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Lewis Research Center
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SUMMARY

Noise measurements and variations of ion production energy per beam ion with magnetic field indicate an onset of anomalous diffusion in the electron-bombardment ion thrustor. Application of the stability theory of the Kadomtsev and Nedospasov kind yielded an expression for the critical magnetic field for the onset of anomalous diffusion and an expression for the oscillatory frequency associated with the anomalous diffusion. The derived behavior of the critical magnetic field is similar to an experimentally derived scaling law for the ion thrustor. Calculated values of the critical magnetic field for mercury and argon propellants agreed well with those obtained from experimental values determined from curves of ion production energy per beam ion. Expressions for ion production energy per beam ion are given for both the classical and the anomalous regions. A theoretical value of the oscillatory frequency was calculated to be in the same frequency regime as an experimental value of a device dominated by a similar type of anomalous diffusion.

INTRODUCTION

The electron-bombardment ion thrustor was developed for use in electric propulsion. This engine has been the subject of many investigations, and its operation and performance are well documented (refs. 1 to 3). The thrustor consists of an ion source in the form of a plasma discharge exhausting into an electrostatic accelerator system. In the ion source, a plasma is created through the electron bombardment of neutral propellant atoms. In the electron bombardment process, electrons are conducted across an axial magnetic field by the application of a radial electric field between a cathode and an anode. The magnetic field serves the purpose of containing the electrons long enough so that they can be utilized most effectively in the electron-bombardment process.

The process of conduction and diffusion of charged particles in a magnetic field has recently been the subject of many investigations (ref. 4). As a result of these studies, an anomalously high value of conduction or diffusion (here and afterwards referred to as anomalous diffusion) has been found in many of the experiments. This anomalous diffusion results in the conduction or diffusion of charged particles at rates much higher than can be expected from classical calculations. Furthermore, this anomalous diffusion frequently tends to start beyond some critical value of magnetic field. Included in these investigations is a wide range of geometries of electric and magnetic fields as well as a wide range of plasma densities.

There are two indications of anomalous diffusion in the plasma geometry associated with an ion thrustor. First, an onset of noise has been found beyond some critical value of magnetic field (refs. 5 and 6); this behavior is typical of the onset of anomalous diffusion. Second, curves of ion production energy per beam ion (here and afterwards referred to as eV/ion) for the ion thrustor show an expected decline with an increasing magnetic field up to approximately this same critical value. Beyond this value of the magnetic field is a transition into either a leveling off region or an increasing region, a result, as it will be shown, that cannot be explained classically.

In this report an application of the stability calculations first introduced by Kadomtsev and Nedospasov (ref. 7) is made to the plasma source of the ion thrustor. (The theory was first applied to the positive column of a high density plasma discharge and later successfully reworked to a low density regime by Guest and Simon (ref. 8).) In this theory an oscillatory type of perturbation is introduced into the number density and electric potential of the plasma source. The condition that the growth of the perturbation becomes undamped, and thus continually increases with time, yields a critical magnetic field. In the study herein, the theory of Kadomtsev and Nedospasov is modified to apply to a type of plasma source having a different relative orientation of electric and magnetic fields from that to which it was originally applied. Also, the theory is applied to a lower plasma pressure regime. The results of this application yield a critical magnetic field and a frequency of oscillation, both associated with the onset of anomalous diffusion. The critical magnetic field can be compared with values obtained from eV/ion curves of the ion thrustor, and the frequency of oscillation obtained can be compared with experiments of a plasma source of similar design.

The curves of eV/ion against the magnetic field of the ion thrustor behave in a classical manner up to a critical magnetic field and thereafter show either a constant value or an increase with increasing magnetic field. The classical behavior, as well as the anomalous behavior, of the variation of eV/ion with magnetic field is described in this report.

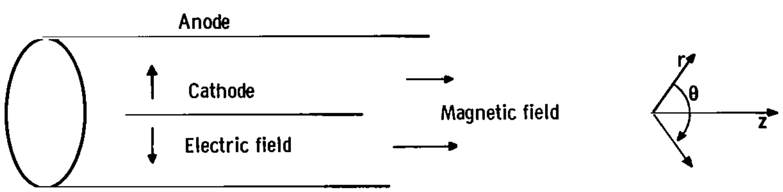
ANALYSIS

Anomalous Diffusion Mechanism

The mechanism of anomalous diffusion introduced in the Kadomtsev and Nedospasov instability theory is based on a macroscopic distortion where the perturbed number density and potential have a θ-direction dependence. As a result of this dependence, anomalous diffusion is caused by a Hall effect which results in increased charged particle currents in the direction transverse to the magnetic field. In the plasma column to which this theory was first applied, the electric field had the same direction (the z-direction) as the longitudinal magnetic field. In the plasma source of the ion thrustor, the applied electric field which can be considered as the driving force is in the r-direction perpendicular to the longitudinal magnetic field. There is an indication that with this configuration the same type of anomalous diffusion mechanism is applicable. First, an analysis in reference 9 considers a similar type mechanism as that of Kadomtsev and Nedospasov and applies it to a plasma source with an axial magnetic field and an inwardly directed radial electric field. The result of this analysis is the prediction of critical magnetic fields which are in agreement with experimental results. Further, an experimental study (ref. 10) of a low density plasma source with magnetic and electric fields perpendicular to each other in a cylindrical geometry resulted in the finding of anomalous diffusion. In this experiment, oscillatory variations as well as a θ-direction dependence of number density were found and shown to cause the anomalous diffusion. In the present report the Kadomtsev and Nedospasov type of anomalous diffusion mechanism is applied with the result that good agreement is found between derived and experimental critical magnetic fields. The mechanism is further applied in this report to the anomalous eV/ion curves of the ion thrustor.

Unperturbed Calculation

The basic geometry of the plasma source in the ion thrustor is shown in sketch (a). The region which is analyzed is a central portion of the plasma source, far from both ends (the long cylinder approximation). The normal magnetic field of an ion thrustor is



(a)

strong enough to affect greatly the motion of the electrons but has negligible effect on the ions. For the electrons the condition

$$\omega_e \tau_e >> 1$$

is satisfied. (Symbols are defined in appendix A.) Ions are produced throughout the region of analysis by electron bombardment of neutral atoms. The equation of motion for electrons is (ref. 8)

$$\frac{kT_e}{m_e n} \nabla n = \frac{e}{m_e} \vec{v} \times \vec{B} + \frac{e}{m_e} \vec{E} - \frac{\vec{v}}{\tau_e} \tag{1}$$

The electrons are tied to the magnetic lines of force and diffuse across the chamber as a result of neutral-electron collisions. The electric-field force on the ions is directed radially inward, but the ions can only reach the cathode by collisions in which there is a loss of angular momentum. For the electrons the equations of motion derived from equation (1) become

$$n_e v_{r,e} = \left(-D_{\perp,e} \frac{\partial n_e}{\partial r} + n_e \mu_{\perp,e} \frac{\partial V}{\partial r}\right) - \omega_e \tau_e \left(-\frac{D_{\perp,e}}{r} \frac{\partial n_e}{\partial \theta} + n_e \frac{\mu_{\perp,e}}{r} \frac{\partial V}{\partial \theta}\right) \tag{2}$$

$$n_e v_{\theta,e} = \left(-\frac{D_{\perp,e}}{r} \frac{\partial n_e}{\partial \theta} + \frac{n_e \mu_{\perp,e}}{r} \frac{\partial V}{\partial \theta}\right) + \omega_e \tau_e \left(-D_{\perp,e} \frac{\partial n_e}{\partial r} + n_e \mu_{\perp,e} \frac{\partial V}{\partial r}\right) \tag{3}$$

$$n_e v_{z,e} = 0 \tag{4}$$

where the Hall currents are taken into account. The motion of the electrons in the z-direction of the thrustor is assumed to be zero. For the ions the equations of motion become

$$n_i v_{r,i} = \left(-D_{\perp,i} \frac{\partial n_i}{\partial r} - n_i \mu_{\perp,i} \frac{\partial V}{\partial r}\right) \tag{5}$$

$$n_i v_{\theta,i} = \left(-\frac{D_{\perp,i}}{r} \frac{\partial n_i}{\partial \theta} - n_i \frac{\mu_{\perp,i}}{r} \frac{\partial V}{\partial \theta}\right) \tag{6}$$

$$n_i v_{z,i} = 0 \tag{7}$$

The Hall currents for ions, because of the weak magnetic field typically used in ion thrustors, are negligible. The motion of the ions in the z-direction should be small since the analysis is restricted to the central portion of the ion thrustor, and the axial potential gradient there is small. Appendix B is included to consider currents in the z-direction. The continuity equations can be written for the present analysis as

$$\frac{\partial n_e}{\partial t} + \nabla \cdot n_e \vec{v}_e = Z n_e \tag{8}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \vec{v}_i = Z n_e \tag{9}$$

where Z is the number of ionization events per electron per unit time. In the system, plasma neutrality is assumed so that

$$n_e = n_i = n$$

Substituting equations (2) to (7) into equations (8) and (9), with the assumption of plasma neutrality, yields the following equations:

$$\frac{\partial n}{\partial t} - D_{\perp,i}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 n}{\partial \theta^2}\right] - \mu_{\perp,i}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rn\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(n\frac{\partial V}{\partial \theta}\right)\right] = Zn \tag{10}$$

$$\frac{\partial n}{\partial t} - D_{\perp,e}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 n}{\partial \theta^2}\right] + \mu_{\perp,e}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rn\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(n\frac{\partial V}{\partial \theta}\right)\right]$$

$$- \frac{\omega_e \tau_e \mu_{\perp,e}}{r}\left(\frac{\partial n}{\partial r}\frac{\partial V}{\partial \theta} - \frac{\partial n}{\partial \theta}\frac{\partial V}{\partial r}\right) = Zn \tag{11}$$

Next, the normal unperturbed state ($n = n_0$) can be deduced with regard to the following conditions from equations (10) and (11). The steady state is recovered by taking $\partial n_0/\partial t$ as zero. Taking n_0 and V_0 (the unperturbed potential V) as being independent of θ results in $\partial n_0/\partial \theta$ and $\partial V_0/\partial \theta$ being equal to zero. The resultant equations are then

$$-\frac{D_{\perp,i}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_0}{\partial r}\right)-\frac{\mu_{\perp,i}}{r}\frac{\partial}{\partial r}\left(rn_0\frac{\partial V_0}{\partial r}\right)=Zn_0 \tag{12}$$

and

$$-\frac{D_{\perp,e}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_0}{\partial r}\right)+\frac{\mu_{\perp,e}}{r}\frac{\partial}{\partial r}\left(rn_0\frac{\partial V_0}{\partial r}\right)=Zn_0 \tag{13}$$

Now eliminating the second term between equations (12) and (13) yields

$$-(D_{\perp,i}\mu_{\perp,e}+D_{\perp,e}\mu_{\perp,i})\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_0}{\partial r}\right)=Z(\mu_{\perp,e}+\mu_{\perp,i})n_0 \tag{14}$$

which is of the form

$$\frac{\partial^2 n_0}{\partial r^2}+\frac{1}{r}\frac{\partial n_0}{\partial r}+An_0=0 \tag{15}$$

where A is a constant (see appendix B for the constant including z-currents). The solution of equation (15) is

$$n_0(r)=n_0'J_0(\alpha_0 r) \tag{16}$$

where $\alpha_0 r$ is the argument of the zero-order Bessel function, and n_0' and α_0 are constants that can be solved for by the insertion of boundary conditions. Taken as a boundary condition, $n_0(a)=0$ (ref. 7) yields

$$\alpha_0=\frac{2.4}{a} \tag{17}$$

where 2.4 is the first zero of the zero-order Bessel function, and $r = a$ is the radius of the anode.

Stability Calculations

Critical magnetic field. - To consider the stability of the plasma source of the ion thrustor, a small perturbation is added to both the unperturbed number density n_0 and the potential V_0 such that

$$n = n_0 + n_1 \tag{18}$$

and

$$V = V_0 + V_1 \tag{19}$$

Substituting these quantities into equations (10) and (11) and ignoring products of perturbations yield the following equations:

$$\frac{\partial n_1}{\partial t} - D_{\perp,i}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 n_1}{\partial \theta^2}\right] - \mu_{\perp,i}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rn_1\frac{\partial V_0}{\partial r} + rn_0\frac{\partial V_1}{\partial r}\right) + \frac{n_0}{r^2}\frac{\partial^2 V_1}{\partial \theta^2}\right] = Zn_1 \tag{20}$$

$$\frac{\partial n_1}{\partial t} - D_{\perp,e}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 n_1}{\partial \theta^2}\right] + \mu_{\perp,e}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rn_1\frac{\partial V_0}{\partial r} + rn_0\frac{\partial V_1}{\partial r}\right) + \frac{n_0}{r^2}\frac{\partial^2 V_1}{\partial \theta^2}\right] - \frac{\omega_e\tau_e\mu_{\perp,e}}{r}\left(\frac{\partial n_0}{\partial r}\frac{\partial V_1}{\partial \theta} - \frac{\partial n_1}{\partial \theta}\frac{\partial V_0}{\partial r}\right) = Zn_1 \tag{21}$$

In equations (20) and (21) the unperturbed equations have been subtracted out and the quantities $\partial n_0/\partial\theta$ and $\partial V_0/\partial\theta$ are again taken equal to zero. Next, the time dependence and angular dependence of the perturbed quantities are taken as

$$n_1 = n_1(r)e^{i\omega t+im\theta} \tag{22}$$

and

$$V_1 = V_1(r)e^{i\omega t+im\theta} \tag{23}$$

When these are substituted into equations (20) and (21), the results are

$$i\omega n_1(r) - Zn_1(r) - D_{\perp,i}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial n_1(r)}{\partial r}\right] - \frac{1}{r^2}m^2 n_1(r)\right\}$$

$$- \mu_{\perp,i}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rn_1(r)\frac{\partial V_0}{\partial r} + rn_0\frac{\partial V_1(r)}{\partial r}\right] - \frac{m^2}{r^2}n_0V_1(r)\right\} = 0 \qquad (24)$$

and

$$i\omega n_1(r) - Zn_1(r) - D_{\perp,e}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial n_1(r)}{\partial r}\right] - \frac{1}{r^2}m^2 n_1(r)\right\}$$

$$+ \mu_{\perp,e}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rn_1(r)\frac{\partial V_0}{\partial r} + rn_0\frac{\partial V_1(r)}{\partial r}\right] - \frac{m^2}{r^2}n_0V_1(r)\right\}$$

$$- \frac{\omega_e\tau_e\mu_{\perp,e}}{r}\left[\frac{\partial n_0}{\partial r}imV_1(r) - imn_1\frac{\partial V_0}{\partial r}\right] = 0 \qquad (25)$$

For the type of instability considered, the fundamental mode m can be taken equal to unity. Solving equations (24) and (25) directly is difficult and, instead, an approximate technique of Kadomtsev and Nedospasov is used. The technique is to take specific functions of r that satisfy the boundary conditions and substitute them into the equations. The test functions used are

$$n_1(r) = n_1'J_1(\alpha_1 r) \qquad (26)$$

and

$$V_1(r) = V_1'J_1(\alpha_1 r) \qquad (27)$$

where $J_1(\alpha_1 r)$ is the first-order Bessel function of argument $\alpha_1 r$. The boundary conditions used to evaluate α_1 are that $n_1(a) = 0$ and $V_1(a) = 0$ so that

$$\alpha_1 = \frac{3.8}{a} \qquad (28)$$

where 3.8 is the first zero of the first-order Bessel function. Substituting equations (26) and (27) into equations (24) and (25) and noting that

$$n_0 = n_0' J_0(\alpha_0 r) \tag{29}$$

$$\frac{\partial^2 J_1(\alpha_1 r)}{\partial r^2} + \frac{1}{r}\frac{\partial J_1(\alpha_1 r)}{\partial r} - \frac{J_1(\alpha_1 r)}{r^2} = -\alpha_1^2 J_1(\alpha_1 r) \tag{30}$$

$$\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x) \tag{31}$$

yield

$$i\omega n_1'(\alpha_1 r) - Z n_1' J_1(\alpha_1 r) + D_{\perp,i}\alpha_1^2 n' J_1(\alpha_1 r)$$

$$- \mu_{\perp,i} n_1' J_0(\alpha_1 r)\frac{\partial V_0}{\partial r}\alpha_1 - \mu_{\perp,i} J_1(\alpha_1 r)\frac{\partial^2 V_0}{\partial r^2} n_1'$$

$$+ \mu_{\perp,i} n_0' V_1' \left\{ -\frac{1}{r}\frac{\partial}{\partial r}\left[r J_0(\alpha_0 r)\frac{\partial J_1(\alpha_1 r)}{\partial r}\right] + \frac{J_0(\alpha_0 r)}{r^2} J_1(\alpha_1 r)\right\} = 0 \tag{32}$$

and

$$i\omega n_1' J_1(\alpha_1 r) - Z n_1' J_1(\alpha_1 r) + D_{\perp,e}\alpha_1^2 n_1' J_1(\alpha_1 r)$$

$$+ \mu_{\perp,e} n_1' J_0(\alpha_1 r)\frac{\partial V_0}{\partial r}\alpha_1 + \mu_{\perp,e} J_1(\alpha_1 r)\frac{\partial^2 V_0}{\partial r^2} n_1'$$

$$- \mu_{\perp,e} n_0' V' \left\{ -\frac{1}{r}\frac{\partial}{\partial r}\left[r J_0(\alpha_0 r)\frac{\partial J_1(\alpha_1 r)}{\partial r}\right] + \frac{J_0(\alpha_0 r)}{r^2} J_1(\alpha_1 r)\right\}$$

$$+ \frac{i\omega_e \tau_e \mu_{\perp,e}}{r}\alpha_0 n_0' V_1' J_1(\alpha_0 r) J_1(\alpha_1 r) + \frac{i\omega_e \tau_e \mu_{\perp,e}}{r} n_1' J_1(\alpha_1 r)\frac{\partial V_0}{\partial r} = 0 \tag{33}$$

For the purpose of simplification, the following quantities are defined:

$$\frac{1}{C}\int_0^a \alpha_1^2 r J_0(\alpha_1 r) J_1(\alpha_1 r) \frac{\partial V_0}{\partial r} d(\alpha_1 r) \equiv L$$

$$\frac{1}{C}\int_0^a \alpha_1 r J_1^2(\alpha_1 r) \frac{\partial^2 V_0}{\partial r^2} d(\alpha_1 r) \equiv M$$

$$\frac{1}{C}\int_0^a -\left(\alpha_1 J_1(\alpha_1 r)\left\{\frac{\partial}{\partial r}\left[r J_0(\alpha_0 r)\frac{\partial J_1(\alpha_1 r)}{\partial r}\right] - \frac{J_0(\alpha_0 r)}{r} J_1(\alpha_1 r)\right\}\right) d(\alpha_1 r) \equiv N$$

$$\frac{1}{C}\int_0^a \alpha_1 J_1^2(\alpha_1 r) \frac{\partial V_0}{\partial r} d(\alpha_1 r) \equiv P$$

$$\frac{1}{C}\int_0^a \alpha_1 \alpha_0 J_1(\alpha_0 r) J_1^2(\alpha_1 r) d(\alpha_1 r) \equiv Q$$

where C is defined as

$$\int_0^a \alpha_1 r J_1^2(\alpha_1 r) d(\alpha_1 r) = C \tag{34}$$

Multiplying equations (32) and (33) by $\alpha_1 r J_1(\alpha_1 r)/C$ and integrating from 0 to a yield

$$\left[i\omega - Z + \alpha_1^2 D_{\perp,i} - \mu_{\perp,i}(L + M)\right]\frac{n_1'}{n_0'} + (\mu_{\perp,i} N) V_1' = 0 \tag{35}$$

and

$$\left[i\omega - Z + \alpha_1^2 D_{\perp,e} + \mu_{\perp,e}(L+M) + i\omega_e \tau_e \mu_{\perp,e} P\right]\frac{n_1'}{n_0'} + (-\mu_{\perp,e} N + i\omega_e \tau_e \mu_{\perp,e} Q) V_1' = 0 \tag{36}$$

The results are two homogeneous algebraic equations. The determinant of these equations is set equal to zero. This yields

$$-N\mu_{\perp,e}\left[i\omega - Z + \alpha_1^2 D_{\perp,i} - \mu_{\perp,i}(L + M)\right] + i\frac{Q}{B}\left[i\omega - Z + \alpha_1^2 D_{\perp,i} - \mu_{\perp,i}(L + M)\right]$$

$$- N\mu_{\perp,i}\left[i\omega - Z + \alpha_1^2 D_{\perp,e} + \mu_{\perp,e}(L + M) + i\frac{P}{B}\right] = 0 \qquad (37)$$

where $\omega_e \tau_e \mu_{\perp,e} = (1/B)$. Taking the real part of equation (37) yields

$$N(\mu_{\perp,e} + \mu_{\perp,i})(-\omega_I - Z) + \alpha_1^2 N(\mu_{\perp,e} D_{\perp,i} + \mu_{\perp,i} D_{\perp,e}) = 0 \qquad (38)$$

where $\omega = \omega_R + i\omega_I$ (the real and imaginary parts of ω) and ω_R/B is small with respect to terms such as $Z\mu_{\perp,i}$. When ω_I goes from negative to positive, the perturbed number density and electric potential grow without limit, and hence the condition that $\omega_I = 0$ is the critical condition. The term ω_I is thus set equal to zero, and the Einstein diffusion relation, $\mu = (eD/kT)$, is used. The equation

$$-Z\left(\frac{eD_{\perp,e}}{kT_e} + \frac{eD_{\perp,i}}{kT_i}\right) + \alpha_1^2 D_{\perp,e} D_{\perp,i}\left(\frac{e}{kT_e} + \frac{e}{kT_i}\right) = 0 \qquad (39)$$

is then obtained from equation (38). Noting that for this analysis the conditions

$$\frac{eD_{\perp,i}}{kT_i} >> \frac{eD_{\perp,e}}{kT_e}$$

and

$$\frac{e}{kT_i} >> \frac{e}{kT_e}$$

are satisfied gives the result that

$$Z = \alpha_1^2 D_{\perp,e} \qquad (40)$$

where $\alpha_1 = (3.8/a)$. The following substitutions can be made into equation (40):

$$Z = n_n \langle \sigma_i' v_e \rangle$$

$$D_{\perp,e} = \frac{1}{3} \nu_{e,n} r_L^2$$

$$\nu_{e,n} = n_n \langle \sigma_{e,n} v_e \rangle$$

$$r_L = \frac{m_e v_{\perp,e}}{eB}$$

where $\langle\,\rangle$ implies averaging over the electron distribution. The quantity $B_c (B_{critical})$ is then

$$B_c = \frac{3.8}{a} \frac{m_e v_{\perp,e}}{e} \sqrt{\frac{1}{3} \frac{\langle \sigma_{e,n} v_e \rangle}{\langle \sigma_i' v_e \rangle}} \tag{41}$$

Rotational frequency. - From the imaginary part of equation (37),

$$-i\omega_R N(\mu_{\perp,e} + \mu_{\perp,i}) - i \frac{Q}{B} \frac{N}{B} \omega_I + i \frac{Q}{B} \frac{N}{B} \left[\alpha_1^2 D_{\perp,i} - \mu_{\perp,i}(L+M) - Z\right] - i \frac{N}{B} \mu_{\perp,i} P = 0 \tag{42}$$

an expression for ω_R, the associated oscillatory frequency, can be obtained. Taking $\omega_I = 0$, noting that $\mu_{\perp,i} >> \mu_{\perp,e}$, and simplifying give

$$-\omega_R \mu_{\perp,i} + \frac{1}{B} \left[\alpha_1^2 D_{\perp,i} - \mu_{\perp,i}(L + M + P) - Z\right] - \frac{\mu_{\perp,i}}{B} P = 0 \tag{43}$$

which, when the Einstein diffusion relation is used, reduces to

$$\omega_R = \frac{1}{B} \left[\left(\frac{\alpha_1^2 k T_i}{e} - \frac{Z}{\mu_{\perp,i}}\right) \frac{Q}{N} - \frac{Q}{N} (L + M) - P\right] \tag{44}$$

From equations (12) and (13) the following expression can be obtained:

$$-\frac{D_{\perp,i} - D_{\perp,e}}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n_0}{\partial r}\right) - \frac{\mu_{\perp,e} + \mu_{\perp,i}}{r} \frac{\partial}{\partial r} \left(r n_0 \frac{\partial V_0}{\partial r}\right) = 0 \tag{45}$$

Integrating equation (45), setting the integration constant equal to zero, and noting that $D_{\perp, i} >> D_{\perp, e}$ and $\mu_{\perp, i} >> \mu_{\perp, e}$ give.

$$\frac{\partial V_0}{\partial r} = - \frac{kT_i}{e} \frac{1}{n_0} \frac{\partial n_0}{\partial r} \tag{46}$$

Equation (46) can be used to evaluate the expressions for L, M, and P. Values of $\frac{Q}{N}$ and $\frac{Q}{N}(L + M) + P$ appearing in equation (44) then become

$$\left. \begin{aligned} \frac{Q}{N} &= 0.9 \\ \frac{Q}{N}(L + M) + P &= 1.5\, \alpha_0^2 \frac{2kT_i}{e} \end{aligned} \right\} \tag{47}$$

Hence, from equations (47) and (40) and the fact that $D_{\perp, i} >> D_{\perp, e}$, equation (44) reduces to the following expression for ω_R:

$$\omega_R = \frac{kT_i}{eB}\left(0.9\, \alpha_1^2 - 1.5\, \alpha_0^2\right) \tag{48}$$

Derivation of Ion Production Energy Per Beam Ion

A classical derivation and a description of the anomalous aspects of the eV/ion for an ion thrustor are set forth in this section. The quantity eV/ion is defined as the power used in the plasma discharge divided by the beam current; hence,

$$\frac{eV}{ion} = \frac{I_e\, \Delta V_I}{I_B} \tag{49}$$

where I_e is the cathode to anode current, ΔV_I is the discharge voltage, and I_B is the ion beam current. The electron current density in the plasma source, with no instability assumed, can be written as

$$J_e = -D_{\perp,e} e \frac{\partial n}{\partial r} + n\mu_{\perp,e} e \frac{\partial V}{\partial r} \tag{50}$$

where the electrons are assumed to carry the current. Substituting the Einstein diffusion relation into equation (50) yields

$$J_e = n\mu_{\perp,e}\left(-\frac{kT_e}{e}\frac{1}{n}\frac{\partial n}{\partial r} + \frac{\partial V}{\partial r}\right) \tag{51}$$

Using equations (16) and (46) and neglecting small terms give

$$J_e = n_0'\mu_{\perp,e} J_1(\alpha_0 r) e \alpha_0 \left(\frac{kT_e}{e}\right) \tag{52}$$

where $J_1(\alpha_0 r)$ is a Bessel function of argument $\alpha_0 r$. Finally, the current I_e is equal to $2\pi a \ell J_e(a)$, where a is the anode radius and ℓ is the length of the plasma.

The beam current density J_B can be written as

$$J_B = env_i' \tag{53}$$

where v_i' is the velocity corresponding to one-half of the electron temperature. This assumption results from the Bohm criteria for stable plasma sheaths (ref. 11). The beam current is then equal to $\int_0^a 2\pi r J_B \, dr$. When equation (16) is used, the beam current becomes

$$I_B = en_0'v_i'ca^2 \tag{54}$$

where c is $2\pi \int_0^a \frac{r}{a} J_0(\alpha_0 r) d\left(\frac{r}{a}\right)$. Taking the classical expression for $\mu_{\perp,e}$ as $m_e/e\tau_e B^2$ gives the expression for the quantity eV/ion as

$$\frac{eV}{ion} = \frac{m_e J_1(\alpha_0 a)\frac{kT_e}{e} 4.8\,\pi\ell\,\Delta V_I}{e\tau_e v_i' c B^2 a^2} \tag{55}$$

where $\alpha_0 a$ is equal to 2.4 as shown in equation (17). It can be seen that for this classical derivation eV/ion behaves as $1/B^2$.

In the case of anomalous diffusion, the current density in the plasma source can be represented by two terms (ref. 12):

$$J_e = J_{class} + J_{anom} \tag{56}$$

where J_{class} is the classical contribution and J_{anom} is the anomalous contribution. The magnitude of J_{anom} with respect to J_{class} depends on the magnitude of the magnetic field. Equation (56) can be rewritten as

$$J_{class} = J_e\left[1 - \eta(B)\right] \tag{57}$$

As the magnetic field becomes large, the quantity J_{anom} becomes much more significant than J_{class}, so that $1 - \eta(B) \rightarrow 0$ (ref. 12). The current density J_e can then be written from equations (57) and (51) as

$$J_e = \frac{n\mu_{\perp,e}e\left(-\frac{kT_e}{e}\frac{1}{n}\frac{\partial n}{\partial r} + \frac{\partial V}{\partial r}\right)}{1 - \eta(B)} \tag{58}$$

Following the foregoing derivation for eV/ion yields

$$\frac{eV}{ion} = \frac{A}{\left[1 - \eta(B)\right]B^2} \tag{59}$$

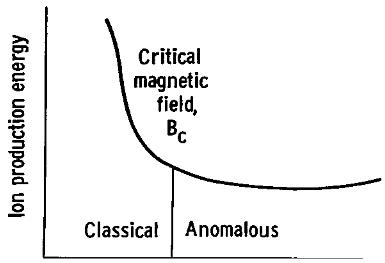
where A is a constant. From the condition that $1 - \eta(B) \rightarrow 0$ for large magnetic field, an expansion of $1 - \eta(B)$ can properly be expressed as

$$1 - \eta(B) = \frac{A_1}{B} + \frac{A_2}{B^2} + \frac{A_3}{B^3} + \ldots \tag{60}$$

Thus, for the proper values of the constants A_1, A_2, A_3, etc., the value of eV/ion can vary as $1/B$ (but not $1/B^2$), can be a constant, or can increase with increasing magnetic field. The results of using equation (56) and the condition $1 - \eta(B) \rightarrow 0$ for large magnetic field are to show, mathematically, that the experimental results can be consistent with this explanation of anomalous diffusion.

RESULTS AND DISCUSSION

A typical experimental variation of eV/ion with magnetic field is shown in sketch (b). The classical derivation given by equation (55) indicates that the variation of eV/ion with magnetic field should behave as $1/B^2$. Typical curves appear to exhibit this behavior (ref. 13) at low magnetic field strengths. In the anomalous description of the eV/ion curve, it was shown that the curve could level off or increase at large values of magnetic field. This behavior is indicated by the right side of sketch (b) and by curves given in reference 14. The transition from the shape of eV/ion curves can be used to obtain experimental values of the critical magnetic field B_c. Values of the critical magnetic field from equation (41) can thus be compared with these experimental values.



(b)

Evaluation of Ion Production Energy Per Beam Ion at Critical Magnetic Field

The magnitude of the quantity eV/ion at the critical magnetic field can be determined from equation (55) with B taken as B_c determined from equation (41). (This quantity is evaluated in the next section.) Selecting the following values of the variables of equation (55) corresponding to a 10-centimeter-diameter ion thrustor with mercury as a propellant

$$\ell = \frac{1}{10}\ \mathrm{m} \qquad B_c = 37\ \mathrm{G}\ (\text{from eq. (41)})$$

$$\Delta V = 50\ \mathrm{V} \qquad v_i' = 1.4\times10^{3}\ \mathrm{m/sec}$$

$$\frac{kT_e}{e} = 5\ \mathrm{V} \qquad a = 0.05\ \mathrm{m}$$

$$\tau_e = 6\times10^{-8}\ \mathrm{sec} \qquad ca^2 = 2\pi\times5.4\times10^{-4}\ \mathrm{m}^2$$

yields a value of eV/ion equal to 320. Typical values of eV/ion for these conditions range from 400 to 600. The agreement is fair, which is all that should be expected from this derivation.

The main objective of the derivation was to differentiate between the classical and anomalous behavior of the eV/ion against magnetic field curve. In achieving this, the behavior of the eV/ion with respect to changes in magnetic field was of uppermost importance. The derivation is not rigorous with respect to all the variables that might affect the eV/ion value of an ion thrustor. First, the blockage of the screens of the accelerator system is not taken into account since it is assumed in equation (53) that all the ions that reach downstream become part of the beam. Since the screens do block some of the ions from getting into the beam (physical blockage, about 50 percent with accelerating electric field effectively reducing this value), the calculated value of eV/ion is lower than the true value. Also, accelerating voltage and propellant utilization affect eV/ion. Neither of these factors is included in the foregoing derivation. It might be noted that thrustors having different anode radii have approximately the same value of eV/ion at B_c. The classical expression, equation (55), predicts this fact since $B_c^2 a^2$ is constant with respect to size, as can be seen from equation (41). This point will be given a more detailed explanation in the following section.

Evaluation of Critical Magnetic Field and Oscillatory Frequency

The equation for B_c, equation (41), can be rewritten as

$$\frac{a}{r_L} = 3.8 \sqrt{\frac{1}{3} \frac{\langle \sigma_{e,n} v_e \rangle}{\langle \sigma_i' v_e \rangle}} \tag{61}$$

For a given propellant and with the electron distribution assumed not to change drastically from one size ion thrustor to another, the right side of equation (61) is a constant. This condition implies that for the critical magnetic field B_c the ratio of the ion thrustor radius to the Larmour radius is a constant. This condition is consistent with the scaling law derived from experimental performance of the electron-bombardment ion thrustor (ref. 3).

Evaluating equation (41) for the critical magnetic field B_c requires a knowledge of the electron distribution in an ion thrustor. Such information is quite scanty. One approach that can be taken with respect to the electron energy distribution is to assume a Maxwellian distribution with an average energy of 5 electron volts, corresponding to the first excitation energy level of mercury. Equation (41) was evaluated for a 10-centimeter-diameter mercury ion thrustor with the result that B_c was 37 gauss. Typical values (refs. 13 and 14) of the critical magnetic field B_c for the 10-centimeter-diameter mercury ion thrustor, evaluated from the transition region of eV/ion against magnetic field curves, range from 30 to 40 gauss. The agreement is thus quite good. As a further check, an experimental electron energy distribution for an ion plasma source with argon propellant, operating at the typical discharge voltage of 50 volts, was evaluated in reference 6. Of the two distributions given in this reference, the one with an average energy of 4 electron volts was selected to simulate a typical operating mercury bombardment ion thrustor operating at a discharge voltage of 50 volts. The value of velocity used in the expression for the Larmour radius corresponded to 4 electron volts, the average energy of the distribution. The value of B_c for a 10-centimeter-diameter mercury ion thrustor was then calculated (by using this distribution) from equation (41), and it was found that B_c was 41 gauss. Finally, using the experimental electron distribution from reference 6 corresponding to an average energy of 10 electron volts (as a maximum that might reasonably be expected) and values of $\sigma_{e,n}$ and σ_i' for argon yielded a value of B_c from equation (41) such that B_c was 23 gauss. From reference 12 the value of B_c for an argon propellant ion thrustor, evaluated from the transition region of the eV/ion against magnetic field curve, is between 20 and 30 gauss.

To determine the critical magnetic field from equation (41), it is necessary to evaluate the behavior of the quantity $\sqrt{\langle\sigma_{e,n}v_e\rangle/3\langle\sigma_i'v_e\rangle}$. The fact that this quantity enters equation (41) as the square root diminishes the sensitivity of the critical magnetic field for changes of propellant as well as for corresponding electron distributions. Furthermore, because the values of σ_i' relative to $\sigma_{e,n}$ for likely ion thrustor propellants do not differ greatly from one atom to another, the effect of propellant change on the critical magnetic field is not great. However, the electron distribution is a more important factor in determining the quantity $\sqrt{\langle\sigma_{e,n}v_e\rangle/3\langle\sigma_i'v_e\rangle}$. The electron energy distributions used in the calculations of B_c had average energies of 4, 5, and 10 volts, corresponding to short to long tail types of electron energy distribution. The results were to obtain values of B_c at 41, 37, and 23 gauss. It is reasonable to conclude that for a few typical propellants and for a variety of possible electron distributions the value of B_c for a 10-centimeter-diameter ion thrustor should fall between 23 and 41 gauss. This is consistent with the experimental values obtained from reference 14.

The value of the oscillatory frequency associated with the anomalous diffusion can be calculated from equation (48). Considering a 10-centimeter-diameter engine with the critical magnetic field equal to 40 gauss and kT_i/e equal to 1/20 electron volt, ω_R is found to be

$$\omega_R = 2.25\times10^4 \text{ sec}^{-1}$$

This is a frequency of 3500 cps. In reference 10 a plasma source with a magnetic field and a perpendicular electric field was described as operating in the region of enhanced conduction. It was also found in this reference that a perturbation of number density existed, and associated with this perturbation was a frequency of approximately 16 000 cps. Inasmuch as equation (48) is for the initiation of anomalous diffusion, rather than for an advanced stage of diffusion, a difference of less than an order of magnitude is felt to be substantial agreement.

CONCLUSIONS

Expressions were derived for the onset of anomalous diffusion in an axially symmetric plasma with an axial magnetic field and a radially outward electric field. This plasma configuration corresponds to the ionization chamber of an electron-bombardment ion thrustor. A theory of plasma stability in a magnetic field first introduced by Kadomtsev and Nedospasov was modified and used in this derivation. As a result of this study, expressions for the critical magnetic field for the onset of anomalous diffusion and for the oscillatory frequency associated with the anomalous diffusion were obtained. The expression derived for the critical magnetic field varied inversely with anode radius in the same manner as an experimentally derived scaling law for the ion thrustor. Values of the critical magnetic field obtained from the derived expression compared well with experimental values obtained from curves of ion production energy per beam ion (eV/ion) against magnetic field. The oscillatory frequency associated with the anomalous diffusion fell within less than an order of magnitude of the frequency of oscillations found experimentally in a plasma source of similar configuration. Expressions for the variation of eV/ion with magnetic field were presented, covering both classical and anomalous regimes. In the classical region of this curve the value of eV/ion of the ion thrustor fell as $1/B^2$, where B is a magnetic field. It would be expected from these classical considerations that an increased magnetic field would contribute to exceptionally low values of eV/ion. It is postulated that this expectation is not realized because of the onset of anomalous diffusion. The effect of this enhanced diffusion is to eliminate completely the effect of the magnetic field in lowering the eV/ion of the ion thrustor beyond the critical value of the magnetic field. In the anomalous region of the eV/ion against magnetic field curve, the variation is either constant or increasing with increasing magnetic field. This behavior corresponds to the increased diffusion of electrons in the anomalous diffusion regime. According to the applied theory, the onset of anomalous diffusion implies a perturbation of number density and potential with a θ-direction dependence. As a result, a Hall effect causes increased conduction in the r-direction and, hence, an increase of current beyond the classical value calculated for the ion thrustor.

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APPENDIX A

SYMBOLS

A	constant
a	anode radius
B	magnetic field
C	defined by eq. (34)
D	diffusion coefficient
E	electric field
e	electron charge
I	current
J	current density
J_0	zero-order Bessel function
J_1	first-order Bessel function
k	Boltzmann constant
ℓ	anode length
m	fundamental mode
m_e	mass of electron
n	number density
r	r-direction in cylindrical coordinates
r_L	electron Larmour radius
T	temperature
t	time
V	potential
ΔV_I	discharge potential
v	velocity
x	argument of Bessel function
Z	ion production coefficient
z	z-direction in cylindrical coordinates
α_0	2.4/a
α_1	3.8/a
$\eta(B)$	J_{anom}/J_E
θ	θ-direction in cylindrical coordinates
μ	mobility coefficient
ν	collision frequency
σ	collisional cross section
σ_i'	ionization cross section
τ	mean free collision time
ω	oscillatory angular frequency
ω_e	electron cyclotron frequency

Subscripts:

B	beam
c	critical
e	electron
I	imaginary
i	ion
n	neutral
R	real
r	radial direction
z	z-direction
θ	θ-direction

0 unperturbed

1 perturbed

$\perp$ perpendicular

Superscript:

$\rightarrow$ vector

APPENDIX B

EFFECTS OF CURRENTS IN z-DIRECTION

If currents in the z-direction are taken into account, equations (4) and (7) become

$$n_e \nu_{e,z} = -D_e \frac{\partial n_e}{\partial z} + \mu_e \frac{\partial v}{\partial z} n_e \tag{B1}$$

$$n_i \nu_{i,z} = -D_i \frac{\partial n_i}{\partial z} + \mu_i \frac{\partial v}{\partial z} n_i \tag{B2}$$

Since there is no applied electric field in the z-direction, and since it is assumed that the analysis is restricted to the central portion of the discharge far from both end walls, the potential gradient in equations (B1) and (B2) can be neglected. As a result of equations (B1) and (B2), equations (10) and (11) have the additional terms $-D_e\ \partial^2 n/\partial z^2$ and $-D_i\ \partial^2 n/\partial z^2$, respectively, on their right sides. As in reference 9, where a similar circumstance is met, n(z) is assumed to vary as $\cos(\pi/L_z)z$, where L_z is some characteristic length and the unperturbed number density is then given as

$$n_0 = n(r)n(z) \tag{B3}$$

Equation (14) then has the additional term $\left[-D_e\mu_{\perp,i}(\pi/L_z)^2 - D_i\mu_{\perp,e}(\pi/L_z)^2\right]n_0$ on the right side. The constant A in equation (15) is now changed, but the solution of equation (15)

$$n_0(r) = n_0' J_0(\alpha_0 r) \tag{B4}$$

still remains intact.

It is necessary to consider the effect of currents in the z-direction on the stability calculation. Equations (20) and (21) have the following additional terms on the right side:

$$D_i \frac{\partial}{\partial z}\left(\frac{\partial n_1}{\partial z}\right) - \mu_i \frac{\partial}{\partial z} n_0\left(\frac{\partial V_1}{\partial z} + n_1 \frac{\partial V_0}{\partial z}\right)$$

and

$$D_e \frac{\partial}{\partial z}\left(\frac{\partial n_1}{\partial z}\right) + \mu_e \frac{\partial}{\partial z} n_0\left(\frac{\partial V_1}{\partial z} + n_1 \frac{\partial V_0}{\partial z}\right)$$

respectively. If, as before, the unperturbed potential gradient in the z-direction $\partial V_0/\partial z$ is neglected for reasons given previously and if the perturbed quantities n_1 and V_1 are as given in this report, that is, as functions of r and θ only, then all the additional terms in this appendix drop out. Taking the perturbed quantities n_1 and V_1 as functions of only r and θ is consistent with statements made in reference 9, where a similar situation is encountered. Thus, since the additional terms of equations (20) and (21) are dropped out, the analysis continues unchanged.

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