DIFFERENTIAL CORRECTION: TIME-VARYING PARTIAL DERIVATIVES WITH RESPECT TO ENERGY-MOMENTA VARIABLES

We now compute the following partial derivatives of time-dependent parameters with respect to the orbital elements a, e, and η_0. We shall later compute the partial derivatives of these same parameters with respect to the remaining orbital elements β_1, β_2, and β_3.

$$\frac{\partial M_s}{\partial a} = 2\pi\left\{\frac{\partial \nu_1}{\partial a}\left(t + \beta_1 - c^2\, \beta_2\, \alpha_2^{-1}\, \eta_0^2\, B_1\, B_2^{-1}\right)\right.$$

$$\left. - \nu_1\, c^2\, \eta_0^2\, \beta_2\left(B_1\, B_2^{-1}\frac{\partial \alpha_2^{-1}}{\partial a} + \alpha_2^{-1}\, B_2^{-1}\frac{\partial B_1}{\partial a} + \alpha_2^{-1}\, B_1\frac{\partial B_2^{-1}}{\partial a}\right)\right\}$$

$$\frac{\partial M_s}{\partial e} = 2\pi\left\{\frac{\partial \nu_1}{\partial e}\left(t + \beta_1 - c^2\, \beta_2\, \alpha_2^{-1}\, \eta_0^2\, B_1\, B_2^{-1}\right)\right.$$

$$\left. - \nu_1\, c^2\, \eta_0^2\, \beta_2\left(B_1\, B_2^{-1}\frac{\partial \alpha_2^{-1}}{\partial e} + \alpha_2^{-1}\, B_2^{-1}\frac{\partial B_1}{\partial e} + \alpha_2^{-1}\, B_1\frac{\partial B_2^{-1}}{\partial e}\right)\right\}$$

$$\frac{\partial M_s}{\partial \eta_0} = 2\pi\left\{\frac{\partial \nu_1}{\partial \eta_0}\left(t + \beta_1 - c^2\, \beta_2\, \alpha_2^{-1}\, \eta_0^2\, B_1\, B_2^{-1}\right)\right.$$

$$- \nu_1\, c^2\, \eta_0^2\, \beta_2\left(B_1\, B_2^{-1}\frac{\partial \alpha_2^{-1}}{\partial \eta_0} + \alpha_2^{-1}\, B_2^{-1}\frac{\partial B_1}{\partial \eta_0} + \alpha_2^{-1}\, B_1\frac{\partial B_2^{-1}}{\partial \eta_0}\right)$$

$$\left. - 2\, \nu_1\, c^2\, \beta_2\, \alpha_2^{-1}\, \eta_0\, B_1\, B_2^{-1}\right\}$$

$$\frac{\partial \psi_s}{\partial a} = 2\pi \left\{ \frac{\partial \nu_2}{\partial a} \left[t + \beta_1 + \beta_2 \alpha_2^{-1} A_2^{-1} (a + b_1 + A_1) \right] \right.$$

$$\left. + \nu_2 \left[\beta_2 (a + b_1 + A_1) \left(\alpha_2^{-1} \frac{\partial A_2^{-1}}{\partial a} + A_2^{-1} \frac{\partial \alpha_2^{-1}}{\partial a} \right) + \beta_2 \alpha_2^{-1} A_2^{-1} \left(1 + \frac{\partial b_1}{\partial a} + \frac{\partial A_1}{\partial a} \right) \right] \right\}$$

$$\frac{\partial \psi_s}{\partial e} = 2\pi \left\{ \frac{\partial \nu_2}{\partial e} \left[t + \beta_1 + \beta_2 \alpha_2^{-1} A_2^{-1} (a + b_1 + A_1) \right] \right.$$

$$\left. + \nu_2 \left[\beta_2 (a + b_1 + A_1) \left(\alpha_2^{-1} \frac{\partial A_2^{-1}}{\partial e} + A_2^{-1} \frac{\partial \alpha_2^{-1}}{\partial e} \right) + \beta_2 \alpha_2^{-1} A_2^{-1} \left(\frac{\partial b_1}{\partial e} + \frac{\partial A_1}{\partial e} \right) \right] \right\}$$

$$\frac{\partial \psi_s}{\partial \eta_0} = 2\pi \left\{ \frac{\partial \nu_2}{\partial \eta_0} \left[t + \beta_1 + \beta_2 \alpha_2^{-1} A_2^{-1} (a + b_1 + A_1) \right] \right.$$

$$\left. + \nu_2 \left[\beta_2 (a + b_1 + A_1) \left(\alpha_2^{-1} \frac{\partial A_2^{-1}}{\partial \eta_0} + A_2^{-1} \frac{\partial \alpha_2^{-1}}{\partial \eta_0} \right) + \beta_2 \alpha_2^{-1} A_2^{-1} \left(\frac{\partial b_1}{\partial \eta_0} + \frac{\partial A_1}{\partial \eta_0} \right) \right] \right\}$$

$$\frac{\partial \mathcal{E}}{\partial a} = \left(\frac{\partial M_s}{\partial a} + \frac{\partial e'}{\partial a} \sin \mathcal{E} \right) (1 - e' \cos \mathcal{E})^{-1}$$

$$\frac{\partial \mathcal{E}}{\partial e} = \left(\frac{\partial M_s}{\partial e} + \frac{\partial e'}{\partial e} \sin \mathcal{E} \right) (1 - e' \cos \mathcal{E})^{-1}$$

$$\frac{\partial \mathcal{E}}{\partial \eta_0} = \left(\frac{\partial M_s}{\partial \eta_0} + \frac{\partial e'}{\partial \eta_0} \sin \mathcal{E} \right) (1 - e' \cos \mathcal{E})^{-1}$$

$$\frac{\partial v_0}{\partial a} = (1 - e^2) \sin \mathcal{E} (\sin v')^{-1} (1 - e \cos \mathcal{E})^{-2} \frac{\partial \mathcal{E}}{\partial a} - \frac{\partial M_s}{\partial a}$$

$$\frac{\partial v_0}{\partial e} = \left[(1 - e^2) \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial e} + \sin^2 \mathcal{E} \right] (\sin v')^{-1} (1 - e \cos \mathcal{E})^{-2} - \frac{\partial M_s}{\partial e}$$

$$\frac{\partial v_0}{\partial \eta_0} = (1 - e^2) \sin \mathcal{E} (\sin v')^{-1} (1 - e \cos \mathcal{E})^{-2} \frac{\partial \mathcal{E}}{\partial \eta_0} - \frac{\partial M_s}{\partial \eta_0}$$

$$\frac{\partial\psi_0}{\partial a} = \psi_0 \left[\left(\alpha_2 \frac{\partial\alpha_2}{\partial a} - \alpha_3 \frac{\partial\alpha_3}{\partial a}\right)(\alpha_2^2-\alpha_3^2)^{-1} - \frac{1}{2}\alpha_1^{-1}\frac{\partial\alpha_1}{\partial a} - B_2^{-1}\frac{\partial B_2}{\partial a}\right]$$

$$+ (-2\alpha_1)^{-1/2}(\alpha_2^2-\alpha_3^2)^{1/2}\eta_0^{-1}B_2^{-1}\left(A_2\frac{\partial v_0}{\partial a} + v_0\frac{\partial A_2}{\partial a}\right)$$

$$\frac{\partial\psi_0}{\partial e} = \psi_0 \left[\left(\alpha_2 \frac{\partial\alpha_2}{\partial e} - \alpha_3 \frac{\partial\alpha_3}{\partial e}\right)(\alpha_2^2-\alpha_3^2)^{-1} - \frac{1}{2}\alpha_1^{-1}\frac{\partial\alpha_1}{\partial e} - B_2^{-1}\frac{\partial B_2}{\partial e}\right]$$

$$+ (-2\alpha_1)^{-1/2}(\alpha_2^2-\alpha_3^2)^{1/2}\eta_0^{-1}B_2^{-1}\left(A_2\frac{\partial v_0}{\partial e} + v_0\frac{\partial A_2}{\partial e}\right)$$

$$\frac{\partial\psi_0}{\partial \eta_0} = \psi_0 \left[\left(\alpha_2 \frac{\partial\alpha_2}{\partial \eta_0} - \alpha_3 \frac{\partial\alpha_3}{\partial \eta_0}\right)(\alpha_2^2-\alpha_3^2)^{-1} - \frac{1}{2}\alpha_1^{-1}\frac{\partial\alpha_1}{\partial \eta_0} - B_2^{-1}\frac{\partial B_2}{\partial \eta_0} - \eta_0^{-1}\right]$$

$$+ (-2\alpha_1)^{-1/2}(\alpha_2^2-\alpha_3^2)^{1/2}\eta_0^{-1}B_2^{-1}\left(A_2\frac{\partial v_0}{\partial \eta_0} + v_0\frac{\partial A_2}{\partial \eta_0}\right)$$

$$\frac{\partial M_1}{\partial a} = -M_1(a+b_1)^{-1}\left(1+\frac{\partial b_1}{\partial a}\right) - (a+b_1)^{-1}\left\{(A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1})\frac{\partial v_0}{\partial a}\right.$$

$$+ v_0\left[\frac{\partial A_1}{\partial a} + c^2\eta_0^2\left(B_1 B_2^{-1}\frac{\partial A_2}{\partial a} + A_2 B_2^{-1}\frac{\partial B_1}{\partial a} + A_2 B_1\frac{\partial B_2^{-1}}{\partial a}\right)\right]$$

$$- \frac{1}{2}c^2\eta_0^3(-2\alpha_1)^{1/2}(\alpha_2^2-\alpha_3^2)^{-1/2}\left[\frac{1}{4}\alpha_1^{-1}\sin(2\psi_s+2\psi_0)\frac{\partial\alpha_1}{\partial a}\right.$$

$$\left.\left. + \cos(2\psi_s+2\psi_0)\left(\frac{\partial\psi_s}{\partial a} + \frac{\partial\psi_0}{\partial a}\right) - \frac{1}{2}(\alpha_2^2-\alpha_3^2)^{-1}\left(\alpha_2\frac{\partial\alpha_2}{\partial a} - \alpha_3\frac{\partial\alpha_3}{\partial a}\right)\sin(2\psi_s+2\psi_0)\right]\right\}$$

$$\frac{\partial M_1}{\partial e} = -M_1(a+b_1)^{-1}\frac{\partial b_1}{\partial e} - (a+b_1)^{-1}\left\{(A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1})\frac{\partial v_0}{\partial e}\right.$$

$$+ v_0\left[\frac{\partial A_1}{\partial e} + c^2\eta_0^2\left(B_1 B_2^{-1}\frac{\partial A_2}{\partial e} + A_2 B_2^{-1}\frac{\partial B_1}{\partial e} + A_2 B_1\frac{\partial B_2^{-1}}{\partial e}\right)\right]$$

$$- \frac{1}{2}c^2\eta_0^3(-2\alpha_1)^{1/2}(\alpha_2^2-\alpha_3^2)^{-1/2}\left[\frac{1}{4}\alpha_1^{-1}\sin(2\psi_s+2\psi_0)\frac{\partial\alpha_1}{\partial e} + \cos(2\psi_s+2\psi_0)\left(\frac{\partial\psi_s}{\partial e} + \frac{\partial\psi_0}{\partial e}\right)\right.$$

$$\left.\left. - \frac{1}{2}(\alpha_2^2-\alpha_3^2)^{-1}\left(\alpha_2\frac{\partial\alpha_2}{\partial e} - \alpha_3\frac{\partial\alpha_3}{\partial e}\right)\sin(2\psi_s+2\psi_0)\right]\right\}$$

$$\frac{\partial M_1}{\partial \eta_0} = - M_1 (a + b_1)^{-1} \frac{\partial b_1}{\partial \eta_0} - (a + b_1)^{-1} \Bigg\{ (A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}) \frac{\partial v_0}{\partial \eta_0}$$

$$+ v_0 \left[\frac{\partial A_1}{\partial \eta_0} + c^2 \eta_0^2 \left(B_1 B_2^{-1} \frac{\partial A_2}{\partial \eta_0} + A_2 B_2^{-1} \frac{\partial B_1}{\partial \eta_0} + A_2 B_1 \frac{\partial B_2^{-1}}{\partial \eta_0} \right) + 2 c^2 \eta_0 A_2 B_1 B_2^{-1} \right]$$

$$- \frac{1}{2} c^2 \eta_0^3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \left[\frac{1}{4} \alpha_1^{-1} \sin (2\psi_s + 2\psi_0) \frac{\partial \alpha_1}{\partial \eta_0} + \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0} \right) \right.$$

$$\left. - \frac{1}{2} (\alpha_2^2 - \alpha_3^2)^{-1} \left(\alpha_2 \frac{\partial \alpha_2}{\partial \eta_0} - \alpha_3 \frac{\partial \alpha_3}{\partial \eta_0} \right) \sin (2\psi_s + 2\psi_0) \right]$$

$$- \frac{3}{4} c^2 \eta_0^2 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \sin (2\psi_s + 2\psi_0) \Bigg\}$$

$$\frac{\partial E_1}{\partial a} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial a} - M_1 (1 - e' \cos \mathcal{E})^{-2} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial a} - \cos \mathcal{E} \frac{\partial e'}{\partial a} \right)$$

$$- M_1 (1 - e' \cos \mathcal{E})^{-3} \sin \mathcal{E} \left[\frac{1}{2} M_1 \frac{\partial e'}{\partial a} - \frac{3}{2} M_1 e' (1 - e' \cos \mathcal{E})^{-1} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial a} - \cos \mathcal{E} \frac{\partial e'}{\partial a} \right) \right.$$

$$\left. + \frac{1}{2} M_1 e' \cot \mathcal{E} \frac{\partial \mathcal{E}}{\partial a} + e' \frac{\partial M_1}{\partial a} \right]$$

$$\frac{\partial E_1}{\partial e} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial e} - M_1 (1 - e' \cos \mathcal{E})^{-2} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial e} - \cos \mathcal{E} \frac{\partial e'}{\partial e} \right)$$

$$- M_1 (1 - e' \cos \mathcal{E})^{-3} \sin \mathcal{E} \left[\frac{1}{2} M_1 \frac{\partial e'}{\partial e} - \frac{3}{2} M_1 e' (1 - e' \cos \mathcal{E})^{-1} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial e} - \cos \mathcal{E} \frac{\partial e'}{\partial e} \right) \right.$$

$$\left. + \frac{1}{2} M_1 e' \cot \mathcal{E} \frac{\partial \mathcal{E}}{\partial e} + e' \frac{\partial M_1}{\partial e} \right]$$

$$\frac{\partial E_1}{\partial \eta_0} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial \eta_0} - M_1 (1 - e' \cos \mathcal{E})^{-2} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial \eta_0} - \cos \mathcal{E} \frac{\partial e'}{\partial \eta_0} \right)$$

$$- M_1 (1 - e' \cos \mathcal{E})^{-3} \sin \mathcal{E} \left[\frac{1}{2} M_1 \frac{\partial e'}{\partial \eta_0} - \frac{3}{2} M_1 e' (1 - e' \cos \mathcal{E})^{-1} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial \eta_0} - \cos \mathcal{E} \frac{\partial e'}{\partial \eta_0} \right) \right.$$

$$\left. + \frac{1}{2} M_1 e' \cot \mathcal{E} \frac{\partial \mathcal{E}}{\partial \eta_0} + e' \frac{\partial M_1}{\partial \eta_0} \right]$$

$$\frac{\partial v_1}{\partial a} = (1-e^2)\sin(\mathcal{E}+\mathbf{E}_1)(\sin v'')^{-1}\,[1-e\cos(\mathcal{E}+\mathbf{E}_1)]^{-2}\left(\frac{\partial\mathcal{E}}{\partial a}+\frac{\partial \mathbf{E}_1}{\partial a}\right) - \left(\frac{\partial M_s}{\partial a}+\frac{\partial v_0}{\partial a}\right)$$

$$\frac{\partial v_1}{\partial e} = \left[(1-e^2)\sin(\mathcal{E}+\mathbf{E}_1)\left(\frac{\partial\mathcal{E}}{\partial e}+\frac{\partial \mathbf{E}_1}{\partial e}\right) + \sin^2(\mathcal{E}+\mathbf{E}_1)\right](\sin v'')^{-1}\,[1-e\cos(\mathcal{E}+\mathbf{E}_1)]^{-2}$$

$$- \left(\frac{\partial M_s}{\partial e}+\frac{\partial v_0}{\partial e}\right)$$

$$\frac{\partial v_1}{\partial \eta_0} = (1-e^2)\sin(\mathcal{E}+\mathbf{E}_1)(\sin v'')^{-1}\,[1-e\cos(\mathcal{E}+\mathbf{E}_1)]^{-2}\left(\frac{\partial\mathcal{E}}{\partial \eta_0}+\frac{\partial \mathbf{E}_1}{\partial \eta_0}\right) - \left(\frac{\partial M_s}{\partial \eta_0}+\frac{\partial v_0}{\partial \eta_0}\right)$$

$$\frac{\partial \psi_1}{\partial a} = \left[\psi_1 - \frac{1}{8}q^2 B_2^{-1}\sin(2\psi_s+2\psi_0)\right]\left[-\frac{1}{2}\alpha_1^{-1}\frac{\partial\alpha_1}{\partial a}\right.$$

$$+ (\alpha_2^2-\alpha_3^2)^{-1}\left(\alpha_2\frac{\partial\alpha_2}{\partial a} - \alpha_3\frac{\partial\alpha_3}{\partial a}\right) - B_2^{-1}\left.\frac{\partial B_2}{\partial a}\right]$$

$$+ (-2\alpha_1)^{-1/2}(\alpha_2^2-\alpha_3^2)^{1/2}\,\eta_0^{-1}\,B_2^{-1}\left[A_2\frac{\partial v_1}{\partial a} + v_1\frac{\partial A_2}{\partial a}\right.$$

$$+ (A_{21}\cos v' + 2A_{22}\cos 2v')\left(\frac{\partial M_s}{\partial a}+\frac{\partial v_0}{\partial a}\right) + \sin v'\frac{\partial A_{21}}{\partial a} + \sin 2v'\left.\frac{\partial A_{22}}{\partial a}\right]$$

$$+ \frac{1}{4}q^2 B_2^{-1}\cos(2\psi_s+2\psi_0)\left(\frac{\partial\psi_s}{\partial a}+\frac{\partial\psi_0}{\partial a}\right)$$

$$+ \frac{1}{4}q B_2^{-1}\sin(2\psi_s+2\psi_0)\frac{\partial q}{\partial a} - \frac{1}{8}q^2 B_2^{-2}\sin(2\psi_s+2\psi_0)\frac{\partial B_2}{\partial a}$$

$$\frac{\partial\psi_1}{\partial e} = \left[\psi_1 - \frac{1}{8} q^2 B_2^{-1} \sin(2\psi_s + 2\psi_0)\right] \left[-\frac{1}{2}\alpha_1^{-1}\frac{\partial\alpha_1}{\partial e} + (\alpha_2^2 - \alpha_3^2)^{-1}\left(\alpha_2\frac{\partial\alpha_2}{\partial e} - \alpha_3\frac{\partial\alpha_3}{\partial e}\right)\right.$$

$$\left. - B_2^{-1}\frac{\partial B_2}{\partial e}\right] + (-2\alpha_1)^{-1/2}(\alpha_2^2 - \alpha_3^2)^{1/2}\eta_0^{-1} B_2^{-1}\left[A_2\frac{\partial v_1}{\partial e} + v_1\frac{\partial A_2}{\partial e}\right.$$

$$\left. + (A_{21}\cos v' + 2A_{22}\cos 2v')\left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right) + \sin v'\frac{\partial A_{21}}{\partial e} + \sin 2v'\frac{\partial A_{22}}{\partial e}\right]$$

$$+ \frac{1}{4} q^2 B_2^{-1}\cos(2\psi_s + 2\psi_0)\left(\frac{\partial\psi_s}{\partial e} + \frac{\partial\psi_0}{\partial e}\right) + \frac{1}{4} q B_2^{-1}\sin(2\psi_s + 2\psi_0)\frac{\partial q}{\partial e}$$

$$- \frac{1}{8} q^2 B_2^{-2}\sin(2\psi_s + 2\psi_0)\frac{\partial B_2}{\partial e}$$

$$\frac{\partial\psi_1}{\partial\eta_0} = \left[\psi_1 - \frac{1}{8} q^2 B_2^{-1}\sin(2\psi_s + 2\psi_0)\right]\left[-\frac{1}{2}\alpha_1^{-1}\frac{\partial\alpha_1}{\partial\eta_0} + (\alpha_2^2 - \alpha_3^2)^{-1}\left(\alpha_2\frac{\partial\alpha_2}{\partial\eta_0} - \alpha_3\frac{\partial\alpha_3}{\partial\eta_0}\right)\right.$$

$$\left. - B_2^{-1}\frac{\partial B_2}{\partial\eta_0} - \eta_0^{-1}\right] + (-2\alpha_1)^{-1/2}(\alpha_2^2 - \alpha_3^2)^{1/2}\eta_0^{-1}B_2^{-1}\left[A_2\frac{\partial v_1}{\partial\eta_0} + v_1\frac{\partial A_2}{\partial\eta_0}\right.$$

$$\left. + (A_{21}\cos v' + 2A_{22}\cos 2v')\left(\frac{\partial M_s}{\partial\eta_0} + \frac{\partial v_0}{\partial\eta_0}\right) + \sin v'\frac{\partial A_{21}}{\partial\eta_0} + \sin 2v'\frac{\partial A_{22}}{\partial\eta_0}\right]$$

$$+ \frac{1}{4} q^2 B_2^{-1}\cos(2\psi_s + 2\psi_0)\left(\frac{\partial\psi_s}{\partial\eta_0} + \frac{\partial\psi_0}{\partial\eta_0}\right) + \frac{1}{4} q B_2^{-1}\sin(2\psi_s + 2\psi_0)\frac{\partial q}{\partial\eta_0}$$

$$- \frac{1}{8} q^2 B_2^{-2}\sin(2\psi_s + 2\psi_0)\frac{\partial B_2}{\partial\eta_0}$$

The following time-dependent partial derivatives with respect to the orbital elements a, e, and η_0 are computed only if the differential correction is carried through terms of second order:

$$\frac{\partial M_2}{\partial a} = -M_2(a + b_1)^{-1}\left(1 + \frac{\partial b_1}{\partial a}\right) - (a + b_1)^{-1}\left\{A_1\frac{\partial v_1}{\partial a} + v_1\frac{\partial A_1}{\partial a} + \sin v'\frac{\partial A_{11}}{\partial a}\right.$$

$$+ (A_{11}\cos v' + 2A_{12}\cos 2v')\left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right) + \sin 2v'\frac{\partial A_{12}}{\partial a}$$

$$- c^2\eta_0^3\left[(-2\alpha_1)^{-1/2}(\alpha_2^2 - \alpha_3^2)^{-1/2}\frac{\partial\alpha_1}{\partial a} + (-2\alpha_1)^{1/2}(\alpha_2^2 - \alpha_3^2)^{-3/2}\left(\alpha_2\frac{\partial\alpha_2}{\partial a} - \alpha_3\frac{\partial\alpha_3}{\partial a}\right)\right]\left[B_1\psi_1\right.$$

$$- \frac{1}{2} \psi_1 \cos (2\psi_s + 2\psi_0) - \frac{1}{8} q^2 \sin (2\psi_s + 2\psi_0)$$

$$+ \frac{1}{64} q^2 \sin (4\psi_s + 4\psi_0)\Bigg] + c^2 \eta_0^3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \Bigg[B_1 \frac{\partial \psi_1}{\partial a} + \psi_1 \frac{\partial B_1}{\partial a}$$

$$- \frac{1}{2} \cos (2\psi_s + 2\psi_0) \frac{\partial \psi_1}{\partial a} + \psi_1 \sin (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right) - \frac{1}{4} q^2 \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right)$$

$$- \frac{1}{4} q \sin (2\psi_s + 2\psi_0) \frac{\partial q}{\partial a} + \frac{1}{16} q^2 \cos (4\psi_s + 4\psi_0) \left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial \alpha}\right)$$

$$+ \frac{1}{32} q \sin (4\psi_s + 4\psi_0) \frac{\partial q}{\partial a}\Bigg]\Bigg\}$$

$$\frac{\partial M_2}{\partial e} = - M_2 (a + b_1)^{-1} \frac{\partial b_1}{\partial e} - (a + b_1)^{-1} \Bigg\{A_1 \frac{\partial v_1}{\partial e} + v_1 \frac{\partial A_1}{\partial e} + \sin v' \frac{\partial A_{11}}{\partial e}$$

$$+ (A_{11} \cos v' + 2 A_{12} \cos 2v') \left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right) + \sin 2v' \frac{\partial A_{12}}{\partial e}$$

$$- c^2 \eta_0^3 \left[(-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \frac{\partial \alpha_1}{\partial e} + (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-3/2} \left(\alpha_2 \frac{\partial \alpha_2}{\partial e} - \alpha_3 \frac{\partial \alpha_3}{\partial e}\right)\right] \Bigg[B_1 \psi_1$$

$$- \frac{1}{2} \psi_1 \cos (2\psi_s + 2\psi_0) - \frac{1}{8} q^2 \sin (2\psi_s + 2\psi_0) + \frac{1}{64} q^2 \sin (4\psi_s + 4\psi_0)\Bigg]$$

$$+ c^2 \eta_0^3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \Bigg[B_1 \frac{\partial \psi_1}{\partial e} + \psi_1 \frac{\partial B_1}{\partial e} - \frac{1}{2} \cos (2\psi_s + 2\psi_0) \frac{\partial \psi_1}{\partial e}$$

$$+ \psi_1 \sin (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right) - \frac{1}{4} q^2 \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right)$$

$$- \frac{1}{4} q \sin (2\psi_s + 2\psi_0) \frac{\partial q}{\partial e} + \frac{1}{16} q^2 \cos (4\psi_s + 4\psi_0) \left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right)$$

$$+ \frac{1}{32} q \sin (4\psi_s + 4\psi_0) \frac{\partial q}{\partial e}\Bigg]\Bigg\}$$

$$\frac{\partial M_2}{\partial \eta_0} = - M_2 (a + b_1)^{-1} \frac{\partial b_1}{\partial \eta_0} - (a + b_1)^{-1} \left\{ A_1 \frac{\partial v_1}{\partial \eta_0} + v_1 \frac{\partial A_1}{\partial \eta_0} + \sin v' \frac{\partial A_{11}}{\partial \eta_0} \right.$$

$$+ (A_{11} \cos v' + 2A_{12} \cos 2v') \left(\frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0} \right) + \sin 2v' \frac{\partial A_{12}}{\partial \eta_0}$$

$$- c^2 \eta_0^2 \left[\eta_0 (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \frac{\partial \alpha_1}{\partial \eta_0} + \eta_0 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-3/2} \left(\alpha_2 \frac{\partial \alpha_2}{\partial \eta_0} - \alpha_3 \frac{\partial \alpha_3}{\partial \eta_0} \right) \right.$$

$$\left. - 3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \right] \left[B_1 \psi_1 - \frac{1}{2} \psi_1 \cos (2\psi_s + 2\psi_0) - \frac{1}{8} q^2 \sin (2\psi_s + 2\psi_0) \right.$$

$$\left. + \frac{1}{64} q^2 \sin (4\psi_s + 4\psi_0) \right] + c^2 \eta_0^3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \left[B_1 \frac{\partial \psi_1}{\partial \eta_0} + \psi_1 \frac{\partial B_1}{\partial \eta_0} - \frac{1}{2} \cos (2\psi_s + 2\psi_0) \frac{\partial \psi_1}{\partial \eta_0} \right.$$

$$+ \psi_1 \sin (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0} \right) - \frac{1}{4} q^2 \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0} \right)$$

$$- \frac{1}{4} q \sin (2\psi_s + 2\psi_0) \frac{\partial q}{\partial \eta_0} + \frac{1}{16} q^2 \cos (4\psi_s + 4\psi_0) \left(\frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0} \right)$$

$$\left. \left. + \frac{1}{32} q \sin (4\psi_s + 4\psi_0) \frac{\partial q}{\partial \eta_0} \right] \right\}$$

$$\frac{\partial E_2}{\partial a} = [1 - e' \cos (\mathcal{E} + E_1)]^{-1} \frac{\partial M_2}{\partial a}$$

$$- M_2 [1 - e' \cos (\mathcal{E} + E_1)]^{-2} \left[e' \sin (\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial a} + \frac{\partial E_1}{\partial a} \right) - \cos (\mathcal{E} + E_1) \frac{\partial e'}{\partial a} \right]$$

$$\frac{\partial E_2}{\partial e} = [1 - e' \cos (\mathcal{E} + E_1)]^{-1} \frac{\partial M_2}{\partial e}$$

$$- M_2 [1 - e' \cos (\mathcal{E} + E_1)]^{-2} \left[e' \sin (\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial e} + \frac{\partial E_1}{\partial e} \right) - \cos (\mathcal{E} + E_1) \frac{\partial e'}{\partial e} \right]$$

$$\frac{\partial E_2}{\partial \eta_0} = [1 - e' \cos (\mathcal{E} + E_1)]^{-1} \frac{\partial M_2}{\partial \eta_0}$$

$$- M_2 [1 - e' \cos (\mathcal{E} + E_1)]^{-2} \left[e' \sin (\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial \eta_0} + \frac{\partial E_1}{\partial \eta_0} \right) - \cos (\mathcal{E} + E_1) \frac{\partial e'}{\partial \eta_0} \right]$$

$$\frac{\partial E}{\partial a} = \frac{\partial \mathcal{E}}{\partial a} + \frac{\partial E_1}{\partial a} + \frac{\partial E_2}{\partial a}$$

$$\frac{\partial \mathcal{E}}{\partial e} = \frac{\partial \mathcal{E}}{\partial e} + \frac{\partial E_1}{\partial e} + \frac{\partial E_2}{\partial e}$$

$$\frac{\partial \mathcal{E}}{\partial \eta_0} = \frac{\partial \mathcal{E}}{\partial \eta_0} + \frac{\partial E_1}{\partial \eta_0} + \frac{\partial E_2}{\partial \eta_0}$$

$$\frac{\partial v_2}{\partial a} = (1 - e^2)\ \sin E\ (\sin v)^{-1}\ (1 - e \cos E)^{-2}\ \frac{\partial E}{\partial a} - \left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a} + \frac{\partial v_1}{\partial a}\right)$$

$$\frac{\partial v_2}{\partial e} = \left[(1 - e^2)\ \sin E\ \frac{\partial E}{\partial e} + \sin^2 E\right] (\sin v)^{-1}\ (1 - e \cos E)^{-2} - \left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e} + \frac{\partial v_1}{\partial e}\right)$$

$$\frac{\partial v_2}{\partial \eta_0} = (1 - e^2)\ \sin E\ (\sin v)^{-1}\ (1 - e \cos E)^{-2}\ \frac{\partial E}{\partial \eta_0} - \left(\frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0} + \frac{\partial v_1}{\partial \eta_0}\right)$$

$$\frac{\partial v}{\partial a} = \frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a} + \frac{\partial v_1}{\partial a} + \frac{\partial v_2}{\partial a}$$

$$\frac{\partial v}{\partial e} = \frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e} + \frac{\partial v_1}{\partial e} + \frac{\partial v_2}{\partial e}$$

$$\frac{\partial v}{\partial \eta_0} = \frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0} + \frac{\partial v_1}{\partial \eta_0} + \frac{\partial v_2}{\partial \eta_0}$$

$$\frac{\partial \psi_2}{\partial a} = \left\{\psi_2 - \frac{1}{4}\, q^2\, B_2^{-1} \left[\psi_1 \cos(2\psi_s + 2\psi_0) + \frac{3}{8}\, q^2 \sin(2\psi_s + 2\psi_0)\right.\right.$$

$$\left.\left. - \frac{3}{64}\, q^2 \sin(4\psi_s + 4\psi_0)\right]\right\} \left[-\frac{1}{2}\,\alpha_1^{-1}\,\frac{\partial \alpha_1}{\partial a} + (\alpha_2^2 - \alpha_3^2)^{-1} \left(\alpha_2\,\frac{\partial \alpha_2}{\partial a} - \alpha_3\,\frac{\partial \alpha_3}{\partial a}\right) - B_2^{-1}\,\frac{\partial B_2}{\partial a}\right]$$

$$+ (-2\alpha_1)^{-1/2}\,(\alpha_2^2 - \alpha_3^2)^{1/2}\,\eta_0^{-1}\,B_2^{-1} \left[A_2\,\frac{\partial v_2}{\partial a} + v_2\,\frac{\partial A_2}{\partial a} + \frac{\partial v_1}{\partial a}\,(A_{21} \cos v' + 2\,A_{22} \cos 2v')\right.$$

$$- (A_{21}\, v_1 \sin v' + 4\,A_{22}\, v_1 \sin 2v' - 3\,A_{23} \cos 3v' - 4\,A_{24} \cos 4v') \left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right)$$

$$\left. + v_1 \cos v'\,\frac{\partial A_{21}}{\partial a} + 2v_1 \cos 2v'\,\frac{\partial A_{22}}{\partial a} + \sin 3v'\,\frac{\partial A_{23}}{\partial a} + \sin 4v'\,\frac{\partial A_{24}}{\partial a}\right]$$

$$+\left[\frac{1}{2} q B_2^{-1}\left(\frac{\partial q}{\partial a}-\frac{1}{2} q B_2^{-1} \frac{\partial B_2}{\partial a}\right)\right]\left[\psi_1 \cos(2\psi_s+2\psi_0)+\frac{3}{8} q^2 \sin(2\psi_s+2\psi_0)\right.$$

$$\left.\frac{3}{64} q^2 \sin(4\psi_s+4\psi_0)\right]+\frac{1}{4} q^2 B_2^{-1}\left\{\cos(2\psi_s+2\psi_0)\frac{\partial\psi_1}{\partial a}-\left[2\psi_1 \sin(2\psi_s+2\psi_0)\right.\right.$$

$$\left.\frac{3}{4} q^2 \cos(2\psi_s+2\psi_0)+\frac{3}{16} q^2 \cos(4\psi_s+4\psi_0)\right]\left(\frac{\partial\psi_s}{\partial a}+\frac{\partial\psi_0}{\partial a}\right)$$

$$\left.+\frac{3}{4} q\left[\sin(2\psi_s+2\psi_0)-\frac{1}{8}\sin(4\psi_s+4\psi_0)\right]\frac{\partial q}{\partial a}\right\}$$

$$\frac{\partial\psi_2}{\partial e}=\left\{\psi_2-\frac{1}{4} q^2 B_2^{-1}\left[\psi_1 \cos(2\psi_s+2\psi_0)+\frac{3}{8} q^2 \sin(2\psi_s+2\psi_0)\right.\right.$$

$$\left.\left.-\frac{3}{64} q^2 \sin(4\psi_s+4\psi_0)\right]\right\}\left[-\frac{1}{2}\alpha_1^{-1}\frac{\partial\alpha_1}{\partial e}+(\alpha_2^2-\alpha_3^2)^{-1}\left(\alpha_2\frac{\partial\alpha_2}{\partial e}-\alpha_3\frac{\partial\alpha_3}{\partial e}\right)-B_2^{-1}\frac{\partial B_2}{\partial e}\right]$$

$$+(-2\alpha_1)^{-1/2}(\alpha_2^2-\alpha_3^2)^{1/2}\eta_0^{-1}B_2^{-1}\left[A_2\frac{\partial v_2}{\partial e}+v_2\frac{\partial A_2}{\partial e}+\frac{\partial v_1}{\partial e}(A_{21}\cos v'+2A_{22}\cos 2v')\right.$$

$$-(A_{21}v_1\sin v'+4A_{22}v_1\sin 2v'-3A_{23}\cos 3v'-4A_{24}\cos 4v')\left(\frac{\partial M_s}{\partial e}+\frac{\partial v_0}{\partial e}\right)$$

$$\left.+v_1\cos v'\frac{\partial A_{21}}{\partial e}+2v_1\cos 2v'\frac{\partial A_{22}}{\partial e}+\sin 3v'\frac{\partial A_{23}}{\partial e}+\sin 4v'\frac{\partial A_{24}}{\partial e}\right]$$

$$+\left[\frac{1}{2} q B_2^{-1}\left(\frac{\partial q}{\partial e}-\frac{1}{2} q B_2^{-1}\frac{\partial B_2}{\partial e}\right)\right]\left[\psi_1 \cos(2\psi_s+2\psi_0)+\frac{3}{8} q^2 \sin(2\psi_s+2\psi_0)\right.$$

$$\left.-\frac{3}{64} q^2 \sin(4\psi_s+4\psi_0)\right]+\frac{1}{4} q^2 B_2^{-1}\left\{\cos(2\psi_s+2\psi_0)\frac{\partial\psi_1}{\partial e}-\left[2\psi_1 \sin(2\psi_s+2\psi_0)\right.\right.$$

$$\left.-\frac{3}{4} q^2 \cos(2\psi_s+2\psi_0)+\frac{3}{16} q^2 \cos(4\psi_s+4\psi_0)\right]\left(\frac{\partial\psi_s}{\partial e}+\frac{\partial\psi_0}{\partial e}\right)$$

$$\left.+\frac{3}{4} q\left[\sin(2\psi_s+2\psi_0)-\frac{1}{8}\sin(4\psi_s+4\psi_0)\right]\frac{\partial q}{\partial e}\right\}$$

$$\frac{\partial\psi_2}{\partial\eta_0} = \left\{\psi_2 - \frac{1}{4} q^2 B_2^{-1} \left[\psi_1 \cos(2\psi_s + 2\psi_0) + \frac{3}{8} q^2 \sin(2\psi_s + 2\psi_0)\right.\right.$$

$$\left.\left. - \frac{3}{64} q^2 \sin(4\psi_s + 4\psi_0)\right]\right\} \left[-\frac{1}{2} \alpha_1^{-1} \frac{\partial\alpha_1}{\partial\eta_0} + (\alpha_2^2 - \alpha_3^2)^{-1} \left(\alpha_2 \frac{\partial\alpha_2}{\partial\eta_0} - \alpha_3 \frac{\partial\alpha_3}{\partial\eta_0}\right) - B_2^{-1} \frac{\partial B_2}{\partial\eta_0} - \eta_0^{-1}\right]$$

$$+ (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} B_2^{-1} \left[A_2 \frac{\partial v_2}{\partial\eta_0} + v_2 \frac{\partial A_2}{\partial\eta_0} + \frac{\partial v_1}{\partial\eta_0} (A_{21} \cos v' + 2 A_{22} \cos 2v')\right.$$

$$- (A_{21} v_1 \sin v' + 4 A_{22} v_1 \sin 2v' - 3 A_{23} \cos 3v' - 4 A_{24} \cos 4v') \left(\frac{\partial M_s}{\partial\eta_0} + \frac{\partial v_0}{\partial\eta_0}\right)$$

$$\left. + v_1 \cos v' \frac{\partial A_{21}}{\partial\eta_0} + 2v_1 \cos 2v' \frac{\partial A_{22}}{\partial\eta_0} + \sin 3v' \frac{\partial A_{23}}{\partial\eta_0} + \sin 4v' \frac{\partial A_{24}}{\partial\eta_0}\right]$$

$$+ \left[\frac{1}{2} q B_2^{-1} \left(\frac{\partial q}{\partial\eta_0} - \frac{1}{2} q B_2^{-1} \frac{\partial B_2}{\partial\eta_0}\right)\right] \left[\psi_1 \cos(2\psi_s + 2\psi_0)\right.$$

$$\left. + \frac{3}{8} q^2 \sin(2\psi_s + 2\psi_0) - \frac{3}{64} q^2 \sin(4\psi_s + 4\psi_0)\right] + \frac{1}{4} q^2 B_2^{-1} \left\{\cos(2\psi_s + 2\psi_0) \frac{\partial\psi_1}{\partial\eta_0}\right.$$

$$- \left[2\psi_1 \sin(2\psi_s + 2\psi_0) - \frac{3}{4} q^2 \cos(2\psi_s + 2\psi_0) + \frac{3}{16} q^2 \cos(4\psi_s + 4\psi_0)\right] \left(\frac{\partial\psi_s}{\partial\eta_0} + \frac{\partial\psi_0}{\partial\eta_0}\right)$$

$$\left. + \frac{3}{4} q \left[\sin(2\psi_s + 2\psi_0) - \frac{1}{8} \sin(4\psi_s + 4\psi_0)\right] \frac{\partial q}{\partial\eta_0}\right\}$$

$$\frac{\partial\psi}{\partial a} = \frac{\partial\psi_s}{\partial a} + \frac{\partial\psi_0}{\partial a} + \frac{\partial\psi_1}{\partial a} + \frac{\partial\psi_2}{\partial a}$$

$$\frac{\partial\psi}{\partial e} = \frac{\partial\psi_s}{\partial e} + \frac{\partial\psi_0}{\partial e} + \frac{\partial\psi_1}{\partial e} + \frac{\partial\psi_2}{\partial e}$$

$$\frac{\partial\psi}{\partial\eta_0} = \frac{\partial\psi_s}{\partial\eta_0} + \frac{\partial\psi_0}{\partial\eta_0} + \frac{\partial\psi_1}{\partial\eta_0} + \frac{\partial\psi_2}{\partial\eta_0}$$

This completes the computation of the partial derivatives of the uniformizing variables E, v, and ψ with respect to the orbital elements a, e, and η_0 when the calculation is followed through terms of the second order. If, however, second-order precision is not necessary, we can eliminate the terms with the subscript "2" (thus omitting all partial derivatives of M_2, E_2, v_2, and ψ_2), and the above partial derivatives of the uniformizing variables reduce to the following:

$$\frac{\partial E}{\partial a} = \frac{\partial \mathcal{E}}{\partial a} + \frac{\partial E_1}{\partial a}$$

$$\frac{\partial E}{\partial e} = \frac{\partial \mathcal{E}}{\partial e} + \frac{\partial E_1}{\partial e}$$

$$\frac{\partial E}{\partial \eta_0} = \frac{\partial \mathcal{E}}{\partial \eta_0} + \frac{\partial E_1}{\partial \eta_0}$$

$$\frac{\partial v}{\partial a} = \frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a} + \frac{\partial v_1}{\partial a}$$

$$\frac{\partial v}{\partial e} = \frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e} + \frac{\partial v_1}{\partial e}$$

$$\frac{\partial v}{\partial \eta_0} = \frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0} + \frac{\partial v_1}{\partial \eta_0}$$

$$\frac{\partial \psi}{\partial a} = \frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a} + \frac{\partial \psi_1}{\partial a}$$

$$\frac{\partial \psi}{\partial e} = \frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e} + \frac{\partial \psi_1}{\partial e}$$

$$\frac{\partial \psi}{\partial \eta_0} = \frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0} + \frac{\partial \psi_1}{\partial \eta_0}$$

We now continue with the necessary equations preparatory to the partial derivatives of the orthogonal co-ordinates X, Y, and Z.

$$\frac{\partial \chi}{\partial a} = (1 - \eta_0^2) \sin \psi (\sin \chi)^{-1} (1 - \eta_0^2 \sin^2 \psi)^{-3/2} \frac{\partial \psi}{\partial a}$$

$$\frac{\partial \chi}{\partial e} = (1 - \eta_0^2) \sin \psi (\sin \chi)^{-1} (1 - \eta_0^2 \sin^2 \psi)^{-3/2} \frac{\partial \psi}{\partial e}$$

$$\frac{\partial \chi}{\partial \eta_0} = (1 - \eta_0^2) \sin \psi (\sin \chi)^{-1} (1 - \eta_0^2 \sin^2 \psi)^{-3/2} \frac{\partial \psi}{\partial \eta_0}$$

$$- \eta_0 \cos \psi \sin^2 \psi (\sin \chi)^{-1} (1 - \eta_0^2 \sin^2 \psi)^{-3/2}$$

$$\frac{\partial \phi}{\partial a} = \left[(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\chi + B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]\left[(\alpha_2^2-\alpha_3^2)^{-1/2}\eta_0\frac{\partial \alpha_3}{\partial a}\right.$$

$$\left.-\alpha_3(\alpha_2^2-\alpha_3^2)^{-3/2}\eta_0\left(\alpha_2\frac{\partial\alpha_2}{\partial a}-\alpha_3\frac{\partial\alpha_3}{\partial a}\right)\right] + \alpha_3(\alpha_2^2-\alpha_3^2)^{-1/2}\eta_0\left[(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\frac{\partial\chi}{\partial a}\right.$$

$$-\eta_2^{-3}(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-3/2}\chi\frac{\partial\eta_2}{\partial a} + B_3\frac{\partial\psi}{\partial a} + \psi\frac{\partial B_3}{\partial a}$$

$$\left.-\frac{3}{8}\eta_0^2\eta_2^{-5}\sin 2\psi\frac{\partial\eta_2}{\partial a} + \frac{3}{16}\eta_0^2\eta_2^{-4}\cos 2\psi\frac{\partial\psi}{\partial a}\right]$$

$$-c^2\left[(-2\alpha_1)^{-1/2}\frac{\partial\alpha_3}{\partial a} + \alpha_3(-2\alpha_1)^{-3/2}\frac{\partial\alpha_1}{\partial a}\right](A_3 v + A_{31}\sin v + A_{32}\sin 2v$$

$$+A_{33}\sin 3v + A_{34}\sin 4v) - c^2\alpha_3(-2\alpha_1)^{-1/2}\left[(A_3 + A_{31}\cos v + 2A_{32}\cos 2v\right.$$

$$+3A_{33}\cos 3v + 4A_{34}\cos 4v)\frac{\partial v}{\partial a} + \sin v\frac{\partial A_{31}}{\partial a} + \sin 2v\frac{\partial A_{32}}{\partial a}$$

$$\left.+\sin 3v\frac{\partial A_{33}}{\partial a} + \sin 4v\frac{\partial A_{34}}{\partial a} + v\frac{\partial A_3}{\partial a}\right]$$

$$\frac{\partial \phi}{\partial e} = \left[(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\chi + B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]\left[(\alpha_2^2-\alpha_3^2)^{-1/2}\eta_0\frac{\partial \alpha_3}{\partial e}\right.$$

$$\left.-\alpha_3(\alpha_2^2-\alpha_3^2)^{-3/2}\eta_0\left(\alpha_2\frac{\partial\alpha_2}{\partial e}-\alpha_3\frac{\partial\alpha_3}{\partial e}\right)\right] + \alpha_3(\alpha_2^2-\alpha_3^2)^{-1/2}\eta_0\left[(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\frac{\partial\chi}{\partial e}\right.$$

$$-\eta_2^{-3}(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-3/2}\chi\frac{\partial\eta_2}{\partial e} + B_3\frac{\partial\psi}{\partial e} + \psi\frac{\partial B_3}{\partial e}$$

$$\left.-\frac{3}{8}\eta_0^2\eta_2^{-5}\sin 2\psi\frac{\partial\eta_2}{\partial e} + \frac{3}{16}\eta_0^2\eta_2^{-4}\cos 2\psi\frac{\partial\psi}{\partial e}\right]$$

$$-c^2\left[(-2\alpha_1)^{-1/2}\frac{\partial\alpha_3}{\partial e} + \alpha_3(-2\alpha_1)^{-3/2}\frac{\partial\alpha_1}{\partial e}\right](A_3 v + A_{31}\sin v + A_{32}\sin 2v$$

$$+A_{33}\sin 3v + A_{34}\sin 4v) - c^2\alpha_3(-2\alpha_1)^{-1/2}\left[(A_3 + A_{31}\cos v + 2A_{32}\cos 2v\right.$$

$$+ 3A_{33}\cos 3v + 4A_{34}\cos 4v)\frac{\partial v}{\partial e} + \sin v\frac{\partial A_{31}}{\partial e} + \sin 2v\frac{\partial A_{32}}{\partial e} + \sin 3v\frac{\partial A_{33}}{\partial e} + \sin 4v\frac{\partial A_{34}}{\partial e} + v\frac{\partial A_3}{\partial e}\Bigg]$$

$$\frac{\partial\phi}{\partial\eta_0} = \left[(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\chi + B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]\Bigg[(\alpha_2^2-\alpha_3^2)^{-1/2}\eta_0\frac{\partial\alpha_3}{\partial\eta_0}$$

$$-\alpha_3(\alpha_2^2-\alpha_3^2)^{-3/2}\eta_0\left(\alpha_2\frac{\partial\alpha_2}{\partial\eta_0} - \alpha_3\frac{\partial\alpha_3}{\partial\eta_0}\right) + \alpha_3(\alpha_2^2-\alpha_3^2)^{-1/2}\Bigg]$$

$$+\alpha_3(\alpha_2^2-\alpha_3^2)^{-1/2}\eta_0\Bigg[(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-1/2}\frac{\partial\chi}{\partial\eta_0} - \eta_2^{-3}(1-\eta_0^2)^{-1/2}(1-\eta_2^{-2})^{-3/2}\chi\frac{\partial\eta_2}{\partial\eta_0} + B_3\frac{\partial\psi}{\partial\eta_0} + \psi\frac{\partial B_3}{\partial\eta_0}$$

$$-\frac{3}{8}\eta_0^2\eta_2^{-5}\sin 2\psi\frac{\partial\eta_2}{\partial\eta_0} + \frac{3}{16}\eta_0^2\eta_2^{-4}\cos 2\psi\frac{\partial\psi}{\partial\eta_0} + \eta_0(1-\eta_0^2)^{-3/2}(1-\eta_2^{-2})^{-1/2}\chi$$

$$+\frac{3}{16}\eta_0\eta_2^{-4}\sin 2\psi\Bigg] - c^2\left[(-2\alpha_1)^{-1/2}\frac{\partial\alpha_3}{\partial\eta_0} + \alpha_3(-2\alpha_1)^{-3/2}\frac{\partial\alpha_1}{\partial\eta_0}\right](A_3 v + A_{31}\sin v$$

$$+ A_{32}\sin 2v + A_{33}\sin 3v + A_{34}\sin 4v) - c^2\alpha_3(-2\alpha_1)^{-1/2}\Big[(A_3 + A_{31}\cos v + 2A_{32}\cos 2v$$

$$+ 3A_{33}\cos 3v + 4A_{34}\cos 4v)\frac{\partial v}{\partial\eta_0} + \sin v\frac{\partial A_{31}}{\partial\eta_0} + \sin 2v\frac{\partial A_{32}}{\partial\eta_0} + \sin 3v\frac{\partial A_{33}}{\partial\eta_0} + \sin 4v\frac{\partial A_{34}}{\partial\eta_0} + v\frac{\partial A_3}{\partial\eta_0}\Big]$$

$$\frac{\partial X}{\partial a} = X\left[(\rho^2+c^2)^{-1}\rho\left(a e\sin E\frac{\partial E}{\partial a} + 1 - e\cos E\right) - (1-\eta_0^2\sin^2\psi)^{-1}\eta_0^2\sin\psi\cos\psi\frac{\partial\psi}{\partial a}\right] - Y\frac{\partial\phi}{\partial a}$$

$$\frac{\partial X}{\partial e} = X\left[(\rho^2+c^2)^{-1}\rho a\left(e\sin E\frac{\partial E}{\partial e} - \cos E\right) - (1-\eta_0^2\sin^2\psi)^{-1}\eta_0^2\sin\psi\cos\psi\frac{\partial\psi}{\partial e}\right] - Y\frac{\partial\phi}{\partial e}$$

$$\frac{\partial X}{\partial\eta_0} = X\Bigg[(\rho^2+c^2)^{-1}\rho a e\sin E\frac{\partial E}{\partial\eta_0}$$

$$-(1-\eta_0^2\sin^2\psi)^{-1}\eta_0\sin\psi\left(\eta_0\cos\psi\frac{\partial\psi}{\partial\eta_0} + \sin\psi\right)\Bigg] - Y\frac{\partial\phi}{\partial\eta_0}$$

$$\frac{\partial Y}{\partial a} = Y\Bigg[(\rho^2+c^2)^{-1}\rho\left(a e\sin E\frac{\partial E}{\partial a} + 1 - e\cos E\right)$$

$$-(1-\eta_0^2\sin^2\psi)^{-1}\eta_0^2\sin\psi\cos\psi\frac{\partial\psi}{\partial a}\Bigg] + X\frac{\partial\phi}{\partial a}$$

$$\frac{\partial Y}{\partial e} = Y\Bigg[(\rho^2+c^2)^{-1}\rho a\left(e\sin E\frac{\partial E}{\partial e} - \cos E\right)$$

$$-(1-\eta_0^2\sin^2\psi)^{-1}\eta_0^2\sin\psi\cos\psi\frac{\partial\psi}{\partial e}\Bigg] + X\frac{\partial\phi}{\partial e}$$

$$\frac{\partial Y}{\partial \eta_0} = Y \left[(\rho^2 + c^2)^{-1} \rho a e \sin E \frac{\partial E}{\partial \eta_0} \right.$$

$$\left. - (1 - \eta_0^2 \sin^2 \psi)^{-1} \eta_0 \sin \psi \left(\eta_0 \cos \psi \frac{\partial \psi}{\partial \eta_0} + \sin \psi \right) \right] + X \frac{\partial \phi}{\partial \eta_0}$$

$$\frac{\partial Z}{\partial a} = \eta_0 (1 - e \cos E) \left(\sin \psi + a \cos \psi \frac{\partial \psi}{\partial a} \right) + a e \eta_0 \sin \psi \sin E \frac{\partial E}{\partial a}$$

$$\frac{\partial Z}{\partial e} = \eta_0 (1 - e \cos E)\, a \cos \psi \frac{\partial \psi}{\partial e} + a\, \eta_0 \sin \psi \left(e \sin E \frac{\partial E}{\partial e} - \cos E \right)$$

$$\frac{\partial Z}{\partial \eta_0} = a(1 - e \cos E) \left(\sin \psi + \eta_0 \cos \psi \frac{\partial \psi}{\partial \eta_0} \right) + a e \eta_0 \sin \psi \sin E \frac{\partial E}{\partial \eta_0}$$

DIFFERENTIAL CORRECTION: TIME-VARYING PARTIAL DERIVATIVES WITH RESPECT TO ANGLE-EPOCH VARIABLES

We now compute partial derivatives of the time-dependent parameters from the orbit generator with respect to the orbital elements β_1, β_2, and β_3. This procedure is analogous to the one followed in the preceding section. Whenever a partial derivative with respect to β_3 is not given, it is assumed to be zero.

$$\frac{\partial M_s}{\partial \beta_1} = 2 \pi \nu_1$$

$$\frac{\partial M_s}{\partial \beta_2} = - 2 \pi \nu_1 c^2 \eta_0^2 \alpha_2^{-1} B_1 B_2^{-1}$$

$$\frac{\partial \psi_s}{\partial \beta_1} = 2 \pi \nu_2$$

$$\frac{\partial \psi_s}{\partial \beta_2} = 2 \pi \nu_2 \alpha_2^{-1} A_2^{-1} (a + b_1 + A_1)$$

$$\frac{\partial \mathcal{E}}{\partial \beta_1} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_s}{\partial \beta_1}$$

$$\frac{\partial \mathcal{E}}{\partial \beta_2} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_s}{\partial \beta_2}$$

$$\frac{\partial v_0}{\partial \beta_1} = (1 - e^2) \sin \mathcal{E} (\sin v')^{-1} (1 - e \cos \mathcal{E})^{-2} \frac{\partial \mathcal{E}}{\partial \beta_1} - \frac{\partial M_s}{\partial \beta_1}$$

$$\frac{\partial v_0}{\partial \beta_2} = (1 - e^2)\sin \mathcal{E}\,(\sin v')^{-1}\,(1 - e\cos \mathcal{E})^{-2}\,\frac{\partial \mathcal{E}}{\partial \beta_2} - \frac{\partial M_s}{\partial \beta_2}$$

$$\frac{\partial \psi_0}{\partial \beta_1} = (-2\alpha_1)^{-1/2}\,(\alpha_2^2 - \alpha_3^2)^{1/2}\,\eta_0^{-1} A_2 B_2^{-1}\,\frac{\partial v_0}{\partial \beta_1}$$

$$\frac{\partial \psi_0}{\partial \beta_2} = (-2\alpha_1)^{-1/2}\,(\alpha_2^2 - \alpha_3^2)^{1/2}\,\eta_0^{-1} A_2 B_2^{-1}\,\frac{\partial v_0}{\partial \beta_2}$$

$$\frac{\partial M_1}{\partial \beta_1} = -(a + b_1)^{-1}\left[(A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1})\,\frac{\partial v_0}{\partial \beta_1}\right.$$

$$\left. - \frac{1}{2} c^2 \eta_0^3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \cos(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1}\right)\right]$$

$$\frac{\partial M_1}{\partial \beta_2} = -(a + b_1)^{-1}\left[(A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1})\,\frac{\partial v_0}{\partial \beta_2}\right.$$

$$\left. - \frac{1}{2} c^2 \eta_0^3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \cos(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2}\right)\right]$$

$$\frac{\partial E_1}{\partial \beta_1} = (1 - e'\cos \mathcal{E})^{-1}\,\frac{\partial M_1}{\partial \beta_1}\,[1 - M_1 e'(1 - e'\cos \mathcal{E})^{-2}\sin \mathcal{E}]$$

$$- M_1 e'(1 - e'\cos \mathcal{E})^{-2}\,\frac{\partial \mathcal{E}}{\partial \beta_1}\,[\sin \mathcal{E}$$

$$+ \frac{1}{2} M_1 (1 - e'\cos \mathcal{E})^{-1}\cos \mathcal{E} - \frac{3}{2} M_1 e' (1 - e'\cos \mathcal{E})^{-2}\sin^2 \mathcal{E}]$$

$$\frac{\partial E_1}{\partial \beta_2} = (1 - e'\cos \mathcal{E})^{-1}\,\frac{\partial M_1}{\partial \beta_2}\,[1 - M_1 e'(1 - e'\cos \mathcal{E})^{-2}\sin \mathcal{E}]$$

$$- M_1 e'(1 - e'\cos \mathcal{E})^{-2}\,\frac{\partial \mathcal{E}}{\partial \beta_2}\left[\sin \mathcal{E} + \frac{1}{2} M_1 (1 - e'\cos \mathcal{E})^{-1}\cos \mathcal{E}\right.$$

$$\left. - \frac{3}{2} M_1 e' (1 - e'\cos \mathcal{E})^{-2}\sin^2 \mathcal{E}\right]$$

$$\frac{\partial v_1}{\partial \beta_1} = (1 - e^2) \sin (\mathcal{E} + E_1)(\sin v'')^{-1} \; [1 - e \cos (\mathcal{E} + E_1)]^{-2} \left(\frac{\partial \mathcal{E}}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1}\right) - \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1}\right)$$

$$\frac{\partial v_1}{\partial \beta_2} = (1 - e^2) \sin(\mathcal{E} + E_1) (\sin v'')^{-1} \; [1 - e \cos(\mathcal{E} + E_1)]^{-2} \left(\frac{\partial \mathcal{E}}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2}\right) - \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2}\right)$$

$$\frac{\partial \psi_1}{\partial \beta_1} = (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} B_2^{-1} \left[A_2 \frac{\partial v_1}{\partial \beta_1} + (A_{21} \cos v' + 2 A_{22} \cos 2v') \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1}\right)\right]$$

$$+ \frac{1}{4} q^2 B_2^{-1} \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1}\right)$$

$$\frac{\partial \psi_1}{\partial \beta_2} = (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} B_2^{-1} \left[A_2 \frac{\partial v_1}{\partial \beta_2} + (A_{21} \cos v' + 2 A_{22} \cos 2v') \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2}\right)\right]$$

$$+ \frac{1}{4} q^2 B_2^{-1} \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2}\right)$$

The following time-dependent partial derivatives with respect to the orbital elements β_1, β_2, and β_3 are computed only if the differential correction is carried through terms of second order:

$$\frac{\partial M_2}{\partial \beta_1} = - (a + b_1)^{-1} \left\{A_1 \frac{\partial v_1}{\partial \beta_1} + (A_{11} \cos v' + 2A_{12} \cos 2v') \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1}\right)\right.$$

$$+ c^2 \eta_0^3 (-2\alpha_1)^{1/2} (\alpha_2^2 - \alpha_3^2)^{-1/2} \left[B_1 \frac{\partial \psi_1}{\partial \beta_1} - \frac{1}{2} \cos (2\psi_s + 2\psi_0) \frac{\partial \psi_1}{\partial \beta_1}\right.$$

$$+ \psi_1 \sin (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1}\right) - \frac{1}{4} q^2 \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1}\right)$$

$$\left.\left. + \frac{1}{16} q^2 \cos (4\psi_s + 4\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1}\right)\right]\right\}$$

$$\frac{\partial M_2}{\partial \beta_2} = -(a+b_1)^{-1}\left\{A_1 \frac{\partial v_1}{\partial \beta_2} + (A_{11}\cos v' + 2A_{12}\cos 2v')\left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2}\right)\right.$$

$$+ c^2 \eta_0^3 (-2\alpha_1)^{1/2}(\alpha_2^2 - \alpha_3^2)^{-1/2}\left[B_1 \frac{\partial \psi_1}{\partial \beta_2} - \frac{1}{2}\cos(2\psi_s + 2\psi_0)\frac{\partial \psi_1}{\partial \beta_2}\right.$$

$$+ \psi_1 \sin(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2}\right) - \frac{1}{4} q^2 \cos(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2}\right)$$

$$\left.\left. + \frac{1}{16} q^2 \cos(4\psi_s + 4\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2}\right)\right]\right\}$$

$$\frac{\partial E_2}{\partial \beta_1} = [1 - e'\cos(\mathcal{E} + E_1)]^{-1}\frac{\partial M_2}{\partial \beta_1}$$

$$- M_2 [1 - e'\cos(\mathcal{E} + E_1)]^{-2} e' \sin(\mathcal{E} + E_1)\left(\frac{\partial \mathcal{E}}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1}\right)$$

$$\frac{\partial E_2}{\partial \beta_2} = [1 - e'\cos(\mathcal{E} + E_1)]^{-1}\frac{\partial M_2}{\partial \beta_2}$$

$$- M_2 [1 - e'\cos(\mathcal{E} + E_1)]^{-2} e' \sin(\mathcal{E} + E_1)\left(\frac{\partial \mathcal{E}}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2}\right)$$

$$\frac{\partial E}{\partial \beta_1} = \frac{\partial \mathcal{E}}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1} + \frac{\partial E_2}{\partial \beta_1}$$

$$\frac{\partial E}{\partial \beta_2} = \frac{\partial \mathcal{E}}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2} + \frac{\partial E_2}{\partial \beta_2}$$

$$\frac{\partial v_2}{\partial \beta_1} = (1 - e^2)\sin E(\sin v)^{-1}(1 - e\cos E)^{-2}\frac{\partial E}{\partial \beta_1} - \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1} + \frac{\partial v_1}{\partial \beta_1}\right)$$

$$\frac{\partial v_2}{\partial \beta_2} = (1 - e^2)\sin E(\sin v)^{-1}(1 - e\cos E)^{-2}\frac{\partial E}{\partial \beta_2} - \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} + \frac{\partial v_1}{\partial \beta_2}\right)$$

$$\frac{\partial v}{\partial \beta_1} = \frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1} + \frac{\partial v_1}{\partial \beta_1} + \frac{\partial v_2}{\partial \beta_1}$$

$$\frac{\partial v}{\partial \beta_2} = \frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} + \frac{\partial v_1}{\partial \beta_2} + \frac{\partial v_2}{\partial \beta_2}$$

$$\frac{\partial \psi_2}{\partial \beta_1} = (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} B_2^{-1} \Bigg[A_2 \frac{\partial v_2}{\partial \beta_1} + \frac{\partial v_1}{\partial \beta_1} (A_{21} \cos v' + 2A_{22} \cos 2v')$$

$$- (A_{21} v_1 \sin v' + 4 A_{22} v_1 \sin 2v' - 3 A_{23} \cos 3v' - 4 A_{24} \cos 4v') \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1} \right) \Bigg]$$

$$+ \frac{1}{4} q^2 B_2^{-1} \Bigg\{ \cos (2\psi_s + 2\psi_0) \frac{\partial \psi_1}{\partial \beta_1} - \Bigg[2\psi_1 \sin (2\psi_s + 2\psi_0)$$

$$- \frac{3}{4} q^2 \cos (2\psi_s + 2\psi_0) + \frac{3}{16} q^2 \cos (4\psi_s + 4\psi_0) \Bigg] \left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1} \right) \Bigg\}$$

$$\frac{\partial \psi_2}{\partial \beta_2} = (-2\alpha_1)^{-1/2} (\alpha_2^2 - \alpha_3^2)^{1/2} \eta_0^{-1} B_2^{-1} \Bigg[A_2 \frac{\partial v_2}{\partial \beta_2} + \frac{\partial v_1}{\partial \beta_2} (A_{21} \cos v' + 2A_{22} \cos 2v')$$

$$- (A_{21} v_1 \sin v' + 4 A_{22} v_1 \sin 2v' - 3 A_{23} \cos 3v' - 4 A_{24} \cos 4v') \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right) \Bigg]$$

$$+ \frac{1}{4} q^2 B_2^{-1} \Bigg\{ \cos (2\psi_s + 2\psi_0) \frac{\partial \psi_1}{\partial \beta_2} - \Bigg[2\psi_1 \sin (2\psi_s + 2\psi_0)$$

$$- \frac{3}{4} q^2 \cos (2\psi_s + 2\psi_0) + \frac{3}{16} q^2 \cos (4\psi_s + 4\psi_0) \Bigg] \left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} \right) \Bigg\}$$

$$\frac{\partial \psi}{\partial \beta_1} = \frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1} + \frac{\partial \psi_1}{\partial \beta_1} + \frac{\partial \psi_2}{\partial \beta_1}$$

$$\frac{\partial \psi}{\partial \beta_2} = \frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} + \frac{\partial \psi_1}{\partial \beta_2} + \frac{\partial \psi_2}{\partial \beta_2}$$

This completes the computation of the partial derivatives of the uniformizing variables E, v, and ψ with respect to the orbital elements β_1, β_2, and β_3 (those with respect to β_3 are all zero) when the calculation is followed through terms of second order. If, however, second-order precision is not necessary, we can eliminate the terms with the subscript "2" (thus omitting all partial derivatives of M_2, E_2, v_2, and ψ_2), and the above partial derivatives of the uniformizing variables reduce to the following:

$$\frac{\partial E}{\partial \beta_1} = \frac{\partial \mathcal{E}}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1}$$

$$\frac{\partial E}{\partial \beta_2} = \frac{\partial \mathcal{E}}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2}$$

$$\frac{\partial v}{\partial \beta_1} = \frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1} + \frac{\partial v_1}{\partial \beta_1}$$

$$\frac{\partial v}{\partial \beta_2} = \frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} + \frac{\partial v_1}{\partial \beta_2}$$

$$\frac{\partial \psi}{\partial \beta_1} = \frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1} + \frac{\partial \psi_1}{\partial \beta_1}$$

$$\frac{\partial \psi}{\partial \beta_2} = \frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} + \frac{\partial \psi_1}{\partial \beta_2}$$

We now continue with the necessary equations preparatory to the partial derivatives of the orthogonal co-ordinates X, Y, and Z.

$$\frac{\partial \chi}{\partial \beta_1} = (1 - \eta_0^2) \sin\psi \, (\sin\chi)^{-1} (1 - \eta_0^2 \sin^2\psi)^{-3/2} \frac{\partial \psi}{\partial \beta_1}$$

$$\frac{\partial \chi}{\partial \beta_2} = (1 - \eta_0^2) \sin\psi \, (\sin\chi)^{-1} (1 - \eta_0^2 \sin^2\psi)^{-3/2} \frac{\partial \psi}{\partial \beta_2}$$

$$\frac{\partial \phi}{\partial \beta_1} = \alpha_3 (\alpha_2^2 - \alpha_3^2)^{-1/2} \eta_0 \left[(1 - \eta_0^2)^{-1/2} (1 - \eta_2^{-2})^{-1/2} \frac{\partial \chi}{\partial \beta_1} \right.$$

$$\left. + B_3 \frac{\partial \psi}{\partial \beta_1} + \frac{3}{16} \eta_0^2 \eta_2^{-4} \cos 2\psi \frac{\partial \psi}{\partial \beta_1} \right]$$

$$- c^2 \alpha_3 (-2\alpha_1)^{-1/2} (A_3 + A_{31} \cos v + 2 A_{32} \cos 2v$$

$$+ 3 A_{33} \cos 3v + 4 A_{34} \cos 4v) \frac{\partial v}{\partial \beta_1}$$

$$\frac{\partial \phi}{\partial \beta_2} = \alpha_3 (\alpha_2^2 - \alpha_3^2)^{-1/2} \eta_0 \left[(1 - \eta_0^2)^{-1/2} (1 - \eta_2^{-2})^{-1/2} \frac{\partial \chi}{\partial \beta_2} \right.$$

$$\left. + B_3 \frac{\partial \psi}{\partial \beta_2} + \frac{3}{16} \eta_0^2 \eta_2^{-4} \cos 2\psi \frac{\partial \psi}{\partial \beta_2} \right]$$

$$- c^2 \alpha_3 (-2\alpha_1)^{-1/2} (A_3 + A_{31} \cos v + 2A_{32} \cos 2v$$

$$+ 3 A_{33} \cos 3v + 4 A_{34} \cos 4v) \frac{\partial v}{\partial \beta_2}$$

$$\frac{\partial \phi}{\partial \beta_3} = 1$$

$$\frac{\partial X}{\partial \beta_1} = X \left[(\rho^2 + c^2)^{-1} \rho \, a \, e \sin E \frac{\partial E}{\partial \beta_1} \right.$$

$$\left. - (1 - \eta_0^2 \sin^2 \psi)^{-1} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_1} \right] - Y \frac{\partial \phi}{\partial \beta_1}$$

$$\frac{\partial X}{\partial \beta_2} = X \left[(\rho^2 + c^2)^{-1} \rho \, a \, e \sin E \frac{\partial E}{\partial \beta_2} \right.$$

$$\left. - (1 - \eta_0^2 \sin^2 \psi)^{-1} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_2} \right] - Y \frac{\partial \phi}{\partial \beta_2}$$

$$\frac{\partial X}{\partial \beta_3} = - Y$$

$$\frac{\partial Y}{\partial \beta_1} = Y \left[(\rho^2 + c^2)^{-1} \rho \, a \, e \sin E \frac{\partial E}{\partial \beta_1} \right.$$

$$\left. - (1 - \eta_0^2 \sin^2 \psi)^{-1} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_1} \right] + X \frac{\partial \phi}{\partial \beta_1}$$

$$\frac{\partial Y}{\partial \beta_2} = Y \left[(\rho^2 + c^2)^{-1} \rho \, a \, e \sin E \frac{\partial E}{\partial \beta_2} \right.$$

$$\left. - (1 - \eta_0^2 \sin^2 \psi)^{-1} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_2} \right] + X \frac{\partial \phi}{\partial \beta_2}$$

$$\frac{\partial Y}{\partial \beta_3} = X$$

$$\frac{\partial Z}{\partial \beta_1} = a \, \eta_0 \left[(1 - e \cos E) \cos \psi \frac{\partial \psi}{\partial \beta_1} + e \sin \psi \sin E \frac{\partial E}{\partial \beta_1} \right]$$

$$\frac{\partial Z}{\partial \beta_2} = a\,\eta_0 \left[(1 - e \cos E) \cos \psi \frac{\partial \psi}{\partial \beta_2} + e \sin \psi \sin E \frac{\partial E}{\partial \beta_2}\right]$$

$$\frac{\partial Z}{\partial \beta_3} = 0$$

THE EQUATIONS OF CONDITION

Now that we have found $\partial X/\partial q_i$, $\partial Y/\partial q_i$, and $\partial Z/\partial q_i$ for the Izsak orbital elements q_i (i = 1, 2, . . . , 6), we can complete the differential correction process by determining the equations of condition. First we expand and substitute into the matrix relation given in the section titled "Analytical Procedure of Differential Correction". The matrix relation, when expanded explicitly, yields the following eighteen equations:

$$\frac{\partial X_M}{\partial a} = \cos \psi_x \frac{\partial X}{\partial a} + \sin \psi_x \frac{\partial Y}{\partial a}$$

$$\frac{\partial Y_M}{\partial a} = -\sin \psi_x \sin \theta_D \frac{\partial X}{\partial a} + \cos \psi_x \sin \theta_D \frac{\partial Y}{\partial a} + \cos \theta_D \frac{\partial Z}{\partial a}$$

$$\frac{\partial Z_M}{\partial a} = \sin \psi_x \cos \theta_D \frac{\partial X}{\partial a} - \cos \psi_x \cos \theta_D \frac{\partial Y}{\partial a} + \sin \theta_D \frac{\partial Z}{\partial a}$$

$$\frac{\partial X_M}{\partial e} = \cos \psi_x \frac{\partial X}{\partial e} + \sin \psi_x \frac{\partial Y}{\partial e}$$

$$\frac{\partial Y_M}{\partial e} = -\sin \psi_x \sin \theta_D \frac{\partial X}{\partial e} + \cos \psi_x \sin \theta_D \frac{\partial Y}{\partial e} + \cos \theta_D \frac{\partial Z}{\partial e}$$

$$\frac{\partial Z_M}{\partial e} = \sin \psi_x \cos \theta_D \frac{\partial X}{\partial e} - \cos \psi_x \cos \theta_D \frac{\partial Y}{\partial e} + \sin \theta_D \frac{\partial Z}{\partial e}$$

$$\frac{\partial X_M}{\partial \eta_0} = \cos \psi_x \frac{\partial X}{\partial \eta_0} + \sin \psi_x \frac{\partial Y}{\partial \eta_0}$$

$$\frac{\partial Y_M}{\partial \eta_0} = -\sin \psi_x \sin \theta_D \frac{\partial X}{\partial \eta_0} + \cos \psi_x \sin \theta_D \frac{\partial Y}{\partial \eta_0} + \cos \theta_D \frac{\partial Z}{\partial \eta_0}$$

$$\frac{\partial Z_M}{\partial \eta_0} = \sin \psi_x \cos \theta_D \frac{\partial X}{\partial \eta_0} - \cos \psi_x \cos \theta_D \frac{\partial Y}{\partial \eta_0} + \sin \theta_D \frac{\partial Z}{\partial \eta_0}$$

$$\frac{\partial X_M}{\partial \beta_1} = \cos \psi_x \frac{\partial X}{\partial \beta_1} + \sin \psi_x \frac{\partial Y}{\partial \beta_1}$$

$$\frac{\partial Y_M}{\partial \beta_1} = - \sin \psi_x \sin \theta_D \frac{\partial X}{\partial \beta_1} + \cos \psi_x \sin \theta_D \frac{\partial Y}{\partial \beta_1} + \cos \theta_D \frac{\partial Z}{\partial \beta_1}$$

$$\frac{\partial Z_M}{\partial \beta_1} = \sin \psi_x \cos \theta_D \frac{\partial X}{\partial \beta_1} - \cos \psi_x \cos \theta_D \frac{\partial Y}{\partial \beta_1} + \sin \theta_D \frac{\partial Z}{\partial \beta_1}$$

$$\frac{\partial X_M}{\partial \beta_2} = \cos \psi_x \frac{\partial X}{\partial \beta_2} + \sin \psi_x \frac{\partial Y}{\partial \beta_2}$$

$$\frac{\partial Y_M}{\partial \beta_2} = - \sin \psi_x \sin \theta_D \frac{\partial X}{\partial \beta_2} + \cos \psi_x \sin \theta_D \frac{\partial Y}{\partial \beta_2} + \cos \theta_D \frac{\partial Z}{\partial \beta_2}$$

$$\frac{\partial Z_M}{\partial \beta_2} = \sin \psi_x \cos \theta_D \frac{\partial X}{\partial \beta_2} - \cos \psi_x \cos \theta_D \frac{\partial Y}{\partial \beta_2} + \sin \theta_D \frac{\partial Z}{\partial \beta_2}$$

$$\frac{\partial X_M}{\partial \beta_3} = \cos \psi_x \frac{\partial X}{\partial \beta_3} + \sin \psi_x \frac{\partial Y}{\partial \beta_3}$$

$$\frac{\partial Y_M}{\partial \beta_3} = - \sin \psi_x \sin \theta_D \frac{\partial X}{\partial \beta_3} + \cos \psi_x \sin \theta_D \frac{\partial Y}{\partial \beta_3}$$

$$\frac{\partial Z_M}{\partial \beta_3} = \sin \psi_x \cos \theta_D \frac{\partial X}{\partial \beta_3} - \cos \psi_x \cos \theta_D \frac{\partial Y}{\partial \beta_3}$$

The last two equations have only two terms on the right-hand side because of the fact that $\partial Z/\partial \beta_3 = 0$. We can now write out explicitly the twelve coefficients to be inserted into the equations of condition:

$$\frac{\partial L_c}{\partial a} = \frac{\partial L_c}{\partial X_M} \frac{\partial X_M}{\partial a} + \frac{\partial L_c}{\partial Y_M} \frac{\partial Y_M}{\partial a} + \frac{\partial L_c}{\partial Z_M} \frac{\partial Z_M}{\partial a}$$

$$\frac{\partial L_c}{\partial e} = \frac{\partial L_c}{\partial X_M} \frac{\partial X_M}{\partial e} + \frac{\partial L_c}{\partial Y_M} \frac{\partial Y_M}{\partial e} + \frac{\partial L_c}{\partial Z_M} \frac{\partial Z_M}{\partial e}$$

$$\frac{\partial L_c}{\partial \eta_0} = \frac{\partial L_c}{\partial X_M} \frac{\partial X_M}{\partial \eta_0} + \frac{\partial L_c}{\partial Y_M} \frac{\partial Y_M}{\partial \eta_0} + \frac{\partial L_c}{\partial Z_M} \frac{\partial Z_M}{\partial \eta_0}$$

$$\frac{\partial L_c}{\partial \beta_1} = \frac{\partial L_c}{\partial X_M} \frac{\partial X_M}{\partial \beta_1} + \frac{\partial L_c}{\partial Y_M} \frac{\partial Y_M}{\partial \beta_1} + \frac{\partial L_c}{\partial Z_M} \frac{\partial Z_M}{\partial \beta_1}$$

$$\frac{\partial L_c}{\partial \beta_2} = \frac{\partial L_c}{\partial X_M} \frac{\partial X_M}{\partial \beta_2} + \frac{\partial L_c}{\partial Y_M} \frac{\partial Y_M}{\partial \beta_2} + \frac{\partial L_c}{\partial Z_M} \frac{\partial Z_M}{\partial \beta_2}$$

$$\frac{\partial L_c}{\partial \beta_3} = \frac{\partial L_c}{\partial X_M}\frac{\partial X_M}{\partial \beta_3} + \frac{\partial L_c}{\partial Y_M}\frac{\partial Y_M}{\partial \beta_3} + \frac{\partial L_c}{\partial Z_M}\frac{\partial Z_M}{\partial \beta_3}$$

$$\frac{\partial M_c}{\partial a} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial a} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial a} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial a}$$

$$\frac{\partial M_c}{\partial e} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial e} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial e} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial e}$$

$$\frac{\partial M_c}{\partial \eta_0} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial \eta_0} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial \eta_0} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial \eta_0}$$

$$\frac{\partial M_c}{\partial \beta_1} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial \beta_1} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial \beta_1} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial \beta_1}$$

$$\frac{\partial M_c}{\partial \beta_2} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial \beta_2} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial \beta_2} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial \beta_2}$$

$$\frac{\partial M_c}{\partial \beta_3} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial \beta_3} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial \beta_3} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial \beta_3}$$

Finally, the two equations of condition corresponding to each observation are given explicitly by the following:

$$\Delta L = L_0 - L_c = \frac{\partial L_c}{\partial a}\Delta a + \frac{\partial L_c}{\partial e}\Delta e + \frac{\partial L_c}{\partial \eta_0}\Delta \eta_0 + \frac{\partial L_c}{\partial \beta_1}\Delta \beta_1 + \frac{\partial L_c}{\partial \beta_2}\Delta \beta_2 + \frac{\partial L_c}{\partial \beta_3}\Delta \beta_3$$

$$\Delta M = M_0 - M_c = \frac{\partial M_c}{\partial a}\Delta a + \frac{\partial M_c}{\partial e}\Delta e + \frac{\partial M_c}{\partial \eta_0}\Delta \eta_0 + \frac{\partial M_c}{\partial \beta_1}\Delta \beta_1 + \frac{\partial M_c}{\partial \beta_2}\Delta \beta_2 + \frac{\partial M_c}{\partial \beta_3}\Delta \beta_3$$

FITTING BY METHOD OF LEAST SQUARES

We have accumulated a set of 2n linear simultaneous equations in six "unknowns," as follows:

$$\left.\begin{aligned} (\Delta L)_i &= (L_0)_i - (L_c)_i = \sum_{j=1}^{6} \left(\frac{\partial L_c}{\partial q_j}\right)_i \Delta q_j \\ (\Delta M)_i &= (M_0)_i - (M_c)_i = \sum_{j=1}^{6} \left(\frac{\partial M_c}{\partial q_j}\right)_i \Delta q_j \end{aligned}\right\} \quad i = 1, 2, \cdots, n$$

where q_j $(j = 1, 2, \ldots, 6) = a, e, \eta_0, \beta_1, \beta_2, \beta_3$. We regard the Δq_j as "unknowns," and the number n of observations in the set is fixed in advance (see above under the section titled, "The Standard Deviation of Fit"). The above equations, written in matrix form, become:

$$\left[\frac{\partial L_c}{\partial q_{ij}}\right] \cdot [\Delta q_j] = [\Delta L_i]$$

$$\left[\frac{\partial M_c}{\partial q_{ij}}\right] \cdot [\Delta q_j] = [\Delta M_i]$$

where the matrices of partial derivatives have n rows and six columns, the matrices of unknowns have six rows and one column, and the matrices of observational residuals have n rows and one column. Recall that, in the general case, the observed direction cosines $(L_0)_i$ and $(M_0)_i$ have associated weighting factors $(w_L)_i$ and $(w_M)_i$ respectively (see above under the section titled, "Computation of Direction Cosines").

For purposes of this section, it is unnecessary to distinguish between direction cosines L and M or between weighting factors w_L and w_M. Further, it is not significant, for the present purpose, that the constant coefficients in the linear simultaneous equations have the form of partial derivatives. In order to simplify the notation in what follows, we combine the two matrix equations, each coefficient matrix having n rows, into a single matrix equation where the coefficient matrix has $m = 2n$ rows. Then the matrix of constant terms (i.e., observational residuals) also has m rows. We rewrite the above two equations in the simple general form:

$$AX = B$$

where $A = [a_{ij}]$ has m rows and six columns and represents the coefficient matrix of partial derivatives, $X = [x_j]$ has six rows and one column and represents the matrix of unknowns, and $B = [b_i]$ has m rows and one column and represents the matrix of observational residuals.

The number m of equations we obtain by expanding the matrix relation is generally much greater than the number (six) of unknowns, and since the observations contain inherent random and possibly systematic errors, no exact solution of the simultaneous set exists. According to the principle of least squares, the values of the unknowns x_j which are preferred are those which cause the sum of the squares of the residuals after the fit to be a minimum. The so-called "residuals after the fit" are calculated by substituting the approximate solution for the x_j in the matrix X, and subtracting the matrix AX from B. When the equations of condition have different weights, the least-squares solution is that which minimizes the sum of the weighted squares of the residuals after the fit, where each square is multiplied by its corresponding weight.

The least-squares criterion is satisfied by reducing the m equations of condition to six equations known as normal equations. This procedure is performed as follows, in which we adopt the usual notation for matrix elements: the first subscript denoting the row number and the second subscript the column number. The first normal equation is obtained by multiplying the first conditional equation by $w_1\, a_{11}$, the second by $w_2\, a_{21}$, the third by $w_3\, a_{31}$, etc. and summing the resulting m equations. The second normal equation is obtained by multiplying the first conditional equation by $w_1\, a_{12}$, the second by $w_2\, a_{22}$, the third by $w_3\, a_{32}$, etc. and summing the resulting m equations. If we repeat this process six times, we obtain the six normal equations. It is seen that this process is equivalent to pre-multiplying the matrix equation $AX = B$ by the weighted transpose of the matrix A, where the rows of the transpose are multiplied by the corresponding weighting factors. The set of normal equations can be represented by the new matrix relation $CX = D$, where

$$c_{ij} = \sum_{k=1}^{m} a_{ki}\, a_{kj}\, w_k \quad (i, j = 1, 2, \cdots, 6)$$

and

$$d_i = \sum_{k=1}^{m} a_{ki}\, b_k\, w_k \quad (i = 1, 2, \cdots, 6)$$

Of course, if the weighting factors are not present, i.e., $w_k = 1$ for all k, the elements c_{ij} are precisely those of A^TA and the elements d_i are precisely those of A^TB. Here the superscript "T" indicates the transposed matrix.

We now have a system of six equations in six unknowns, since C is a square (and symmetric) matrix. In order to solve this system, we use a method known variously as the Gaussian elimination method or the method of pivotal condensation. This has the effect of reducing the square matrix to an upper triangular matrix (i.e., all elements below the principal diagonal are zero) which represents the same solution for the x_j.

To begin this process, we choose the element of the first column of matrix C greatest in absolute value, say c_{k1}. We then divide all the elements of the k^{th} row (the "pivotal row") by the so-called dominant element (or "pivot"), c_{k1}. This done, we exchange the corresponding elements of the pivotal row with those of the first row. The leading element c_{11} of the matrix is now unity. We now replace all the elements in each row beginning with the second row by the following procedure: multiply all elements in the first (pivotal) row by the element in the first column of each row successively and subtract this product from the corresponding element of the successive rows. Mathematically, this is indicated by:

$$c_{ij} = c_{ij} - c_{i1}\, c_{1j} \quad (i = 2, 3, \cdots, 6\, ;\; j = 1, 2, \cdots, 6)$$

Since $c_{11} = 1$, it is obvious from this equation that $c_{i1} = 0$ for all $i = 2, 3, \ldots, 6$. That is, all elements in the first column except for the first (diagonal) element are replaced by zeros. Essentially, we have added suitable multiples of the pivotal row to all the other rows so that in each resulting row the element in the first column vanishes.

Consider the matrix with five rows and five columns obtained by deleting the first (pivotal) row and the first column. Now select as a new pivotal element the largest element in absolute value in the new first column of the five-by-five matrix, and repeat the entire process with respect to the square matrix of order five.

Continuing in this manner, we have finally a single non-zero (diagonal) element in the last row. The procedure is completed by dividing this final row by the diagonal element. The result is an upper triangular matrix with ones along the principal diagonal. Note that all operations described above to be performed on the original square matrix C are elementary row operations (i.e., an operation belonging to one of the three following types: the interchange of any two rows; the multiplication of a row by any non-zero constant; the addition of any multiple of one row to any other row). Thus, the triangularization process does not change the solution to the simultaneous set of linear equations as long as the operations performed on matrix C are performed in an analogous manner on the elements of the column matrix D. This can most readily be done by augmenting the six-by-six matrix C by a seventh column composed of the elements of D. In practice, the six-by-six matrix C is further augmented by a six-by-six identity matrix placed in columns eight through thirteen. The purpose of this is to determine ultimately the inverse of the coefficient matrix C, from which we may easily find the standard errors of the least-squares solution for the Δq_j. Note that the various columns of C^{-1} can be found in succession by solving the matrix equation $CX = I_i$ for the column matrix X, where I_i represents the i^{th} column ($i = 1, 2, \ldots, 6$) of the identity matrix of order six. We can thus view the six columns of the identity matrix placed in the augmented six-by-thirteen matrix as constant right-hand side column matrices replacing B in successive least-squares solutions of the matrix equation. These successive solutions are determined simultaneously in the Gaussian elimination method simply by forming the augmented matrix E and performing the elementary row operations on all thirteen columns. The augmented matrix E appears as follows, after the normal equations are determined, but before the elementary row operations are begun:

$$E = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & d_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & d_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & d_3 & 0 & 0 & 1 & 0 & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & d_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & d_5 & 0 & 0 & 0 & 0 & 1 & 0 \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & d_6 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

After the triangularization process, the augmented matrix E is transformed to a matrix (call it F) of the following form:

$$F = \begin{bmatrix} 1 & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} \\ 0 & 1 & f_{23} & f_{24} & f_{25} & f_{26} & g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} \\ 0 & 0 & 1 & f_{34} & f_{35} & f_{36} & g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} \\ 0 & 0 & 0 & 1 & f_{45} & f_{46} & g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} \\ 0 & 0 & 0 & 0 & 1 & f_{56} & g_{51} & g_{52} & g_{53} & g_{54} & g_{55} & g_{56} & g_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & g_{66} & g_{67} \end{bmatrix}$$

The first six columns of F represent the triangularized coefficient matrix, and the remaining seven columns represent successive constant right-hand side matrices, each of which is associated with a particular column solution matrix. At this point, it is only natural that we augment the column solution matrix X (corresponding to the seventh column of F only) to a six-by-seven solution matrix Y, which contains X as its first column. The remaining columns of Y will contain the inverse of the coefficient matrix C of the normal equations.

We can now write explicitly the set of linear simultaneous equations in the triangularized form.

$$\begin{aligned} y_{1i} + f_{12}\, y_{2i} + f_{13}\, y_{3i} + f_{14}\, y_{4i} + f_{15}\, y_{5i} + f_{16}\, y_{6i} &= g_{1i} \\ y_{2i} + f_{23}\, y_{3i} + f_{24}\, y_{4i} + f_{25}\, y_{5i} + f_{26}\, y_{6i} &= g_{2i} \\ y_{3i} + f_{34}\, y_{4i} + f_{35}\, y_{5i} + f_{36}\, y_{6i} &= g_{3i} \\ y_{4i} + f_{45}\, y_{5i} + f_{46}\, y_{6i} &= g_{4i} \\ y_{5i} + f_{56}\, y_{6i} &= g_{5i} \\ y_{6i} &= g_{6i} \end{aligned}$$

In the above, the subscript "i" assumes values from one to seven, corresponding to various solutions for the seven right-hand side sets of constants.

The construction of this triangular system of equations is known as the forward solution, and the process of obtaining its solution is called back-substitution. The last equation in the triangular system gives the value for y_{6i} directly. If we insert its value in the previous equation, we can obtain y_{5i}, and so on for the remainder of the unknowns. Mathematically, the relation is:

$$y_{6i} = g_{6i}$$

and

$$y_{ji} = g_{ji} - \sum_{k=j+1}^{6} f_{jk}\, y_{ki}$$

where $j = 5, 4, 3, 2, 1$ (in that order) and $i = 1, 2, \ldots, 7$ (in any order).

We have now completed the determination of the $\Delta q_j = y_{j1}$ by the least-squares principle. Theoretically, this procedure may always be followed to a successful conclusion provided that the m original equations of condition are independent; that is, provided that the determinant of the coefficient matrix C does not vanish.

The formal solution by the method of least squares is now concluded, but ordinarily a measure of the "goodness" of the least-squares fit is desirable. The residuals after the fit are assembled in the so-called residual matrix U, equal to $B - AX$. In terms of elements:

$$u_i = b_i - \sum_{j=1}^{6} a_{ij}\, x_j \quad (i = 1, 2, \cdots, m)$$

From this, it is obvious that the sum of the squares (unweighted) of the residuals after the fit is given by:

$$\sum_{i=1}^{m} u_i^2$$

We can now easily find the so-called variance-covariance matrix of the fit from the inverse C^{-1} of the coefficient matrix in the normal equations. Recall that C^{-1} occupies columns two through seven of matrix Y. The variance-covariance matrix is obtained simply by multiplying each element in C^{-1} by the sum of the squares of the residuals after the fit and dividing this product by m-6 (the excess of simultaneous equations of condition over the number of independent unknowns). If we represent the variance-covariance matrix by V, then we have that:

$$v_{ij} = y_{i,j+1}\, \mu_f^2 \quad (i, j = 1, 2, \cdots, 6)$$

where

$$\mu_f = \left(\frac{1}{m-6} \sum_{i=1}^{m} u_i^2 \right)^{1/2}$$

By comparison with computations performed above in the section titled "The Standard Deviation of Fit," we can see that the quantity μ_f is a standard deviation of fit. More precisely, μ_f is the standard deviation of the residuals after the fit, or the standard deviation of the least squares fit. It is not to be confused with

$$\sigma_f = \left(\frac{1}{m-6} \sum_{i=1}^{m} b_i^2 \right)^{1/2} = \sqrt{\frac{1}{2n-6} \sum_{i=1}^{n} \left[(\Delta L_i)^2 + (\Delta M_i)^2 \right]}$$

in the earlier notation, which is the standard deviation of the observational residuals, or the standard deviation of the observational fit.

Finally, we can find the so-called standard errors μ_j of the six unknowns $\Delta q_j = y_{j1}$. These are simply equal to the square-roots of the diagonal elements in the variance-covariance matrix, or

$$\mu_j = \sqrt{v_{jj}} = \mu_f \sqrt{y_{j,j+1}} \quad (j = 1, 2, \cdots, 6)$$

where $y_{j,j+1}$ is the term on the principal diagonal of the inverse of the matrix C, corresponding to the unknown $x_j = y_{j1}$.

ITERATIVE LEAST-SQUARES PROCEDURE

The procedure for producing a mean set of orbital elements is essentially an iterated least-squares fitting of the first-order Taylor differential expansion of the conditional equations to numerous observational values. Using the values for the Δq_j determined by the method of least squares, as described in the preceding section, we can calculate the corrected Izsak orbital elements.

$$a' = a + \Delta q_1 = a + \Delta a$$

$$e' = e + \Delta q_2 = e + \Delta e$$

$$\eta_0' = \eta_0 + \Delta q_3 = \eta_0 + \Delta \eta_0$$

$$\beta_1' = \beta_1 + \Delta q_4 = \beta_1 + \Delta \beta_1$$

$$\beta_2' = \beta_2 + \Delta q_5 = \beta_2 + \Delta \beta_2$$

$$\beta_3' = \beta_3 + \Delta q_6 = \beta_3 + \Delta \beta_3$$

At this point, it is useful to check that the improved or corrected elements are physically meaningful. For instance, it should be ascertained that the semi-major axis $a' > 1$ earth radius, that the eccentricity $e' \geq 0$, that the sine of the inclination η_0' is not greater than unity in absolute value, and so on.

It is now necessary to update the other parameters used in the differential correction process, based upon the improved orbital elements. Accordingly, the various parameters included under the heading, "Prime Constants II" are re-evaluated using the improved set of elements. This done, the various parameters included under "Mutual Constants" are similarly re-evaluated. Now, assuming that the times of the various observations in the data deck are available as needed, the Orbit Generator may be used to produce the required calculated values of the position and velocity components. From these components, we calculate the local co-ordinates of the satellite and then the computed values of the direction cosines (refer to section titled, "Computation of Direction Cosines"). Finally, the observational residuals are calculated. Thus, with each observation time in the data deck are associated corresponding principal time-dependent quantities as follows:

(a) position and velocity components $X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$;

(b) local co-ordinates of the satellite X_M, Y_M, Z_M;

(c) computed values of the direction cosines L_c, M_c; and

(d) observational residuals $\Delta L, \Delta M$.

Next, the statistical analysis is repeated (refer to the section titled, "The Standard Deviation of Fit") wherein the following quantities are determined: the average observational residual, the standard deviation of the observational residuals from their mean value, the standard deviation of (the observational) fit, the upper and lower range limits for the observational residuals, and the standard deviation of fit of the accepted observational residuals. Once these quantities are found, the differential correction may be repeated. Of course, the time-independent partial derivatives are computed once only, while the time-varying partial derivatives, both with respect to the energy-momenta variables and to the angle-epoch variables, are computed for each observation time in the data deck. A new set of equations of condition can then be assembled and the fitting by least squares repeated.

In summary, the following sequence of steps represents the iterative least-squares procedure in producing a mean set of orbital elements for a given time span represented by a set of observation points:

1. Correct the six Izsak orbital elements utilizing the values determined by the method of fitting the equations of condition by least squares.

2. Update the parameters included in Prime Constants II.

3. Update the parameters included in Mutual Constants.

4. Produce sets of position and velocity components for each observation time using the Orbit Generator.

5. Calculate the local co-ordinates of the satellite at each observation time.

6. Compute the direction cosines of the satellite at each observation time.

7. Determine the observational residuals for each time point.

8. Perform a statistical analysis of the observational residuals to find various standard deviations and a statistically valid range within which observational residuals must fall for inclusion in the fitting process.

9. Begin the differential correction process by evaluating the time-independent partial derivatives. Then evaluate the time-varying partial derivatives for each observation point.

10. Assemble the set of equations of condition.

11. Fit the equations of condition by the method of least squares. First determine the six normal equations, then triangularize the system by the Gaussian elimination method, and finally use the back-substitution method to find the solution.

12. Measure the "goodness" of the least-squares fitting by finding the residuals after the fit, the variance-covariance matrix, and the standard errors of the unknowns.

Return to step number one.

DEFINITIVE ORBITAL PARAMETERS

The iterative least-squares procedure is generally terminated in one of two ways. Either the total number of iterations through the least squares routine is prescribed in advance, or the standard deviation of the observational fit is used as the criterion in halting the iterative method. If this standard deviation falls below a value prescribed in advance during a given iteration, then the precision of the differential correction is deemed sufficient at that point. Of course, both methods of terminating computation can be used concurrently; i.e., if the standard deviation does not meet the prescribed criterion by the p-th iteration, then the differential correction process is halted.

At the conclusion of the differential correction, the following definitive orbital parameters may be found:

The semi-major axis is found by multiplying a by the proper length conversion constant (3963.339 mi./Earth radius or 6378.388 km./Earth radius).

The eccentricity of the orbit is given by e.

The inclination of the orbital plane to the equator is given by arcsin η_0 $(0° \leq \eta_0 < 180°)$.

The time of passage through the perigee point is found by multiplying $-\beta_1$ by the proper time conversion constant (13.4472 min./Vanguard unit of time or 806.832 sec./Vanguard unit of time). The time of perigee passage is given with respect to the reference (or epoch) time t_0, which is that used as a basis for the observational times and that corresponding to the initial position and velocity components $X_i, Y_i, Z_i, \dot{X}_i, \dot{Y}_i, \dot{Z}_i$.

The argument of perigee (measured in the orbit plane from the node to the perigee point) is found by multiplying β_2 by the angular conversion constant 57.295780 deg./rad.

The right ascension (measured in the equatorial plane from the vernal equinox) of the ascending node is found by multiplying β_3 by the angular conversion constant. (Note that these last two parameters are angles usually given as greater than or equal to 0° and less than 360°, so that some multiple of 360° may have to be added or subtracted to bring the values into this principal range).

The height of the perigee point above the Earth's surface is found by multiplying $a(1-e) - 1$ by one of the length conversion constants given above.

The height of the apogee point above the Earth's surface is found by multiplying $a(1+e) - 1$ by one of the length conversion constants.

The anomalistic mean motion is found by multiplying $a^{-3/2}$ by the angular conversion constant and dividing by one of the time conversion constants (this gives the mean motion in deg./min. or deg./sec.).

The anomalistic period is found by multiplying $2\pi a^{3/2}$ by one of the time conversion constants.

The mean anomaly (at the time of perigee passage) is found by multiplying $-\beta_1 a^{-3/2}$ by the angular conversion constant. This expression assumes that the reference (epoch) time t_0 is zero; in general, the mean anomaly is found by multiplying $-a^{-3/2}(\beta_1 + t_0)$ by the angular conversion constant.

RESULTS OF PRELIMINARY APPLICATIONS

Both the orbit generator portion and the differential correction process by least-squares fitting have been tested independently by application to actual satellite orbits. Primarily, use has been made of two relatively close-in yet drag-free satellite orbits, so that neither atmospheric drag nor luni-solar perturbing forces would exert major disturbing influences. The ANNA 1B satellite (international designation 1962 BM 1; N.A.S.A. identification number 56017) was launched in October, 1962 under the project direction of the U. S. Navy from the Atlantic Missile Range into a near-circular orbit of medium inclination. Its purpose was predominantly that of geodetic investigation. The Relay 2 satellite (international designation 1964 3A; N.A.S.A. identification number 64031) was launched in January, 1964 under the project direction of the National Aeronautics and Space Administration from the renamed Eastern Test Range into a relatively high-eccentricity orbit. Its function was that of active-repeater communications satellite. Initial orbital parameters for both these satellites are given in Table 1. The observational data for the Relay 2 satellite consist of direction-cosine pairs reported from fifteen tracking stations in the Minitrack network operated by the N.A.S.A., while the data for ANNA 1B consist of right ascensions and declinations reported from twelve stations in the optical camera network operated by the Smithsonian Astrophysical Observatory. It might be noted that no weighting factors were associated with any of the sets of observational data for either the Relay 2 or the ANNA 1B satellite in the applications described in this section.

Table 1
Initial Orbital Parameters for Satellites Used in Preliminary Applications

	ANNA 1B	Relay 2
Perigee (statute miles)	670	1298
Apogee (statute miles)	728	4606
Period (minutes)	107.8	194.7
Inclination to Earth's equator (degrees)	50.1	46.0
Semi-major axis (units of Earth's equatorial radius)	1.1764	1.7448
Eccentricity	0.00622	0.23918
Sine of the inclination	0.7672	0.7193

In order to gauge the intrinsic accuracy of the orbit generator, a double-precision ninth-order Cowell numerical integration program was utilized. Two numerically integrated comparison ephemerides were produced: one using recently determined geodetic values for the zonal harmonic coefficients in the expansion of the geopotential and the other using the corresponding values for these coefficients based upon the Vinti potential. Refer to Table 2. The numerically integrated ephemeris produced by the geodetic values of the zonal harmonic coefficients was used as a basis for comparison with both the numerically integrated ephemeris produced by Vinti values of the zonal harmonic coefficients and the ephemeris produced by the orbit generator based upon the Vinti potential function. Figure 1(a) illustrates the residuals of the X-co-ordinate between (1) the Vinti ephemeris and the numerically integrated ephemeris produced by geodetic values, and (2) the numerically integrated ephemeris using the Vinti values and the numerically integrated ephemeris produced by geodetic values. Figures 1(b) and 1(c) do likewise for the residuals of the Y-co-ordinate and Z-co-ordinate, respectively.

The comparisons illustrated in Figure 1 are based on the implicit assumption that the initial position and velocity conditions do not contain any inaccuracies. In actual practice, such inaccuracies are always present, and they must be removed by utilizing observational data in the differential correction. Figure 2 illustrates the determination of a mean set of Izsak orbital elements by an iterated least-squares fitting of the differential solution to observational data for the ANNA 1B satellite. In all cases, the total number of iterations through the least squares fitting routine is prescribed in advance to be ten. This number is sufficient to attain convergence within a very small tolerance. In the graph of each of the six orbital elements, three

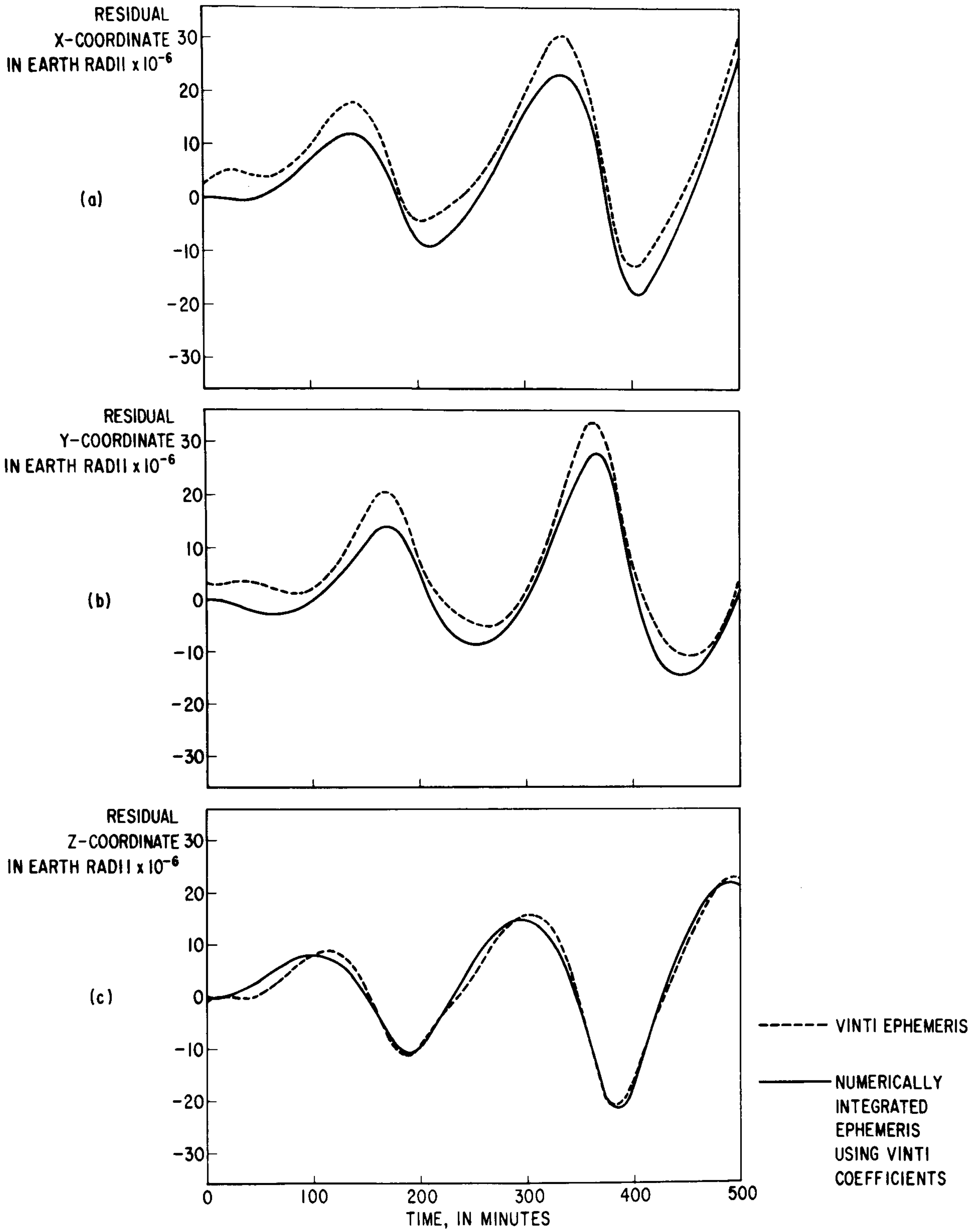

Figure 1—Co-ordinate residuals for Vinti potential ephemeris and for numerically integrated ephemeris using zonal harmonic coefficients of the Vinti potential each compared with numerically integrated ephemeris produced by geodetic values.

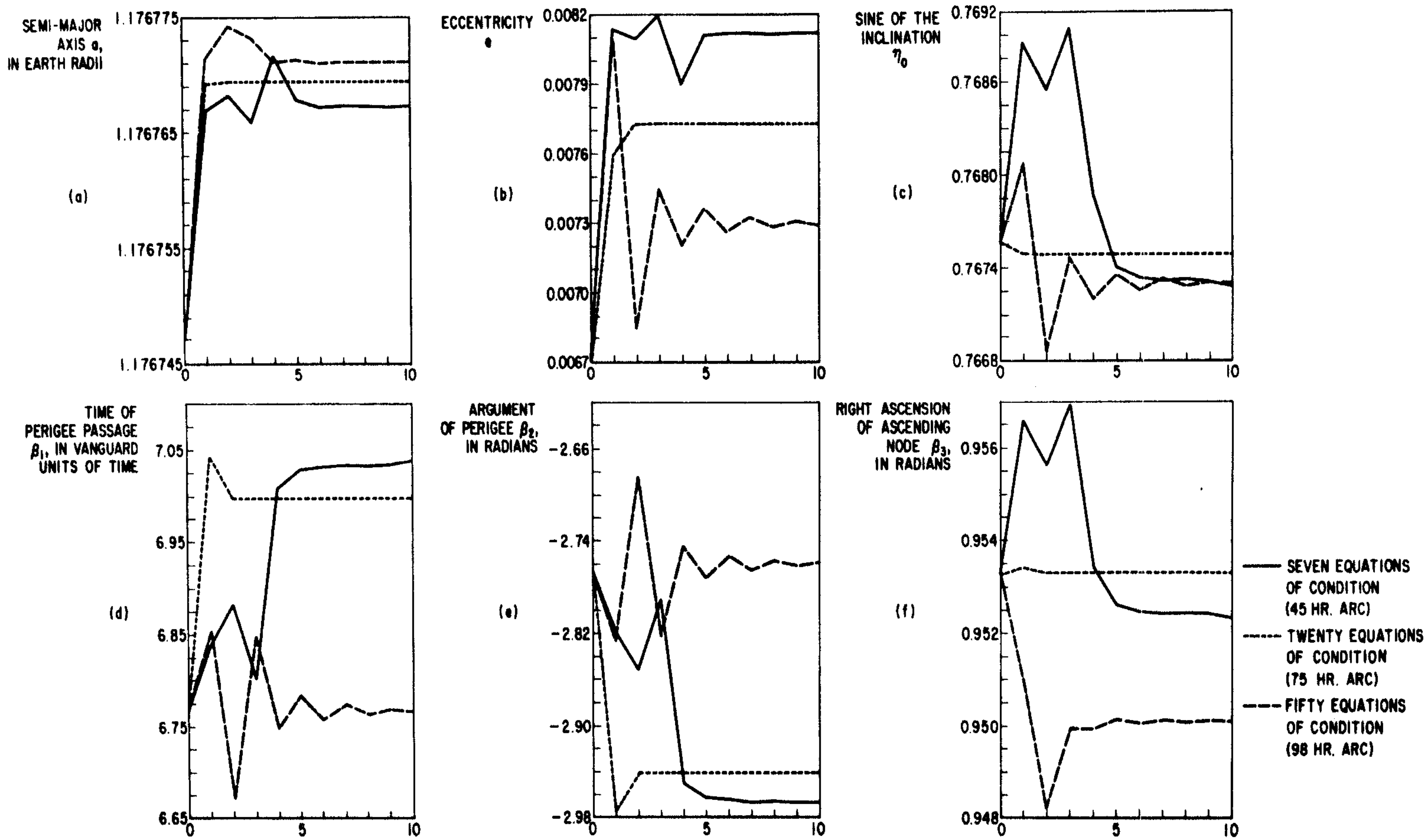

Figure 2–Convergence of a mean set of Izsak orbital elements in ten iterated least-squares fittings of the differential solution to observational data for the ANNA 1B satellite, for various numbers of equations of condition.

Table 2
Zonal Harmonic Coefficients in the Geopotential Function Used in Generation of Numerically Integrated Comparison Ephemerides

Coefficient	Geodetic value	Vinti potential value*
J_2	1.0823×10^{-3}	1.0823×10^{-3}
J_3	-2.3×10^{-6}	0
J_4	-1.8×10^{-6}	-1.2×10^{-6}
J_n $(n \geq 5)$	$< 1 \times 10^{-6}$	$< 1 \times 10^{-8}$

*The zonal harmonic coefficients for the Vinti potential function are obtained from the relations: $J_{2k} = (-1)^{k+1} J_2^k$ and $J_{2k+1} = 0$.

"curves" (actually a sequence of connected line segments) are shown, corresponding to various numbers of observations included in the fitting. An equation of condition results, of course, from a "semi-observation": either a single right ascension or a single declination value. One curve represents the minimum number of equations of condition for a true least-square fitting, viz., seven. This is associated with an observational arc length of approximately 45 hours. A second curve represents twenty equations of condition, or an addition of eleven equations, extending the observational arc length to approximately 75 hours. The third curve represents fifty equations of condition, or a further addition of thirty equations, extending the observational arc length to a total of approximately 98 hours. The starting point of each of the three arcs is the same, so that they overlap in time. Notice that each observational arc produces a somewhat different set of mean orbital elements, depending upon the additional observational values introduced. Physically, this may be explained as the resultant effect of forces not accounted for in the analytical theory. For example, electromagnetic disturbances, solar radiation pressures, aerodynamic drag, meteoric bombardment, etc., all influence the mean set of orbital elements to the extent that they are reflected in the observational values. In performing the iterated least-square fittings, all the residuals corresponding to the pre-selected observation times were accepted at each fitting. That is, the acceptable range of values for the residuals constituted infinitely wide bands on either side of the mean value of the residuals. Mathematically, using symbols introduced in the section titled, "The Standard Deviation of Fit,"

$$[r_1, r_2] = \lim_{j \to \infty} [\bar{R} - j\sigma, \bar{R} + j\sigma].$$

Figure 3 illustrates the determination of a mean set of Izsak orbital elements by an iterated least-squares fitting of the differential solution to observational data for the Relay 2 satellite. In this case, however, the observational arc length and the total number of observations are held fixed, while the acceptability criterion for the observational residuals is varied. The arc length in all cases is one week, representing a total of eighty observations or a maximum of 160 possible conditional equations. In each of the six graphs, one curve corresponds to a "three-sigma" criterion, i.e.,

$$[r_1, r_2] = [\bar{R} - 3\sigma, \bar{R} + 3\sigma]$$

where, of course, σ is the standard deviation of the observational residuals from their mean value. A second curve corresponds to a two-sigma criterion, and the third curve to a one-sigma criterion. Each curve is terminated when convergence of the orbital element is attained. Notice that convergence appears to be a slower process with a one-sigma criterion than with either a two-sigma or a three-sigma criterion. The rate of convergence in these latter two

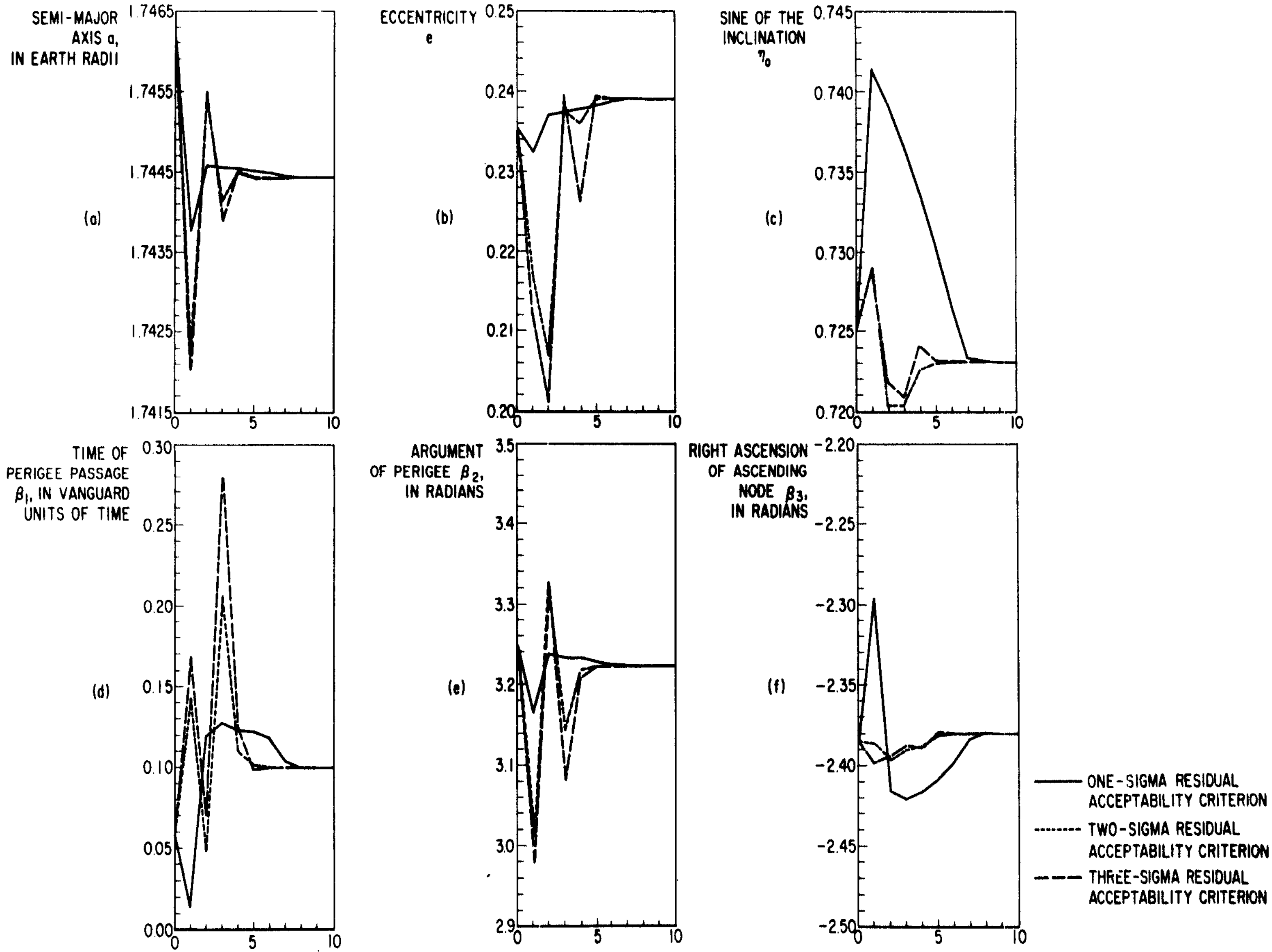

Figure 3—Convergence of a mean set of Izsak orbital elements in ten iterated least-squares fittings of the differential solution to observational data for the Relay 2 satellite, for various observational residual acceptability half-width criteria.

cases seems generally about the same. One effect of a wider acceptance range appears to be greater fluctuations in the value of an orbital element early in the iterated fitting procedure, although this is not always true. Also, despite the fact that differing numbers of conditional equations are accepted in the fittings depending upon the criterion for the residuals, the values of the orbital elements at convergence are remarkably similar. Refer to Table 3 for precise values, including the required number of iterations to attain convergence in each case. The uncertainties in the final significant figures (stated as ±X) are estimates based upon slight fluctuations in the values of the orbital elements in least-square fittings after convergence is attained.

Table 3
Values at Convergence of Izsak Orbital Elements for Varying Observational Residual Acceptability Half-Widths

Orbital Elements	One Sigma Criterion	Two Sigma Criterion	Three Sigma Criterion
Semi-major axis a	1.7444277 ± 1 (12)	1.7444278 ± 0 (7)	1.7444278 ± 0 (7)
Eccentricity e	0.2391624 ± 4 (21)	0.2391728 ± 3 (7)	0.2391818 ± 2 (7)
Sine of inclination η_0	0.7231110 ± 2 (21)	0.7231020 ± 4 (7)	0.7231015 ± 2 (9)
Time of perigee passage β_1	0.099885 ± 6 (22)	0.099985 ± 7 (9)	0.099942 ± 9 (8)
Argument of perigee β_2	3.222265 ± 5 (21)	3.222232 ± 5 (9)	3.222278 ± 6 (7)
Rt. asc. of ascending node β_3	-2.380274 ± 3 (22)	-2.380244 ± 4 (8)	-2.380241 ± 4 (8)

Note: Units of all elements are canonical (a in Earth equatorial radii; β_1 in Vanguard units of time; β_2 and β_3 in radians). The integers in parentheses refer to the number of iterations required to attain the converged value given.

Table 4 presents the same information relative to the orbital elements as Table 3, but for an acceptance criterion fixed at two sigma, with the maximum possible number of conditional equations varied. The observational arc length remains one week, but the 160 maximum possible number of conditional equations are first reduced to one hundred, and then this number is in turn reduced to forty. An attempt was made to maintain an even distribution of the observations throughout the seven-day period, while still operating on the "subset principle" (i.e., the set of twenty observations is a subset of the set of fifty observations, which is in turn a subset of the original set of eighty observations).

Table 5 again records the same information relating to the orbital elements, but this time the parameter involving the order of precision in the differential correction is varied. Here the maximum possible number of conditional equations covering the one-week observational arc is held constant at forty, and the acceptance criterion is fixed at two sigma. The inexact designations "first order" and "second order" indicate whether or not terms of purely second order are retained in the differential correction. (Refer to the section titled, "Analytical Procedure of Differential Correction.") It is seen that retaining terms of purely second order adds immeasurably to the precision of the final converged results in all cases, and, similarly, does not affect the rate of least-squares convergence.

Table 4
Values at Convergence of Izsak Orbital Elements for Varying Numbers of Observational Points

Orbital Elements	40 Conditional Equations	100 Conditional Equations	160 Conditional Equations
Semi-major axis a	1.7444276 ± 0 (10)	1.7444279 ± 0 (10)	1.7444278 ± 0 (7)
Eccentricity e	0.2391425 + 5 (11)	0.2391736 ± 2 (9)	0.2391728 ± 3 (7)
Sine of inclination η_0	0.7231009 ± 3 (12)	0.7230975 ± 3 (10)	0.7231020 ± 4 (7)
Time of perigee passage β_1	0.100116 ± 9 (12)	0.100002 ± 7 (11)	0.099985 ± 7 (9)
Argument of perigee β_2	3.222097 ± 5 (11)	3.222261 ± 6 (9)	3.222232 ± 5 (9)
Rt. asc. of ascending node β_3	-2.380249 ± 3 (12)	-2.380248 ± 3 (10)	-2.380244 ± 4 (8)

Note: Units of all elements are canonical (a in Earth equatorial radii; β_1 in Vanguard units of time; β_2 and β_3 in radians). The integers in parentheses refer to the number of iterations required to attain the converged value given.

Table 5
Values at Convergence of Izsak Orbital Elements for Varying Orders of Precision in Differential Correction

Orbital Elements	First Order	Second Order
Semi-major axis a	1.7444276 ± 0 (10)	1.7444276 ± 0 (10)
Eccentricity e	0.2391427 ± 8 (11)	0.2391425 ± 5 (11)
Sine of inclination η_0	0.7231010 ± 3 (12)	0.7231009 ± 3 (12)
Time of perigee passage β_1	0.100115 ± 8 (12)	0.100116 ± 9 (12)
Argument of perigee β_2	3.222098 ± 6 (11)	3.222097 ± 5 (11)
Rt. asc. of ascending node β_3	-2.380250 ± 4 (12)	-2.380249 ± 3 (12)

Note: Units of all elements are canonical (a in Earth equatorial radii; β_1 in Vanguard units of time; β_2 and β_3 in radians). The integers in parentheses refer to the number of iterations required to attain the converged value given.

The remainder of the Figures display the convergence of perhaps the most significant single parameter in evaluating the efficacy of the differential correction process, viz., the standard deviation of fit. Actually, there are two standard deviations shown in each graph. The upper curve corresponds to a standard deviation of fit which includes all of the observational residuals, while the lower curve corresponds to a standard deviation of fit which includes only the observational residuals accepted at each fitting. Plotted on the same abscissa is a curve showing the number of equations of condition (or, equivalently, the number of observational residuals) accepted at each iteration of the fitting process.

Figure 4 illustrates the standard deviations for a maximum of forty possible conditional equations covering a one-week observational arc for the Relay 2 satellite. Note that convergence using a two-sigma criterion for the residuals, as shown in Figure 4(b), is much more rapid than the convergence using a one-sigma criterion shown in Figure 4(a). However, the convergence is not so smoothly monotonic in the case of the wider acceptance range. Both these facts confirm what was said earlier about the convergence of the orbital elements.

Figure 5 illustrates the standard deviations for a maximum of one hundred possible conditional equations and Figure 6 for a maximum of 160 possible conditional equations, both covering the same one-week observational arc for Relay 2. Similar remarks apply to these Figures as to Figure 4. Table 6 supplies the values of the standard deviations at convergence for the various runs illustrated in Figures 4, 5, and 6, as well as several others not graphed. It also gives the number of accepted residuals at convergence, and the number of iterated fittings required to achieve convergence in each case.

Table 6
Values of Standard Deviations of Fit and Number of Accepted Conditional Equations at Convergence for Various Runs Covering a One-Week Observational Arc

Description of Run			Std. Dev. of Fit (all)	Std. Dev. of Fit (accepted)	Accepted Residuals	Percentage of Total	Iterations
Total Conditional Equations	Residual Criterion	Order of D.C.					
40	1σ	2nd	0.432	0.157	37	92.5	26
40	2σ	2nd	0.440	0.177	38	95	12
40	2σ	1st	0.438	0.175	38	95	12
100	1σ	2nd	0.373	0.160	95	95	11
100	2σ	2nd	0.377	0.169	97	97	9
160	1σ	1st	0.307	0.132	141	88.1	24
160	1σ	2nd	0.307	0.132	140	87.5	22
160	2σ	2nd	0.313	0.172	157	98.1	8
160	3σ	2nd	0.315	0.187	158	98.75	9

Note: All standard deviations of fit are given in mils (i.e., in units of 10^{-3}). The parenthetical word "all" signifies that all of the observational residuals, ΔL and ΔM, were included in determining the standard deviation of fit; "accepted" means that only the observational residuals corresponding to the accepted conditional equations were included in determining the standard deviation of fit.

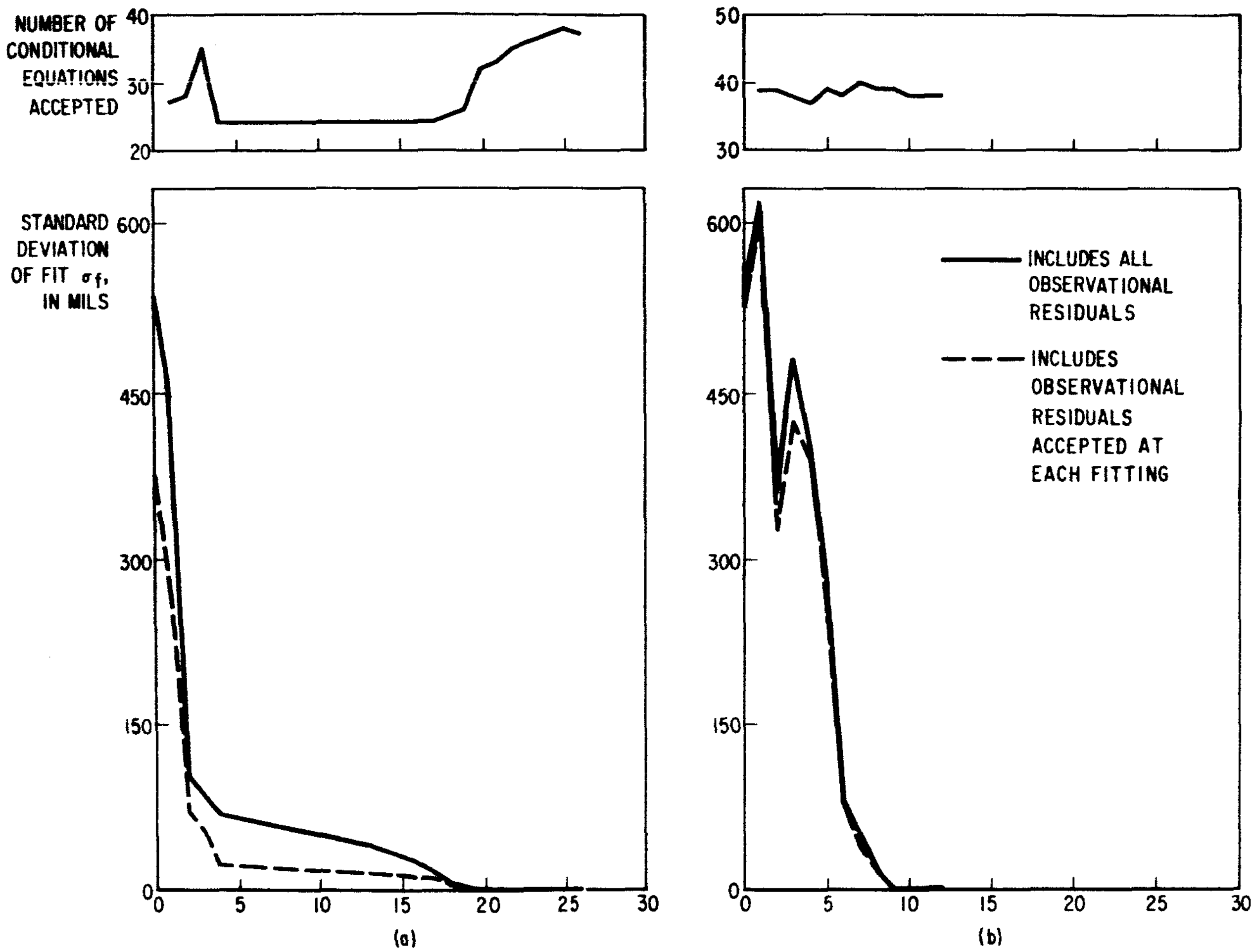

Figure 4—Standard deviations of the observational residuals and the number of equations of condition accepted at each iteration of the fitting process for a maximum of 40 possible conditional equations covering a one-week observational arc for the Relay 2 satellite.

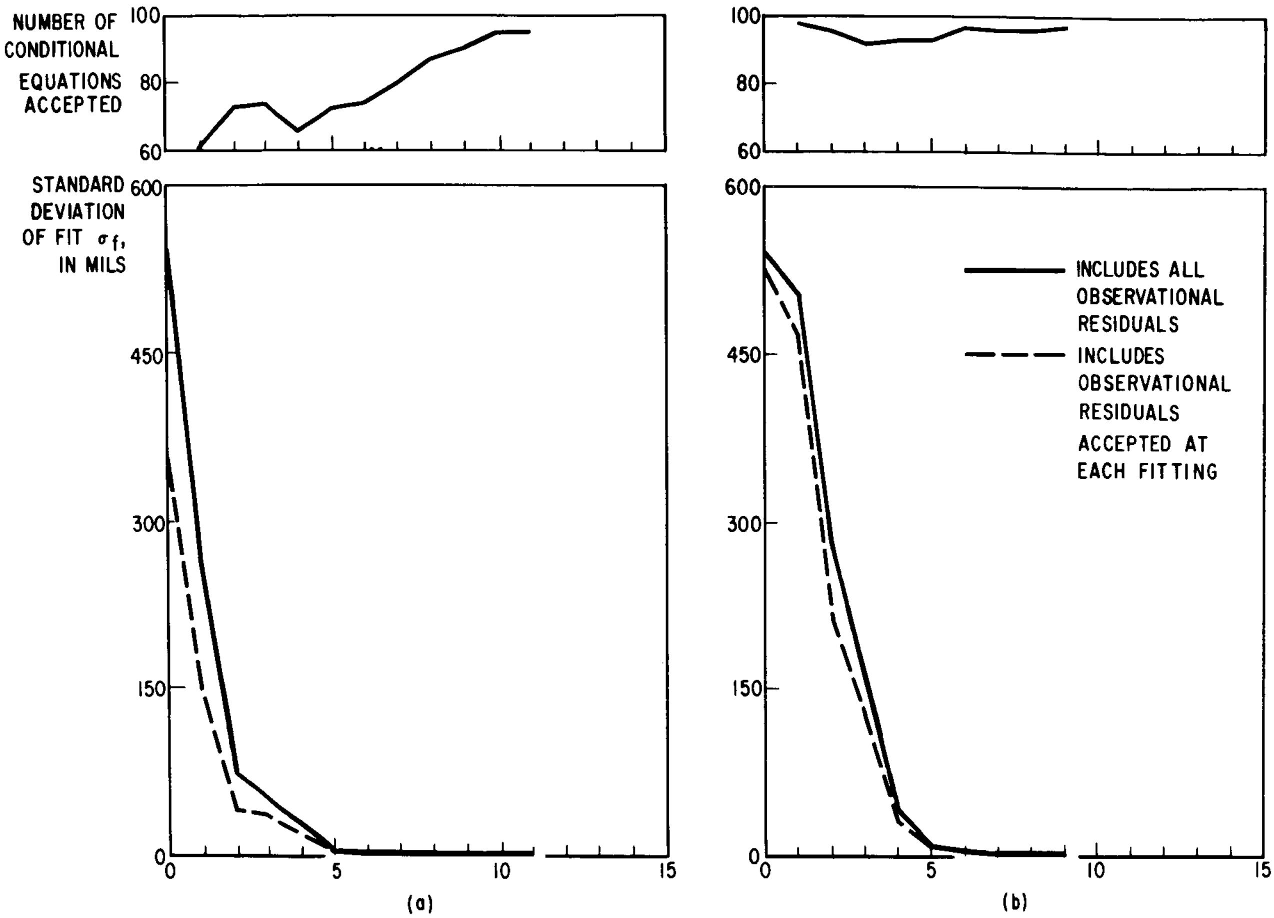

Figure 5—Standard deviations of the observational residuals and the number of equations of condition accepted at each iteration of the fitting process for a maximum of 100 possible conditional equations covering a one-week observational arc for the Relay 2 satellite.

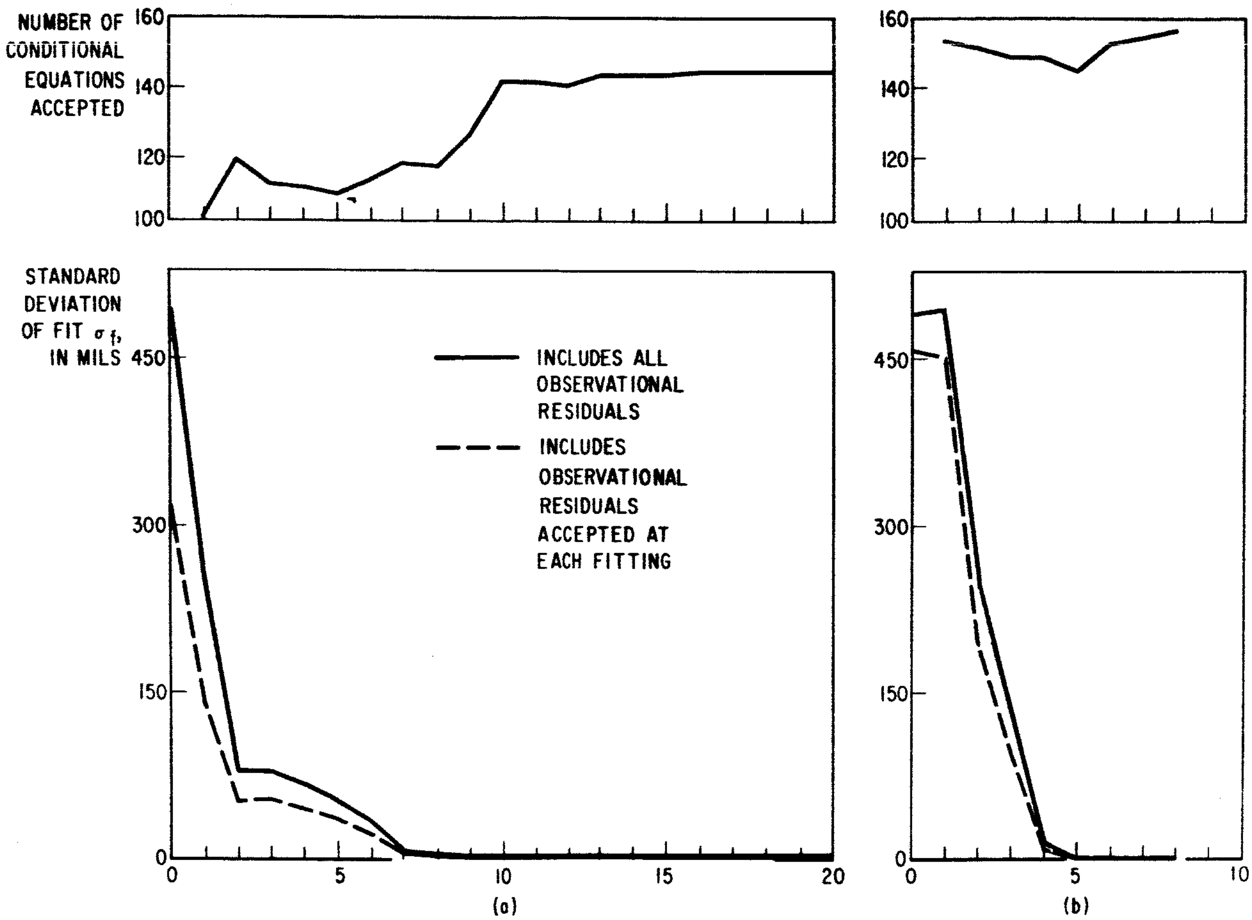

Figure 6—Standard deviations of the observational residuals and the number of equations of condition accepted at each iteration of the fitting process for a maximum of 160 possible conditional equations covering a one-week observational arc for the Relay 2 satellite.

Figure 7 illustrates the standard deviations for a maximum of one hundred possible conditional equations covering an observational arc of only three hours for Relay 2. This is the three-hour period immediately following insertion of the satellite into orbit, when observations are recorded at very frequent intervals in order to insure a wealth of data for the real-time differential correction. Here, using a one-sigma criterion, convergence of the orbital elements occurs after only four (in some cases, five) iterations. The standard deviations of fit converge after three iterations to values of 0.425×10^{-3} (all one hundred observational residuals) and 0.145×10^{-3} (including seventy-seven accepted observational residuals). The graph shows that the standard deviations remain essentially constant after the third iteration, and this is confirmed by the insignificant fluctuations in the orbital elements after the third iteration, although a total of ten iterations through the least squares fitting routine was prescribed in advance.

Figure 8 illustrates the standard deviations for a maximum of one hundred possible conditional equations for two distinct non-overlapping observational arcs for the ANNA 1B satellite. Figure 8(a) covers an arc of approximately nine days and fifteen hours, while Figure 8(b) covers an arc of approximately six days and nineteen hours. Both use a one-sigma criterion, and convergence of the standard deviations occurs after five iterations in both cases. The values are a relatively large 38.379 milliradians (all one hundred observational residuals) and 10.919 mrad (including eighty-eight accepted observational residuals) for Figure 8(a). For the somewhat shorter arc in Figure 8(b), the values are 6.684 mrad (all one hundred observational residuals) and 0.726 mrad (including ninety-five accepted observational residuals).

The totality of data presented herein represents a small sampling of the preliminary applications by which the orbit generator and differential correction have been tested. Yet this sampling is indicative of the utility of the spheroidal method for artificial satellite orbits.

CONCLUDING REMARKS

The method of solution for unretarded satellite orbits discussed in this paper has been programmed, primarily in the FORTRAN language, for use on the I.B.M. 7094 digital electronic computer. It requires a relatively small number of computer storage locations, and the analytical nature of the entire procedure assures a very rapid computational process. Extensive tests have indicated a capacity for generating co-ordinate and velocity points, based upon a set of empirically estimated initial conditions, in impressively short intervals of computer operating time.

Presently, work is underway on slightly modifying the accurate reference orbit to account for the effects of the most important perturbations of the neglected zonal harmonics, notably the third and the residual fourth. The inclusion of these perturbative effects by a procedure described in a recent paper by Vinti (see references) is expected to improve the accuracy of the method so as to provide computed values agreeing with observation over a longer interval of time.

In the future, a method of modifying the spheroidal potential for an oblate planet in order to permit the exact inclusion of the effects of the third zonal harmonic in the reference orbit is anticipated. Preliminary investigations are also being conducted into accounting for the lunisolar forces and aerodynamic drag. Further results will be published as they become available.

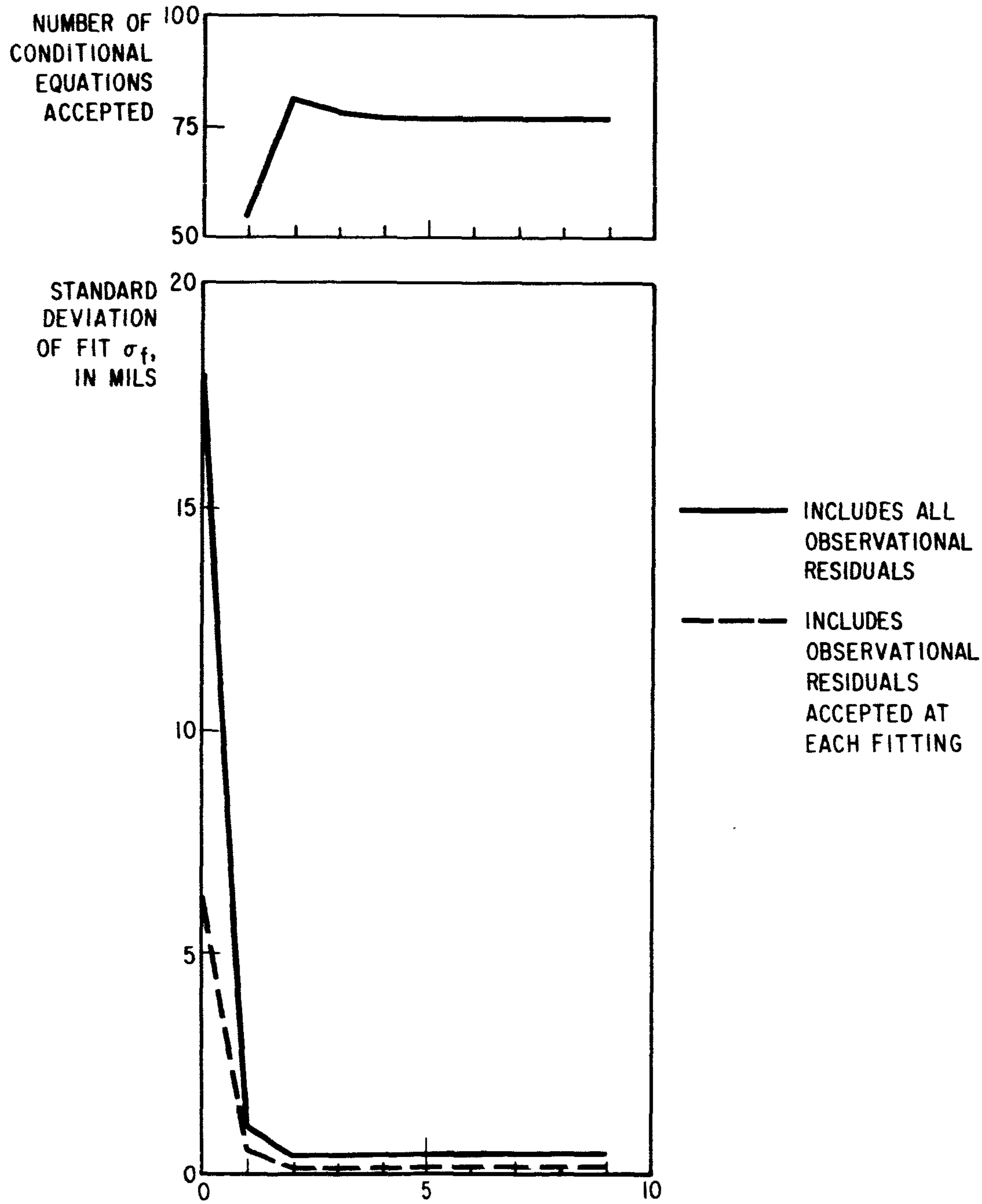

Figure 7—Standard deviations of the observational residuals and the number of equations of condition accepted at each iteration of the fitting process for a maximum of 100 possible conditional equations covering a three-hour observational arc for the Relay 2 satellite.

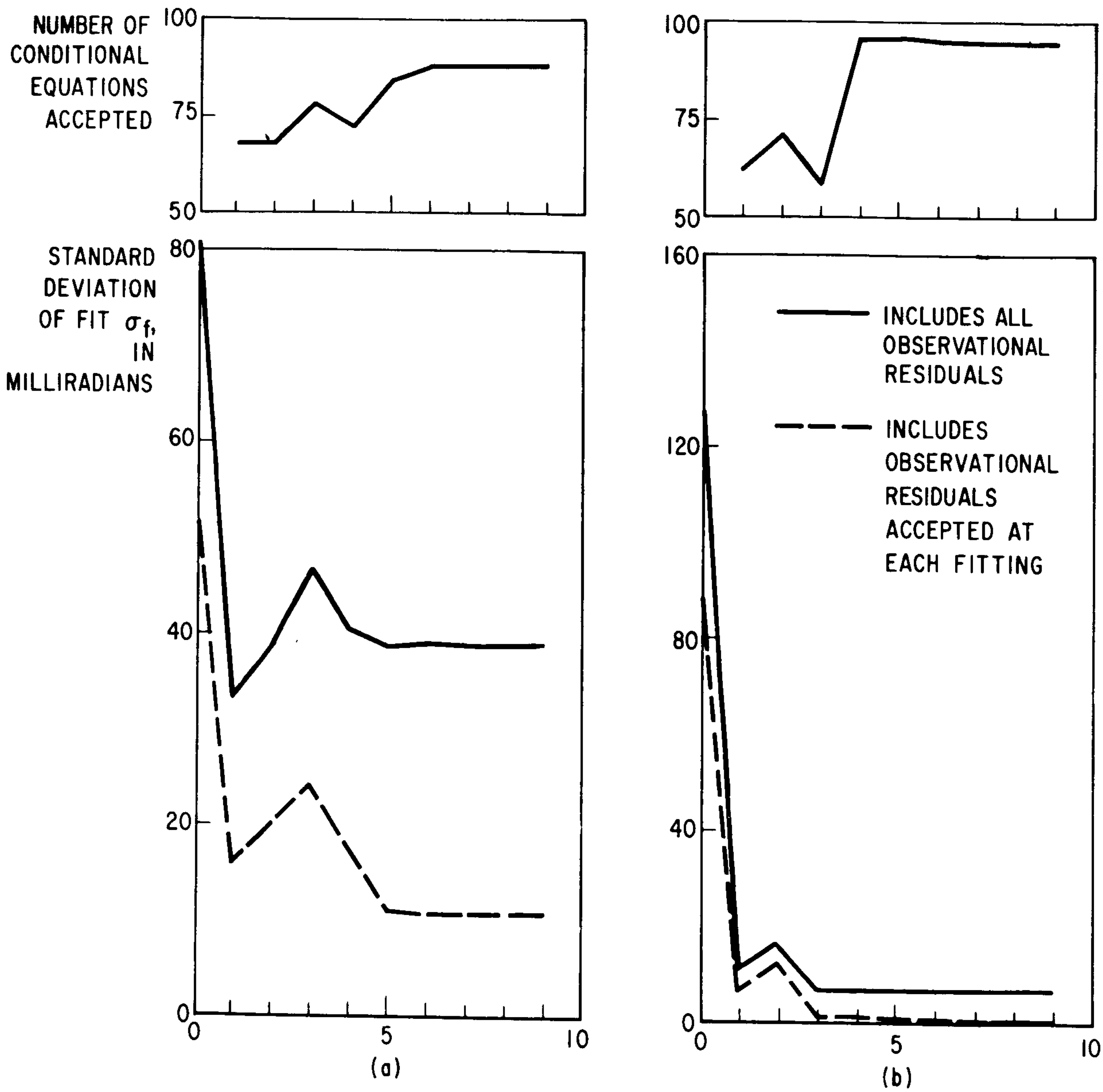

Figure 8—Standard deviations of the observational residuals and the number of equations of condition accepted at each iteration of the fitting process for a maximum of 100 possible conditional equations covering two distinct observational arcs for the ANNA 1B satellite.

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The author wishes to express gratitude for the substantial contribution of his colleague, J. Stan Watson of the Advanced Projects Branch, to the mathematical analysis and programming effort which preceded the publication of this report. Special acknowledgment is due to N. L. Bonavito of the Theory and Analysis Office for his helpful advice throughout the preparation of this report.

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APPENDIX

Herein we present the modifications which must be introduced in order to utilize an alternate form of satellite tracking data known as right ascension-declination data. Such data are recorded, for instance, by the optical Baker-Nunn cameras of the Astrophysical Observatory of the Smithsonian Institution. The modifications to be described replace the material presented in the main body of this report in the section titled, "Computation of Direction Cosines."

A set of observation data of the right ascension-declination type includes the following parameters for each recorded spacecraft observation:

t', the date and time of observation. The same remarks about removing reference to the calendar in transforming t' to the relative time t apply here as included in the main body of this report.

k, the code number for the tracking station reporting the observation.

α_0, the observed right ascension, measured in radians eastward from the vernal equinox $(0 \leq \alpha_0 < 2\pi)$.

δ_0, the observed declination in radians, measured as positive north of the equator and as negative south of the equator $(-\pi/2 \leq \delta_0 \leq +\pi/2)$.

w_α and w_δ, the weighting factors corresponding to observations α_0 and δ_0, respectively. This information is optional; if not provided, then it is assumed that w_α and w_δ are each unity.

The co-ordinate system employed for the observation data is centered at the tracking station on the Earth's surface, and, unlike the system used for recording direction-cosine data, its three co-ordinate axes are parallel to the respective axes of the inertial system. That is, the Z-axis is parallel to the Earth's polar axis, and the X-Y plane is parallel to the equatorial plane of the Earth, with the X-axis extending toward the vernal equinox. The Y-axis extends orthogonally to the east to form a right-handed system.

The differential correction process requires the same data to be available as listed in the main body of this report, viz., the Earth's flattening coefficient f, the Earth's rotational rate ω, the geodetic longitudes λ_E of the stations, the geodetic latitudes θ_D of the stations, the altitudes H of the stations, the angular distances λ_0 from the vernal equinox to the Greenwich meridian at midnight Greenwich mean time for each day in the observational arc, and the reference time t_0.

Computations follow the same scheme given in the main body of this report for the following parameters: the auxiliary functions C and S, the geocentric latitude θ_G, the geocentric distance $\hat{\rho}$ of the station, the angular distance δ between the vernal equinox and the observation meridian plane, and the inertial geocentric co-ordinates X_T, Y_T, Z_T of the station. However, the angle ψ_x between the vernal equinox and the tracking station's X-co-ordinate axis (measured in the observation latitude plane) is zero, since the topocentric and inertial co-ordinate systems are parallel. No rotations are necessary to bring the two systems into coincidence; a single translation will suffice. Hence, the relations for the topocentric or local co-ordinates of the satellite are simply:

$$X_M = X - X_T$$

$$Y_M = Y - Y_T$$

$$Z_M = Z - Z_T$$

where X, Y, Z are the inertial geocentric co-ordinates of the satellite predicted by the orbit generator. The above simplified relations are obtained from those of the direction-cosine-data case by the artificial device of setting $\psi_x = 0$ and $\theta_D = \pi/2$ in the corresponding equations for X_M, Y_M, Z_M given in the main body of this report. (Refer to the note at the end of this appendix.)

The computed values of the right ascension and the declination may now be found in terms of the local co-ordinates:

$$\alpha_c = \arctan\left(\frac{Y_M}{X_M}\right)$$

$$\delta_c = \arctan\left[\frac{Z_M}{(X_M^2 + Y_M^2)^{1/2}}\right]$$

It is important that the angles α_c and δ_c be placed in the proper quadrant for comparison purposes with the angles α_0 and δ_0. In the case of the right ascension, this is done by examining the signs of X_M and Y_M separately. The following list presents all possible combinations (note that the range for α_0 is $0 \le \alpha_0 < 2\pi$).

$$X_M > 0, \quad Y_M > 0: \quad 0 < \alpha_c < \frac{\pi}{2}$$

$$X_M > 0, \quad Y_M < 0: \quad \frac{3\pi}{2} < \alpha_c < 2\pi$$

$$X_M > 0, \quad Y_M \;\square\; 0: \quad \alpha_c = 0$$

$$X_M < 0, \quad Y_M > 0: \quad \frac{\pi}{2} < \alpha_c < \pi$$

$$X_M < 0, \quad Y_M < 0: \quad \pi < \alpha_c < \frac{3\pi}{2}$$

$$X_M < 0, \quad Y_M = 0: \quad \alpha_c = \pi$$

$$X_M = 0, \quad Y_M > 0: \quad \alpha_c = \frac{\pi}{2}$$

$$X_M = 0, \quad Y_M < 0: \quad \alpha_c = \frac{3\pi}{2}$$

$$X_M = 0, \quad Y_M = 0: \quad \alpha_c \text{ indeterminate}$$

In the case of the declination, the signs of the numerator and denominator of the arctangent argument are examined separately. The following list presents all possible combinations (note that the range for δ_0 is $-\pi/2 \le \delta_0 \le +\pi/2$).

$$(X_M^2 + Y_M^2)^{1/2} > 0, \quad Z_M > 0: \quad 0 < \delta_c < \frac{\pi}{2}$$

$$(X_M^2 + Y_M^2)^{1/2} > 0, \quad Z_M < 0: \quad -\frac{\pi}{2} < \delta_c < 0$$

$$(X_M^2 + Y_M^2)^{1/2} > 0, \quad Z_M = 0: \quad \delta_c = 0$$

$$(X_M^2 + Y_M^2)^{1/2} = 0, \quad Z_M > 0: \quad \delta_c = \frac{\pi}{2}$$

$$(X_M^2 + Y_M^2)^{1/2} = 0, \quad Z_M < 0: \quad \delta_c = -\frac{\pi}{2}$$

Of course, the case $(X_M^2 + Y_M^2)^{1/2} = Z_M = 0$ is not physically possible.

The observational residuals are now found:

$$\Delta\alpha = \alpha_0 - \alpha_c$$

$$\Delta\delta = \delta_0 - \delta_c$$

Here too, care must be exercised. There is one instance where simple subtraction in finding the observational residual will yield a misleading result. If one of the right ascensions (either observed or computed) is in the first quadrant and very nearly zero and the other right ascension is in the fourth quadrant and very nearly 2π, then direct subtraction will provide an erroneous result near to 2π, whereas the intended difference is near to zero. This situation can be rectified by the following logical steps:

If $|\alpha_0 - \alpha_c| \leq \pi$, then $\Delta\alpha = \alpha_0 - \alpha_c$ (as above).
If $|\alpha_0 - \alpha_c| > \pi$, then $\Delta\alpha = \text{sgn}\,(\alpha_0 - \alpha_c)\ [2\pi - |\alpha_0 - \alpha_c|]$.

Equivalently, whenever $|\alpha_0 - \alpha_c| > \pi$, use the following:

(1) if $\alpha_0 > \alpha_c$, then $\Delta\alpha = 2\pi - \alpha_0 + \alpha_c > 0$.

(2) if $\alpha_0 < \alpha_c$, then $\Delta\alpha = \alpha_c - \alpha_0 - 2\pi < 0$.

The statistical analysis of the observational residuals follows the procedure given in the main body of this report in the section titled, "The Standard Deviation of Fit" except that the observational residuals are given by $\Delta\alpha_i$ and $\Delta\delta_i$, rather than by ΔL_i and ΔM_i. Hence, the average residual is given by:

$$\bar{R} = \frac{1}{2n} \sum_{i=1}^{n} (\Delta\alpha_i + \Delta\delta_i)$$

The standard deviation of the residuals from their mean value is found from:

$$\sigma = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \left[(\Delta\alpha_i - \bar{R})^2 + (\Delta\delta_i - \bar{R})^2\right]}$$

The standard deviation of fit is given by:

$$\sigma_f = \sqrt{\frac{1}{2n-6} \sum_{i=1}^{n} \left[(\Delta\alpha_i)^2 + (\Delta\delta_i)^2\right]}$$

Modifications are now presented to supplant the material from the main body of this report in the section titled, "Analytical Procedure of Differential Correction."

The first-order Taylor series expansion of the equations of condition may be written:

$$\Delta\alpha = \sum_{i=1}^{6} \frac{\partial \alpha_c}{\partial q_i} \Delta q_i$$

$$\Delta\delta = \sum_{i=1}^{6} \frac{\partial \delta_c}{\partial q_i} \Delta q_i$$

where q_i $(i = 1, 2, \cdots, 6)$ are the mean or Izsak orbital elements. Expanding the above partial derivatives by the chain rule yields:

$$\frac{\partial \alpha_c}{\partial q_i} = \frac{\partial \alpha_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial \alpha_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial \alpha_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i}$$

$$\frac{\partial \delta_c}{\partial q_i} = \frac{\partial \delta_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial \delta_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial \delta_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i}$$

From the equations for α_c and δ_c in terms of the local co-ordinates, we find:

$$\frac{\partial \alpha_c}{\partial X_M} = - Y_M (X_M^2 + Y_M^2)^{-1}$$

$$\frac{\partial \alpha_c}{\partial Y_M} = + X_M (X_M^2 + Y_M^2)^{-1}$$

$$\frac{\partial \alpha_c}{\partial Z_M} = 0$$

$$\frac{\partial \delta_c}{\partial X_M} = - X_M Z_M (X_M^2 + Y_M^2)^{-1/2} (X_M^2 + Y_M^2 + Z_M^2)^{-1}$$

$$\frac{\partial \delta_c}{\partial Y_M} = - Y_M Z_M (X_M^2 + Y_M^2)^{-1/2} (X_M^2 + Y_M^2 + Z_M^2)^{-1}$$

$$\frac{\partial \delta_c}{\partial Z_M} = + (X_M^2 + Y_M^2)^{1/2} (X_M^2 + Y_M^2 + Z_M^2)^{-1}$$

Since the station co-ordinates X_T, Y_T, Z_T are independent of orbital parameters (and merely geodesic functions), the following simple relations hold:

$$\frac{\partial X_M}{\partial q_i} = \frac{\partial X}{\partial q_i}$$

$$\frac{\partial Y_M}{\partial q_i} = \frac{\partial Y}{\partial q_i}$$

$$\frac{\partial Z_M}{\partial q_i} = \frac{\partial Z}{\partial q_i}$$

The method for calculating the partial derivatives $\partial X/\partial q_i$, $\partial Y/\partial q_i$, and $\partial Z/\partial q_i$ is identical to that presented in the differential correction scheme in the main body of this report. Then the equations of condition are formulated in a precisely analogous manner to that given for the direction-cosine data (see the section titled, "The Equations of Condition"), and there is little need to repeat the explicit form of these equations.

NOTE: The fact that the observational co-ordinate system is independent of the latitude and longitude of the tracking station for right ascension-declination data (as is not the case for direction-cosine data) leads to certain possible simplifications in the determination of the computed co-ordinates of the satellite, α_c and δ_c. First, recall that the equations for the Cartesian inertial co-ordinates of the observation point are given by:

$$X_T = \hat{\rho} \cos \theta_G \cos \delta$$

$$Y_T = \hat{\rho} \cos \theta_G \sin \delta$$

$$Z_T = \hat{\rho} \sin \theta_G$$

where $\delta = (\lambda_0)_d + \omega(\Delta T) + \lambda_E$. Here the terms $(\lambda_0)_d$ and $\omega(\Delta T)$ depend upon the time of the observation only, while the term λ_E is a function of the location of the observation point. Let us denote:

$$\delta' = (\lambda_0)_d + \omega(\Delta T)$$

Then we can expand the above equations as:

$$\begin{aligned} X_T &= \hat{\rho} \cos \theta_G \cos (\delta' + \lambda_E) \\ &= \hat{\rho} \cos \theta_G \cos \lambda_E \cos \delta' - \hat{\rho} \cos \theta_G \sin \lambda_E \sin \delta' \end{aligned}$$

$$Y_T = \hat{\rho} \cos \theta_G \sin (\delta' + \lambda_E)$$

$$= \hat{\rho} \cos \theta_G \cos \lambda_E \sin \delta' + \hat{\rho} \cos \theta_G \sin \lambda_E \cos \delta'$$

$$Z_T = \hat{\rho} \sin \theta_G$$

Now denote:

$$X_0 = \hat{\rho} \cos \theta_G \cos \lambda_E$$

$$Y_0 = \hat{\rho} \cos \theta_G \sin \lambda_E$$

$$Z_0 = \hat{\rho} \sin \theta_G$$

so that:

$$\begin{bmatrix} X_T \\ Y_T \\ Z_T \end{bmatrix} = \begin{bmatrix} \cos \delta' & -\sin \delta' & 0 \\ \sin \delta' & \cos \delta' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

This represents, in matrix form, the fact that the co-ordinates X_T, Y_T, Z_T are obtained from X_0, Y_0, Z_0 by a simple rotation about the inertial Z-axis through an angle δ'. Here δ' is the angle between the vernal equinox and the Greenwich meridian at observation time. The rectangular co-ordinates X_0, Y_0, Z_0, obtained directly from the spherical geocentric co-ordinates $\hat{\rho}$, θ_G, λ_E of the station, represent the Cartesian inertial geocentric co-ordinates of the tracking station at a time when the Greenwich meridian and the first point of Aries (the vernal equinox) coincide. If the co-ordinates X_0, Y_0, Z_0 (dependent upon the station location only) are provided as input parameters rather than λ_E, θ_D, and H, then the computations leading up to X_T, Y_T, Z_T are simplified considerably. We need not first compute C, S, θ_G, $\hat{\rho}$, and δ. Instead, find δ' from parameters relating to the time of observation, and then compute directly:

$$X_T = X_0 \cos \delta' - Y_0 \sin \delta'$$

$$Y_T = X_0 \sin \delta' + Y_0 \cos \delta'$$

$$Z_T = Z_0$$

Note that this simplified procedure cannot be adopted efficiently with direction-cosine data because the rotation matrix involved in computing X_T, Y_T, Z_T is a function of θ_D, the geodetic latitude, and ψ_x, an angular parameter dependent upon λ_E, the geodetic longitude.