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A CLASS OF FUNCTIONAL EQUATIONS OF NEUTRAL TYPE

by

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A

I. INTRODUCTION

In the study of weakly nonlinear systems, the most useful elements from the theory of linear non-homogeneous ordinary differential equations with autonomous homogeneous part are 1) the variation of constants formula, 2) the decomposition of Euclidean space into the direct sum of subspaces which are invariant with respect to the solutions of the homogeneous system (the Jordan canonical form) and 3) sharp exponential bounds on the growth of solutions on these invariant subspaces. Once these facts are well understood, many problems in the theory of stability, asymptotic behavior and nonlinear oscillations can be discussed. For delay differential equations of retarded type these three concepts have been developed and applied to problems of the above type (see, for example, [1],[2],[3,4],[5,6],[7]).

For delay differential equations of neutral type, the theory is not so well developed even though some results are contained in the book of Bellman and Cooke [1]. In equations of neutral type, the first difficulty arises because the derivative of a solution occurs with a retardation. This leaves much freedom in the choice of the topology on the solution space as well as on the space of initial conditions. The topology must be chosen in such a way as to obtain solutions which are at least continuous with respect to the initial data. That such a choice is not obvious may be easily seen by consulting the papers of Driver [8,9] where a general existence and uniqueness theorm is given for a rather broad class of neutral equations.

Our approach in this paper is to investigate a class of functional integral equations in the space of continuous functions. This class includes certain types of equations of neutral type and does include some equations which arise in the applications. For this class of equations, we obtain precise analogues of the above stated properties of ordinary differential equations. Furthermore, the decomposition of our space into invariant subspaces is given in a way that is amenable to computations. As specific applications of the theory, we give a stability theorem and extend the method of averaging to these systems.

The symbol [ ] indicates references in the bibliography, Roman numerals refer to sections and Arabic to formulas.

Let R^n be a real or complex n-dimensional linear space of column vectors with norm $|\cdot|$ and let $C([a,b],R^n)$ denote the Banach space of continuous functions from $[a,b]$ into R^n with norm $\|\cdot\|_{[a,b]}$ given by $\|\varphi\|_{[a,b]} = \sup\{|\varphi(\theta)| : \theta \in [a,b]\}$. Let r be a fixed non-negative number and let $C = C([-r,0],R^n)$ and $\|\cdot\| = \|\cdot\|_{[-r,0]}$.

Let $\mathcal{L}_p([a,b],R^n)$, $1 \leq p < \infty$, be the set of Lebesque integral functions from $[a,b]$ into R^n with the norm of any φ in $\mathcal{L}_p([a,b],R^n)$ defined by $[\int_a^b |\varphi(s)|^p ds]^{1/p}$. Also let $\mathcal{L}_\infty([a,b],R^n)$ denote the set of essentially bounded measurable functions from $[a,b]$ into R^n, with the norm of any φ in $\mathcal{L}_\infty([a,b],R^n)$ given by $\text{ess.sup}\,|\varphi(\theta)|$. We shall also use the space $\mathcal{L}_\infty([a,b],R^{n^2})$ of essentially bounded measurable functions into the space of $n \times n$ matrices with the norm defined in the obvious way.

Suppose τ is a given real number. We allow $\tau = -\infty$ and in this case the interval $[\tau,\infty)$ denotes the interval $(-\infty,\infty)$. Let g and f be continuous functions from $[\tau,\infty) \times C$ into R^n such that for each $t \in [\tau,\infty)$ the functions $f(t,\cdot)$ and $g(t,\cdot)$ are linear operators and there exist positive continuous functions K and L defined for all $t \geq \tau$ such that

$$(1) \qquad |g(t,\varphi)| \leq K(t)\|\varphi\| \quad \text{and} \quad |f(t,\varphi)| \leq L(t)\|\varphi\|$$

for all $\varphi \in C$ and $t \in [\tau,\infty)$.

By the Riesz representation theorem there exists $n \times n$ matrix

valued functions μ and η defined on $[\tau,\infty) \times [-r,0]$ such that

$$(2)\qquad \begin{aligned} g(t,\varphi) &= \int_{-r}^{0} [d_\theta \mu(t,\theta)]\varphi(\theta) \\ f(t,\varphi) &= \int_{-r}^{0} [d_\theta \eta(t,\theta)]\varphi(\theta) \end{aligned}$$

for all $\varphi \in C$. Moreover for each fixed t the functions $\mu(t,\cdot)$ and $\eta(t,\cdot)$ are of bounded variation in $[-r,0]$.

For any $x \in C([-r,A),R^n)$, $A > 0$, define x_t, $0 \leqq t < A$, as the element of C given by $x_t(\theta) = x(t + \theta)$; that is, x_t is the restriction of x to the interval $[t-r,t]$ shifted to $[-r,0]$.

For any $\varphi \in C$ and any σ in $[\tau,\infty)$ define $\gamma(\sigma,\varphi) = \varphi(0)-g(\sigma,\varphi)$. For any h, $h \in \mathcal{L}_1([\sigma,\nu),R^n)$ for every ν in $[\sigma,\infty)$, consider the following functional integral equation

$$(3)\qquad \begin{aligned} &\text{a)}\quad x_\sigma = \varphi \\ &\text{b)}\quad x(t) = \gamma(\sigma,\varphi)+g(t,x_t)+\int_\sigma^t f(s,x_s)ds+\int_\sigma^t h(s)ds\ ,\ t \in [\sigma,\infty). \end{aligned}$$

By a solution of (3) we shall mean an element of $C([\sigma-r,A),R^n)$, $\sigma < A \leqq \infty$, that satisfies the relations in (3). We shall refer to φ as the initial function and to σ as the initial time.

If f and g are independent of t then (3) will be called autonomous and otherwise non-autonomous. If $h = 0$ the equation (3) will be called homogeneous and otherwise non-homogeneous.

If $g \equiv 0$ then (3) is equivalent to the functional differential equation of retarded type

$$\dot{x}(t) = f(t,x_t) + h(t)$$

with initial function at $t = \sigma$ given by φ.

If $f \equiv 0$ and $h \equiv 0$ then equation (3) is a functional difference equation of retarded type, and in particular, includes difference equations. For both f and g not identically zero, equation (3) corresponds to a functional differential equation of neutral type. Indeed, formal differentiation of the equation yields

$$\dot{x}(t) = g(t,\dot{x}_t) + \hat{f}(t,x_t) + h(t) \ , \tag{4}$$

where $\hat{f} = \partial g/\partial t + f$ and $\dot{x}_t$ is defined by $\dot{x}_t(\theta) = \dot{x}(t+\theta)$, $-r \leqq \theta \leqq 0$. Also, if one begins with (4) and defines a solution with initial function φ at σ to be a continuous function satisfying (4) almost everywhere, then an integration yields (3) with $\gamma(\sigma,\varphi) = \varphi(0)-g(\sigma,\varphi)$.

Notice that all differential difference equations of neutral type with variable coefficients and constant retardations can be written in the form (3) provided the coefficients of the terms involving the derivatives have an integrable first derivative.

Also, equation (3) contains as a special case some differential difference equations of neutral type with variable lags provided that the lags are bounded and satisfy some other reasonable conditions. For

example, the equation $\dot{x}(t) = \dot{x}(\beta(t)) + \dot{x}(\gamma(t))$ can be written in the form (3) if $\beta, \dot{\beta} > 0$, γ are continuous, $\ddot{\beta}$ is integrable and there is a constant $r \geqq 0$ such that $t-r \leqq \beta(t) \leqq t$, $t-r \leqq \gamma(t) \leqq t$.

These last remarks are precisely the reason for considering equation (3). It one attempts to discuss the equation (4) directly, then the first problem encountered are precise definitions of a solution and the topology to be used on the space in which the solutions lie. To discuss (4) the topology must include information about the derivatives of functions whereas (3) can be discussed in the simpler space C.

Equation (3) would also include equations of advanced type unless some further restriction is made on the function g. This is due to the fact that the measure $\mu(t,\theta)$ in (2) may have a jump at $\theta = 0$ equal to the identity for some values of t. To avoid this difficulty, we shall assume that the measure μ is uniformly nonatomic at zero. More precisely, we assume that there exists a nonnegative, continuous, nondecreasing function δ defined on $[0,\varepsilon_0]$ for some $0 < \varepsilon_0 \leqq r$ such that

$$(5) \qquad \delta(0) = 0 \quad \text{and} \quad \left| \int_{-s}^{0} [d\mu(t,\theta)]\varphi(\theta) \right| \leqq \delta(s)\|\varphi\|_{[-s,0]}$$

for all $\varphi \in C$, $t \in [\tau,\infty)$ and all $s \in [0,\varepsilon_0]$. In some cases it will be necessary to further restrict μ.

Observe that the solution $x(t,\sigma,\varphi)$ of (3) with initial function φ at σ satisfies

$$x(t,\sigma,\varphi) = x(t,s,x_s(\cdot,\sigma,\varphi)) \qquad t \geqq s \geqq \sigma \tag{6}$$

provided all the above solutions exist and are uniquely defined by initial values.

Also, at times it will be necessary to consider solutions of (3) that are matrix valued. In this case we define the action of f and g by (2) when φ is a continuous $n \times n$ matrix valued function of the scalar θ, $\theta \in [-r,0]$.

II. THE GENERAL LINEAR EQUATION.

This section deals with the general non-autonomous equation I(3). Existence and uniqueness of solutions and variation of constants formula are discussed.

THEOREM 1. *For any given* $\varphi \in C$, $\sigma \in [\tau,\infty)$ *and* h, *where* $h \in \mathcal{L}_1([\sigma,\nu),R^n)$ *for every* ν *in* $[\sigma,\infty)$, *there exists a unique function* $x(\sigma,\varphi)$ *defined and continuous on* $[\sigma-r,\infty)$ *that satisfies* I(3).

PROOF. Suppose $K(t)$, $L(t)$ are defined by I(1) and $\delta(s)$, s in $[0,\varepsilon_0]$ is defined by I(5). Let $\beta > \sigma$ be any fixed positive number and let K_β and L_β be the supremum on $[\sigma,\beta]$ of $K(t)$ and $L(t)$, respectively. Choose $A > 0$ so that $\delta(A)+L_\beta A < 1$ and $\sigma+A < \beta$, $A < \varepsilon_0$. Let $\Gamma = \{y \in C([\sigma-r,\sigma+A],R^n) : y_\sigma = \varphi\}$, and for any y in Γ, define

$$(Iy)(t) = \begin{cases} \varphi(t-\sigma) & \text{for } \sigma-r \leq t \leq \sigma \\ \gamma(\sigma,\varphi)+g(t,y_t)+\int_\sigma^t f(s,y_s)ds+\int_\sigma^t h(s)ds, & \sigma < t \leq \sigma+A \end{cases}$$

Clearly $I\Gamma \subset \Gamma$. For any y and z in Γ

$$|Iy(t)-Iz(t)| \leq |g(t,y_t-z_t)| + \int_\sigma^t |f(s,y_s-z_s)|ds$$

$$\leq [\delta(A) + L_\beta A]\|y-z\|_{[\sigma-r,\sigma+A]}$$

and so I is contracting in Γ. Thus, I has a unique fixed point in Γ, which implies $I(3)$ has a unique continuous solution defined on $[\sigma-r,\sigma+A]$. But A is a constant independent of the norm of φ and the solution can be extended to $[\sigma-r,\beta]$ by use of the above and relation $I(6)$. Since β was arbitrary the theorem is proved.

If the operators f and g do not increase too fast with t we would expect that the solutions of $I(3)$ are exponentially bounded. Indeed one has

<u>LEMMA 1.</u> <u>Suppose</u> $|g(t,\varphi)| \leq K\|\varphi\|$ <u>and</u> $|f(t,\varphi)| \leq L\|\varphi\|$ <u>for all</u> $\varphi \in C$ <u>and all</u> $t \in [\tau,\infty)$ <u>where</u> K <u>and</u> L <u>are constants. Then there exist constants</u> a, b <u>and</u> c <u>such that for any</u> σ <u>in</u> $[\tau,\infty)$

$$\|x_t(\sigma,\varphi)\| \leq \{a\|\varphi\| + b\int_\sigma^t |h(s)|\,ds\}e^{c(t-\sigma)}, \quad t \geq \sigma.$$

PROOF. In this proof, we let x_t designate $x_t(\sigma,\varphi)$. Let M be such that $K+M > 1$, $|\gamma(t,\varphi)| \leq M\|\varphi\|$ for all $t \in [\tau,\infty)$, $\varphi \in C$, and let A be a positive constant such that $1-\delta(A) > 0$. Define $b = (1-\delta(A))^{-1}$ and $a = (K+M)(1-\delta(A))^{-1}$. For any $t \in [\sigma,\sigma+A]$ one has $|g(t,x_t)| \leq K\|\varphi\| + \delta(A)\|x_t\|$ and so

$$|x(t)| \leq (M+K)\|\varphi\| + \delta(A)\|x_t\| + L\int_\sigma^t \|x_s\|ds + \int_\sigma^t |h(s)ds, \quad t \geq \sigma.$$

Since $K+M > 1$ and $x_\sigma = \varphi$, the right-hand side is an upper bound for

$\|x_t\|$. Solving the resulting inequality for $\|x_t\|$ and applying Gronwall's inequality, we obtain

$$\|x_t\| \leqq \{a\|\varphi\| + b\int_\sigma^t |h(s)|\,ds\}e^{bL(t-\sigma)} \quad \text{for} \quad t \in [\sigma,\sigma+A].$$

We shall now show by an induction argument that the above inequality is valid for all $t \geqq \sigma$ provided bL is replaced by a larger constant. Let c be so large that $ae^{(bL-c)A} \leqq 1$ and $c > bL$. Assume that

$$\|x_t\| \leqq \{a\|\varphi\|+b\int_\sigma^t |h(s)|\,ds\}e^{c(t-\sigma)} \quad \text{for} \quad t \in [\sigma,\sigma+kA].$$

From the above, this assumption is true if $k = 1$. If $t \in [\sigma+kA,\sigma+(k+1)A]$, then the above estimate yields

$$\|x_t\| \leqq \{a\|x_{t-A}\| + b\int_{t-A}^t |h(s)|\,ds\}e^{bLA}$$

and by the induction hypothesis

$$\|x_t\| \leqq \{a[a\|\varphi\| + b\int_\sigma^{t-A} |h(s)|\,ds]e^{c(t-\sigma-A)} + \int_{t-A}^t |h(s)|\,ds\}e^{bLA}$$

$$\leqq \{a\|\varphi\| + b\int_\sigma^t |h(s)|\,ds\}e^{c(t-\sigma)} .$$

This completes the proof of the lemma.

COROLLARY 1. *Let* $x(\cdot,\sigma,\varphi,h)$ *be the unique solution of* I(3) *with initial function* φ *at* σ *and forcing function* $h \in \mathcal{L}_1([\sigma,t_1),R^n)$. *For fixed* t_1 *and* σ, $x(t_1,\sigma,\cdot,\cdot)$ *is a continuous function from* $C \times \mathcal{L}_1([\sigma,t_1),R^n)$ *into* R^n.

PROOF. The corollary is obvious from lemma 1 if f and g admit a constant bound as required by the lemma. Since changing f and g for $t \geq t_1$ does not effect the value of the solution in $[\sigma,t_1]$ one can define new f' and g' to be identical to f and g for $\sigma \leq t \leq t_1$ and to equal $f(t_1,\cdot)$ and $g(t_1,\cdot)$ for $t \geq t_1$. Applying the above theorem to equation I(3) with f and g replaced by f' and g' yields the result.

The next problem is to obtain a variation of constants formula for the solutions of I(3). This is accomplished by observing that the solutions of I(3) are linear operators on the forcing function h. In particular we have:

THEOREM 2. (*Variation of Constants Formula*). *If* $x(\sigma,\varphi,h)$ *is the solution of* I(3) *with forcing function* h, *where* $h \in \mathcal{L}_1([\sigma,\nu),R^n)$, *for all* $\nu \geq \sigma$, *and initial value* φ *in* C *at* σ, *then*

$$(1) \qquad x(\sigma,\varphi,h)(t) = x(\sigma,\varphi,0)(t) + \int_\sigma^t U(t,s)h(s)ds, \qquad t \geq \sigma,$$

where $U(t,s)$ *is defined for* $\tau \leq s \leq t+r$, $U(t,\cdot) \in \mathcal{L}_\infty([\sigma,t],R^{n^2})$ *for each* t, $U(t,s) = +\partial W(t,s)/\partial s$ *a.e.*, *where* $W(t,s)$ *is the unique*

solution of

(2a) $$W_s(\cdot,s) = 0$$

(2b) $$W(t,s) = \int_{-r}^{0} \{d_\theta \mu(t,\theta)\} W(t+\theta,s) + \int_s^t \int_{-r}^{0} \{d_\theta \mu(\zeta,\theta)\} W(\theta+\zeta,s)\, d\zeta - (t-s)I \quad \text{for} \quad \tau \leqq s \leqq t.$$

PROOF. Let $h \in \mathcal{L}_1([\sigma,t],R^n)$ and let $u(\cdot,\sigma,h)$ be the solution of I(3) that satisfies $u_\sigma = 0$. For fixed t and σ it follows from Corollary 1 that $u(t,\sigma,\cdot)$ is a continuous linear operator from $\mathcal{L}_1([\sigma,t],R^n)$ into R^n. So there exists (see [10]) an $n \times n$ matrix valued function $U^*(t,\sigma,\cdot) \in \mathcal{L}_\infty([\sigma,t],R^{n^2})$, $t \geqq \sigma$, such that

$$u(t,\sigma,h) = \int_\sigma^t U^*(t,\sigma,\theta)h(\theta)d\theta \ .$$

Let α be in $[\sigma,t]$ and let k be any element of $\mathcal{L}_1([\sigma,t],R^n)$ that satisfies $k(\theta) = 0$ for $\theta \in [\sigma,\alpha]$. Then $u(t,\sigma,k) = u(t,\alpha,k)$, $t \geqq \alpha$, and $U^*(t,\sigma,\theta) = U^*(t,\alpha,\theta)$ a.e. Since α is an arbitrary element of $[\sigma,t]$, it follows that U^* is independent of σ. Define $U(t,\theta) = U^*(t,\sigma,\theta)$, $t \in [\tau,\infty)$, $\theta \in [\tau,t]$, $U(t,\theta) = 0$ for $t \leqq \theta \leqq t+r$. For any s in $[\tau,\infty)$, let $W(t,s) = -\int_s^t U(t,\theta)d\theta$ for $t \geqq s$ and $W(t,s) = 0$ for $t \in [s-r,s]$. Clearly W satisfies (2a), (2b) and U is given as stated in the theorem.

COROLLARY 2. *If* f *and* g *are independent of* t *then*

$$x(\sigma,\varphi,h)(t) = x(\sigma,\varphi,0)(t) + \int_{\sigma}^{t} U(t-s)h(s)ds \tag{3}$$

where U *is defined on* $[-r,\infty)$, $U \in \mathcal{L}_{\infty}([-r,t),R^{n^2})$, *for each* t *in* $[-r,\infty)$, $U(t) = -dW(t)/dt$ *a.e.* *and* W *satisfies*

$$W_o = 0 \tag{4a}$$

$$W(t) = g(W_t) + \int_{o}^{t} f(W_s)ds + tI, \quad t \in [0,\infty). \tag{4b}$$

III. THE AUTONOMOUS, HOMOGENEOUS EQUATION.

In this section we study equation I(3) when f and g are independent of t and $h \equiv 0$. Since, for the autonomous case it is no restriction to choose the initial time $\sigma = 0$, we consider

$$(1) \qquad \begin{aligned} &\text{a)} \quad x_0 = \varphi \\ &\text{b)} \quad x(t) = \gamma(\varphi) + g(x_t) + \int_0^t f(x_s)ds \quad \text{for} \quad t \geqq 0 \end{aligned}$$

with $\varphi \in C$, $\gamma(\varphi) = \varphi(0) - g(\varphi)$ and

$$(2) \qquad g(\varphi) = \int_{-r}^{0} \{d\mu(\theta)\}\varphi(\theta), \quad f(\varphi) = \int_{-r}^{0} \{d\eta(\theta)\}\varphi(\theta),$$

where μ and η are functions of bounded variation in $[-r,0]$.

The aim of this section is to study the behavior of the solutions in C. By some general results from functional analysis we are able to introduce coordinates in C in such a way that the behavior of the solution of 1) on certain finite dimensional subspaces are determined by ordinary differential equations. An explicit characterization of these subspaces is given that is amenable to computations.

If φ is any given function in C and $x(\varphi)$ is the unique solution of (1) with initial function φ at zero then we define a mapping $T(t)\colon C \to C$, for each fixed t, by the relation

(3) $$T(t)\varphi = x_t(\varphi) .$$

The following lemma is an immediate consequence of the discussion in section II.

LEMMA 1. The family $\{T(t)\}_{t \in [0,\infty]}$ forms a strongly continuous, semi-group of bounded linear operators from C into itself for all $t \geqq 0$.

Since $T(t)$ is strongly continuous we may define the infinitesimal operator A of $T(t)$ (see Hille and Phillips [11], p.306) as

(4) $$A\varphi = \lim_{t \to 0} \frac{1}{t} [T(t)\varphi - \varphi]$$

whenever this limit exists in the norm topology of C. The infinitesimal generator of $T(t)$ is the smallest closed extension of A. By the strong continuity of $T(t)$ on $[0,\infty)$ it follows that the infinitesimal generator and infinitesimal operator are the same (see corollary, p. 344 and Theorem 10.61, p. 322 of Hille and Phillips [11]). From the above remarks and Theorem 10.3.1 of Hille and Phillips, page 307, the domain $\mathcal{D}(A)$ of A, is dense in C and the range $\mathcal{R}(A)$ of A is C. These remarks allow us to compute A directly from (4). In fact, we have

LEMMA 2. The infinitesimal generator A of the semi-group $\{T(t)\}_{t\in[0,\infty)}$

<u>and its domain</u> $\mathcal{D}(A)$ <u>are given by</u>

(5) a) $A\varphi(\theta) = \frac{d}{d\theta}\varphi(\theta) = \dot{\varphi}(\theta)$

b) $\mathcal{D}(A) = \{\varphi \in C : \dot{\varphi} \in C,\ \dot{\varphi}(0) = g(\dot{\varphi}) + f(\varphi)\}$.

<u>Moreover,</u> $\mathcal{D}(A)$ <u>is dense in</u> C <u>and, for</u> $\varphi \in \mathcal{D}(A)$,

(6) $$\frac{d}{dt}T(t)\varphi = T(t)A\varphi = AT(t)\varphi .$$

PROOF. Suppose φ is in $\mathcal{D}(A)$. Since $T(t)\varphi(\theta) = \varphi(t+\theta)$ when $-r \leq t+\theta \leq 0$, it follows directly from the definition (4) that $(A\varphi)(\theta) = \dot{\varphi}(\theta^+)$ for $\theta \in [-r,0)$, where $\dot{\varphi}(\theta^+)$ is the right-hand derivative of φ at θ.

Since $\lim_{t \to 0^+}[T(t)\varphi-\varphi]/t$ exists for φ in $\mathcal{D}(A)$, there are constants α and β such that $\|T(t)\varphi-\varphi\| \leq \beta t$ for $t \in [0,\alpha)$. Thus $|x(t+\theta)-\varphi(\theta)| \leq \beta t$ for $t \in [0,\alpha)$ and $\theta \in [-r,0]$. This implies

$$\frac{1}{t}\int_{-r}^{0} d\mu(\theta)\{x(t+\theta)-\varphi(\theta)\} = \int_{-r}^{-t} d\mu(\theta)\left\{\frac{\varphi(t+\theta)-\varphi(\theta)}{t}\right\} + \int_{-t}^{0} d\mu(\theta)\left\{\frac{x(t+\theta)-\varphi(\theta)}{t}\right\}$$

tends to $\int_{-r}^{0} d\mu(\theta)\dot{\varphi}(\theta^+)$ as $t \to 0^+$ since

$$\left|\int_{-t}^{0} d\mu(\theta)\left\{\frac{x(t+\theta)-\varphi(\theta)}{t}\right\}\right| \leq \delta(t)\beta \to 0, \text{ as } t \to 0^+ .$$

From 1(b), it follows immediately that

$$\mathcal{A}\varphi(0) = g(\dot{\varphi}^{+}) + f(\varphi) \ .$$

Since $\mathcal{A}\varphi$ must be in C it follows that $d\varphi(\theta)/d\theta$ exists and is continuous. The rest of the lemma follows by Theorem 10.3.3 of Hille and Phillips [11], page 308.

We shall now proceed to analyze the spectrum of $\mathcal{A}$. Let B be any linear operator of a Banach space $\mathcal{B}$ into itself. The <u>resolvent set</u> $\rho(B)$ is defined as the set of λ in the complex plane for which $(\lambda I-B)$ has a bounded inverse in all of $\mathcal{B}$. The complement of $\rho(B)$ in the λ-plane is called the <u>spectrum of B</u> and is denoted by $\sigma(B)$. The <u>point spectrum</u>, $P\sigma(B)$, consists of those λ in $\sigma(B)$ for which $(\lambda I-B)$ does not have an inverse. The points of $P\sigma(B)$ are called <u>eigenvalues</u> of B and the nonzero $\varphi \in \mathcal{B}$ such that $(\lambda I-B)\varphi = 0$ are called <u>eigenvectors</u> of $\mathcal{A}$. The <u>null space</u> $\mathfrak{N}(B)$ of B is the set of all $\varphi \in \mathcal{B}$ for which $B\varphi = 0$. For any given $\lambda \in \sigma(B)$ the <u>generalized eigenspace of λ</u> is defined to be the smallest closed subspace of $\mathcal{B}$ containing the subspaces $\mathfrak{N}(\lambda I-B)^k$, $k = 1,2,\ldots,$ and will be denoted by $\mathfrak{M}_\lambda(B)$.

One of our objects is to determine the nature of $\sigma(A)$ and $\sigma(T(t))$. We would hope to discuss most of the properties of $T(t)$ by using only properties of the known operator A.

<u>THEOREM 1. Let A be defined as in Lemma 2, then $\sigma(A) = P\sigma(A)$ and $\lambda \in \sigma(A)$ if and only if λ satisfies the characteristic equation</u>

$$(7) \qquad \det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda I - \int_{-r}^{0} \lambda e^{\lambda\theta} d\mu(\theta) - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta) .$$

<u>The roots of</u> (7) <u>have real parts bounded above and for any</u> $\lambda \in \sigma(A)$, <u>the generalized eigenspace</u> $\mathfrak{M}_\lambda(A)$ <u>is finite dimensional. Finally if</u> λ <u>is a root of</u> (7) <u>of multiplicity</u> k, <u>then</u> $\mathfrak{M}_\lambda(A) = \mathfrak{N}(\lambda I - A)^k$ <u>and</u> $C = \mathfrak{N}(A-\lambda I)^k \oplus \mathcal{R}(A-\lambda I)^k$, <u>where</u> $\oplus$ <u>is the direct sum. Moreover</u> $T(t)$ <u>is completely reduced by the two linear manifolds</u> $\mathfrak{M}_\lambda(A)$ <u>and</u> $\mathcal{R}_\lambda(A)$; <u>that is,</u> $T(t)\mathfrak{M}_\lambda(A) \subset \mathfrak{M}_\lambda(A)$, $T(t)\mathcal{R}_\lambda(A) \subset \mathcal{R}_\lambda(A)$ <u>for all</u> $t \geq 0$.

PROOF. To prove that $\sigma(A) = P\sigma(A)$, we show that the resolvent set $\rho(A)$ consists of all λ except those that satisfy (7) and then show that any λ satisfying (7) is in $P\sigma(A)$. The constant λ will be in $\rho(A)$ if and only if the equation

$$(8) \qquad (A-\lambda I)\varphi = \psi$$

has a solution φ in $\mathcal{D}(A)$ for all ψ in C and the solution depends continuously on ψ. Thus, we must have $\dot{\varphi}(\theta) - \lambda\varphi(\theta) = \psi(\theta)$, $\theta \in [-r,0]$; that is,

$$(9) \qquad \varphi(\theta) = e^{\lambda\theta} b + \int_0^\theta e^{\lambda(\theta-\xi)} \psi(\xi) d\xi \quad , \quad \theta \in [-r,0].$$

But, φ will be in $\mathcal{D}(A)$ if and only if $\dot{\varphi}(0) = g(\dot{\varphi}) + f(\varphi)$ and this yields

$$(10) \qquad \Delta(\lambda)b = \{-\psi(0) + \int_{-r}^{0} d\mu(\theta)[\frac{d}{d\theta} \int_{0}^{\theta} e^{\lambda(\theta-\xi)}\psi(\xi)d\xi] + $$

$$+ \int_{-r}^{0} d\eta(\theta) \int_{0}^{\theta} e^{\lambda(\theta-\xi)}\psi(\xi)d\xi\} .$$

Thus, if $\det \Delta(\lambda) \neq 0$, (9) and (10) show that (8) has a solution for any ψ in C and the solution is a continuous linear operator on C. This operator, called the resolvent operator, will be denoted by $(A-\lambda I)^{-1}$ and is given by

$$(11) \qquad [(A-\lambda I)^{-1}\psi](\theta) = e^{\lambda\theta}b + \int_{0}^{\theta} e^{\lambda(\theta-\xi)}\psi(\xi)d\xi \qquad \theta \in [-r,0]$$

where b is given by (10) and $\det \Delta(\lambda) \neq 0$. Hence $\rho(A) \supset \{\lambda : \det \Delta(\lambda) \neq 0\}$.

If $\det \Delta(\lambda) = 0$, then (9) and (10) imply there exists a nonzero solution of (8) for $\psi = 0$; that is, λ is in $P\sigma(A)$. This proves the first part of the theorem.

As we have seen, if λ is such that $\det \Delta(\lambda) = 0$ and b is such that $\Delta(\lambda)b = 0$, then $be^{\lambda\theta}$ is an eigenvector of A and every eigenvector is of this form. But then $x(t) = e^{\lambda t}b$ is a solution of (1) and hence by Lemma II(1) the real parts of the roots of (7) are bounded above.

For fixed k, any element of $\mathfrak{N}(A-\lambda I)^k$ is of the form $\sum_{i=0}^{k-1} \theta^i e^{\lambda\theta}\alpha_i$ and since there are only a finite number of linearly independent vectors α_i the space $\mathfrak{N}(A-\lambda I)^k$ is finite dimensional.

Since $\det \Delta(\lambda)$ is an entire function of λ is follows that

$(A-\lambda I)^{-1}$ is a meromorphic function with poles only at the zeros of $\det \Delta(\lambda)$. Thus we can apply Theorem 5.8-A of Taylor [12] to conclude that if λ is a zero of order $k > 0$ of $\det \Delta(\lambda)$ then $C = \mathfrak{N}(A-\lambda I)^k \oplus \mathcal{R}(A-\lambda I)^k$. Furthermore, since A and $T(t)$ commute for all $t \geq 0$ it follows that $T(t)$ is completely reduced by the two linear manifolds $\mathfrak{N}(A-\lambda I)^k$ and $\mathcal{R}(A-\lambda I)^k$. Thus the theorem is proved.

Now let us consider these spaces in more detail. Let $\varphi_1^\lambda,\ldots,\varphi_d^\lambda$ be a basis for $\mathfrak{M}_\lambda(A) = \mathfrak{N}(A-\lambda I)^k$ and let $\Phi_\lambda = (\varphi_1^\lambda,\ldots,\varphi_d^\lambda)$. Since $A\mathfrak{M}_\lambda(A) \subseteq \mathfrak{M}_\lambda(A)$, there exists a $d \times d$ matrix B_λ such that $A\Phi_\lambda = \Phi_\lambda B_\lambda$ and the only eigenvalue of B_λ is λ. From the definition of A and the relation $A\Phi_\lambda = \Phi_\lambda B_\lambda$ it follows that $\Phi_\lambda(\theta) = \Phi_\lambda(0)e^{B_\lambda\theta}$. From this fact and (6), one obtains

$$T(t)\Phi_\lambda = \Phi_\lambda e^{B_\lambda t}, \quad t \in [0,\infty),$$

(12)

$$[T(t)\Phi_\lambda](\theta) = \Phi_\lambda(0)e^{B_\lambda(t+\theta)}, \quad \theta \in [-r,0),\ t \in [0,\infty).$$

This relation permits one to define $T(t)$ on $\mathfrak{M}_\lambda(A)$ for all values of $t \in (-\infty,\infty)$, and so on a generalized eigenspace the equation (1) has the same structure as an ordinary differential equation. By repeated application of the same process one obtains

<u>COROLLARY 1</u>. <u>Suppose</u> Λ <u>is a finite set</u> $\{\lambda_1,\ldots,\lambda_p\}$ <u>of eigenvalues</u>

of (1) *and let* $\Phi_\Lambda = (\Phi_{\lambda_1},\ldots,\Phi_{\lambda_p})$, $B_\Lambda = \mathrm{diag}(B_{\lambda_1},\ldots,B_{\lambda_p})$, *where* Φ *and is a basis for* $\mathfrak{M}_{\lambda_i}(A)$ *and* B_{λ_i} *is the matrix defined by* $A\Phi_{\lambda_i} = \Phi_{\lambda_i}B_{\lambda_i}$, $i = 1,2,\ldots,p$. *Then the only eigenvalue of* B_{λ_i} *is* λ_i *and for any vector* a *of the same dimension as* Φ_Λ, *the solution* $T(t)\Phi a$ *with initial value* $\Phi_\Lambda a$ *at* $t = 0$ *may be defined on* $(-\infty,\infty)$ *by the relation*

$$(13) \qquad T(t)\Phi_\Lambda a = \Phi_\Lambda e^{B_\Lambda t} a\ , \quad \Phi_\Lambda(\theta) = \Phi_\Lambda(0)e^{B_\Lambda \theta}, \ \theta \in [-r,0]\ .$$

Furthermore there exists a subspace Q_Λ *of* C *such that* $T(t)Q_\Lambda \subseteq Q_\Lambda$ *for all* $t \geqq 0$ *and*

$$(14) \qquad C = P_\Lambda \oplus Q_\Lambda\ , \quad P_\Lambda = \{\varphi \in C\colon \varphi = \Phi_\Lambda a\ ,\ \textit{for some fixed vector}\ a\}.$$

This corollary gives a very clear picture of the behavior of the solutions of (1). In fact on the generalized eigenspaces the system behaves much like an ordinary differential equation. The above decomposition of C allows one to introduce a coordinate system in C which plays the same role as the Jordan canonical form in ordinary differential equations.

Before obtaining estimates for $T(t)$ on the complementary subspace Q_Λ, we give an explicit characterization for Q_Λ. This could be obtained from the general theory of linear operators, by means of a contour integral, but we prefer to give this representation in terms

of an operator "adjoint" to A relative to a certain bilinear form. This method leads to ease in computations and also provides a language more familiar to differential equationists. Let $C^* = C([0,r],R^{n*})$ where R^{n*} is the n-dimensional linear vector space of row vectors. For any φ in C, define

$$(15) \qquad (\alpha,\varphi) = \alpha(0)\varphi(0) - \int_{-r}^{0} \left[\frac{d}{d\zeta} \int_{0}^{\zeta} \alpha(s-\zeta)d\mu(\theta)\varphi(s)ds\right]_{\zeta=\theta} - \int_{-r}^{0}\int_{0}^{\theta} \alpha(s-\theta)d\eta(\theta)\varphi(s)ds$$

for all those α in C^* for which this expression is meaningful. In particular, (α,φ) will have meaning if α is continuously differentiable. The motivation for this bilinear form is not easy to understand, but it was first encountered in the proof of Theorem 1. In fact, equations (8), (9), (10) show that $(A-\lambda I)\varphi = \psi$ has a solution if and only if $(ae^{-\lambda\cdot}I,\psi) = 0$ for all row vectors a for which $a\Delta(\lambda) = 0$.

Without further ado, we use this bilinear form to try to determine an operator A^* with domain dense in C^* such that

$$(16) \qquad (\alpha,A\varphi) = (A^*\alpha,\varphi), \quad \text{for } \varphi \text{ in } \mathcal{D}(A),\ \alpha \text{ in } \mathcal{D}(A^*).$$

If we suppose α has a continuous first derivative and perform the standard type of calculations using an integration by parts, one shows that (16) is satisfied if A^* and the domain $\mathcal{D}(A^*)$ of A^* are defined by

$$(17a) \qquad (A^*\alpha)(s) = -d\alpha(s)/ds\ , \quad 0 \leq s \leq r$$

(17b) $\mathcal{D}(A^*) = \{\alpha \in C^*;\ \dot{\alpha} \in C^*,$

$$-\dot{\alpha}(0) = -\int_{-r}^{0} \dot{\alpha}(-\theta)d\mu(\theta) + \int_{-r}^{0} \alpha(-\theta)d\eta(\theta)\} .$$

Hereafter, we will take (17) as the defining relation for A^* and refer to A^* as the adjoint of A relative to the bilinear form (15).

For any α in C^*, consider the equation

(18a) $y(s) = \alpha(s)\ ,\quad 0 \leqq s \leqq r,$

(18b) $$y(s) = \alpha(0) - \int_{-r}^{0} \alpha(-\theta)d\mu(\theta) + \int_{-r}^{0} y(s-\theta)d\mu(\theta) - \int_{0}^{s}[\int_{-r}^{0} y(u-\theta)d\eta(\theta)]du,\quad s \leqq 0.$$

If we let y^s be the element of C^* defined by $y^s(\nu) = y(s+\nu)$, $0 \leqq \nu \leqq r$ and designate the solution of (18) by $y(\alpha)$, then the family of operators $T^*(s)$, $s \leqq 0$, defined by $y^s(\alpha) = T^*(s)\alpha$, $s \leqq 0$, is a strongly continuous semigroup for which $(-A^*)$ is the infinitesimal generator. We shall refer to (18) as the equation adjoint to (1).

Observe that α in $\mathcal{D}(A^*)$ implies that the solution $y(\alpha)$ of (18) on $(-\infty,r]$ is continuously differentiable and

(19) $$\dot{y}(s) = \int_{-r}^{0} \dot{y}(s-\theta)d\mu(\theta) - \int_{-r}^{0} y(s-\theta)d\eta(\theta)$$

for $s \leqq 0$.

LEMMA 3. <u>Suppose</u> $y(\alpha)$, $\alpha \in \mathcal{D}(A^*)$, <u>is the solution of</u> (18) <u>on</u> $(-\infty,r]$

and $x(\varphi)$ *is the solution of the nonhomogeneous equation*

(20a) $\quad x_\sigma = \varphi$

(20b) $\quad x(t) = \gamma(\varphi) + g(x_t) + \int_\sigma^t f(x_s)ds + \int_\sigma^t h(s)ds, \quad t \geqq \sigma.$

Then for any $\nu \geqq \sigma$,

(21) $\quad (y^{t-\nu}(\alpha), x_t(\varphi)) = (y^{\sigma-\nu}(\alpha), \varphi) + \int_\sigma^t y(s-\nu)h(s)ds, \quad \sigma \leqq t \leqq \nu.$

PROOF: For simplicity in notation, let $z^t = y^{t-\nu}(\alpha)$, $t \leqq \nu$, $x_t = x_t(\varphi)$, $t \geqq 0$. Since α is in $\mathcal{D}(A^*)$, $z(t)$ is continuously differentable and satisfies (19) for $t \leqq \nu$. From the definition (15) and the fact that $x(\varphi)$ satisfies (20), one shows very easily that, for $0 \leqq t \leqq \nu$,

$$(z^t, x_t) = z(t)[\gamma(\varphi) + \int_\sigma^t f(x_s)ds + \int_\sigma^t h(s)ds] +$$

$$+ \int_{-r}^{o}\int_{t}^{t+\theta} \dot{z}(u-\theta)d\mu(\theta)x(u)du - \int_{-r}^{o}\int_{t}^{t+\theta} z(u-\theta)d\eta(\theta)x(u)du.$$

Consequently, (z^t, x_t) is differentiable in t and a simple calculation yields $d(z^t, x_t)/dt = z(t)h(t)$, $0 \leqq t \leqq \nu$. Integrating this expression from o to ν yields the formula (21) which proves Lemma 3.

LEMMA 4. λ *is in* $\sigma(A)$ *if and only if* λ *is in* $\sigma(A^*)$. *The operator* A^* *has only point spectrum and for any* λ *in* $\sigma(A^*)$,

<u>the generalized eigenspace of</u> λ <u>is finite dimensional.</u>

PROOF: The last part of the lemma is proved exactly as in Lemma 2 and the first part follows from the observation that λ is in $\sigma(A^*)$ if and only if $\alpha(\theta) = e^{-\lambda\theta}b$ where b is a nonzero row vector satisfying $b\Delta(\lambda) = 0$.

<u>LEMMA 5. A necessary and sufficient condition for the equation</u>

$$(22) \qquad (A-\lambda I)^k\varphi = \psi$$

<u>to have a solution</u> φ <u>in</u> C, <u>or, equivalently, that</u> ψ <u>is in</u> $\mathcal{R}(A-\lambda I)^k$ <u>is that</u> $(\alpha,\psi) = 0$ <u>for all</u> α <u>in</u> $\mathfrak{N}(A^*-\lambda I)^k$. <u>Also,</u> $\dim \mathfrak{N}(A-\lambda I)^k = \dim \mathfrak{N}(A^*-\lambda I)^k$ <u>for every</u> k.

PROOF: First, we introduce some notation. With the matrix $\Delta(\lambda)$ given in (7), we define the matrices P_j as

$$(23) \qquad P_{j+1} = P_{j+1}(\lambda) = \frac{\Delta^{(j)}(\lambda)}{j!}, \quad \Delta^{(j)}(\lambda) = \frac{d^j\Delta(\lambda)}{d\lambda^j}, \quad j = 0,1,2,\ldots,k$$

and the matrices A_k of dimension $kn \times kn$ as

$$(24) \qquad A_k = \begin{bmatrix} P_1 & P_2 & \cdots & P_k \\ 0 & P_1 & \cdots & P_{k-1} \\ \vdots & & & \\ 0 & 0 & \cdots & P_1 \end{bmatrix}$$

Let us also define functions β_j by

$$(25) \qquad \beta_j(s) = \frac{(-s)^{k-j}}{(k-j)!} e^{-\lambda s}, \quad 0 \leqq s \leqq r, \; j = 1,2,\ldots,k .$$

If (22) is to have a solution, then necessarily $(\frac{d}{d\theta} - \lambda)^k \varphi(\theta) = \psi(\theta)$, $-r \leqq \theta \leqq 0$, or

$$\varphi(\theta) = \sum_{j=0}^{k-1} \gamma_{j+1} \beta_{k-j}(-\theta) + \int_0^\theta \beta_1(\xi-\theta)\psi(\xi)d\xi ,$$

where the γ_{j+1} are arbitrary n-dimensional column vectors which must be determined so that φ belongs to $\mathcal{D}(A-\lambda I)^k$. We now derive these conditions on the γ_j.

A simple induction argument on m shows that

$$\varphi^{(m)}(\theta) \overset{\text{def}}{=} (\frac{d}{d\theta} - \lambda)^m \varphi(\theta) = \sum_{j=0}^{k-m-1} \gamma_{m+j+1} \beta_{k-j}(-\theta) + \int_0^\theta \beta_{m+1}(\xi-\theta)\psi(\xi)d\xi$$

for $0 \leqq m \leqq k-1$.

Next, observe that φ belongs to $\mathcal{D}(A-\lambda I)^k$ if and only if $\varphi^{(m)}$ belongs to $\mathcal{D}(A-\lambda I)$, $m = 0,1,\ldots,k-1$. Since a continuously differentiable φ belongs to $\mathcal{D}(A)$ if and only if $\dot{\varphi}(0) = g(\dot{\varphi})+f(\varphi)$, it follows from the definition of the function $\varphi^{(m)}$ and the matrices P_j that $\varphi^{(m)}$, $m < k-1$, belongs to $\mathcal{D}(A)$ if and only if

$$P_1\gamma_{m+1} + P_2\gamma_{m+2} + \cdots + P_{k-m}\gamma_k = -(\beta_{m+1} I_n, \psi)$$

where I_n is the $n \times n$ identity matrix and $(\, , \,)$ is the bilinear form defined in (15). Since $\dot{\varphi}^{(k-1)}(0) = \lambda\gamma_k + \psi(0)$, it follows that $\varphi^{(k-1)}$ belongs to $\mathcal{D}(A)$ if and only if

$$P_1\gamma_k = -(\beta_k I_n, \psi) .$$

If we introduce the additional notation $\gamma = \mathrm{col}(\gamma_1, \ldots, \gamma_k)$, $B = \mathrm{diag}(\beta_1 I_n, \ldots, \beta_n I_n)$, then equation (22) has a solution if and only if γ satisfies the equation $A_k\gamma = -(B, \psi)$. But this equation has a solution if and only if $b(B, \psi) = (bB, \psi) = 0$ for all row vectors b satisfying $bA_k = 0$. On the other hand, calculations very similar to the ones above show that a function α in C^* belongs to $\mathfrak{N}(A^*-\lambda I)^k$ if and only if $\alpha = bB$ for some b satisfying $bA_k = 0$. It is clear from the above that $\dim \mathfrak{N}(A-\lambda I)^k = \dim \mathfrak{N}(A^*-\lambda I)^k$ for every k and this completes the proof of the lemma.

In the proof of the above lemma, we have actually characterized $\mathfrak{N}(A-\lambda I)^k$, $\mathfrak{N}(A^*-\lambda I)^k$ in a manner which is convenient for computations. In fact,

$$\text{(26a)} \qquad \mathfrak{N}(A-\lambda I)^k = \{\varphi \in C: \varphi(\theta) = \textstyle\sum_{j=0}^{k-1} \gamma_{j+1}\beta_{k-j}(-\theta),\ -r \leq \theta \leq 0,$$
$$A_k\gamma = 0,\ \gamma = \mathrm{col}(\gamma_1, \ldots, \gamma_k)\},$$

$$\text{(26b)} \qquad \mathfrak{N}(A^*-\lambda I)^k = \{\psi \in C^*: \psi(s) = \textstyle\sum_{j=1}^{k} \delta_j\beta_j(s),\ 0 \leq s \leq r$$
$$\delta A_k = 0,\ \delta = \mathrm{row}(\delta_1, \ldots, \delta_k)\},$$

where A_k, β_j, $j = 1,2,\ldots,k$, are defined by (23),(24),(25).

An important implication of the preceding lemma is

THEOREM 2. *For* λ *in* $\sigma(A)$, *let* $\Psi_\lambda = \mathrm{col}(\psi_1,\ldots,\psi_p)$, $\Phi_\lambda = (\varphi_1,\ldots,\varphi_p)$ *be bases for* $\mathfrak{M}_\lambda(A)$, $\mathfrak{M}_\lambda(A^*)$, *respectively, and let* $(\Psi_\lambda,\Phi_\lambda) = (\psi_i,\varphi_j)$, $i,j = 1,2,\ldots,p$. *Then* $(\Psi_\lambda,\Phi_\lambda)$ *is nonsingular and may be taken to be the identity. The decomposition of* C *given by Lemma 2 may be written explicitly as*

$$C = P_\lambda \oplus Q_\lambda$$

$$Q_\lambda = \{\varphi \in C\colon (\Psi_\lambda,\varphi) = 0\}$$

$$P_\lambda = \{\varphi \in C\colon \varphi = \Phi_\lambda(\Psi_\lambda,\varphi)\}.$$

PROOF: If k is the smallest integer for which $\mathfrak{M}_\lambda(A) = \mathfrak{N}(A-\lambda I)^k$ then Lemma 5 implies that $\mathcal{R}(A-\lambda I)^k = Q_\lambda$. If there is a p-vector a such that $0 = (\Psi_\lambda,\Phi_\lambda)a = (\Psi_\lambda,\Phi_\lambda a)$, then $\Phi_\lambda a$ belongs to both $\mathfrak{N}(A-\lambda I)^k$ and $\mathcal{R}(A-\lambda I)^k$ which implies by Lemma 3 that $\Phi_\lambda a = 0$ and, thus, $a = 0$. Consequently, $(\Psi_\lambda,\Phi_\lambda)$ is nonsingular and a change of the basis Ψ_λ will result in the identity matrix for $(\Psi_\lambda,\Phi_\lambda)$. The remaining statements in the lemma are obvious.

It is interesting to note that $(\Psi_\lambda,\Phi_\lambda) = I$ and $A^*\Psi_\lambda = B_\lambda^*\Psi_\lambda$, $A\Phi_\lambda = \Phi_\lambda B_\lambda$ implies $B_\lambda^* = B_\lambda$. In fact,

$$(\Psi_\lambda, A\Phi_\lambda) = (\Psi_\lambda, \Phi_\lambda B_\lambda) = (\Psi_\lambda, \Phi_\lambda)B_\lambda = B_\lambda$$

$$= (A^*\Psi_\lambda, \Phi_\lambda) = (B_\lambda^*\Psi_\lambda, \Phi_\lambda) = B_\lambda^*(\Psi_\lambda, \Phi_\lambda) = B_\lambda^* .$$

The following lemma is also convenient.

<u>LEMMA 6.</u> <u>If</u> $\lambda \neq \mu$, $\lambda, \mu \in \sigma(A)$, <u>then</u> $(\psi,\varphi) = 0$ <u>for all</u> ψ <u>in</u> $\mathfrak{M}_\mu(A^*)$, $\varphi \in \mathfrak{M}_\lambda(A)$.

The proof of this is not difficult but tedious and may be supplied as in [5].

If $\Lambda = \{\lambda_1, \ldots, \lambda_p\}$ is a finite set of characteristic values of (1); that is, $\lambda_j \in \sigma(A)$, we let P_Λ be the linear extension of the $\mathfrak{M}_{\lambda_j}(A)$, $\lambda_j \in \Lambda$ and refer to this set as the generalized eigenspace of (1) associated with Λ. In a similar manner we define $P_\Lambda^* = \mathfrak{M}_{\lambda_1}(A^*) \oplus \ldots \oplus \mathfrak{M}_{\lambda_p}(A^*)$ as the generalized eigenspace of the adjoint equation (18) associated with Λ. If $\Phi_\Lambda, \Psi_\Lambda$ are bases for P_Λ, P_Λ^*, respectively, $(\Psi_\Lambda, \Phi_\Lambda) = I$, then

$$\begin{aligned} C &= P_\Lambda \oplus Q_\Lambda \\ P_\Lambda &= \{\varphi \in C: \varphi = \Phi_\Lambda b \text{ for some vector } b\} \\ Q_\Lambda &= \{\varphi \in C: (\Psi_\Lambda, \varphi) = 0\} \end{aligned} \tag{27}$$

and, therefore, for any φ in C

$$\varphi = \varphi^{P_\Lambda} \oplus \Lambda^{Q_\Lambda}$$

(28)

$$\varphi^{P_\Lambda} = \Phi_\Lambda(\Psi_\Lambda, \varphi).$$

When this particular decomposition of C is used, we shall briefly express this by saying that C is <u>decomposed by Λ</u>.

Our next objective is to perform the above decomposition on the variation of constants formula for the solution of (20). From Corollary II.2, we know that the solution of (20) can be written as

$$\begin{aligned} x(t+\theta,\sigma,\varphi,h) &= x(t+\theta,\sigma,\varphi,0) + \int_\sigma^{t+\theta} U(t+\theta-s)h(s)ds \\ &= x(t+\theta,\sigma,\varphi,0) + \int_\sigma^{t+\theta} [d_s W(t+\theta-s)]h(s), \quad t+\theta \geqq \sigma. \end{aligned}$$

If we use our notation $x(t+\theta,\sigma,\varphi,0) = x(t+\theta-\sigma,0,\varphi,0) = [T(t-\sigma)\varphi](\theta)$ and the fact that $W_0 = 0$, then

$$x_t(\sigma,\varphi,h)(\theta) = [T(t-\sigma)\varphi](\theta) + \int_\sigma^t [d_s W_{t-s}(\theta)]h(s), \quad -r \leqq \theta \leqq 0.$$

For simplicity we suppress the explicit dependence on θ and write this as

$$\begin{aligned} x_t(\sigma,\varphi,h) &= T(t-\sigma)\varphi + \int_\sigma^t [d_s W_{t-s}]h(s) \\ &= T(t-\sigma)\varphi + \int_\sigma^t U_{t-s}h(s)ds \end{aligned} \tag{29}$$

where U_t is defined in the obvious way.

Now, suppose that Λ is a finite set of characteristic values of (1) and C is decomposed by Λ as in formulas (27),(28). For simplitity in notation, let $\Phi = \Phi_\Lambda$, $\Psi = \Psi_\Lambda$ and let B be the matrix defined by $A\Phi = \Phi B$. We have remarked before that $(\Psi,\Phi) = I$ implies that $A^*\Psi = B\Psi$. Consequently, the matrix $e^{-Bt}\Psi(0)$ is a solution of the adjoint equation (18) on $(-\infty,\infty)$. If we let $x_t = x_t(\sigma,\varphi,h) = x_t^{P_\Lambda} + x_t^{Q_\Lambda}$ and apply Lemma 3, it therefore follows that

$$
\begin{aligned}
x_t^{P_\Lambda} &\overset{\text{def}}{=} \Phi(\Psi,x_t) = \Phi e^{Bt}(e^{-Bt}\Psi,x_t) \\
&= \Phi e^{Bt}[(e^{-B\sigma}\Psi,\varphi) + \int_\sigma^t e^{-Bs}\Psi(0)h(s)ds] \\
&= T(t-\sigma)\Phi(\Psi,\varphi) + \int_\sigma^t \Phi e^{B(t-s)}\Psi(0)h(s)ds \\
&= T(t-\sigma)\varphi^P + \int_\sigma^t [d_s(-\int_0^{t-s} \Phi e^{Bu}\Psi(0))]h(s)
\end{aligned}
\tag{30}
$$

If $W_t = W_t^P + W_t^Q$, $W_t^P = \Phi(\Psi,W_t)$, $t \geq 0$, then by the same type of argument as above making use of Lemma 3 and the fact that W satisfies II(4), we obtain

$$
W_t^P \overset{\text{def}}{=} \Phi(\Psi,W_t) = -\int_0^t \Phi e^{B(t-s)}\Psi(0)ds = -\int_0^t \Phi e^{Bu}\Psi(0)du .
$$

Using this fact, equation (29),(30) and the formulas $x_t^Q = x_t - x_t^P$,

$\varphi^Q = \varphi - \varphi^P$, we have

$$x_t^P(\sigma,\varphi,h) = T(t-\sigma)\varphi^P + \int_\sigma^t [d_s W_{t-s}^P]h(s) \ , \tag{31a}$$

$$x_t^Q(\sigma,\varphi,h) = T(t-\sigma)\varphi^Q + \int_\sigma^t [d_s W_{t-s}^Q]h(s) \ , \quad t \geq 0. \tag{31b}$$

From formula (29), it is obvious that if $x_t^P(\sigma,\varphi,h) = \Phi y(t)$, then $y(t)$ satisfies the ordinary differential equation

$$\dot{y}(t) = B_\Lambda y(t) + \Psi(0)h(t) \ , \qquad t \geq 0. \tag{32}$$

<u>THEOREM 3</u>. <u>If</u> Λ <u>is a finite set of characteristic values of</u> (1) <u>and</u> C <u>is decomposed by</u> Λ <u>as in</u> (27),(28), <u>then the solution</u> $x(\sigma,\varphi,h)$ <u>of</u> (20) <u>satisfies</u> (31). <u>Furthermore, if</u> $x_t^P(\sigma,\varphi,h) = \Phi_\Lambda y(t)$, <u>then</u> $y(t)$ <u>satisfies</u> (32).

We now give an example to clarify the concepts discussed in this section. An easier illustration could be given by considering only a retarded equation, but the example to be given will be used later for other applications of the theory. Consider the homogeneous scalar equation

$$\dot{x}(t) = \alpha_o \dot{x}(t-r) - \beta x(t) - \alpha_o \gamma x(t-r) \tag{33}$$

where $r > 0$, α_o, β, γ are constants and the associated nonhomogeneous

equation

$$(34) \qquad \dot{x}(t) = \alpha_o \dot{x}(t-r) - \beta x(t) - \alpha_o \gamma x(t-r) + h$$

where h is some given function. For simplicity in notation, we are writing these equations in differential form, but it is always understood that solutions are defined by specifying a continuous initial function on an interval $[\sigma-r,\sigma]$ and solving the integrated form of the equation for x on $t \geq \sigma$.

The characteristic equation for (33) is

$$(35) \qquad \lambda - \alpha_o \lambda e^{-\lambda r} + \beta + \alpha_o \gamma e^{-\lambda r} = 0$$

and the associated bilinear form is

$$(36) \qquad (\psi,\varphi) = \psi(0)\varphi(0) - \alpha_o \psi(0)\varphi(-r) - \alpha_o \int_{-r}^{o} \dot{\psi}(\theta+r)\varphi(\theta)d\theta - \alpha_o \gamma \int_{-r}^{o} \psi(\theta+r)\varphi(\theta)d\theta.$$

Equation (34) was encountered by Brayton [13] in the study of transmission lines and he showed that for $\gamma > \beta > 0$ there are an infinite set of real pairs (α_o,ω_o), $\omega_o > 0$, $\alpha_o^2 < 1$, such that $\pm i\omega_o$ are simple roots of (35) and ω_o,α_o are related by the formulas

$$(37) \qquad \sin \omega_o r = \frac{\omega_o}{\alpha_o} \cdot \frac{\gamma + \beta}{\omega_o^2+\gamma^2} \quad , \quad \cos \omega_o r = \frac{1}{\alpha_o} \cdot \frac{\omega_o^2 - \gamma\beta}{\omega_o^2+\gamma^2} \quad .$$

Let us assume that α_o is such a real number and compute the decomposition of C assording to the set $\Lambda = \{+ i\omega_o, - i\omega_o\}$.

If $\Phi = (\varphi_1,\varphi_2)$, $\varphi_1(\theta) = \sin \omega_o\theta$, $\varphi_2(\theta) = \cos \omega_o\theta$, $-r \leqq \theta \leqq 0$, then Φ is a basis for the generalized eigenspace of (33) associated with Λ since we are assuming these eigenvalues are simple. Furthermore, $A\Phi = \Phi B$ implies

$$B = (b_{ij}),\ b_{11} = b_{22} = 0,\ b_{12} = -\omega_o = -b_{21}. \tag{38}$$

The equation adjoint to (33) is

$$\dot{y}(t) = \alpha_o\dot{y}(t+r)+\beta y(t)+\alpha_o \gamma y(t+r) \tag{39}$$

and $\Psi^* = \mathrm{col}(\psi_1^*,\psi_2^*)$, $\psi_1^*(\theta) = \sin \omega_o\theta$, $\psi_2^*(\theta) = \cos \omega_o\theta$, $0 \leqq \theta \leqq r$ is a basis for the generalized eigenspace of (39) associated with Λ.

After some straightforward but tedious calculations using (37) one obtains

$$(\psi_1^*,\varphi_1) = (\psi_2^*,\varphi_2) = \frac{1}{2(\omega_o^2+\gamma^2)}\,[\gamma(\gamma+\beta)+r\beta(\gamma^2+\omega_o^2)]$$

$$(\psi_2^*,\varphi_1) = -(\psi_1^*,\varphi_2) = \frac{\omega_o}{2(\omega_o^2+\gamma^2)}\,[\gamma+\beta+r(\gamma^2+\omega_o^2)]\ .$$

If we now define $\Psi = (\Psi^*,\Phi)^{-1}\Psi^*$, then $(\Psi,\Phi) = I$, the identity

and we are in a position to make our decomposition of C by Λ. Our main interest lies in formulas (31),(32) and in particular (31b) and (32). Consequently, we only need $\Psi(0)$ which is easily calculated from the above formulas and found to be

$$\Psi(0) = \mathrm{col}\left[\frac{D}{C^2+D^2}, \frac{C}{C^2+D^2}\right]$$

(40)
$$C = \frac{1}{2(\omega_o^2+\gamma^2)}\left[\gamma(\gamma+\beta)+r\beta(\gamma^2+\omega_o^2)\right]$$

$$D = \frac{\omega_o}{2(\omega_o^2+\gamma^2)}\left[\gamma+\beta+r(\gamma^2+\omega_o^2)\right] .$$

Finally, equation (34) is equivalent to the following system

$$x_t = \Phi y(t) + x_t^Q$$

(41)
$$\dot{y}(t) = By(t) + \Psi(0)h$$

$$x_t^Q = T(t-\sigma)\varphi^Q + \int_\sigma^t [d_s W_{t-s}]h , \qquad t \geq \sigma$$

where $\Psi(0)$ is given in (40) and B is defined in (38).

IV. THE CHARACTERISTIC EQUATION AND EXPONENTIAL BOUNDS.

In this section the zeros of the characteristic equation are discussed and estimates are obtained for the growth of the solutions on the compliment of the generalized eigenspaces.

In order to analyze the characteristic equation it is necessary to further restrict the functional g or equivalently the measure μ. It is known [10] that every function of bounded variation can be decomposed into three summands 1) a saltus function (essentially a step function with a countable number of discontinuities) 2) an absolutely continuous function and 3) a "singular function" that is a continuous function of bounded variation whose derivative is zero almost everywhere. We shall assume that the measure μ is without singular part.

Specifically, assume that

$$(1) \qquad g(\varphi) = \sum_{k=1}^{\infty} A_k \varphi(-\omega_k) + \int_{-r}^{0} A(\theta)\varphi(\theta)d\theta \quad , \quad \varphi \in C([-r,0],R^n)$$

where the A_k are $n \times n$ constant matrices with $\sum_1^{\infty} A_k$ absolutely convergent, the ω_k are a countable sequence of real numbers with $0 < \omega_k \leq r$ for all k and $A(\theta) \in \mathcal{L}_1([-r,0],R^{n^2})$.

Under the above assumption $\Delta(\lambda)$ has the form

$$(2) \qquad \Delta(\lambda) = \lambda\{H_1(\lambda) + H_2(\lambda)\} + H_3(\lambda)$$

where

$$\text{a) } H_1(\lambda) = I - \sum_{k=1}^{\infty} A_k e^{-\omega_k \lambda}$$

$$(3) \qquad \text{b) } H_2(\lambda) = -\int_{-r}^{0} A(\theta)e^{\lambda\theta}d\theta$$

c) $H_3(\lambda) = -\int_{-r}^{0} e^{\lambda\theta} d\eta(\theta)$.

Moreover $\det \Delta(\lambda) = \lambda^n h_1(\lambda) + h_2(\lambda)$ where $h_1(\lambda) = \det H_1(\lambda)$ and $h_2(\lambda) = \det \Delta(\lambda) - \lambda^n h_1(\lambda)$.

For any pair of real numbers α,β $(\alpha \leq \beta)$ let $[\alpha,\beta] = \{\lambda: \alpha \leq \operatorname{Re} \lambda \leq \beta\}$. In any $[\alpha,\beta]$ the elements of $H_3(\lambda)$ are bounded and the elements of $H_2(\lambda)$ tend uniformly to zero as $|\lambda| \to \infty$. Thus $h_2(\lambda) = o(\lambda^n)$ as $|\lambda| \to \infty$ in $[\alpha,\beta]$.

<u>LEMMA 1</u>. <u>If</u> $\{\lambda_k\}$ <u>is a sequence of zeros of</u> h_1 <u>in</u> $[\alpha + \delta, \beta - \delta]$, $\delta > 0$, <u>with</u> $|\lambda_n| \to \infty$, <u>then there exists a sequence</u> $\{\lambda_k'\}$ <u>of zeros of</u> $\det \Delta(\lambda)$ <u>in</u> $[\alpha,\beta]$ <u>with the property that</u> $|\lambda_k - \lambda_k'| \to 0$, <u>as</u> $k \to \infty$.

<u>LEMMA 2</u>. <u>Let</u> a <u>be a real number such that only a finite number of zeros of</u> $\det \Delta(\lambda)$ <u>have real part greater than</u> $a - \varepsilon$ <u>for some</u> $\varepsilon > 0$. <u>Then there exists an</u> a^* <u>and a</u> $K > 0$ <u>such that</u> $a - \frac{1}{2}\varepsilon \leq a^* \leq a$ <u>and</u> $\|\Delta(a^* + i\xi)^{-1}\| \leq K/(1 + |\xi|)$ for ξ real.

PROOFS. The function $h_1(\lambda)$ is an analytic almost periodic function for all λ. Then by a theorem in [14], page 351 there exists a number N such that the number of zeros of $h_1(\lambda)$ in the box $\mathcal{B}(\alpha + \delta, \beta - \delta, t^*) = \{\lambda: \alpha + \delta \leq \operatorname{Re}\lambda \leq \beta - \delta,\ t^* - 1/2 \leq \operatorname{Im} \lambda \leq t^* + 1/2\}$ does not exceed N for any real t^*. Moreover for each $r > 0$ there exists an $m(r) > 0$ such that for all λ in $[\alpha,\beta]$ at a distance greater than r from a zero of $h_1(\lambda)$ the inequality $|h_1(\lambda)| \geq m(r)$ holds.

Thus Lemma 1 follows by applying Rouché's Theorem.

Now let a be as in Lemma 2. Since $h(\lambda)$ has only finitely many zeros with real part greater than $a-\varepsilon$ for some $\varepsilon > 0$ it follows from Lemma 1 that $h_1(\lambda)$ has only finitely many zeros with real part greater than $a-\varepsilon/2$. Therefore there exists an a^*, $a-\varepsilon/2 \leq a^* \leq a$, and a $K_2 > 0$ such that $|h_1(\lambda)| \geq K_2$ for all $\lambda = a^* + i\xi$, ξ, real. Thus $|h(a^* + i\xi)^{-1}| = O(\xi^{-n})$ as $|\xi| \to \infty$, ξ real.

Since $\Delta(\lambda)^{-1} = (h(\lambda)^{-1})\operatorname{adj} \Delta(\lambda)$ and $\|\operatorname{adj}\Delta(a^* + i\xi)\| = O(\xi^{n-1})$ as $|\xi| \to \infty$, ξ real, Lemma 2 follows.

With the aid of Lemma 2 one can now estimate the growth of the solutions on the space Q_Λ. Let Λ be a finite set of eigenvalues of A with the property that all other eigenvalues of A have real part less that $a-\varepsilon$ for a fixed real number a and some $\varepsilon > 0$.

Let $u(\cdot,\sigma,h)$ be the solution of the nonhomogeneous equation that satisfies $u_\sigma = 0$, i.e., the solution given by the integral in the Corollary 1 of Section III. Let u_t^Q be the projection of u_t on the space Q_Λ and $u(t)^Q = u_t^Q(0)$.

Let C^1 denote the set of continuously differentiable function from $[-r,0]$ into R^n with the norm $\|\varphi\|^1 = \sup_{\theta\in[-r,0]} \{|\varphi(\theta)| + |\dot{\varphi}(\theta)|\}$.

<u>THEOREM 1</u>. <u>Let</u> $\varphi \in C^1$. <u>Then there exist constants</u> M <u>and</u> N <u>such that</u>

(4) $$\| T(t)\varphi^Q \| \leq Me^{at} \|\varphi\|^1$$

<u>and</u>

(5) $$\|u_t^Q\| \leq N \{\int_\sigma^t |e^{a(t-\tau)}h(\tau)|^2 d\tau\}^{1/2}$$

PROOF: In the proof of this theorem the fact that the formulas

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(y)dy \ ; \ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} g(y)dy$$

define a unitary transformation of the space $L_2(-\infty,\infty)$ and its inverse will be used several times (see [10]). In the formulas $\int_{-\infty}^{\infty} = \lim_{T\to\infty} \int_{-T}^{T}$.

By standard Laplace transform methods

(6) $$u(t)^Q = \int_c e^{\lambda t} \Delta(\lambda)^{-1} \{\int_0^t e^{-\lambda\tau} h(\tau)d\tau\} d\lambda$$

where $\int_c = \lim_{T\to\infty} \int_{a^*-iT}^{a^*+iT}$ and a^* is in Lemma 2. Now (6) can be written

(7) $$u(t)^Q = i \int_{-\infty}^{\infty} e^{i\xi t} \Delta(a^*+i\xi)^{-1} \{\int_0^t e^{-i\xi\tau} [e^{a(t-\tau)} h(\tau)d\tau]\} d\xi \ .$$

The function in the braces is an L^2 function of ξ for each t and $\Delta(a^*+i\xi)^{-1}$ is an L_2 function of ξ by Lemma 2. Applying Schwartz's inequality yields

$$|u(t)^Q| \leq M_1\{\int_0^t |e^{a^*(t-\tau)}h(\tau)|^2 d\tau\}^{1/2}$$

from which the inequality (5) follows at once.

Let $\varphi \in \mathscr{D}(A)$. Then

$$(8) \qquad T(t)\varphi^Q = \int_c e^{\lambda t}[\Delta(\lambda)^{-1}\{-\varphi(0) + \int_{-r}^{0} d\mu(\theta)(\frac{d}{d\theta}\int_0^\theta e^{\lambda(\theta-\alpha)}\varphi(\alpha)d\alpha)$$

$$+ \int_{-r}^{0} d\eta(\theta) \int_0^\theta e^{\lambda(\theta-\alpha)}\varphi(\alpha)d\alpha\} + \int_0^\theta e^{\lambda(\theta-\alpha)}\varphi(\alpha)d\alpha]d\lambda$$

by [11].

The term containing $\int_0^\theta e^{\lambda(\theta-\alpha)}\varphi(\alpha)d\alpha$ contributes nothing since it is an entire function of λ and the contour can be shifted to $-\infty$.

Now

$$\int_{-r}^{0} d\mu(\theta)(\frac{d}{d\theta}\int_0^\theta e^{\lambda(\theta-\alpha)}\varphi(\alpha))d\alpha =$$

$$= \{\int_{-r}^{0} d\mu(\theta)e^{\lambda\theta}\}\varphi(0) + \int_{-r}^{0} d\mu(\theta)\int_0^\theta d^{\lambda(\theta-\alpha)}\dot{\varphi}(\alpha)d\alpha .$$

From the matrix identity $\{\lambda B + C\}^{-1}B = \lambda^{-1}\{I-(\lambda B+C)^{-1}B\}$ one obtains

$$\int_c e^{\lambda t}\Delta(\lambda)^{-1}\{-I + \int_{-r}^{0} d\mu(\theta)e^{\lambda\theta}\}\varphi(0) =$$

$$\{-\int_c \frac{e^{\lambda t}}{\lambda}.d\lambda + \int_c e^{\lambda t}\frac{\Delta(\lambda)^{-1}}{\lambda}[\int_{-r}^{0} d\eta(\theta)e^{\lambda\theta}]d\lambda\}\varphi(0).$$

The first integral is integrable and is known to admit an estimate of the form

$$|\int_c \frac{e^{\lambda t}}{\lambda} d\lambda| \leq M_2 e^{a^* t}.$$

The second integral is absolutely convergent since $\lambda^{-1}\Delta(\lambda)^{-1}$ is like λ^{-2} on c, and thus

$$|\int_c e^{\lambda t}\frac{\Delta(\lambda)^{-1}}{\lambda}[\int_{-r}^{0} d\eta(\theta)e^{\lambda\theta}]d\lambda| \leq e^{a^* t}M_3 .$$

For $\lambda = a^*+i\xi$, we have

$$\int_{-r}^{0} d\mu(\theta)\int_{0}^{\theta} e^{\lambda(\theta-\alpha)}\phi(\alpha)d\alpha = \int_{0}^{-r} d\beta \int_{-r}^{\beta} d\mu(\theta)\{e^{\lambda\beta}\phi(\theta-\beta)\}$$

$$= \int_{0}^{-r} e^{i\xi\beta}[e^{a^*\beta}\int_{-r}^{\beta} d\mu(\theta)\phi(\theta-\beta)]d\beta .$$

As a function of ξ the above is an element of $L_2(-\infty,\infty)$ whose norm can be estimated by $M_4\|\varphi\|^1$. By applying Schwartz's inequality we obtain

$$|\int_c e^{\lambda t}\Delta(\lambda)^{-1}\{\int_{-r}^{0} d\mu(\theta)\int_0^{\theta} e^{\lambda(\theta-\alpha)}\dot{\varphi}(\alpha)d\alpha\}| \leq M_5 e^{a^*t}\|\varphi\|^1.$$

In the same way as the above

$$|\int_c e^{\lambda t}\Delta(\lambda)^{-1}\{\int_{-r}^{0} d\eta(\theta)\int_0^{\theta} e^{\lambda(\theta-\alpha)}\varphi(\alpha)d\alpha\ |\} \leq M_5 e^{a^*t}\|\varphi\| \ .$$

Thus the estimate (4) is obtained for all $\varphi \in \mathcal{D}(A)$. The estimate (4) remains true for all continuously differentiable φ since $\mathcal{D}(A)$ is dense in C^1.

COROLLARY 1. If $g \equiv 0$ in III(1)b) then

$$\|T(t)\varphi\| \leq Me^{at}\|\varphi\| \quad \text{for all } \varphi \in C \ .$$

V. APPLICATIONS: STABILITY AND INTEGRAL MANIFOLDS.

In this section two applications are given to illustrate how the previously developed theory of linear equations can be used to study weakly nonlinear systems. It is hoped that this section will indicate the possibility of further extensions and applications. The first application is the analogue of a well known stability theorem by first approximation for ordinary differential equations. The second is an extension of the method of integral manifolds to this new class of equations.

The general outline of the proofs given below is the same as in the case of ordinary differential equations, but certain technical details are markedly different.

V.1. Stability

Our proof of the stability theorem is modeled on the standard proof using Gronwall's inequality (see [15] and [16]). For this we need the following:

LEMMA 1. *There exists a constant* $K > 0$ *independent of* $\alpha, \beta > 0$, *such that any function* u *that is continuous for all* $t \geq 0$ *and satisfies*

$$u(t) \leq \alpha + \beta\{\int_0^t u(s)^2 ds\}^{1/2} \quad \text{for} \quad t \geq 0$$

also satisfies the inequality $u(t) \leq \alpha K \exp \beta^2 t/2$.

PROOF. Note that there is no loss in generality by taking $\alpha = 1$. Consider the continuous linear operator I from $C([0,E],R)$ into itself for each $E > 0$ defined by

$$(Iu)(t) = 1 + \beta\{\int_0^t u(s)^2 ds\}^{1/2}$$

Observe that I has the following property: if $u(t) \leq v(t)$ for $t \in [0,E]$, $E \geq 0$, then $(Iu)(t) \leq (Iv)(t)$ for $t \in [0,E]$. Hence by [1], p. 61, it follows that any function w continuous for $t \geq 0$ will dominate functions satisfying (1) if $(Iw)(t) > w(t)$ for $t \geq 0$. That is if w satisfies $(Iw)(t) > w(t)$ for all $t \geq 0$ and $u(t)$ satisfies (1) then $u(t) \leq w(t)$ for $t \geq 0$.

Observe that if v satisfies $Iv = v$; $(Iv)(t) = v(t)$, $t \geq 0$; then $w = Bv$, $B > 1$ satisfies $(Iw)(t) > w(t)$ for $t \geq 0$. Hence we must only analyze the equation $Iv = v$.

By a simple application of the contracting mapping principle one finds that I has a fixed point in $C([0,E],R)$ for E sufficiently small. Denote this fixed point by u and then u satisfies the differential equation

$$(2) \qquad \dot{u} = \frac{\beta^2}{2}\{\frac{u^2}{u-1}\} = \frac{\beta^2}{2}u\{\frac{1}{1-u^{-1}}\} \quad \text{for} \quad 0 < t < E.$$

Clearly u can be shown to exist for $t \geq E$ and hence for $t \geq 0$. Moreover it is clear from (2) that u admits an estimate of the form $u(t) \leq K \exp(\beta^2 t/2)$.

Now consider the equation

$$(3) \qquad x(t) = \gamma(\varphi) + g(x_t) + \int_\sigma^t f(x_s)ds + \int_\sigma^t F(s,x_s)ds$$

$$x_\sigma = \varphi \ , \ \varphi \in C.$$

where F is a continuous mapping from $[\tau,\infty) \times S_E$ into R^n where $S_E = \{\varphi \in C: \|\varphi\| < E\}$ and also $\tau \leq \sigma$. Also assume F is Lipschitzian in the second argument on all of $[\tau,\infty) \times S_E$ and let $|F(t,\varphi)| = o(\|\varphi\|)$ uniformly in t as $\|\varphi\| \to 0$.

Furthermore let g be such that the estimates of section IV apply and let A be the infinitesimal generator of the semigroup generated by (3) with $F \equiv 0$.

<u>THEOREM 1.</u> <u>Let all the eigenvalues of</u> A <u>have real parts less than</u> $-a < 0$, <u>let</u> $\varphi \in C^1([-r,0],R^n)$, <u>and let</u> $x(\varphi)$ <u>be the solution of</u> (3) <u>with</u> $x_\sigma(\varphi) = \varphi$. <u>Then for any</u> $\varepsilon > 0$, $0 < \varepsilon < a$, <u>there exists a pair of constants</u> ρ <u>and</u> L <u>such that</u>

$$(4) \qquad \|x_t(\varphi)\| \leq L\|\varphi\|^1 e^{-(a-\varepsilon)(t-\sigma)} \ , \ t \geq \sigma$$

<u>provided</u> $\|\varphi\|^1 \leq \rho$.

REMARK. Existence and uniqueness of a solution to equation (3) can be established in a manner similar to that found in section II. The present problem is slightly more complicated since the application of the contracting mapping principle gives the existence of solutions over an interval whose length depends on the norm of the initial condition. This difficulty can be overcome by using a continuation argument as in ordinary differential equations. Indeed it can be shown that a solution of (3) can be extended either for all $t \geq 0$ or until it reaches the boundary of S_E.

PROOF. Let x be the solution of (3) corresponding to the continuously differentiable inital function $\varphi \in S_E$. As long as $x(\varphi)$ satisfies (3) then

$$(5) \qquad x(t) = T(t)\varphi + \int_\sigma^t \{d_s W(t-s)\} F(s, x_s)$$

From the results of section IV there exist constants M and N such that

$$(6) \qquad \|x_t\| \leq M(\|\varphi\|^1) e^{-a(t-\sigma)} + N\{\int_\sigma^t e^{-2a(t-s)} |F(s,x_s)|^2 ds\}^{1/2}$$

and since $|F(s,\varphi)| = o(\|\varphi\|)$ we can choose a $\rho > 0$ such that $|F(s,\varphi)| \leq N^{-1}\sqrt{2\varepsilon}\, \|\varphi\|$ for all $\|\varphi\| < \rho$ and so

$$e^{a(t-\sigma)}\|x_t\| \leq M\|\varphi\|^1 + \sqrt{2\varepsilon}\, \{\int_\sigma^t [e^{a(s-\sigma)}\|x_s\|]^2 ds\}^{1/2}$$

and so by Lemma 1 $e^{a(t-\sigma)}\|x_t\| \leq KM\|\varphi\|^1 e^{\mathcal{E}(t-\sigma)}$ or $\|x_t\| \leq KM\|\varphi\|^1 e^{-(a-\mathcal{E})(t-\sigma)}$ for $t \geq \sigma$. The last estimate holds for all $t \geq \sigma$ provided ρ is sufficiently small since the above estimate implies that the solution does not leave S_E.

V.2. Averaging and integral manifolds.

In this section, we shall show how the results of the previous pages together with generalizations of well known perturbational methods of ordinary differential equations can be used to discuss the existence and stability of periodic solutions and integral manifolds of perturbed linear systems where the nonlinear term is of a special type. The hypotheses are unnecessarily restrictive and the presentation is given in this way for simplicity only. Generalizations will be obvious to the reader acquainted with the theory of oscillations for ordinary differential equations.

Consider the linear system

$$\text{(7)} \quad \begin{aligned} &\text{a)} \quad x_\sigma = \varphi \quad \text{where} \quad \varphi \in C, \\ &\text{b)} \quad x(t) - \gamma(\varphi,\mathcal{E}) + g(x_t,\mathcal{E}) + \int_\sigma^t f(x_\tau,\mathcal{E})d\tau, \quad t \geq \sigma \end{aligned}$$

where $\mathcal{E} \geq 0$ is a parameter, $\gamma(\varphi,\mathcal{E}) = \varphi(0)-g(\varphi,\mathcal{E})$, $g(\varphi,\mathcal{E})$, $f(\varphi,\mathcal{E})$ are linear in φ and continuous in φ, for all φ in C,

$0 \leq \varepsilon \leq \varepsilon_0$ with the continuity in φ being uniform in ε. Furthermore, suppose $g(\varphi,\varepsilon)$ has the nonatomic property I(5) uniformly in ε. The characteristic equation of (7) is

$$(8) \qquad \begin{aligned} &\det \Delta(\lambda,\varepsilon) = 0 \\ &\Delta(\lambda,\varepsilon) = \lambda[I-g(e^{\lambda\cdot},\varepsilon)] - f(e^{\lambda\cdot},\varepsilon). \end{aligned}$$

We shall always assume that equation (8) has two simple roots $\varepsilon\nu(\varepsilon) \pm i\omega(\varepsilon)$, $\omega(\varepsilon) = \omega_0 + \varepsilon\omega_1(\varepsilon)$, $\omega_0 > 0$, $\nu(\varepsilon)$, $\omega(\varepsilon)$ continuous in ε, $0 \leq \varepsilon \leq \varepsilon_0$, and the remaining roots have real parts $\leq -\delta < 0$. Notice that for $\varepsilon = 0$, this hypothesis implies that (7) has a two parameter family of periodic solutions of period $2\pi/\omega_0$ to which all other solutions (with smooth enough initial data) approach as $t \to \infty$. For $\varepsilon > 0$, there is a two parameter family of solutions [corresponding to the characteristic roots $\varepsilon\nu(\varepsilon) \pm i\omega(\varepsilon)$] which are exponentially stable. We shall let $\Phi_\varepsilon = (\varphi_{1\varepsilon},\varphi_{2\varepsilon})$ be a basis for the solutions in C generated by the roots $\Lambda = \{\varepsilon\nu(\varepsilon) \pm i\omega(\varepsilon)\}$ and $\Psi_\varepsilon = \mathrm{col}\,(\psi_{1\varepsilon},\psi_{2\varepsilon})$ a corresponding basis for the solutions of the adjoint equation, $(\Psi_\varepsilon,\Phi_\varepsilon) = I$.

Suppose $F: R \times C \to R^n$ is continuous and $F(t,\varphi)$, $t \in R$, $\varphi \in C$ has continuous second derivatives with respect to φ and consider the nonlinear equation

$$(9) \qquad \begin{aligned} &\text{a) } x(t) = \varphi(t-\sigma),\ \sigma-r \leq t \leq \sigma, \\ &\text{b) } x(t) = \gamma(\varphi,\varepsilon)+g(x_t,\varepsilon)+\int_\sigma^t f(x_\tau,\varepsilon)d\tau + \varepsilon\int_0^t F(\tau,x_\tau)d\tau, t \geq \sigma. \end{aligned}$$

Notice that formal differentiation of this equation with respect to t yields

$$\dot{x}(t) = g(\dot{x}_t,\varepsilon) + f(x_t,\varepsilon) + \varepsilon F(t,x_t); \tag{10}$$

that is, an equation of neutral type where the nonlinearity does not involve the derivative of x. An equation of this type with $F(t,\varphi)$ independent of t was encountered by Miranker [17] and Brayton [13] in the theory of transmission lines. Similar equations have also been studied by Marchenko and Rubanik [18] in connection with some mechanical vibration problems.

If the space C is decomposed by $\Lambda = \{\varepsilon\nu(\varepsilon) \pm i\omega(\varepsilon)\}$, then the theory of section 3 shows that system (3) is equivalent to the system

$$\begin{aligned} &\text{a)} \quad x_t = \Phi_\varepsilon y(t) + x_t^Q \ , \quad y(t) = (\Psi_\varepsilon, x_t) \\ (11) \qquad &\text{b)} \quad \dot{y}(t) = B_\varepsilon y(t) + \varepsilon\Psi_\varepsilon(0)F(t,\Phi_\varepsilon y(t)+x_t^Q) \ , \\ &\text{c)} \quad x_t^Q = T_\varepsilon(t-\sigma)x_\sigma^Q + \varepsilon\int_\sigma^t [d_s W^Q_{\varepsilon,t-s}]F(s,\Phi_\varepsilon y(s)+x_s^Q)ds, \ t \geq \sigma, \end{aligned}$$

where the eigenvalues of B_ε are $\{\varepsilon\nu(\varepsilon) \pm i\omega(\varepsilon)\}$; B_ε is determined by $\Phi_\varepsilon(\theta) = \Phi_\varepsilon(0)\exp B_\varepsilon\theta$, $-r \leq \theta \leq 0$, $T_\varepsilon(t)$, $t \geq 0$ designates the semigroup of transformations associated with (7) and $W_{\varepsilon,t}$ is the kernel function associated with the variation of constants formula II(3); that is, $W_\varepsilon(t)$ satisfies II(4) for $0 \leq \varepsilon \leq \varepsilon_0$. The

matrix B_ε can actually be chosen as

$$B_\varepsilon = \begin{bmatrix} \varepsilon\nu(\varepsilon) & -\omega(\varepsilon) \\ \omega(\varepsilon) & \varepsilon\nu(\varepsilon) \end{bmatrix}$$

The above hypotheses on the characteristic equation (8) and the estimates of section 4 imply that there are positive constants K, c such that

$$(12)\quad \begin{aligned} &\text{a)}\ \left|\int_\sigma^t [d_s W^Q_{\varepsilon,t-s}]h(s)ds\right| \leq K\left(\int_\sigma^t (e^{-c(t-s)}|h(s)|)^2\right)^{1/2} \\ &\text{b)}\ \|T_\varepsilon(t)\varphi^Q\| \leq Ke^{-ct}\|\varphi^Q\|^1 \quad , \qquad t \geq 0, \end{aligned}$$

for all bounded functions $h(s)$ and $0 \leq \varepsilon \leq \varepsilon_o$.

If $y = \mathrm{col}(y_1, y_2)$, $y_1 = \rho\cos\zeta$, $y_2 = \rho\sin\zeta$, then equations (11b),(11c) are equivalent to

$$(13)\quad \begin{aligned} &\text{a)}\ \dot{\zeta} = \omega(\varepsilon) + \varepsilon\, Z(t,\zeta,\rho,x^Q_t,\varepsilon) \\ &\text{b)}\ \dot{\rho} = \varepsilon R(t,\zeta,\rho,x^Q_t,\varepsilon) \\ &\text{c)}\ x^Q_t = T_\varepsilon(t-\sigma)x^Q + \varepsilon\int_\sigma^t [d_s W_{\varepsilon,t-s}]\tilde{F}(s,\zeta(s),\rho(s),x^Q_s,\varepsilon)ds,\ t \geq \sigma, \end{aligned}$$

where

$$\text{a)}\ \tilde{F}(t,\zeta,\rho,\varphi,\varepsilon) = F[t,\rho(\varphi_{1\varepsilon}\cos\zeta + \varphi_{2\varepsilon}\sin\zeta) + \varphi]$$

$$(14)\qquad \text{b)}\ Z(t,\zeta,\rho,\varphi,\varepsilon) = \frac{1}{\rho}[-\psi_{1\varepsilon}(0)\sin\zeta+\psi_{2\varepsilon}(0)\cos\zeta]\tilde{F}(t,\zeta,\rho,\varphi,\varepsilon)$$

$$\text{c)}\ R(t,\zeta,\rho,\varphi,\varepsilon) = \nu(\varepsilon)\rho+[\psi_{1\varepsilon}(0)\cos\zeta+\psi_{2\varepsilon}(0)\sin\zeta]\tilde{F}(t,\zeta,\rho,\varphi,\varepsilon).$$

Suppose that the functions $\tilde{F}$, Z, R are almost periodic in t uniformly with respect to the other variables [$F(t,\varphi)$ in (10) almost periodic in t uniformly with respect to φ will imply this] and suppose that

$$(15)\qquad \begin{aligned} &\text{a)}\ Z_o(\rho,\varepsilon) \overset{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\int_o^T Z(t+s,\ \zeta+s,\ \rho,0,\varepsilon)ds \\ &\text{b)}\ R_o(\rho,\varepsilon) \overset{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\int_o^T R(t+s,\ \zeta+s,\ \rho,0,\varepsilon)ds \quad ; \end{aligned}$$

that is, the mean values of Z, R are independent of t, ζ. Notice that these mean values are computed slightly differently then in ordinary differential equations. As in [ ], we have put $x_t^Q = 0$ and this is the basic fact that allows the theory to go through in a simple way. On the other hand, it makes some estimates more delicate as we shall see below.

Following the same type of reasoning as in ordinary differential equations (see [6] or [19]), there is a transformation of variables

$$(16)\qquad \zeta \to \zeta + \varepsilon u(t,\zeta,\rho,\varepsilon),\quad \rho \to \rho + \varepsilon v(t,\zeta,\rho,\varepsilon)$$

such that system (13) is equivalent to the system

$$
(17)\qquad
\begin{aligned}
&\text{a)}\quad \dot{\zeta} = \omega(\varepsilon) + \varepsilon\, Z_o(\rho,\varepsilon) + \varepsilon\, Z_1(t,\zeta,\rho,x_t^Q,\varepsilon)\\
&\text{b)}\quad \dot{\rho} = \varepsilon\, R_o(\rho,\varepsilon) + \varepsilon\, R_1(t,\zeta,\rho,x_t^Q,\varepsilon)\\
&\text{c)}\quad x_t^Q = T_\varepsilon(t-\sigma)x_\sigma^Q + \varepsilon\int_\sigma^t [d_s W^Q_{\varepsilon,t-s}]F_1(s,\zeta(s),\rho(s),x_s^Q,\varepsilon)ds
\end{aligned}
$$

where $F_1(t,\zeta,\rho,\varphi,\varepsilon) = \tilde{F}(t,\zeta+\varepsilon u,\rho+\varepsilon v,\varphi,\varepsilon)$, the functions Z_1, R_1 have the same smoothness properties as Z, R, are almost periodic in t uniformly with respect to the other variables, periodic in ζ of period 2π, and the functions $Z_1(t,\zeta,\rho,0,\varepsilon)$, $R_1(t,\zeta,\rho,0,\varepsilon)$ as well as their lipschitz constants with respect to ζ,ρ approach zero as $\varepsilon \to 0$.

Equations of type (17) can arise from system (9) without the severe restrictions made above on the characteristic equation (8). In fact, there could be any number of roots of (8) with zero real parts for $\varepsilon = 0$. The main part of the assumption that we have used is the dependence of the roots on ε near $\varepsilon = 0$. In this case, various transformations on (11b) yield equation of the form (17) with ζ,ρ vectors of not necessarily the same dimension. Also, some roots (a finite number) of (8) could have positive real parts for $\varepsilon = 0$. This adds an extra equation to (17) which can be easily discussed. For the sake of generality in the applications, we will assume that ζ,ρ are vectors of dimension p,q, respectively, and the functions in (17) are 2π-periodic in the components of the vector $\zeta = (\zeta_1,\ldots,\zeta_p)$.

If $\alpha : R \times R^p \to R^q$, $\beta : R \times R^p \to C$ are given functions, we say that the set

$$(18) \qquad S(\alpha,\beta) = \{(t,\zeta,\rho,\varphi) : \rho = \alpha(t,\zeta),\ \varphi = \beta(t,\zeta),\ t \in R,\ \zeta \in R^p\}$$

is an <u>integral manifold</u> of (17) if for every θ in R^p, σ in R and $\zeta(t) = \zeta(t,\sigma,\theta)$, $\zeta(\sigma,\sigma,\theta) = \theta$, the solution of (11a) with ρ, x_t^Q replaced by $\alpha(t,\zeta)$, $\beta(t,\zeta)$, respectively, it follows that the triple $\zeta(t)$, $\rho(t) = \alpha(t,\zeta(t))$, $x_t^Q = \beta(t,\zeta(t))$ is a solution of (17).

<u>THEOREM 1. Suppose</u> $W^Q_{\varepsilon,t}$, $T_\varepsilon(t)\varphi^Q$ <u>satisfy</u> (12) <u>and there is a</u> ρ_o <u>such that</u> $R_o(\rho_o,0) = 0$ <u>and the eigenvalues of</u> $\partial R_o(\rho_o,0)/\partial\rho$ <u>have nonzero real parts. Then there is an</u> $\varepsilon_o > 0$ <u>and functions</u> $\alpha_\varepsilon : R \times R^p \to R^q$, $\beta_\varepsilon : R \times R^p \to C$, $\alpha_\varepsilon(t,\zeta)$, $\beta_\varepsilon(t,\zeta)$ <u>continuous in</u> t,ζ,ε <u>for</u> $t \in R$, $\zeta \in R^p$, $0 \leq \varepsilon \leq \varepsilon_o$, <u>almost periodic in</u> t <u>uniformly with respect to</u> ζ, <u>periodic in the components of</u> ζ <u>of period</u> 2π, $\alpha_o = \rho_o$, $\beta_o = 0$ <u>such that</u> $S(\alpha_\varepsilon,\beta_\varepsilon)$ <u>in</u> (18) <u>is an integral manifold of</u> (17) <u>for</u> $0 \leq \varepsilon \leq \varepsilon_o$. <u>Furthermore, if</u> $\gamma_\varepsilon = (\alpha_\varepsilon,\beta_\varepsilon)$, <u>then</u> $\partial\gamma_\varepsilon(t,\zeta)/\partial t^{\beta_o}\partial\zeta_1^{\beta_1}\ldots\partial\zeta_p^{\beta_p}$ <u>exists and is continuous for</u> $\beta_o \leq k$, $\beta_o+\beta_1+\ldots+\beta_p \leq k+\ell$ <u>if the functions in</u> (17) <u>have</u> k <u>lipschitz continuous derivatives with respect to</u> t <u>and</u> $(k+\ell)$ <u>lipschitz continuous derivatives with respect to</u> (ζ,ρ,φ^Q). <u>Finally, the manifold</u> $S(\alpha_\varepsilon,\beta_\varepsilon)$ <u>is asymptotically stable</u>* <u>if the matrix</u> $\partial R_o(\rho_o)/\partial\rho$

*The stability here is the same sense as in Section V.1; namely C^1 perturbations in the initial data.

has all eigenvalues with negative real parts and unstable if there is one eigenvalue with a positive real part.

Sketch of the proof: We only give the main elements of the proof of Theorem 2 since it is so similar to the usual ones in the theory of ordinary differential equations. Also, to avoid so many formulas, we assume all eigenvalues of $E \overset{\text{def}}{=} \partial R_o(\rho_o,0)/\partial\rho$ have negative real parts and $|\exp Et| \leq K\exp(-ct)$, $t \geq 0$. Letting $\rho \to \rho_o + \rho$, the equations (17) become

$$
\begin{aligned}
&\text{a)} \quad \dot{\zeta} = \omega_2(\varepsilon) + \varepsilon\, \overline{Z}_1(t,\zeta,\rho,x_t^Q,\varepsilon) \\
(19) \qquad &\text{b)} \quad \dot{\rho} = \varepsilon E\rho + \varepsilon\, \overline{R}_1(t,\zeta,\rho,x_t^Q,\varepsilon) \\
&\text{c)} \quad x_t^Q = T_\varepsilon(t-\sigma)x_\sigma^Q + \varepsilon \int_\sigma^t [d_s W^Q_{\varepsilon,t-s}]\overline{F}_1(s,\zeta(s),\rho(s),x_s^Q,\varepsilon)ds
\end{aligned}
$$

where $\omega_2(0) = \omega_o$, $\overline{F}_1(t,\zeta,\rho,\varphi,\varepsilon) = F_1(t,\zeta,\rho_o+\rho,\varphi,\varepsilon)$ and $\overline{Z}_1,\overline{R}_1$ satisfy the following properties. For any given $r > 0$, $\varepsilon_1 > 0$, $H > 0$, there exist a constant $K_1 > 0$ and a continuous nondecreasing function $\nu(\varepsilon)$, $0 \leq \varepsilon \leq \varepsilon_1$ such that $\nu(0) = 0$ and

$$
\begin{aligned}
&|\overline{Z}_1(t,\zeta,0,0,\varepsilon)| \leq \nu(\varepsilon),\ |\overline{R}_1(t,\zeta,0,0,\varepsilon)| \leq \nu(\varepsilon), \\
&\qquad |\overline{F}_1(t,\zeta,0,0,\varepsilon)| \leq K_1, \\
&|\overline{Z}_1(t,\zeta,\rho,\varphi,\varepsilon)-\overline{Z}_1(t,\zeta,\rho_1,\varphi_1,\varepsilon)| \leq \\
&\qquad \leq [\nu(\varepsilon) + k_1H][|\zeta-\zeta_1| + |\rho-\rho_1|] + K_1\|\varphi-\varphi_1\|,
\end{aligned}
$$

(20)

$$|\bar{R}_1(t,\zeta,\rho,\varphi,\varepsilon)-\bar{R}_1(t,\zeta_1,\rho_1,\varphi_1,\varepsilon)| \leq$$

$$\leq [\nu(\varepsilon)+K_1H]|\zeta-\zeta_1|+[\nu(r)+\nu(\varepsilon)+K_1H]|\rho-\rho_1|+K_1\|\varphi-\varphi_1\|,$$

$$|\bar{F}_1(t,\zeta,\rho,\varphi,\varepsilon)-\bar{F}_1(t,\zeta,\rho,\varphi,\varepsilon)| \leq K_1[|\zeta-\zeta_1|+|\rho-\rho_1|+\|\varphi-\varphi_1\|]$$

for $t \in R$, ζ, $\zeta_1 \in R^p$, $\rho,\rho_1 \in R^q$, $|\rho|,|\rho_1| \leq r$, $\varphi,\varphi_1 \in C$, $\|\varphi\|,\|\varphi_1\| \leq H$ and $0 \leq \varepsilon \leq \varepsilon_1$. Of course, all functions are almost periodic in t and periodic in ζ.

Let $\mathscr{C}_1(\Delta_1,D_1)$ be the class of continuous functions $\alpha : R \times R^p \to R^q$ which are bounded by D_1 and have lipschitz constant Δ_1 with respect to the second variable. Similarly, let $\mathscr{C}_2(\Delta_2,D_2)$ be the class of $\beta : R \times R^p \to C$. We introduce the uniform norm in these spaces and designate the norm by $\|\alpha\|_{\mathscr{C}_1}$, $\|\beta\|_{\mathscr{C}_2}$, $\alpha \in \mathscr{C}_1(\Delta_1,D_1)$, $\beta \in \mathscr{C}_2(\Delta_2,D_2)$.

It is convenient to introduce some notation. For $\alpha \in \mathscr{C}_1(\Delta_1,D_1)$, $\beta \in \mathscr{C}_2(\Delta_2,D_2)$, we abbreviate the collection $(t,\zeta,\alpha(t,\zeta),\beta(t,\zeta),\varepsilon)$ by $(t,\zeta,\alpha,\beta,\varepsilon)$. Also let

(21)

$$a = (\nu(\varepsilon)+K_1D_2)(1+\Delta_1)+K_1\Delta_2$$

$$b = \nu(\varepsilon) + K_1D_2$$

and then it follows from (14) that

$$
(22)\quad
\begin{aligned}
|\bar{R}_1(t,\zeta,\alpha,\beta,\varepsilon)| &\leq \nu(\varepsilon) + [b+\nu(D_1)]D_1+K_1D_2 \\
|\bar{F}_1(t,\zeta,\alpha,\beta,\varepsilon)| &\leq K_1(1+D_1+D_2) \\
|\bar{Z}_1(t,\zeta,\alpha,\beta,\varepsilon)-\bar{Z}_1(t,\zeta_1,\alpha_1,\beta_1,\varepsilon)| &\leq a|\zeta-\zeta_1| +b\|\alpha-\alpha_1\|_{\mathcal{C}_1} +K_1\|\beta-\beta_1\|_{\mathcal{C}_2} \\
|\bar{R}_1(t,\zeta,\alpha,\beta,\varepsilon)-\bar{R}_1(t,\zeta_1,\alpha_1,\beta_1,\varepsilon)| &\leq [a+\nu(D_1)\Delta_1]|\zeta-\zeta_1| +[b+\nu(D_1)]\|\alpha-\alpha_1\|_{\mathcal{C}_1} + \\
&\quad + K_1\|\beta-\beta_1\|_{\mathcal{C}_2} \\
|\bar{F}_1(t,\zeta,\alpha,\beta,\varepsilon)-\bar{F}_1(t,\zeta_1,\alpha_1,\beta_1,\varepsilon)| &\leq K_1\{(1+\Delta_1+\Delta_2)|\zeta-\zeta_1| +\|\alpha_1-\alpha_2\|_{\mathcal{C}_1} +\|\beta_1-\beta_2\|_{\mathcal{C}_2}\}
\end{aligned}
$$

With the constants defined as above choose $\varepsilon_1 > 0$ and continuous $\Delta_j(\varepsilon), D_j(\varepsilon)$, $0 \leq \varepsilon \leq \varepsilon_1$, $\Delta_j(\varepsilon)$, $D_j(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that, for $0 \leq \varepsilon \leq \varepsilon_1$,

$$
(23)\quad
\begin{aligned}
&\nu(\varepsilon) + [b+\nu(D_1)]D_1+ K_1D_2 \leq D_1c/K \ ; \\
&\varepsilon K_1(1+D_1+D_2) \leq D_2\sqrt{2c}/K \ ; \\
&a+b\Delta_1+2\nu(D_1)\Delta_1+K_1\Delta_2 \leq \Delta_1c/2K \ ; \\
&2\sqrt{3}\,\varepsilon KK_1/\sqrt{c} \leq \Delta_2 \leq \min\,[KK_1/4c,\ 1/4] \ ; \\
&c-a > c/2;\ c-\varepsilon a > c/2 \ ;\ \Delta_1+K < K/4;\ 2b^2+1 \leq 4 \\
&\Delta_1ab+K(b+\nu(D_1)) < c/4;\ 1+8K_1^2\varepsilon^2/3c^2 \leq 4
\end{aligned}
$$

Let $\mathcal{C}(\Delta,D) = \mathcal{C}_1(\Delta_1,D_1) \times \mathcal{C}_2(\Delta_2,D_2)$ and for any γ in $\mathcal{C}(\Delta,D)$, $\gamma = (\alpha,\beta)$, define $\|\gamma\| = \|\alpha\|_{\mathcal{C}_1} + KK_1\|\beta\|_{\mathcal{C}_2}/4$. For any $\gamma = (\alpha,\beta)$ in $\mathcal{C}(\Delta,D)$, let $\zeta(t,\sigma,\theta,\gamma)$, $\zeta(\sigma,\sigma,\theta,\gamma) = \theta$, be the solution of

(19a) with (ρ, x_t) replaced by $\gamma(t,\zeta)$ and define a transformation $T\gamma$ by

$$\text{a)}\quad T\gamma = (T_1\gamma, T_2\gamma)$$

$$\text{(24)}\qquad \text{b)}\quad (T_1\gamma)(\sigma,\theta) = \varepsilon \int_{-\infty}^{0} e^{-\varepsilon E u}\, \overline{R}_1(u+\sigma, \zeta(u+\sigma,\sigma,\theta,\gamma), \gamma, \varepsilon)\,du$$

$$\text{c)}\quad (T_2\gamma)(\sigma,\theta) = \varepsilon \int_{-\infty}^{0} [d_u W^{Q}_{\varepsilon,-u}]\overline{F}_1(u+\sigma,\sigma,\theta,\gamma), \gamma, \varepsilon)\,du\ .$$

We shall show that this equation has a unique solution in $\mathcal{C}(\Delta, D)$ for $0 \leq \varepsilon \leq \varepsilon_0$. This will prove the existence of an integral manifold. From (12),(22) and (23), we have $\|T_1\gamma\|_{\mathcal{C}_1} \leq D_1$, $\|T_2\gamma\|_{\mathcal{C}_2} \leq D_2$.

From the Lipschitz constant of $\overline{Z}_1$ in (22) and (19a), we obtain

$$|\zeta(u+\sigma,\sigma,\theta_1,\gamma_1) - \zeta(u+\sigma,\sigma,\theta_2,\gamma_2)| \leq$$

$$\leq e^{-\varepsilon a u}|\theta_1-\theta_2| + (e^{-\varepsilon a u}-1)[b\|\alpha_1-\alpha_2\|_{\mathcal{C}_1} + \frac{K_1}{a}\|\beta_1-\beta_2\|_{\mathcal{C}_2}]$$

for $-\infty < u \leq 0$.

Using this fact and the estimates (22),(23), we have

$$|(T_1\gamma_1)(\sigma,\theta_1) - (T_1\gamma_2)(\sigma,\theta_2)| \leq \Delta_1|\theta_1-\theta_2| + \|\gamma_1-\gamma_2\|_{\mathcal{C}}/4,$$

$$|(T_2\gamma_1)(\sigma,\theta_1) - (T_2\gamma_2)(\sigma,\theta_2)| \leq \Delta_2|\theta_1-\theta_2| + \|\gamma_1-\gamma_2\|_{\mathcal{C}}/KK_1$$

for $0 \leq \varepsilon \leq \varepsilon_1$. This implies $T : \mathcal{C}(\Delta,D) \to \mathcal{C}(\Delta,D)$ and is a

contraction since $\|T\gamma_1 - T\gamma_2\| \leq \|\gamma_1 - \gamma_2\|_{\mathscr{C}}/4$ for $0 \leq \varepsilon \leq \varepsilon_1$. This completes the proof of the existence of an integral manifold and also shows that the integral manifold is lipschitzian in ζ.

To obtain the smoothness properties of the manifold $S(\alpha_\varepsilon, \beta_\varepsilon)$ one proceeds in exactly the same manner as above except making use of a different class of functions $\mathscr{C}_1, \mathscr{C}_2$. For example, to show that $\alpha_\varepsilon, \beta_\varepsilon$ have continuous first derivatives with respect to ζ if the functions in (17) have continuous first derivatives with respect to ζ, ρ, φ^Q one defines $\mathscr{C}_1(\Delta_1, D_1)$ to be a class of functions $\alpha : R \times R^p \to R^q$ such that $|\alpha(t,\zeta)| \leq D_1$, $|\partial\alpha(t,\zeta)/\partial\zeta| \leq D_2$ for all t, ζ. The class $\mathscr{C}_2(\Delta_2, D_2)$ is defined in the same manner. Using the same definition of T as in (24), one shows by a proper choice of $\Delta_j(\varepsilon), D_j(\varepsilon) \to 0$ as $\varepsilon \to 0$ that T has a unique fixed point in $\mathscr{C}_1 \times \mathscr{C}_2$. The other derivatives are analyzed in exactly the same manner.

We will not prove the stability result since it again involves complicated estimates of the above type and the reader can easily supply the details by following the standard procedure in the method of integral manifolds in ordinary differential equations together with the lemma 1 of section V.1.

It is clear that Theorem 2 has an interpretation in the original equation (9) at the beginning of this section. For simplicity, we state an important corollary for the special case when F in (9) is independent of t. The notations are the ones given at the beginning of this section.

<u>COROLLARY 1</u>. Suppose $F(t,\varphi) = \overline{F}(\varphi)$ for all t and let

$$(25) \qquad G(\rho) = \nu(0)\rho + \\ + \frac{1}{2\pi}\int_0^{2\pi} [\psi_{1o}(0)\cos s+\psi_{2o}(0)\sin s]\overline{F}[\rho(\varphi_{1o}\cos s+\varphi_{2o}\sin s)]ds.$$

If there is a ρ_o such that $G(\rho_o) \neq 0$, $dG(\rho_o)/d\rho \neq 0$, then there is an $\varepsilon_1 > 0$, a constant $\omega^*(\varepsilon)$ and a function $x^*(t,\varepsilon)$, continuous in t,ε and having a continuous derivative with respect to t, $-\infty < t < \infty$, $0 \leqq \varepsilon \leqq \varepsilon_1$,

$$x^*(t,0) = \rho_o[\varphi_{1o}(0)\cos\omega_o t+\varphi_{2o}(0)\sin\omega_o t],$$

$\omega^*(0) = \omega_o$, $x^*(t+\omega^*(\varepsilon),\varepsilon) = x^*(t,\varepsilon)$ such that $x^*(t,\varepsilon)$ satisfies (9) and since it is differentiable satisfies (10). The periodic solution $x^*(\cdot,\varepsilon)$ is orbitally asymptotically stable* if $dG(\rho_o)/d\rho < 0$ and unstable if $dG(\rho_o)/d\rho > 0$.

As an example, consider the equation

$$(26) \qquad \dot{x}(t) = \alpha\dot{x}(t-r)-\beta x(t) - \alpha\gamma x(t-r) + \varepsilon F(x_t)$$

where $\varepsilon \geqq 0$, $r > 0$, $\gamma > \beta > 0$, $\alpha = \alpha(\varepsilon) = \alpha_o(1+\varepsilon)$, where α_o is

---

* A periodic solution $x(t)$ of (10) is called asymptotically orbitally stable if the orbit, $U_t x_t$, of x in C is asymptotically stable in the sense of C^1 perturbations.

the unique real number in $(0,1)$ such that the characteristic equation III(35) for the linear system III(33) has two purely imaginary roots $\pm i\omega_o$, $\omega_o > 0$, and the remaining roots have real parts $< -\delta < 0$. Brayton [13] has shown that such an α_o exists. This implies that there is an $\varepsilon_1 > 0$ such that the equation

$$(27) \qquad \lambda - \alpha(\varepsilon)\lambda e^{-\lambda r} + \beta + \alpha(\varepsilon)\gamma e^{-\lambda r} = 0$$

has two simple roots $\varepsilon\nu(\varepsilon) \pm i\omega(\varepsilon)$, $\omega(0) = \omega_o$, $\nu(\varepsilon), \omega(\varepsilon)$ continuous in $0 \leqq \varepsilon \leqq \varepsilon_1$, and the remaining roots have real parts $< -\delta < 0$ for $0 \leqq \varepsilon \leqq \varepsilon_1$. We are writing the equation (26) in differential form for simplicity in notation but it always understood that solutions are defined by means of the integrated form of this equation.

In the discussion of this example, we use the notations introduced at the end of section III. A straightforward computation on the characteristic equation (27) shows that

$$(28) \qquad 2\nu(0) = \frac{\beta C + \omega_o D}{C^2 + D^2} > 0 ,$$

where C,D are defined in III(40). Using the formula for $\Psi(0)$ in III(40), it is easily seen that the function $G(\rho)$ in (25) is given by

$$G(\rho) = \frac{\beta C+\omega_o D}{C^2+D^2} G^*(\rho)$$

(29)

$$G^*(\rho) = \frac{\rho}{2} + \frac{1}{\beta C+\omega_o D} \cdot \frac{1}{2\pi} \int_o^{2\pi} (D\cos s+C\sin s)F(\rho(\varphi_{1o}\cos s+\varphi_{2o}\sin s))ds$$

From Corollary 1, we can now state the following result: equation (26) will have an asymptotically orbitally stable periodic solution if there exists a ρ_o such that $G^*(\rho_o) = 0$, $dG^*(\rho_o)/d\rho < 0$ and an unstable one if $G^*(\rho_o) = 0$, $dG^*(\rho_o)/d\rho > 0$.

In the particular case where $F(x_t) = h(x(t))$ relation (29) yields

$$G^*(\rho) = \frac{\rho}{2} + \frac{C}{\beta C+\omega_o D} \cdot \frac{1}{2\pi} \int_o^{2\pi} h(\rho\cos \zeta)\cos \zeta d\zeta$$

and the criterion for existence of a periodic solution is the same as the one obtained by Brayton [13]. However, we can also say something about the stability of the solution. In the particular case, when $h(x) = -x^3$, an easy computation yields $G^*(\rho) = (\rho/2)[1-3C\rho^2/4(\beta C+\omega_o D)]$ and $G^*(\rho_o) = 0$, $dG^*(\rho_o)/d\rho = -1$ for $\rho_o^2 = 4(\beta C+\omega_o D)/3C$. Thus, the equation has an asymptotically orbitally stable periodic solution.

As another illustration, suppose $F(x_t) = -x^3(t-s)$, $0 \leq s \leq r$. Then

$$(2/\rho)G^*(\rho) = 1 - \frac{3\rho^2}{4(\beta C+\omega_o D)}[C \cos \omega_o s - D \sin \omega_o s] .$$

As before, if $C \cos \omega_o s - D \sin \omega_o s > 0$, the we obtain an asymptotically orbitally stable periodic solution. To find the limitations on s for which this inequality remains valid is difficult since ω_o depends upon all parameters in the linear differential equation III(33).

This example illustrates the application of the general theory to autonomous systems, but it is clear that Theorem 2 is equally applicable to nonautonomous equation.

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