NASA TECHNICAL NOTE

NASA TN D-3855

C.1

NASA TN D-3855

DEFORMATION OF SHALLOW SPHERICAL SANDWICH SHELLS UNDER LOCAL LOADING

by John N. Rossettos

Langley Research Center

Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • FEBRUARY 1967

NASA TN D-3855

DEFORMATION OF SHALLOW SPHERICAL SANDWICH SHELLS UNDER LOCAL LOADING

By John N. Rossettos

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $2.00

DEFORMATION OF SHALLOW SPHERICAL SANDWICH SHELLS UNDER LOCAL LOADING

By John N. Rossettos
Langley Research Center

SUMMARY

Analytical solutions are obtained for the small symmetrical deformation of shallow spherical sandwich caps under both uniform and ring-type local loading with various types of edge support. The governing equations, which include transverse shear deformation, are reduced to a single complex second-order differential equation. The solution is obtained in terms of Kelvin functions and a parameter which measures the shear flexibility of the sandwich core. Explicit formulas for calculating the stresses and displacements at all points in the cap have been obtained.

Typical results are presented. They indicate that for localized loading, where the radius of the local load is one-tenth of the shell edge radius or less, transverse shear deformation plays an important role over a wide range of sandwich core stiffness. In general, the stresses and displacements for uniform localized loading are greater than those for a corresponding ring load and depend on the loading area. For all cases where the edge of the cap is clamped and/or restrained against radial expansion, the maximum lateral deflection at the center of the cap varies by at most 20 percent, depending on the precise edge condition. The deflections for these cases, however, differ substantially (by 60 to 100 percent) from the deflection for the case where the edge is both simply supported and unrestrained against radial expansion.

INTRODUCTION

The utilization of sandwich-type shells in aerospace vehicles provides structures with high strength-weight ratios. One shell component that is widely used is the shallow spherical sandwich cap (e.g., the aft heat shield of the Apollo command module and bulkheads for pressurized tankage). Such components are often subjected to various types of local loading distributed over a small finite area. Furthermore, the degree of edge restraint varies and is largely dictated by the support structure in a specific design configuration. The present study was undertaken to determine the influence of such factors on the spherical sandwich cap.

References 1 to 4 describe previous studies which involve sandwich-type shells, and also include additional bibliography and discussion of past work. Some sample numerical calculations, chosen for their simplicity for purposes of evaluating the influence of transverse shear deformation in sandwich plates and cylindrical shells, are given in reference 2. In general, however, much of the literature applicable to the detailed deformation behavior of sandwich shells of revolution has been concerned to a large extent with establishing appropriate governing equations for use in applications. (See ref. 4.)

The work herein is concerned with obtaining analytical solutions to the linear equations for the symmetrical deformation of spherical caps, taking into account transverse shear deformations, as is essential for sandwich shells. These solutions provide explicit formulas for calculating the stresses and deformations for all points in the shell. They should be directly useful for design as well as for checking large computer programs for solution of more complex sandwich-shell problems. An assessment is also made of the effects of two types of local load and various edge supports. In order to bracket realistic cases of local loading (for example, impact pressure distributions on the Apollo command module) two extreme types – a uniformly distributed local load and a circular line (ring-type) load – are considered. In conjunction with such loadings, simple and clamped edge supports and elastic edge-restraint conditions are treated.

In the present analysis, the relevant equilibrium equations and boundary conditions are derived directly for the shallow spherical shell rather than being obtained as special cases of more general theory such as that presented in reference 1. Analytical solutions are obtained in terms of Kelvin functions and their first derivatives. These functions have been extensively tabulated in the literature. Furthermore, accurate polynomial approximations to these functions are available. (See, for instance, ref. 5.) The present solutions exhibit certain well-defined singularities as the area of local loading approaches zero. These singularities are obtained from the leading terms of the series representation of the Kelvin functions for small values of their argument, and reduce to those obtained for conventional spherical caps with no transverse shear deformation.

Calculated values for stresses and displacements which are obtained by means of the present solution are given for various conditions of loading and edge support. In particular, the effect that the area and type (i.e., uniform or ring) of local loading has on deflections and stresses is evaluated. Transverse shear deformation is seen to play an important role in the results, especially when the loading is highly localized. This influence is depicted by specifying values of a physical parameter which is a measure of the transverse shear stiffness of the sandwich shell (this parameter appears explicitly in the general solution). The results also indicate important quantitative differences between practical types of edge support and provide an understanding of how sensitive the various quantities are to the precise edge condition. Calculations of deflection and moment distributions for several areas of localized loading show that only a small portion of the cap near the load is sensitive to the precise size of the area of local load. Finally, the role of curvature is established quantitatively by obtaining results in a range which includes the flat plate as a limiting case. Even slight curvature is found to modify substantially the flat-plate values for deflection and bending moment.

SYMBOLS

$A_s = \left(2E_s t_s\right)^{-1}$

a radius of circle described by cap edge (see fig. 1)

$\left.\begin{array}{l}\mathrm{bei}\\ \mathrm{ber}\end{array}\right\}$ Kelvin functions

C_u constant defined by equation (50)

C_φ constant defined by equation (52)

$c, c_1, \ldots, c_4$ constants of integration

$\bar{c}$ radius of circular local load (see fig. 1)

D_Q transverse shear stiffness of isotropic sandwich cap, $G_c\left(h + t_s\right)$

D_s flexural stiffness of isotropic sandwich cap, $\dfrac{E_s t_s h^2}{2\left(1 - \nu^2\right)}$

d constant of integration

$\bar{d} = \dfrac{1}{1 + \alpha^2} \dfrac{d}{\sqrt{\mu}\, A_s}$

E_s Young's modulus of face sheets of isotropic sandwich cap

$e_s = \dfrac{3E_s t_s h}{D_Q R}\left[12\left(1 - \nu^2\right)\right]^{-1/2}$ (see eq. (27b))

e_{sp} sandwich core shear stiffness parameter, $\dfrac{e_s}{a^2/Rh} = \dfrac{3E_s t_s}{\sqrt{12\left(1 - \nu^2\right)}D_Q} \dfrac{h^2}{a^2}$

F	stress function equal to rN_r
f	complex function, $F + \lambda\varphi$
G_c	shear modulus of sandwich core
$H_1, \ldots, H_4$	functions defined in appendix B
H_e	horizontal prescribed edge stress resultant
h	depth of isotropic sandwich cap measured between middle surface of faces
$J_1, \ldots, J_8$	functions defined in appendix B
$k_1, \ldots, k_8$	constants defined by equation (71)
kei, ker	Kelvin functions
M_r	radial bending moment
M_θ	circumferential bending moment
m_r	nondimensional radial bending moment (see eq. (76))
m_θ	nondimensional circumferential bending moment (see eq. (76))
N_r	radial stress resultant
N_θ	circumferential stress resultant
n_r	nondimensional radial stress resultant (see eq. (76))
n_θ	nondimensional circumferential stress resultant (see eq. (76))
P	magnitude of local load
p	intensity of local load in pressure units

p_H horizontal component of surface loading

p_V vertical component of surface loading

Q transverse shear stress resultant

q nondimensional shear stress resultant (see eq. (76))

R radius of curvature of spherical cap

r radial coordinate

r_1, r_2 values of r at inner and outer edges of a spherical shell

t_s thickness of faces of sandwich cap

u radial displacement

V vertical stress resultant defined by equation (17a)

V_e vertical prescribed edge stress resultant

W nondimensional vertical displacement (see eq. (76))

w vertical displacement

x argument of functions in appendix B, $\sqrt{\mu}\,\frac{r}{R}$

$$x_a = \sqrt{\mu}\,\frac{a}{R} = \left[\frac{4}{3}\left(3 - e_s^{\,2}\right)\left(1 - \nu^2\right)\right]^{1/4}\left(\frac{a^2}{Rh}\right)^{1/2}$$

$$x_{\bar{c}} = \sqrt{\mu}\,\frac{\bar{c}}{R} = \frac{\bar{c}}{a}\,x_a$$

α transverse shear parameter, $e_s\left(3 - e_s^{\,2}\right)^{-1/2}$

γ transverse shear strain

ϵ_r radial strain

ϵ_θ	circumferential strain
θ	circumferential coordinate
κ_r	radial curvature
κ_θ	circumferential curvature
$\Lambda_1, \ldots, \Lambda_8$	functions defined in appendix B
λ	complex quantity given by equation (27), $\frac{R}{A_s \mu} \frac{i - \alpha}{1 + \alpha^2}$

$\mu = \frac{1}{3}\left(3 - e_s{}^2\right)^{1/2} \left[12\left(1 - \nu^2\right)\right]^{1/2} \frac{R}{h}$

ν	Poisson's ratio
φ	rotation of the normal to middle surface of cap
ψ	rotation of the tangent to middle surface of cap
$\Omega_1, \ldots, \Omega_8$	functions defined in appendix B

Subscripts:

a	annular region, $\bar{c} \leqq r \leqq a$ (see fig. 1)
e	edge value
e_1, e_2	inner and outer edge of shell (see eq. (7))
i	inner region, $r \leqq \bar{c}$ (see fig. 1)

ANALYSIS

Fundamental Equations

The governing linear equations for shallow spherical shells undergoing axially symmetric displacements, including deformation due to transverse shear, are derived with the notation and geometry in figure 1. All quantities (stress resultants, moments, and displacements) are positive as shown. The relevant strain-displacement equations are consistent with the Donnell-Mushtari-Vlasov approximation for shallow shells. (See ref. 6.) The appropriate strain-displacement relations, which include transverse shear deformation (see ref. 1), are

$$\epsilon_r = \frac{du}{dr} + \frac{r}{R}\frac{dw}{dr} \tag{1}$$

$$\epsilon_\theta = \frac{u}{r} \tag{2}$$

$$\kappa_r = \frac{d\varphi}{dr} \tag{3}$$

$$\kappa_\theta = \frac{\varphi}{r} \tag{4}$$

$$\psi = \frac{dw}{dr} \tag{5}$$

$$\gamma = \varphi - \psi \tag{6}$$

The rotation of the normal to the middle surface of the shell is denoted by φ, and the rotation of the tangent to the middle surface is denoted by ψ.

The principle of virtual work is written as follows:

$$\begin{aligned}\int_0^{2\pi}\int_{r_1}^{r_2}&\left(N_r\ \delta\epsilon_r + N_\theta\ \delta\epsilon_\theta + M_r\ \delta\kappa_r + M_\theta\ \delta\kappa_\theta + Q\ \delta_\gamma\right) r\ dr\ d\theta \\ &= \int_0^{2\pi}\int_{r_1}^{r_2}\left(p_V\ \delta w + p_H\ \delta u\right) r\ dr\ d\theta + \int_0^{2\pi}\left(V_{e1}\ \delta w + H_{e1}\ \delta u\right) r_1\ d\theta \\ &\quad + \int_0^{2\pi}\left(V_{e2}\ \delta w + H_{e2}\ \delta u\right) r_2\ d\theta\end{aligned} \tag{7}$$

The equilibrium equations and boundary conditions which are a consequence of equation (7) and equations (1) to (6) are

$$\frac{d}{dr}\left[\left(\frac{r}{R}\right) rN_r - rQ\right] = -rp_V \tag{8}$$

$$\frac{d}{dr}\left(rN_r\right) - N_\theta = -rp_H \tag{9}$$

$$\frac{d}{dr}\left(rM_r\right) - M_\theta - rQ = 0 \tag{10}$$

The boundary conditions at the edges r_1 or r_2 are to prescribe

$$u \quad \text{or} \quad N_r \tag{11}$$

$$w \quad \text{or} \quad \frac{r}{R} N_r - Q \tag{12}$$

$$\varphi \quad \text{or} \quad M_r \tag{13}$$

The stress-strain relations for isotropic sandwich shells (see, for instance, ref. 3) in this case become

$$\epsilon_r = A_s\left(N_r - \nu N_\theta\right) \tag{14a}$$

$$\epsilon_\theta = A_s\left(N_\theta - \nu N_r\right) \tag{14b}$$

$$M_r = D_s\left(\kappa_r + \nu\kappa_\theta\right) \tag{15a}$$

$$M_\theta = D_s\left(\kappa_\theta + \nu\kappa_r\right) \tag{15b}$$

$$Q = D_Q\gamma \tag{16}$$

Auxiliary quantities, a vertical stress resultant V and a stress function F, are defined as follows:

$$V = \frac{r}{R} N_r - Q \tag{17a}$$

$$F = rN_r \tag{17b}$$

In the present study, the horizontal component of applied surface force p_H is taken equal to zero. Equations (9) and (17) then yield

$$N_r = \frac{F}{r} \tag{18}$$

$$N_\theta = \frac{dF}{dr} \tag{19}$$

$$Q = \frac{F}{R} - V \tag{20}$$

The problem can now be reduced to the solution of two coupled second-order differential equations for the variables F and φ. First the vertical equilibrium equation (8) is integrated, and equation (17a) is substituted into the result to yield

$$rV = -\int r\, p_V(r)\, dr + c \tag{21}$$

where c is a constant of integration. From equations (1), (2), and (5) the following compatibility equation is obtained:

$$\epsilon_r - \frac{r}{R}\psi = \frac{d}{dr}\left(r\epsilon_\theta\right) \tag{22}$$

Equation (22) can be written in terms of F and φ, by substituting from equations (6), (14), (16), and (18) to (20), to obtain

$$r^2 \frac{d^2F}{dr^2} + r\frac{dF}{dr} - F + \frac{r^2}{A_sR}\varphi - \frac{r^2}{R^2A_sD_Q}F = -\frac{r^2V}{RA_sD_Q} \tag{23}$$

A second equation relating F and φ is obtained from the moment equation (10) by substitution from equations (3), (4), (15), and (20):

$$r^2\frac{d^2\varphi}{dr^2} + r\frac{d\varphi}{dr} - \varphi - \frac{r^2}{RD_s}F = -\frac{r^2}{D_s}V \tag{24}$$

The two coupled equations (23) and (24) can finally be combined to form a single complex second-order differential equation in which the dependent variable is a complex function:

$$f = F + \lambda\varphi \tag{25}$$

The complex quantity λ is chosen so that when equation (24) is multiplied by λ and added to equation (23) the following equation for f is obtained:

$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} - \left[1 + \frac{r^2}{R^2}\mu(i + \alpha)\right]f = -\frac{r^2}{R}\mu(i + \alpha)V \tag{26}$$

The appropriate expression for λ is found to be

$$\lambda = \frac{R}{A_s \mu} \frac{i - \alpha}{1 + \alpha^2} \tag{27}$$

where

$$\alpha = e_s\left(3 - e_s^{\,2}\right)^{-1/2} \tag{27a}$$

$$e_s = \frac{3E_s t_s h}{D_Q R}\left[12\left(1 - \nu^2\right)\right]^{-1/2} \tag{27b}$$

$$\mu = \frac{1}{3}\left(3 - e_s^{\,2}\right)^{1/2}\left[12\left(1 - \nu^2\right)\right]^{1/2} \frac{R}{h} \tag{27c}$$

It is noted that the parameter α $\left(\text{or } e_s\right)$ is a measure of the influence of transverse shear deformation (that is, $\alpha \rightarrow 0$ and $e_s \rightarrow 0$ for no transverse shear deformation).

General Solution of Governing Equations

Particular integrals.- A particular integral of equation (26) can be determined once the loading p_V is specified. For a uniformly distributed load on the cap, $p_V = p$. Then, from equation (21), the expression for rV for regions under a uniform load is

$$rV = -\frac{pr^2}{2} + c \tag{28}$$

With V given by equation (28), a particular solution of equation (26) for regions under a uniform load is

$$f = \frac{cR}{r} - \frac{prR}{2} \tag{29}$$

For regions on which no load acts, $p = 0$. A particular solution of equation (26) in this case is

$$f = \frac{cR}{r} \tag{30}$$

Homogeneous solution.- The homogeneous solution to equation (26) can be written in terms of the modified Bessel functions $I_1\left(\sqrt{i+\alpha}\; x\right)$ and $K_1\left(\sqrt{i+\alpha}\; x\right)$, in which $x = \sqrt{\mu}\,\frac{r}{R}$. (See ref. 5.) As shown in appendix A, I_1 and K_1 can be expanded in a series of positive powers of the transverse shear parameter α (values of α are small for most practical cases). The coefficients of the various powers of α are expressed in terms of Kelvin functions with argument x and their first derivatives. These functions are extensively tabulated and polynomial approximations of high accuracy have also been developed to represent them. (See ref. 5.) The following representations of $I_1\left(\sqrt{i+\alpha}\, x\right)$ and $K_1\left(\sqrt{i+\alpha}\, x\right)$, as developed and discussed in appendix A, are to be used in the present study:

$$i^{1/2} I_1\left(\sqrt{i+\alpha}\, x\right) = \text{ber}' x + \frac{\alpha}{2} x\left(\text{ber}\, x - \frac{\text{bei}' x}{x}\right) + \frac{\alpha^2}{8}\left(x^2\, \text{bei}' x - 2x\, \text{bei}\, x - 3\, \text{ber}' x\right)$$
$$+ i\left[\text{bei}' x + \frac{\alpha x}{2}\left(\text{bei}\, x + \frac{\text{ber}' x}{x}\right) + \frac{\alpha^2}{8}\left(-x^2\, \text{ber}' x + 2x\, \text{ber}\, x - 3\, \text{bei}' x\right)\right] \tag{31}$$

$$-i^{1/2} K_1\left(\sqrt{i+\alpha}\, x\right) = \text{ker}' x + \frac{\alpha x}{2}\left(\text{ker}\, x - \frac{\text{kei}' x}{x}\right) + \frac{\alpha^2}{8}\left(x^2\, \text{kei}' x - 2x\, \text{kei}\, x - 3\, \text{ker}' x\right)$$
$$+ i\left[\text{kei}' x + \frac{\alpha x}{2}\left(\text{kei}\, x + \frac{\text{ker}' x}{x}\right) + \frac{\alpha^2}{8}\left(-x^2\, \text{ker}' x + 2x\, \text{ker}\, x - 3\, \text{kei}' x\right)\right] \tag{32}$$

where primes denote differentiation with respect to the argument x. The homogeneous solution to equation (26) is then written as follows:

$$f = \left(c_1 - ic_2\right)\left[i^{1/2} I_1\left(\sqrt{i+\alpha}\, x\right)\right] + \left(c_3 - ic_4\right)\left[-i^{1/2} K_1\left(\sqrt{i+\alpha}\, x\right)\right] \tag{33}$$

where c_1, c_2, c_3, and c_4 are real constants. The particular solution (eq. (29)) is then added to equation (33) to obtain the complete solution for f. If the definition of f (eq. (25)) is taken into account, and real and imaginary parts are equated by using equations (29), (31), (32), and (33), the following solutions for the stress function F and the rotation of the normal to the middle surface φ are obtained:

$$F = c_1\Lambda_1(\alpha,x) + c_2\Lambda_2(\alpha,x) + c_3\Lambda_3(\alpha,x) + c_4\Lambda_4(\alpha,x) + \frac{cR}{r} - \frac{prR}{2} \tag{34}$$

$$\frac{1}{1+\alpha^2}\frac{R}{\mu A_s}\varphi = c_1\Lambda_5(\alpha,x) - c_2\Lambda_6(\alpha,x) + c_3\Lambda_7(\alpha,x) - c_4\Lambda_8(\alpha,x) \tag{35}$$

The functions $\Lambda_1(\alpha,x), \ldots, \Lambda_8(\alpha,x)$ that appear in equations (34) and (35) depend on both the parameter α and x, involve Kelvin functions and their first derivatives, and are written out in appendix B.

Stresses and displacements.- The stress resultants N_r, N_θ, and Q and the moments M_r and M_θ can now be obtained by using the expressions for F and φ (eqs. (34) and (35)) and previous relations which provide a means of expressing all quantities in terms of F and φ. The arguments of the functions Λ_1, Λ_2, etc., and their derivatives with respect to x which appear in what follows are omitted for convenience. The results for the stresses are therefore written as

$$N_r = \frac{F}{r} = c_1 \frac{\Lambda_1}{r} + c_2 \frac{\Lambda_2}{r} + c_3 \frac{\Lambda_3}{r} + c_4 \frac{\Lambda_4}{r} + \frac{cR}{r^2} - \frac{pR}{2} \tag{36}$$

$$N_\theta = \frac{dF}{dr} = c_1 \frac{\sqrt{\mu}}{R} \Lambda_1' + c_2 \frac{\sqrt{\mu}}{R} \Lambda_2' + c_3 \frac{\sqrt{\mu}}{R} \Lambda_3' + c_4 \frac{\sqrt{\mu}}{R} \Lambda_4' - \frac{cR}{r^2} - \frac{pR}{2} \tag{37}$$

$$Q = \frac{F}{R} - V = c_1 \frac{\Lambda_1}{R} + c_2 \frac{\Lambda_2}{R} + c_3 \frac{\Lambda_3}{R} + c_4 \frac{\Lambda_4}{R} \tag{38}$$

$$\begin{aligned} M_r = D_s\left(\frac{d\varphi}{dr} + \nu \frac{\varphi}{r}\right) = \left(1 + \alpha^2\right)\frac{\mu A_s D_s}{R}\Bigg[\left(c_1 \frac{\sqrt{\mu}}{R} \Lambda_5' - c_2 \frac{\sqrt{\mu}}{R} \Lambda_6' + c_3 \frac{\sqrt{\mu}}{R} \Lambda_7' - c_4 \frac{\sqrt{\mu}}{R} \Lambda_8'\right) \\ + \frac{\nu}{r}\left(c_1\Lambda_5 - c_2\Lambda_6 + c_3\Lambda_7 - c_4\Lambda_8\right)\Bigg] \end{aligned} \tag{39}$$

$$M_\theta = D_s\left(1 - \nu^2\right)\frac{\varphi}{r} + \nu M_r = \left(1 + \alpha^2\right)\left(1 - \nu^2\right)\frac{\mu A_s D_s}{Rr}\left(c_1\Lambda_5 - c_2\Lambda_6 + c_3\Lambda_7 - c_4\Lambda_8\right) + \nu M_r \tag{40}$$

The displacement w can be determined by integrating equation (5). First, if equations (6) and (16) are taken into account the expression for $\frac{dw}{dr}$ can be written as

$$\frac{dw}{dr} = \varphi - \frac{Q}{D_Q} \tag{41}$$

Equation (41) is then integrated by using equations (35) and (38), and w is written as

$$w = \left(1 + \alpha^2\right)\sqrt{\mu}\, A_s \int \left(c_1\Lambda_5 - c_2\Lambda_6 + c_3\Lambda_7 - c_4\Lambda_8\right)dx$$

$$- \frac{1}{\sqrt{\mu}\, D_Q} \int \left(c_1\Lambda_1 + c_2\Lambda_2 + c_3\Lambda_3 + c_4\Lambda_4\right)dx + d \tag{42}$$

or

$$w = \left(1 + \alpha^2\right)\sqrt{\mu}\, A_s\left(c_1H_1 + c_2H_2 + c_3H_3 + c_4H_4\right) + d \tag{43}$$

The functions $H_1, \ldots, H_4$ are defined in appendix B, and d is a constant of integration.

SPECIFIC SOLUTIONS

The general solution is utilized to obtain results for two cases of local loading which will bracket other intermediate situations. In one case the local load is uniformly distributed over a circular area of radius $\bar{c}$ with the center at the pole of the spherical cap (fig. 1(b)). In the other case a ring load, also of radius $\bar{c}$, is applied concentric with the cap pole (fig. 1(c)). In each case, various edge-support conditions are treated.

Uniformly Distributed Local Load

The boundary conditions which are to be imposed in the case of the uniformly distributed local load are now outlined. In order to do this conveniently, two regions of the cap are defined. The inner region covered by the local load is defined by $r \leqq \bar{c}$, and the annular region, where $p = 0$, is defined by $\bar{c} \leqq r \leqq a$. Subscripts i (for inner) and a (for annular) will be used to associate each quantity with its respective region. In order to avoid repetition in writing equations, it is noted that the expressions for the stress function and stress resultants in equations (34), (36), and (37), which hold for the inner region, can also be used for the annular region by setting p equal to zero.

Boundary conditions.- The following conditions are to be applied. At $r = 0$, N_r and N_θ must be finite, and also

$$\varphi_i = 0 \tag{44a}$$

$$u_i = 0 \tag{44b}$$

At $r = \bar{c}$, the continuity conditions which apply are given by

$$V_i = V_a \tag{45a}$$

$$N_{r,i} = N_{r,a} \tag{45b}$$

$$u_i = u_a \tag{45c}$$

$$w_i = w_a \tag{45d}$$

$$\varphi_i = \varphi_a \tag{45e}$$

$$M_{r,i} = M_{r,a} \tag{45f}$$

The boundary conditions at $r = a$ involve the following alternatives:

Set 1 (no radial restraint, simple support) –

$$N_{r,a} = 0 \qquad M_{r,a} = 0 \qquad w_a = 0 \tag{46}$$

Set 2 (no radial restraint, clamped) –

$$N_{r,a} = 0 \qquad \varphi_a = 0 \qquad w_a = 0 \tag{47}$$

Set 3 (radial restraint, clamped) –

$$u_a = 0 \qquad \varphi_a = 0 \qquad w_a = 0 \tag{48}$$

Set 4 (radial restraint, simple support) –

$$u_a = 0 \qquad M_{r,a} = 0 \qquad w_a = 0 \tag{49}$$

Equations (46) to (49) are actually limiting cases of the more general elastic edge-support conditions. In particular, for elastic edge restraint against radial expansion the following relation holds at $r = a$:

$$N_{r,a} = C_u u_a \tag{50}$$

where the constant C_u involves knowledge of the edge influence coefficients for the supporting structure, which for the present problem can be any shell of revolution. By means of equations (2) and (14), equation (50) may be written as

$$N_{\theta,a} - \left(\nu + \frac{1}{aA_sC_u}\right)N_{r,a} = 0 \tag{51}$$

For the case of elastic edge restraint against rotation the following relation holds at $r = a$:

$$M_{r,a} = -C_\varphi \varphi_a \tag{52}$$

where the constant C_φ also depends on the supporting structure. If equations (3), (4), and (15) are used, equation (52) may be written as

$$\frac{d\varphi_a}{dr} + \left(\frac{\nu}{a} + \frac{C_\varphi}{D_s}\right)\varphi_a = 0 \tag{53}$$

It is noted that equations (51) and (53) lead to conditions (46) to (49) when C_u and C_φ take on their limiting values of 0 and ∞.

Evaluation of constants of integration.- The conditions at $r = 0$ are now applied. The condition $\varphi_i(0) = 0$ with the use of equation (35) leads to

$$\lim_{x \to 0} \left(c_{1,i}\Lambda_5 - c_{2,i}\Lambda_6 + c_{3,i}\Lambda_7 - c_{4,i}\Lambda_8\right) = 0 \tag{54}$$

where the constants $c_{1,i}, \ldots, c_{4,i}$ refer to the solution valid in the region $r \leqq \bar{c}$. The condition $u_i(0) = 0$, with the use of equations (2), (14), (36), and (37) leads to

$$\lim_{x \to 0} \left[x\left(N_{\theta,i} - \nu N_{r,i}\right)\right] = 0 \tag{55}$$

where

$$N_{\theta,i} = \frac{\sqrt{\mu}}{R}\left(c_{1,i}\Lambda_1' + c_{2,i}\Lambda_2' + c_{3,i}\Lambda_3' + c_{4,i}\Lambda_4' - \frac{c_iR}{r^2} - \frac{pR}{2}\right) \tag{56}$$

$$N_{r,i} = \frac{1}{r}\left(c_{1,i}\Lambda_1 + c_{2,i}\Lambda_2 + c_{3,i}\Lambda_3 + c_{4,i}\Lambda_4 + \frac{c_iR}{r} - \frac{pRr}{2}\right) \tag{57}$$

It is also required that at $r = 0$, N_θ and N_r are to remain bounded. These conditions, together with equations (54) to (57), which involve limiting procedures as $x \to 0$ (that is, $r \to 0$), can be evaluated by making use of the expressions for the Kelvin functions of small argument which are given in appendix B and which appear in the various functions needed for the calculation. When this is done, it is found that the following constants must vanish:

$$c_i = c_{3,i} = c_{4,i} = 0 \tag{58}$$

The boundary conditions at the edge $r = a$ are considered next, for which the four alternatives are denoted by set 1, set 2, set 3, and set 4. Each set of edge conditions is given by three equations which are now discussed.

Consider the conditions given by set 1 (no radial restraint, simple support). In this case equations (46) are to be applied. Use is made of the general solutions (eqs. (34) to (43)) with $p = 0$ since solutions which are valid in the region $\bar{c} \leqq r \leqq a$ are employed here. The argument of the functions involved is $\sqrt{\mu}\,\frac{a}{R}$ (the value of x at $r = a$). For brevity the symbol x_a, defined by $x_a = \sqrt{\mu}\,\frac{a}{R}$, is employed. Equations (46) then lead to the following three equations:

$$c_{1,a}\Lambda_1(x_a) + c_{2,a}\Lambda_2(x_a) + c_{3,a}\Lambda_3(x_a) + c_{4,a}\Lambda_4(x_a) + \frac{c_a R}{a} = 0 \tag{59}$$

$$c_{1,a}\Omega_5(x_a) - c_{2,a}\Omega_6(x_a) + c_{3,a}\Omega_7(x_a) - c_{4,a}\Omega_8(x_a) = 0 \tag{60}$$

$$c_{1,a}H_1(x_a) + c_{2,a}H_2(x_a) + c_{3,a}H_3(x_a) + c_{4,a}H_4(x_a) + \bar{d}_a = 0 \tag{61}$$

The symbols $\Omega_5, \ldots, \Omega_8$ and $H_1, \ldots, H_4$ are defined in appendix B, and the constant $\bar{d}_a$ equals $\frac{1}{1 + \alpha^2}\,\frac{d_a}{\sqrt{\mu}\,A_s}$. The constants $c_{1,a}, \ldots, c_{4,a}, c_a, \bar{d}_a$ apply of course to the solution valid in $\bar{c} \leqq r \leqq a$.

Consider the conditions given by set 2 (no radial restraint, clamped). Two of the equations in this case are identical to equations (59) and (61). Equation (60), however, must now be replaced by

$$c_{1,a}\Lambda_5(x_a) - c_{2,a}\Lambda_6(x_a) + c_{3,a}\Lambda_7(x_a) - c_{4,a}\Lambda_8(x_a) = 0 \tag{62}$$

Consider the conditions given by set 3 (radial restraint, clamped). In this case the second and third equations are given by equations (62) and (61). The first equation to be used here is

$$c_{1,a}\Omega_1(x_a) + c_{2,a}\Omega_2(x_a) + c_{3,a}\Omega_3(x_a) + c_{4,a}\Omega_4(x_a) - \frac{1+\nu}{a} c_a R = 0 \tag{63}$$

Consider the conditions given by set 4 (radial restraint, simple support). The three equations to be used in this case are given by equations (63), (60), and (61), respectively.

For each set of the possible edge conditions just described, continuity conditions which are given by equations (45a) to (45f) are to be applied at $r = \bar{c}$. These conditions can be written in terms of the various functions defined in appendix B. First, if equation (45a) is applied and use is made of equation (28), the following relation is obtained:

$$-\frac{p\bar{c}}{2} + \frac{c_i}{\bar{c}} = \frac{c_a}{\bar{c}} \tag{64}$$

Since $c_i = 0$, the constant c_a can be determined from equation (64) and is

$$c_a = -\frac{p\bar{c}^2}{2} = -\frac{P}{2\pi} \tag{65}$$

where P is the magnitude of the local load. Now, in the five remaining continuity conditions, equations (45b) to (45f), the argument of the functions which appear is $\sqrt{\mu}\,\frac{\bar{c}}{R}$. For brevity the symbol $x_{\bar{c}}$, defined by $x_{\bar{c}} = \sqrt{\mu}\,\frac{\bar{c}}{R}$, is used in what is to follow. Then, equations (45b) to (45f) lead, respectively, to the following equations:

$$c_{1,i}\Lambda_1(x_{\bar{c}}) + c_{2,i}\Lambda_2(x_{\bar{c}}) - c_{1,a}\Lambda_1(x_{\bar{c}}) - c_{2,a}\Lambda_2(x_{\bar{c}}) - c_{3,a}\Lambda_3(x_{\bar{c}}) - c_{4,a}\Lambda_4(x_{\bar{c}}) = 0 \tag{66}$$

$$c_{1,i}\Omega_1(x_{\bar{c}}) + c_{2,i}\Omega_2(x_{\bar{c}}) - c_{1,a}\Omega_1(x_{\bar{c}}) - c_{2,a}\Omega_2(x_{\bar{c}}) - c_{3,a}\Omega_3(x_{\bar{c}}) - c_{4,a}\Omega_4(x_{\bar{c}}) = \frac{PR}{\pi\bar{c}} \tag{67}$$

$$c_{1,i}H_1(x_{\bar{c}}) + c_{2,i}H_2(x_{\bar{c}}) - c_{1,a}H_1(x_{\bar{c}}) - c_{2,a}H_2(x_{\bar{c}}) - c_{3,a}H_3(x_{\bar{c}}) - c_{4,a}H_4(x_{\bar{c}}) + \bar{d}_i - \bar{d}_a = 0 \tag{68}$$

$$c_{1,i}\Lambda_5(x_{\bar{c}}) - c_{2,i}\Lambda_6(x_{\bar{c}}) - c_{1,a}\Lambda_5(x_{\bar{c}}) + c_{2,a}\Lambda_6(x_{\bar{c}}) - c_{3,a}\Lambda_7(x_{\bar{c}}) + c_{4,a}\Lambda_8(x_{\bar{c}}) = 0 \tag{69}$$

$$c_{1,i}\Omega_5(x_{\bar{c}}) - c_{2,i}\Omega_6(x_{\bar{c}}) - c_{1,a}\Omega_5(x_{\bar{c}}) + c_{2,a}\Omega_6(x_{\bar{c}}) - c_{3,a}\Omega_7(x_{\bar{c}}) + c_{4,a}\Omega_8(x_{\bar{c}}) = 0 \tag{70}$$

The symbols $\Omega_1, \ldots, \Omega_8$ and $H_1, \ldots, H_4$ are defined in appendix B.

Now, with the constant c_a given by equation (65), the remaining eight constants of integration, $c_{1,i}$, $c_{2,i}$, $c_{1,a}$, $c_{2,a}$, $c_{3,a}$, $c_{4,a}$, d_i, and d_a, can be determined by using eight equations. These are the three equations for any set of the edge conditions (set 1, set 2, set 3, or set 4) and the five equations (66) to (70). The values for these constants then comprise four different sets of solutions for the case of uniformly distributed local loading. The constants are now redefined for convenience as follows:

$$c_{1,i},\ c_{2,i},\ c_{1,a},\ c_{2,a},\ c_{3,a},\ c_{4,a},\ \bar{d}_i,\ \bar{d}_a = \frac{PR}{\pi a}\left(k_1,\ k_2,\ k_3,\ k_4,\ k_5,\ k_6,\ k_7,\ k_8\right) \tag{71}$$

It is observed next that the constants k_7 and k_8 appear in only equations (61) and (68) of the eight equations involved in each set of solutions for the constants. This means that only six equations need be solved simultaneously for the six unknown constants $k_1, \ldots, k_6$ in each case, and the evaluation of only a 6×6 determinant is necessary. Once these six constants are determined, k_7 and k_8 can be calculated by using equations (61) and (68), and they are given by

$$-k_8 = k_3H_1(x_a) + k_4H_2(x_a) + k_5H_3(x_a) + k_6H_4(x_a) \tag{72}$$

$$k_7 = k_8 - (k_1 - k_3)H_1(x_{\bar{c}}) - (k_2 - k_4)H_2(x_{\bar{c}}) + k_5H_3(x_{\bar{c}}) + k_6H_4(x_{\bar{c}}) \tag{73}$$

For each alternative set of edge conditions the six relevant equations which are to be used to determine $k_1, \ldots, k_6$ are as follows: The first two equations for each of the sets 1, 2, 3, and 4 are given by equations (59, 60), (59, 62), (63, 62), and (63, 60), respectively; the last four equations are the same in each case and are given by equations (66), (67), (69), and (70).

The matrix equation associated with set 1 (no radial restraint, simple support) for calculating $k_1, \ldots, k_6$ is therefore

$$\begin{bmatrix} 0 & 0 & \Lambda_1(x_a) & \Lambda_2(x_a) & \Lambda_3(x_a) & \Lambda_4(x_a) \\ 0 & 0 & \Omega_5(x_a) & -\Omega_6(x_a) & \Omega_7(x_a) & -\Omega_8(x_a) \\ \Lambda_1(x_{\bar{c}}) & \Lambda_2(x_{\bar{c}}) & 0 & 0 & -\Lambda_3(x_{\bar{c}}) & -\Lambda_4(x_{\bar{c}}) \\ \Omega_1(x_{\bar{c}}) & \Omega_2(x_{\bar{c}}) & 0 & 0 & -\Omega_3(x_{\bar{c}}) & -\Omega_4(x_{\bar{c}}) \\ \Lambda_5(x_{\bar{c}}) & -\Lambda_6(x_{\bar{c}}) & 0 & 0 & -\Lambda_7(x_{\bar{c}}) & \Lambda_8(x_{\bar{c}}) \\ \Omega_5(x_{\bar{c}}) & -\Omega_6(x_{\bar{c}}) & 0 & 0 & -\Omega_7(x_{\bar{c}}) & \Omega_8(x_{\bar{c}}) \end{bmatrix} \begin{bmatrix} k_{13} \\ k_{24} \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{a}{c} \\ 0 \\ 0 \end{bmatrix} \qquad (74)$$

where $k_{13} = k_1 - k_3$ and $k_{24} = k_2 - k_4$, which are used to calculate k_1 and k_2. In order to obtain the appropriate matrix equation for the other cases, the following changes are to be effected in equation (74). For set 2 (no radial restraint, clamped) the six elements of the second row of the square matrix should be replaced by 0, 0, $\Lambda_5(x_a)$, $-\Lambda_6(x_a)$, $\Lambda_7(x_a)$, and $-\Lambda_8(x_a)$, respectively. For set 3 (radial restraint, clamped) the first row in the square matrix of equation (74) is to be replaced by 0, 0, $\Omega_1(x_a)$, $\Omega_2(x_a)$, $\Omega_3(x_a)$, and $\Omega_4(x_a)$; the second row is to be replaced by 0, 0, $\Lambda_5(x_a)$, $-\Lambda_6(x_a)$, $\Lambda_7(x_a)$, and $-\Lambda_8(x_a)$; and the first element of the column matrix on the right-hand side of the equation is to be replaced by $-\frac{1}{2}(1 + \nu)$. For set 4 (radial restraint, simple support) the first row in the square matrix of equation (74) is to be replaced by 0, 0, $\Omega_1(x_a)$, $\Omega_2(x_a)$, $\Omega_3(x_a)$, and $\Omega_4(x_a)$, and the first element of the column matrix on the right-hand side of the equation is to be replaced by $-\frac{1}{2}(1 + \nu)$.

Stresses and displacements.- The constants $k_1, \ldots, k_8$ are determined by means of equations (72) and (73) and the alternative forms of equation (74) described in the preceding paragraph (the constants, of course, take on different values within each case for various values of the parameters a^2/Rh, $\bar{c}/a$, and α). Equation (74) is solved on a digital computer in which polynomial approximations of the Kelvin functions, accurate to the eighth decimal place (ref. 5), are used. Once the constants are known, the stresses and displacements may be determined from equations (34) to (43) by making use of equations (58), (65), and (71) and the fact that p is to be set equal to zero in equations (34), (36), and (37) when values of quantities are desired in the region $\bar{c} \leqq r \leqq a$. In this manner, analytical expressions are determined for the quantities M_r, M_θ, N_r, N_θ, Q, and w which are valid in each of the two regions $r \leqq \bar{c}$ and $\bar{c} \leqq r \leqq a$. The quantities are first nondimensionalized as follows:

$$\left.\begin{aligned} m_r &= \frac{2\pi\sqrt{\mu}\,a}{(1+\nu)PR} M_r & n_r &= \frac{\pi a}{\sqrt{\mu}\,P} N_r & q &= \frac{\pi a}{P} Q \\ m_\theta &= \frac{\pi\sqrt{\mu}\,a}{\left(1-\nu^2\right)PR} M_\theta & n_\theta &= \frac{\pi a}{\sqrt{\mu}\,P} N_\theta & W &= \frac{\pi a}{\left(1+\alpha^2\right)\sqrt{\mu}\,A_s PR} w \end{aligned}\right\} \qquad (75)$$

where m_r, m_θ, n_r, n_θ, q, and W are nondimensional. The expressions for m_r and m_θ are then

$$m_r(x) = \frac{2}{(1+\nu)x}\left[k_1\Omega_5(x) - k_2\Omega_6(x)\right] \qquad (r \leqq \bar{c}) \qquad (76)$$

$$m_r(x) = \frac{2}{(1+\nu)x}\left[k_3\Omega_5(x) - k_4\Omega_6(x) + k_5\Omega_7(x) - k_6\Omega_8(x)\right] \qquad (\bar{c} \leqq r \leqq a) \qquad (77)$$

$$m_\theta(x) = \frac{\nu m_r}{2(1-\nu)} + \frac{1}{x}\left[k_1\Lambda_5(x) - k_2\Lambda_6(x)\right] \qquad (r \leqq \bar{c}) \qquad (78)$$

$$m_\theta(x) = \frac{\nu m_r}{2(1-\nu)} + \frac{1}{x}\left[k_3\Lambda_5(x) - k_4\Lambda_6(x) + k_5\Lambda_7(x) - k_6\Lambda_8(x)\right] \qquad (\bar{c} \leqq r \leqq a) \qquad (79)$$

Expressions for the stress resultants n_r, n_θ, and q are given by

$$n_r(x) = \frac{1}{x}\left[k_1\Lambda_1(x) + k_2\Lambda_2(x) - \frac{xa}{2\sqrt{\mu}\,R}\left(\frac{R}{\bar{c}}\right)^2\right] \qquad (r \leqq \bar{c}) \qquad (80)$$

$$n_r(x) = \frac{1}{x}\left[k_3\Lambda_1(x) + k_4\Lambda_2(x) + k_5\Lambda_3(x) + k_6\Lambda_4(x) - \frac{a\sqrt{\mu}}{2Rx}\right] \qquad (\bar{c} \leqq r \leqq a) \qquad (81)$$

$$n_\theta(x) = k_1\Lambda_1'(x) + k_2\Lambda_2'(x) - \frac{a}{2R\sqrt{\mu}}\left(\frac{R}{\bar{c}}\right)^2 \qquad (r \leqq \bar{c}) \qquad (82)$$

$$n_\theta(x) = k_3\Lambda_1'(x) + k_4\Lambda_2'(x) + k_5\Lambda_3'(x) + k_6\Lambda_4'(x) + \frac{a\sqrt{\mu}}{2Rx^2} \qquad (\bar{c} \leqq r \leqq a) \qquad (83)$$

$$q(x) = k_1\Lambda_1(x) + k_2\Lambda_2(x) \qquad (r \leqq \bar{c}) \qquad (84)$$

$$q(x) = k_3\Lambda_1(x) + k_4\Lambda_2(x) + k_5\Lambda_3(x) + k_6\Lambda_4(x) \qquad (\bar{c} \leqq r \leqq a) \qquad (85)$$

The vertical displacement W is given by

$$W(x) = k_1H_1(x) + k_2H_2(x) + k_7 \qquad (r \leqq \bar{c}) \qquad (86)$$

$$W(x) = k_3H_1(x) + k_4H_2(x) + k_5H_3(x) + k_6H_4(x) + k_8 \qquad (\bar{c} \leqq r \leqq a) \qquad (87)$$

It is useful to have formulas for computing the vertical displacement $W(0)$ and the meridional bending moment $m_r(0)$ at the origin, $r = 0$ (or $x = 0$). To determine the displacement $W(0)$, use is made of equation (86) and of expressions for $H_1(x)$ and $H_2(x)$ for small values of x, which can be obtained from appendix B. It turns out that $H_1(0) = -\frac{3\alpha}{2}$ and $H_2(0) = -1 + \frac{11\alpha^2}{8}$ so that

$$W(0) = k_7 - \frac{3\alpha}{2}k_1 - \left(1 - \frac{11\alpha^2}{8}\right)k_2 \qquad (88)$$

For the bending moment $m_r(0)$, use is made of equation (76). In this case the following expressions are needed for small x:

$$\left.\begin{aligned} \Omega_5(x) &= \frac{1+\nu}{2}x\left(1 + \frac{\alpha^2}{8}\right) + 0\left(x^2\right) \\ \Omega_6(x) &= \frac{\alpha(1+\nu)}{4}x + 0\left(x^2\right) \end{aligned}\right\} \qquad (89)$$

Equations (89) can be used to evaluate the following limit:

$$\lim_{x\to 0}\frac{1}{x}\left[k_1\Omega_5(x) - k_2\Omega_6(x)\right] = \frac{1+\nu}{2}\left[k_1\left(1 + \frac{\alpha^2}{8}\right) - \frac{\alpha}{2}k_2\right] \qquad (90)$$

If the result in equation (90) is used, equation (76) gives

$$m_r(0) = k_1\left(1 + \frac{\alpha^2}{8}\right) - \frac{\alpha}{2}k_2 \qquad (91)$$

which provides a means of calculating the bending moment m_r at the origin.

Local Ring Loading

In the case of local ring loading concentric with the pole of the cap (see fig. 1(c)), expressions for the stresses and displacements can be obtained by making appropriate changes in the previous results for the case of uniformly distributed local load. Actually, very few changes are necessary. First, p must be set equal to zero in equations (34), (36), and (37) because the vertical component of surface loading is zero everywhere in the case of a ring load. Another change involves the continuity condition given by equation (45a), which must now be replaced by

$$V_a(\bar{c}) = V_i(\bar{c}) - \frac{P}{2\pi\bar{c}} \tag{92}$$

For the case of ring loading, the conditions at the origin, $r = 0$, do not differ from those for the case of local uniform loading, so that equation (58) applies here also $\left(\text{that is, } c_i = c_{3,i} = c_{4,i} = 0\right)$. Then on using equation (28) with $p = 0$, equation (92) leads to the following result for the constant c_a:

$$c_a = -\frac{P}{2\pi} \tag{93}$$

The equations which describe the edge boundary conditions, equations (46) to (49) and (59) to (63), do not change in the present case of ring loading. Also, the only changes in the equations for the matching conditions (eqs. (66) to (70)) occur in the right-hand sides of equations (66) and (67). After p is set equal to zero in equations (36) and (37), equations (45b) and (45c) are used to derive the following two equations, which replace equations (66) and (67), respectively:

$$c_{1,i}\Lambda_1\left(x_{\bar{c}}\right) + c_{2,i}\Lambda_2\left(x_{\bar{c}}\right) - c_{1,a}\Lambda_1\left(x_{\bar{c}}\right) - c_{2,a}\Lambda_2\left(x_{\bar{c}}\right) - c_{3,a}\Lambda_3\left(x_{\bar{c}}\right) - c_{4,a}\Lambda_4\left(x_{\bar{c}}\right) = -\frac{PR}{2\pi\bar{c}} \tag{94}$$

$$c_{1,i}\Omega_1\left(x_{\bar{c}}\right) + c_{2,i}\Omega_2\left(x_{\bar{c}}\right) - c_{1,a}\Omega_1\left(x_{\bar{c}}\right) - c_{2,a}\Omega_2\left(x_{\bar{c}}\right) - c_{3,a}\Omega_3\left(x_{\bar{c}}\right) - c_{4,a}\Omega_4\left(x_{\bar{c}}\right) = \frac{(1 + \nu)PR}{2\pi\bar{c}} \tag{95}$$

This means that the column vector on the right-hand side of equation (74) is to be replaced by

$$
\begin{bmatrix} 1/2 \\ 0 \\ -a/2\bar{c} \\ (1 + \nu)a/2\bar{c} \\ 0 \\ 0 \end{bmatrix} \qquad (96)
$$

and the discussion which follows equation (74) for the various edge conditions applies in this case also. Finally, the stresses and displacements can be obtained for the case of ring loading by using equations (76) to (87) and by deleting the last term in equations (80) to (83). As a result, the formulas for $W(0)$ and $m_r(0)$ given by equations (88) and (91) also hold for the case of ring loading. In these formulas, the constants $k_1, \ldots, k_8$ will differ from those for uniform local loading, since the matrix equation used to calculate them differs.

RESULTS AND DISCUSSION

The analytical solutions derived herein are used to calculate stresses and displacements for the various cases under consideration. The functions which appear in these solutions are written out in appendix B, and the constants of integration are determined by evaluating at most a 6×6 determinant. This is done for each case of local loading, each type of edge support, and several values of core stiffness.

In this manner, representative results are obtained to show the effects of the type of local loading (ring or uniformly distributed load), the area over which it acts, and the type of edge support involved. (See figs. 2 to 11.) The results also show the influence of the transverse shear flexibility of the sandwich core as measured by values of the parameter α (see "Symbols"), which involves the curvature and thickness of the shell as well as the transverse shear stiffness of the core. Values for this parameter are selected to cover a practical range from strong to relatively weak cores (for honeycomb and corrugated cores, for instance, α may range from 0.01 to 0.1). It should be pointed out that cases other than those calculated for purposes of the present discussion can be studied without difficulty by making use of the solutions within the framework developed in this paper. It is noted that when $\alpha = 0$ the present results reduce to well-established results for conventional shells (see, for example, ref. 7). Also, for isotropic, homogeneous conventional shells with transverse shear deformation, the parameter becomes

$$\alpha = \left[\frac{1 + \nu}{12(1 - \nu)}\right]^{1/2} \frac{h}{R}$$

Center Deflection and Center and Edge Bending Moments

The displacement at the center of the spherical cap, $w(0)$, is shown in figures 2 to 4 for a fixed value of the curvature parameter a^2/Rh, which defines the degree of shallowness and relative thickness of the shell. In figure 2, $w(0)$ is plotted for various values of α against the quantity $\bar{c}/a$, which is a measure of the area of local loading. The edge conditions are $M_r = N_r = w = 0$; that is, simple support and no restraint against radial expansion. As the loading becomes highly localized the deflection reaches rather large magnitudes, and also the influence of the parameter α becomes more pronounced. It is noted that for the case of uniform loading over the entire cap surface ($\bar{c}/a = 1$), values of $w(0)$ for $\alpha = 0.001$ and $\alpha = 0.2$ differ by about 20 percent, while for $\bar{c}/a = 0.05$ the value of $w(0)$ for $\alpha = 0.2$ is nearly twice that for $\alpha = 0.001$.

Figure 3 shows the effect that different edge conditions have on the deflection at the cap center for a given value of α. It is clear that edge restraint is decisive in lowering the deflection, since sets 2, 3, and 4, which involve either edge restraint against rotation ($\varphi = 0$) or edge restraint against expansion ($u = 0$), have much lower deflections than set 1, which involves simple support $(M_r = 0)$ and no radial restraint $(N_r = 0)$. Figure 4 shows the influence of the type of loading (ring or uniform) on the deflection. Here greater differences in the two curves occur when the load is less localized, as would be expected.

The effects of edge condition, type of loading, and core stiffness on the bending moment M_r are indicated in figures 5 to 7 for fixed a^2/Rh. In figure 5, $M_r(0)$ is plotted against $\bar{c}/a$ for two extreme values of α and two sets of edge conditions for uniform loading. The curves indicate that for the larger values of α (i.e., weaker core) the bending moment is smaller. This result implies, of course, that the shearing action which contributes to the displacement alleviates somewhat the tendency toward increased curvature changes during deformation. The effect is small for stiffer sandwich cores. The curves also indicate that for the given boundary conditions, the conditions which involve greater edge restraint correspond to smaller bending moments. This again is due to the relative inhibition of curvature changes.

Figure 6 demonstrates the effect that the type of loading has on the bending moment $M_r(0)$. There are substantial differences in the computed values over the entire range of $\bar{c}/a$, and even for $\bar{c}/a = 0.05$ the curves differ by about 20 percent.

In figure 7 the bending moment at the edge ($r = a$) of the shell is given for the fully clamped condition in the cases of ring and uniformly distributed loading for various values of α. Note that for each value of α, both ring and uniform loading approach the same limiting value as the load becomes more localized, while differences occur for larger values of $\bar{c}/a$. Also, the influence of core stiffness as implied by different values of α is shown to be significant for the entire range of $\bar{c}/a$.

Deflection and Bending-Moment Distributions

In figures 8 and 9 the displacement distribution $w(r/a)$ and bending-moment distribution $M_r(r/a)$ are plotted for several highly localized loading conditions ($\bar{c}/a \leqq 0.1$) and several values of α characteristic of typical core stiffnesses. It is noted that in figure 8 the influence of the degree of localized loading (values of $\bar{c}/a$) on the deflection is restricted to a region in the spherical cap which is approximately within a radius of $r = 0.1a$. The effect of different values of the parameter α, however, is felt over a greater region of the cap. In figure 9, the major influence of the highly localized loading condition on bending moments is again evidenced for $r/a \leqq 0.1$, with a smaller corresponding influence of α.

Effect of Curvature

In figure 10 the center deflection $w(0)$ caused by a uniform local load over a small circular area of radius $\bar{c} = 0.05a$ is plotted against the curvature parameter a^2/Rh for various edge conditions. The parameter e_{sp} is related to α (see "Symbols"), but in contrast to α, it is independent of the curvature parameter a^2/Rh. The solid curves have a value of e_{sp} equal to 0.01, which is representative of honeycomb sandwich. For the edge conditions described by sets 1 and 4 (simple support, without and with restraint against radial expansion, respectively), it can be seen that even slight curvature has a significant effect in reducing the flat-plate $\left(a^2/Rh = 0\right)$ deflection. This is true to a lesser degree for edge conditions given by sets 2 and 3 (clamped, without and with restraint against radial expansion, respectively). In general, as expected, the shells with edges having less restraint experience larger deflections. However, for a range of values of a^2/Rh between 5 and 10 the deflections $w(0)$ are the same for set 3 and set 4. The differences for all cases begin to disappear for shells with high curvature. Results corresponding to a weaker core $\left(e_{sp} = 0.02\right)$ are given in figure 10 only for set 1 (although the same conclusions hold for the other edge conditions), and indicate that the shells of higher curvature are more sensitive to transverse shear effects on deflections. The example in figure 10 shows a 7-percent increase in deflection for the flat plate $\left(a^2/Rh = 0\right)$ and a corresponding 21-percent increase for the deeper shell where $a^2/Rh = 20$.

In figure 11 the meridional bending moment at the edge of the spherical cap (r = a) is given for various edge conditions and two values of e_{sp}. The marked effect of even slight curvature is again present. It is interesting to note that for the fully restrained shell (set 3) the edge moment changes sign as the curvature increases beyond a value of $a^2/Rh \approx 6.5$.

CONCLUDING REMARKS

Analytical solutions are obtained for the linear equations governing the deformation of shallow spherical sandwich caps under symmetrical local loading and various edge conditions. Results for stresses and displacements typical of a practical range of parameter values of edge restraint are presented. Calculations for other cases can be made readily, and involve the evaluation of a 6×6 determinant. The results indicate that for localized loading, where the radius of the local load is 1/10 of the shell edge radius or less, transverse shear deformation plays an important role over a wide range of sandwich core stiffness. In general, the stresses and displacements for uniform localized loading are greater than those for a corresponding ring load. For the cases where the edge of the cap is either clamped or restrained against radial expansion, or both, the maximum deflection differs by at most 20 percent, depending on the precise edge condition. The deflection, however, differs substantially (60 to 100 percent) from that for the case where the edge is both simply supported and unrestrained against radial expansion.

The effect of curvature on deflections and bending moments is established by obtaining results in a range which includes the flat plate as a limiting case. The results show that even slight curvature modifies substantially the flat-plate values for deflection and bending moment.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., September 26, 1966,
124-08-06-17-23.

APPENDIX A

EXPANSION OF BESSEL FUNCTIONS

In this appendix the modified Bessel functions $I_1\left(\sqrt{i+\alpha}\,x\right)$ and $K_1\left(\sqrt{i+\alpha}\,x\right)$ are expanded in series of positive powers of the small parameter α. The coefficients of the various powers of α are expressed in terms of Kelvin functions of argument x and their first derivatives. The function $I_1\left(\sqrt{i+\alpha}\,x\right)$ is treated first, and its series representation as defined in reference 5 (p. 379) leads to

$$I_1\left(\sqrt{i+\alpha}\,x\right) = \sum_{r=0}^{\infty} \frac{\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1}}{r!\,\Gamma(r+2)} \left[1 - \frac{i\alpha}{2} + \frac{\alpha^2}{8} + \frac{i\alpha^3}{16} + 0\left(\alpha^4\right)\right]^{2r+1} \tag{A1}$$

and

$$I_1\left(\sqrt{i}\,x\right) = \sum_{r=0}^{\infty} \frac{\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1}}{r!\,\Gamma(r+2)} = i^{-1/2}(\text{ber}'\,x + i\,\text{bei}'\,x) \tag{A2}$$

The term in brackets in equation (A1) can be expanded to give

$$\left[1 - \frac{i\alpha}{2} + \frac{\alpha^2}{8} + \frac{i\alpha^3}{16} + 0\left(\alpha^4\right)\right]^{2r+1} = 1 + (2r+1)\left(-\frac{i\alpha}{2} + \frac{\alpha^2}{8} + \frac{i\alpha^3}{16}\right) - \frac{(2r+1)(2r)}{2!}\left(\frac{\alpha^2}{4} + \frac{i\alpha^3}{8}\right)$$
$$+ \frac{(2r+1)(2r)(2r-1)}{3!}\,\frac{i\alpha^3}{8} + 0\left(\alpha^4\right) \tag{A3}$$

Next, the following identities are easily established by using equation (A2):

$$x\,\frac{dI_1\left(\sqrt{i}\,x\right)}{dx} = \sum_{r=0}^{\infty} \frac{(2r+1)\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1}}{r!\,\Gamma(r+2)} \tag{A4a}$$

$$x^2\,\frac{d^2I_1\left(\sqrt{i}\,x\right)}{dx^2} = \sum_{r=0}^{\infty} \frac{(2r+1)(2r)}{r!\,\Gamma(r+2)}\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1} \tag{A4b}$$

$$x^3 \frac{d^3 I_1(\sqrt{i}\,x)}{dx^3} = \sum_{r=0}^{\infty} \frac{(2r+1)(2r)(2r-1)}{r!\ \Gamma(r+2)} \left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1} \tag{A4c}$$

When equations (A3) and (A4) are taken into account, equation (A1) leads to the following result:

$$I_1(\sqrt{i+\alpha}\,x) = I_1(\sqrt{i}\,x) - \frac{i\alpha}{2} x \frac{dI_1(\sqrt{i}\,x)}{dx} + \frac{\alpha^2}{8}\left[x \frac{dI_1(\sqrt{i}\,x)}{dx} - x^2 \frac{d^2 I_1(\sqrt{i}\,x)}{dx^2}\right]$$

$$+ \frac{i\alpha^3}{16}\left[x \frac{dI_1(\sqrt{i}\,x)}{dx} - x^2 \frac{d^2 I_1(\sqrt{i}\,x)}{dx^2} + \frac{x^3}{3} \frac{d^3 I_1(\sqrt{i}\,x)}{dx^3}\right] + (\alpha^4 \text{ terms}) \tag{A5}$$

An expression similar in form to equation (A5) may also be established for the function $K_1(\sqrt{i+\alpha}\,x)$ by first using its series definition (see ref. 5) to write

$$K_1(\sqrt{i+\alpha}\,x) = \frac{1}{\sqrt{i}\,x}\left[1 + \frac{i\alpha}{2} - \frac{3\alpha^2}{8} - \frac{i5\alpha^3}{16} + 0(\alpha^4)\right] + \sum_{r=0}^{\infty} \frac{\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1}}{r!\ \Gamma(r+2)} S_r B_r \tag{A6}$$

where

$$S_r = 1 + (2r+1)\left(-\frac{i\alpha}{2} + \frac{\alpha^2}{8} + \frac{i\alpha^3}{16}\right) - (2r+1)(2r)\left(\frac{\alpha^2}{8} + \frac{i\alpha^3}{16}\right) + (2r+1)(2r)(2r-1)\frac{i\alpha^3}{48} + 0(\alpha^4)$$

$$B_r = \log \frac{(\sqrt{i}\,x)}{2} - \frac{1}{2}\left[\Psi(r+1) + \Psi(r+2)\right] - \frac{i\alpha}{2} + \frac{\alpha^2}{4} + \frac{i\alpha^3}{6} + 0(\alpha^4)$$

$$\Psi(r+1) = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}\right) - 0.5772157$$

and

$$K_1(\sqrt{i}\,x) = \frac{1}{\sqrt{i}\,x} + \sum_{r=0}^{\infty} \frac{\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1}}{r!\ \Gamma(r+2)} B_{r,o} = -i^{-1/2}(\text{ker}'\,x + i\,\text{kei}'\,x) \tag{A7}$$

where

$$B_{r,o} = \log\left(\frac{\sqrt{i}\,x}{2}\right) - \frac{1}{2}\left[\Psi(r+1) + \Psi(r+2)\right]$$

By using equation (A7), the following identities are established in this case:

$$x\frac{d}{dx}\left[K_1\left(\sqrt{i}\,x\right) - \frac{1}{\sqrt{i}\,x}\right] = \sum_{r=0}^{\infty} \frac{(2r+1)}{r!\,\Gamma(r+2)}\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1} B_{r,o} + I_1\left(\sqrt{i}\,x\right) \tag{A8a}$$

$$x^2\frac{d^2}{dx^2}\left[K_1\left(\sqrt{i}\,x\right) - \frac{1}{\sqrt{i}\,x}\right] = \sum_{r=0}^{\infty} \frac{(2r+1)(2r)}{r!\,\Gamma(r+2)}\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1} B_{r,o} + 2x\frac{dI_1\left(\sqrt{i}\,x\right)}{dx} - I_1\left(\sqrt{i}\,x\right) \tag{A8b}$$

$$x^3\frac{d^3}{dx^3}\left[K_1\left(\sqrt{i}\,x\right) - \frac{1}{\sqrt{i}\,x}\right] = \sum_{r=0}^{\infty} \frac{(2r+1)(2r)(2r-1)}{r!\,\Gamma(r+2)}\left(\frac{\sqrt{i}\,x}{2}\right)^{2r+1} B_{r,o} + 3x^2\frac{d^2I_1\left(\sqrt{i}\,x\right)}{dx^2}$$
$$- 3x\frac{dI_1\left(\sqrt{i}\,x\right)}{dx} + 2I_1\left(\sqrt{i}\,x\right) \tag{A8c}$$

Equation (A6) can now be expanded by using equations (A7) and (A8). After many terms cancel out, the following result is obtained:

$$K_1\left(\sqrt{i+\alpha}\,x\right) = K_1\left(\sqrt{i}\,x\right) - \frac{i\alpha}{2}x\frac{dK_1\left(\sqrt{i}\,x\right)}{dx} + \frac{\alpha^2}{8}\left[x\frac{dK_1\left(\sqrt{i}\,x\right)}{dx} - x^2\frac{d^2K_1\left(\sqrt{i}\,x\right)}{dx^2}\right]$$
$$+ \frac{i\alpha^3}{16}\left[x\frac{dK_1\left(\sqrt{i}\,x\right)}{dx} - x^2\frac{d^2K_1\left(\sqrt{i}\,x\right)}{dx^2} + \frac{x^3}{3}\frac{d^3K_1\left(\sqrt{i}\,x\right)}{dx^3}\right] + \left(\alpha^4 \text{ terms}\right) \tag{A9}$$

Certain relations which can be determined from the properties of Kelvin functions (see ref. 5) are to be used next. These are

$$\left.\begin{aligned}
\mathrm{ber}''\,x &= -\mathrm{bei}\,x - \frac{\mathrm{ber}'\,x}{x} & \mathrm{bei}''\,x &= \mathrm{ber}\,x - \frac{\mathrm{bei}'\,x}{x} \\
\mathrm{ber}'''\,x &= -\mathrm{bei}'\,x + \frac{\mathrm{bei}\,x}{x} + \frac{2}{x^2}\,\mathrm{ber}'\,x & \mathrm{bei}'''\,x &= \mathrm{ber}'\,x - \frac{\mathrm{ber}\,x}{x} + \frac{2}{x^2}\,\mathrm{bei}'\,x
\end{aligned}\right\} \tag{A10}$$

$$\left.\begin{aligned}
\operatorname{ker}'' x &= -\operatorname{kei} x - \frac{\operatorname{ker}' x}{x} & \operatorname{kei}'' x &= \operatorname{ker} x - \frac{\operatorname{kei}' x}{x} \\
\operatorname{ker}''' x &= -\operatorname{kei}' x + \frac{\operatorname{kei} x}{x} + \frac{2}{x^2}\operatorname{ker}' x & \operatorname{kei}''' x &= \operatorname{ker}' x - \frac{\operatorname{ker} x}{x} + \frac{2}{x^2}\operatorname{kei}' x
\end{aligned}\right\} \tag{A11}$$

Finally, equations (A10) and (A11) are used together with equations (A2), (A5), (A7), and (A9) to obtain

$$\begin{aligned}
i^{1/2}I_1\left(\sqrt{i+\alpha}\, x\right) = {} & \operatorname{ber}' x + \frac{\alpha}{2} x\left(\operatorname{ber} x - \frac{\operatorname{bei}' x}{x}\right) + \frac{\alpha^2}{8}\left(x^2 \operatorname{bei}' x - 2x \operatorname{bei} x - 3 \operatorname{ber}' x\right) \\
& + i\left[\operatorname{bei}' x + \frac{\alpha x}{2}\left(\operatorname{bei} x + \frac{\operatorname{ber}' x}{x}\right) + \frac{\alpha^2}{8}\left(-x^2 \operatorname{ber}' x + 2x \operatorname{ber} x - 3 \operatorname{bei}' x\right)\right]
\end{aligned} \tag{A12}$$

and

$$\begin{aligned}
-i^{1/2}K_1\left(\sqrt{i+\alpha}\, x\right) = {} & \operatorname{ker}' x + \frac{\alpha}{2} x\left(\operatorname{ker} x - \frac{\operatorname{kei}' x}{x}\right) + \frac{\alpha^2}{8}\left(x^2 \operatorname{kei}' x - 2x \operatorname{kei} x - 3 \operatorname{ker}' x\right) \\
& + i\left[\operatorname{kei}' x + \frac{\alpha x}{2}\left(\operatorname{kei} x + \frac{\operatorname{ker}' x}{x}\right) + \frac{\alpha^2}{8}\left(-x^2 \operatorname{ker}' x + 2x \operatorname{ker} x - 3 \operatorname{kei}' x\right)\right]
\end{aligned} \tag{A13}$$

It is noted that the terms containing α^3 are dropped in equations (A12) and (A13) for purposes of the present analysis (they may be calculated if desired). For practical shallow sandwich shells in the range where $0 < \alpha < \frac{1}{4}$ and where $\frac{\alpha^2 a^2}{Rh} < \frac{1}{2}$, an error of less than 1 percent is incurred in equations (A12) and (A13) when the series in powers of α is truncated to exclude α^3 terms.

APPENDIX B

FUNCTIONS WHICH APPEAR IN THE SOLUTIONS

The following functions of the parameter α and independent variable $x = \sqrt{\mu}\,\frac{r}{R}$ appear in the solutions for stresses and displacements given in the text. They are written here for ready reference.

$$\Lambda_1(\alpha,x) = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{ber}'\,x + \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{bei}'\,x + \frac{\alpha x}{2}\,\mathrm{ber}\,x + \frac{\alpha^2 x}{4}\,\mathrm{bei}\,x$$

$$\Lambda_2(\alpha,x) = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{bei}'\,x - \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{ber}'\,x + \frac{\alpha x}{2}\,\mathrm{bei}\,x - \frac{\alpha^2 x}{4}\,\mathrm{ber}\,x$$

$$\Lambda_3(\alpha,x) = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{ker}'\,x + \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{kei}'\,x + \frac{\alpha x}{2}\,\mathrm{ker}\,x + \frac{\alpha^2 x}{4}\,\mathrm{kei}\,x$$

$$\Lambda_4(\alpha,x) = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{kei}'\,x - \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{ker}'\,x + \frac{\alpha x}{2}\,\mathrm{kei}\,x - \frac{\alpha^2 x}{4}\,\mathrm{ker}\,x$$

$$\Lambda_5(\alpha,x) = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{bei}'\,x + \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{ber}'\,x + \frac{\alpha x}{2}\,\mathrm{bei}\,x + \frac{\alpha^2 x}{4}\,\mathrm{ber}\,x$$

$$\Lambda_6(\alpha,x) = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{ber}'\,x - \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{bei}'\,x + \frac{\alpha x}{2}\,\mathrm{ber}\,x - \frac{\alpha^2 x}{4}\,\mathrm{bei}\,x$$

$$\Lambda_7(\alpha,x) = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{kei}'\,x + \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{ker}'\,x + \frac{\alpha x}{2}\,\mathrm{kei}\,x + \frac{\alpha^2 x}{4}\,\mathrm{ker}\,x$$

$$\Lambda_8(\alpha,x) = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{ker}'\,x - \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{kei}'\,x + \frac{\alpha x}{2}\,\mathrm{ker}\,x - \frac{\alpha^2 x}{4}\,\mathrm{kei}\,x$$

$$\Lambda_1'(\alpha,x) = -\left(1 - \frac{\alpha^2}{8}\right)\mathrm{bei}\,x + \left(\alpha + \frac{\alpha^2x^2}{8}\right)\mathrm{ber}\,x - \left(1 - \frac{\alpha x^2}{2} + \frac{\alpha^2}{8}\right)\frac{\mathrm{ber}'\,x}{x} - \left(\frac{\alpha}{2} - \frac{3\alpha^2x^2}{8}\right)\frac{\mathrm{bei}'\,x}{x}$$

$$\Lambda_2'(\alpha,x) = \left(1 - \frac{\alpha^2}{8}\right)\mathrm{ber}\,x + \left(\alpha + \frac{\alpha^2x^2}{8}\right)\mathrm{bei}\,x - \left(1 - \frac{\alpha x^2}{2} + \frac{\alpha^2}{8}\right)\frac{\mathrm{bei}'\,x}{x} + \left(\frac{\alpha}{2} - \frac{3\alpha^2x^2}{8}\right)\frac{\mathrm{ber}'\,x}{x}$$

$$\Lambda_3'(\alpha,x) = -\left(1 - \frac{\alpha^2}{8}\right)\mathrm{kei}\,x + \left(\alpha + \frac{\alpha^2x^2}{8}\right)\mathrm{ker}\,x - \left(1 - \frac{\alpha x^2}{2} + \frac{\alpha^2}{8}\right)\frac{\mathrm{ker}'\,x}{x} - \left(\frac{\alpha}{2} - \frac{3\alpha^2x^2}{8}\right)\frac{\mathrm{kei}'\,x}{x}$$

$$\Lambda_4'(\alpha,x) = \left(1 - \frac{\alpha^2}{8}\right)\mathrm{ker}\ x + \left(\alpha + \frac{\alpha^2x^2}{8}\right)\mathrm{kei}\ x - \left(1 - \frac{\alpha x^2}{2} + \frac{\alpha^2}{8}\right)\frac{\mathrm{kei}'\ x}{x} + \left(\frac{\alpha}{2} - \frac{3\alpha^2x^2}{8}\right)\frac{\mathrm{ker}'\ x}{x}$$

$$\Lambda_5'(\alpha,x) = \left(1 - \frac{\alpha^2}{8}\right)\mathrm{ber}\ x + \frac{\alpha^2x^2}{8}\ \mathrm{bei}\ x - \left(1 - \frac{\alpha x^2}{2} - \frac{3\alpha^2}{8}\right)\frac{\mathrm{bei}'\ x}{x} - \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\frac{\mathrm{ber}'\ x}{x}$$

$$\Lambda_6'(\alpha,x) = -\left(1 - \frac{\alpha^2}{8}\right)\mathrm{bei}\ x + \frac{\alpha^2x^2}{8}\ \mathrm{ber}\ x - \left(1 - \frac{\alpha x^2}{2} - \frac{3\alpha^2}{8}\right)\frac{\mathrm{ber}'\ x}{x} + \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\frac{\mathrm{bei}'\ x}{x}$$

$$\Lambda_7'(\alpha,x) = \left(1 - \frac{\alpha^2}{8}\right)\mathrm{ker}\ x + \frac{\alpha^2x^2}{8}\ \mathrm{kei}\ x - \left(1 - \frac{\alpha x^2}{2} - \frac{3\alpha^2}{8}\right)\frac{\mathrm{kei}'\ x}{x} - \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\frac{\mathrm{ker}'\ x}{x}$$

$$\Lambda_8'(\alpha,x) = -\left(1 - \frac{\alpha^2}{8}\right)\mathrm{kei}\ x + \frac{\alpha^2x^2}{8}\ \mathrm{ker}\ x - \left(1 - \frac{\alpha x^2}{2} - \frac{3\alpha^2}{8}\right)\frac{\mathrm{ker}'\ x}{x} + \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\frac{\mathrm{kei}'\ x}{x}$$

$$J_1(\alpha,x) = \int \Lambda_1 dx = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{ber}\ x + \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{bei}\ x + \frac{\alpha x}{2}\ \mathrm{bei}'\ x$$

$$J_2(\alpha,x) = \int \Lambda_2 dx = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{bei}\ x - \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{ber}\ x - \frac{\alpha x}{2}\ \mathrm{ber}'\ x$$

$$J_3(\alpha,x) = \int \Lambda_3 dx = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{ker}\ x + \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{kei}\ x + \frac{\alpha x}{2}\ \mathrm{kei}'\ x$$

$$J_4(\alpha,x) = \int \Lambda_4 dx = \left(1 + \frac{\alpha^2}{8}\right)\mathrm{kei}\ x - \left(\frac{\alpha}{2} + \frac{\alpha^2x^2}{8}\right)\mathrm{ker}\ x - \frac{\alpha x}{2}\ \mathrm{ker}'\ x$$

$$J_5(\alpha,x) = \int \Lambda_5 dx = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{bei}\ x + \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{ber}\ x - \frac{\alpha x}{2}\ \mathrm{ber}'\ x + \frac{\alpha^2 x}{2}\ \mathrm{bei}'\ x$$

$$J_6(\alpha,x) = \int \Lambda_6 dx = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{ber}\ x - \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{bei}\ x + \frac{\alpha x}{2}\ \mathrm{bei}'\ x + \frac{\alpha^2 x}{2}\ \mathrm{ber}'\ x$$

$$J_7(\alpha,x) = \int \Lambda_7 dx = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{kei}\ x + \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{ker}\ x - \frac{\alpha x}{2}\ \mathrm{ker}'\ x + \frac{\alpha^2 x}{2}\ \mathrm{kei}'\ x$$

$$J_8(\alpha,x) = \int \Lambda_8 dx = \left(1 - \frac{3\alpha^2}{8}\right)\mathrm{ker}\ x - \left(\frac{\alpha}{2} - \frac{\alpha^2x^2}{8}\right)\mathrm{kei}\ x + \frac{\alpha x}{2}\ \mathrm{kei}'\ x + \frac{\alpha^2 x}{2}\ \mathrm{ker}'\ x$$

APPENDIX B

Definitions:

$$\Omega_1 = x\Lambda_1' - \nu\Lambda_1 \qquad \Omega_5 = x\Lambda_5' + \nu\Lambda_5$$

$$\Omega_2 = x\Lambda_2' - \nu\Lambda_2 \qquad \Omega_6 = x\Lambda_6' + \nu\Lambda_6$$

$$\Omega_3 = x\Lambda_3' - \nu\Lambda_3 \qquad \Omega_7 = x\Lambda_7' + \nu\Lambda_7$$

$$\Omega_4 = x\Lambda_4' - \nu\Lambda_4 \qquad \Omega_8 = x\Lambda_8' + \nu\Lambda_8$$

$$H_1 = J_5 - 2\alpha J_1 \qquad H_3 = J_7 - 2\alpha J_3$$

$$H_2 = -J_6 - 2\alpha J_2 \qquad H_4 = -J_8 - 2\alpha J_4$$

For small values of the argument of the functions ber, bei, ker, and kei, the following series are useful (ref. 5):

$$\text{ber } x = 1 - \frac{x^4}{2^2 \cdot 4^2} + \ldots$$

$$\text{bei } x = \frac{x^2}{4} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

$$\text{ber}' x = -\frac{x^3}{16} + \ldots$$

$$\text{bei}' x = \frac{x}{2} - \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \ldots$$

$$\text{ker } x = -\ln x + 0.1159 + \frac{\pi x^2}{16} + \ldots$$

$$\text{kei } x = -\frac{x^2}{4} \ln x - \frac{\pi}{4} + 1.1159 \frac{x^2}{4} + \ldots$$

$$\text{ker}' x = -\frac{1}{x} + \frac{\pi x}{8} + \frac{x^3}{16} \ln x + \ldots$$

$$\text{kei}' x = -\frac{x}{2} \ln x - \frac{x}{4} + 0.558x + \ldots$$

REFERENCES

1. Naghdi, P. M.: The Effect of Transverse Shear Deformation on the Bending of Elastic Shells of Revolution. Quart. Appl. Math., vol. XV, no. 1, Apr. 1957, pp. 41-52.

2. Reissner, Eric: Small Bending and Stretching of Sandwich-Type Shells. NACA Rept. 975, 1950.

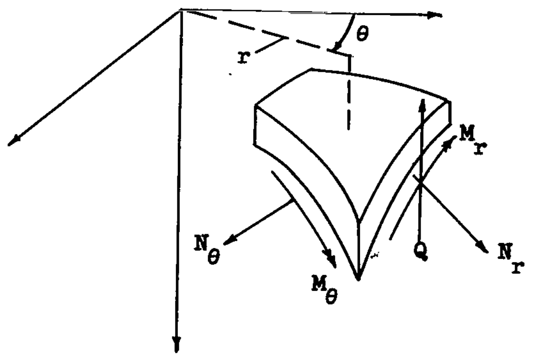
3. Stein, Manuel; and Mayers, J.: A Small-Deflection Theory for Curved Sandwich Plates. NACA Rept. 1008, 1951. (Supersedes NACA TN 2017.)

4. Habip, Lui M.: A Survey of Modern Developments in the Analysis of Sandwich Structures. Appl. Mech. Rev., vol. 18, no. 2, Feb. 1965, pp. 93-98.

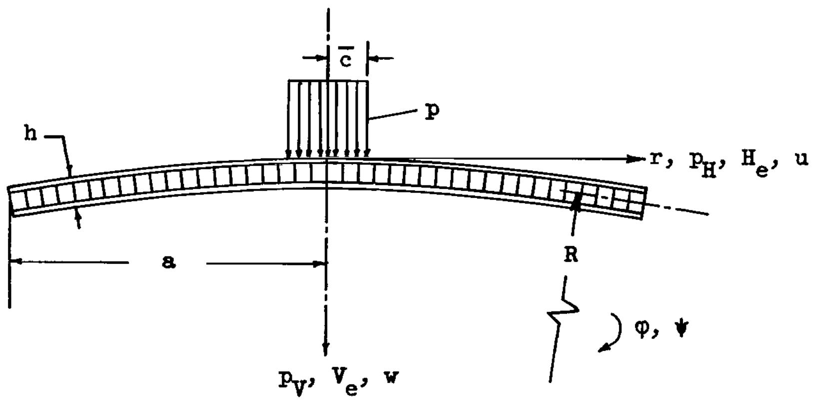
5. Abramowitz, Milton; and Stegun, Irene A., eds.: Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables. Natl. Bur. Std. Appl. Math. Ser. 55, U.S. Dept. Com., June 1964, pp. 354-389.

6. Mushtari, Kh. M.; and Galimov, K. Z.: Non-Linear Theory of Thin Elastic Shells. NASA TT F-62, The Israel Program for Scientific Translations, 1961, pp. 79-85.

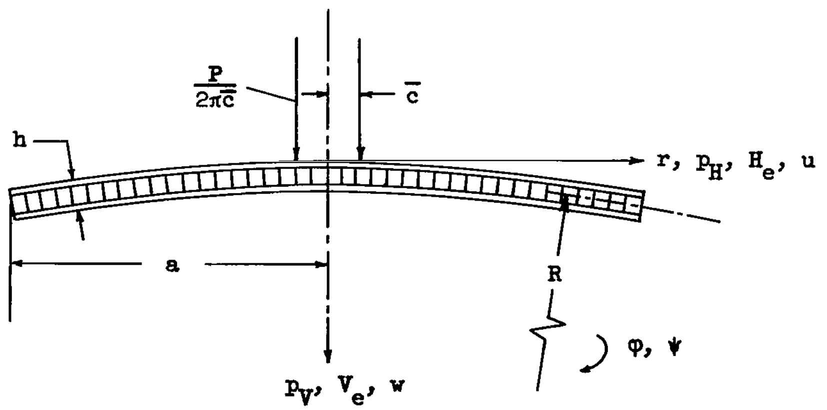
7. Reissner, Eric: Stresses and Small Displacements of Shallow Spherical Shells. J. Math. Phys., vol. XXV, no. 4, Jan. 1947, pp. 279-300.



(a) Typical element.



(b) Uniform local loading.



(c) Local ring loading.

Figure 1.- Geometry and notation.

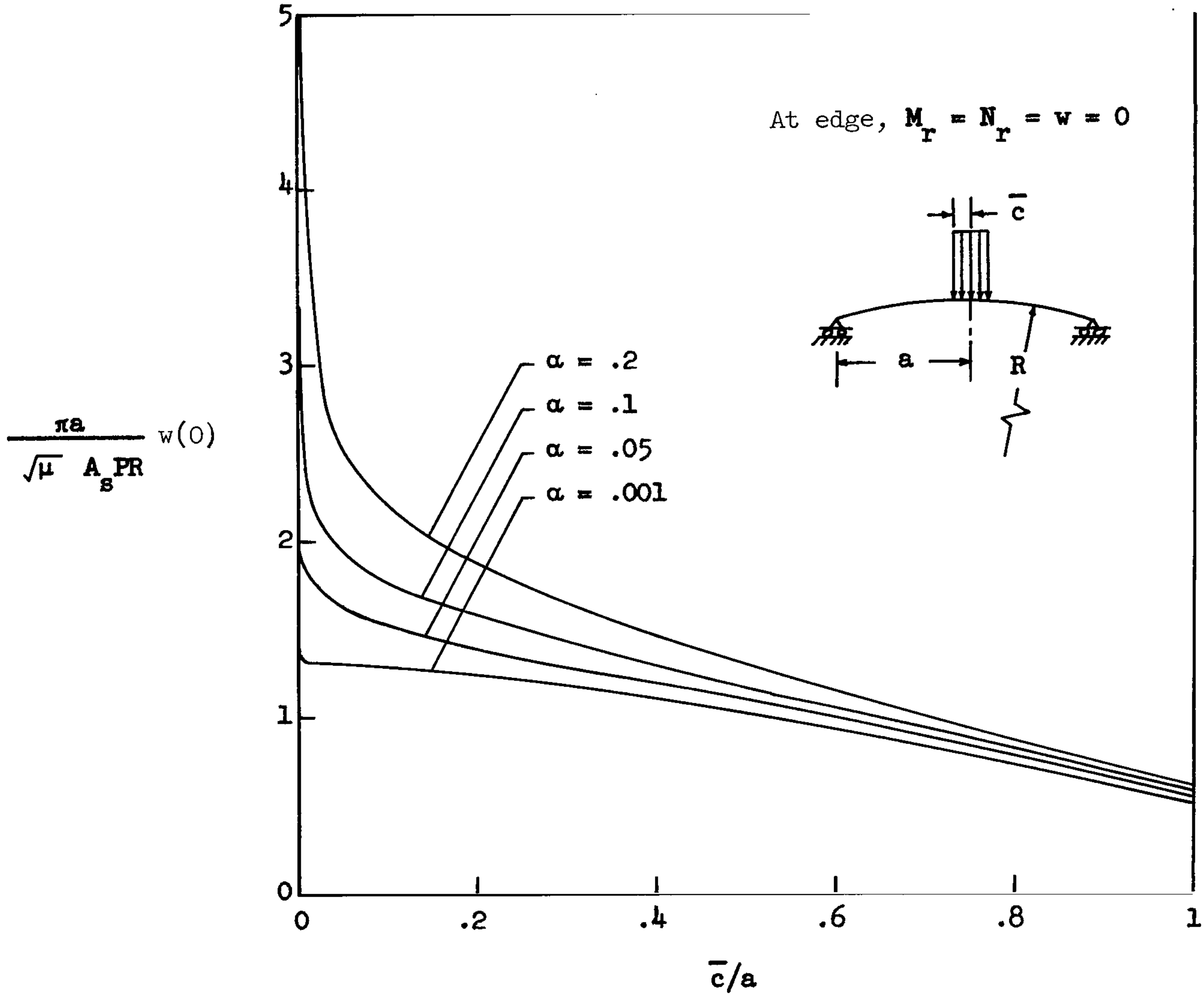


Figure 2.- Center displacement w(0) under uniform local loading for various values of α. $a^2/Rh = 2.5$; $\nu = 0.3$.

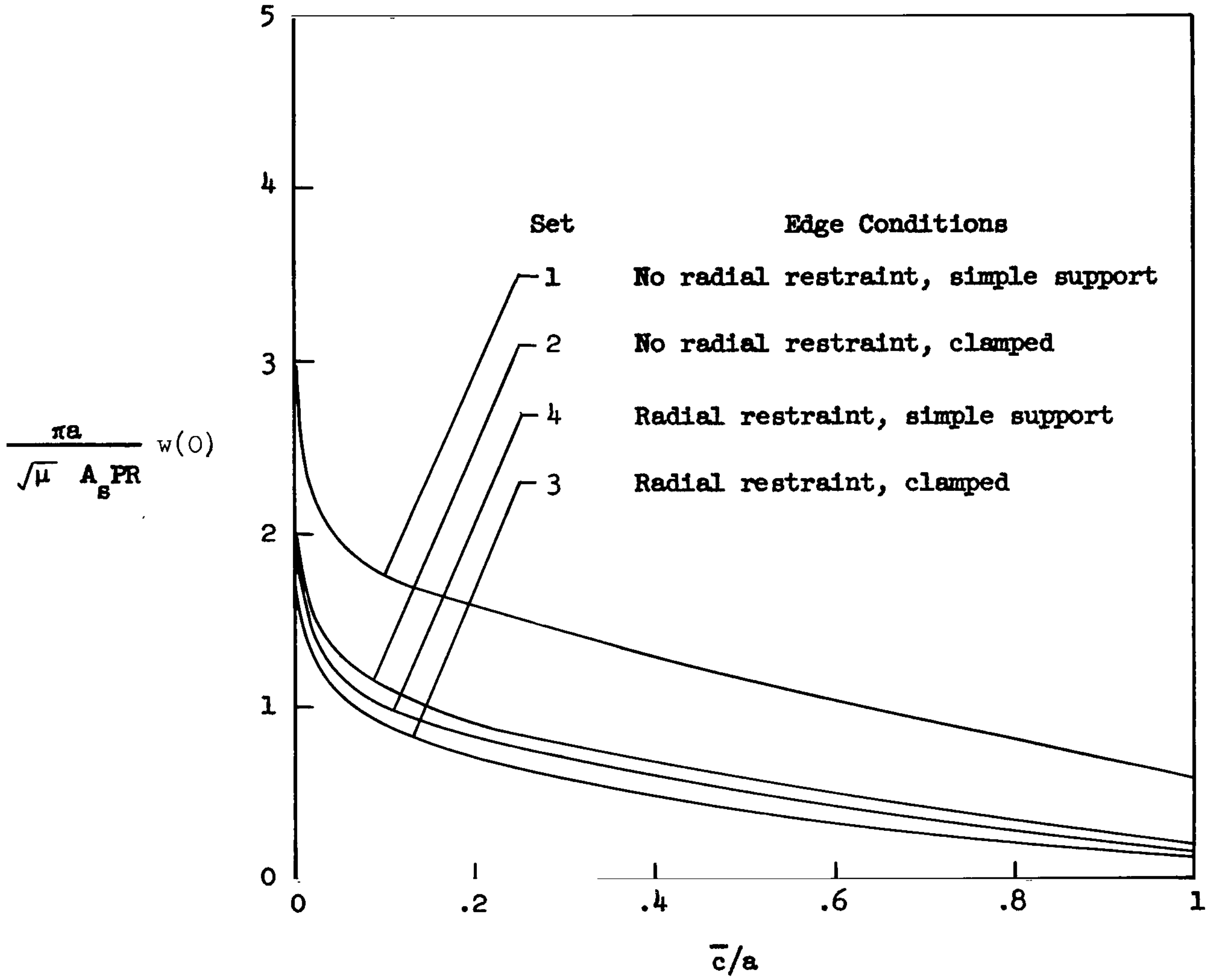


Figure 3.- Displacement w(0) under uniform local loading and various edge-support conditions. $a^2/Rh = 2.5$; $\alpha = 0.1$; $\nu = 0.3$.

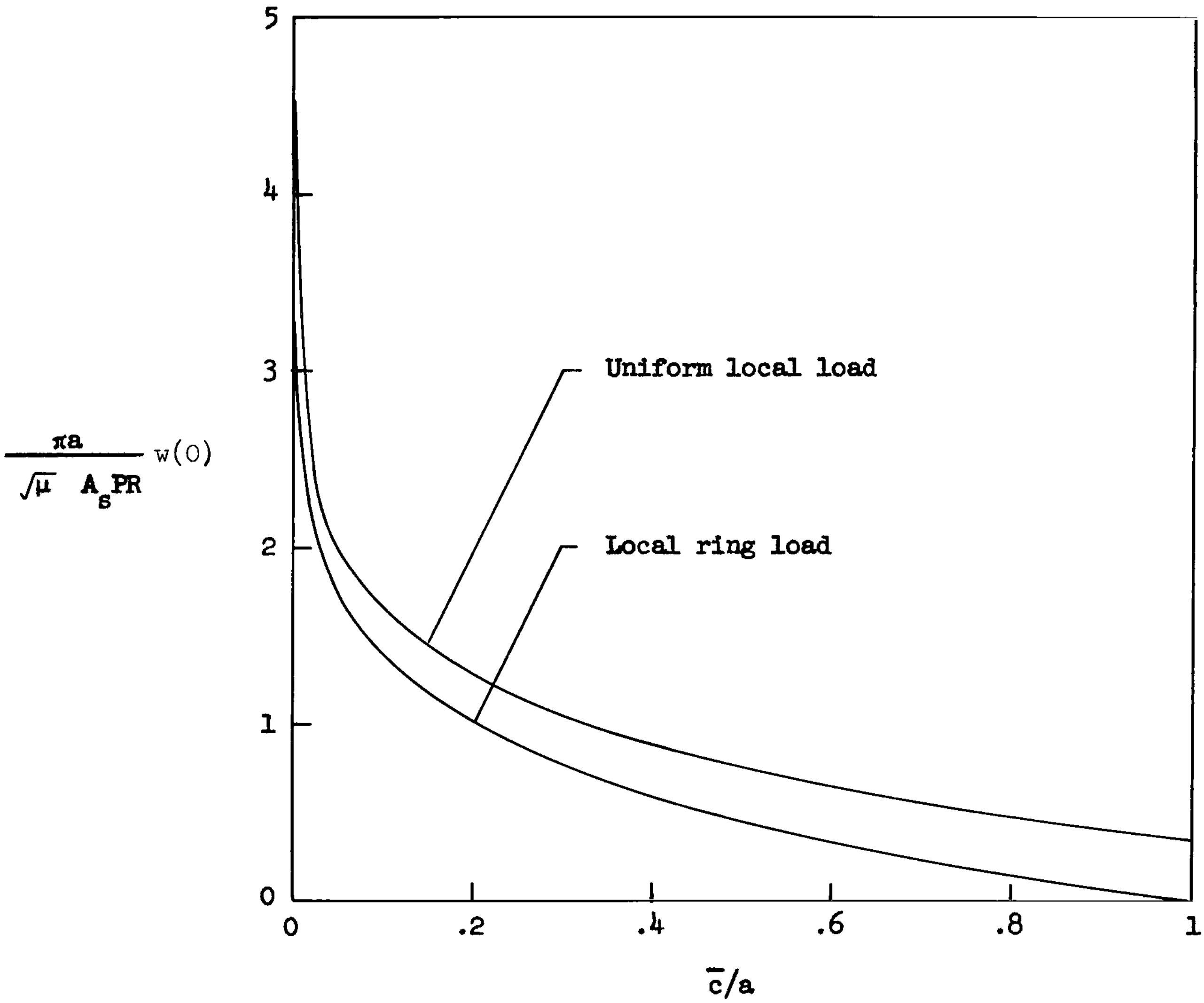


Figure 4.- Comparison of center deflection w(0) for uniform and local ring loading. The total load equals P in each case. $a^2/Rh' = 2.5$; $\alpha = 0.2$.

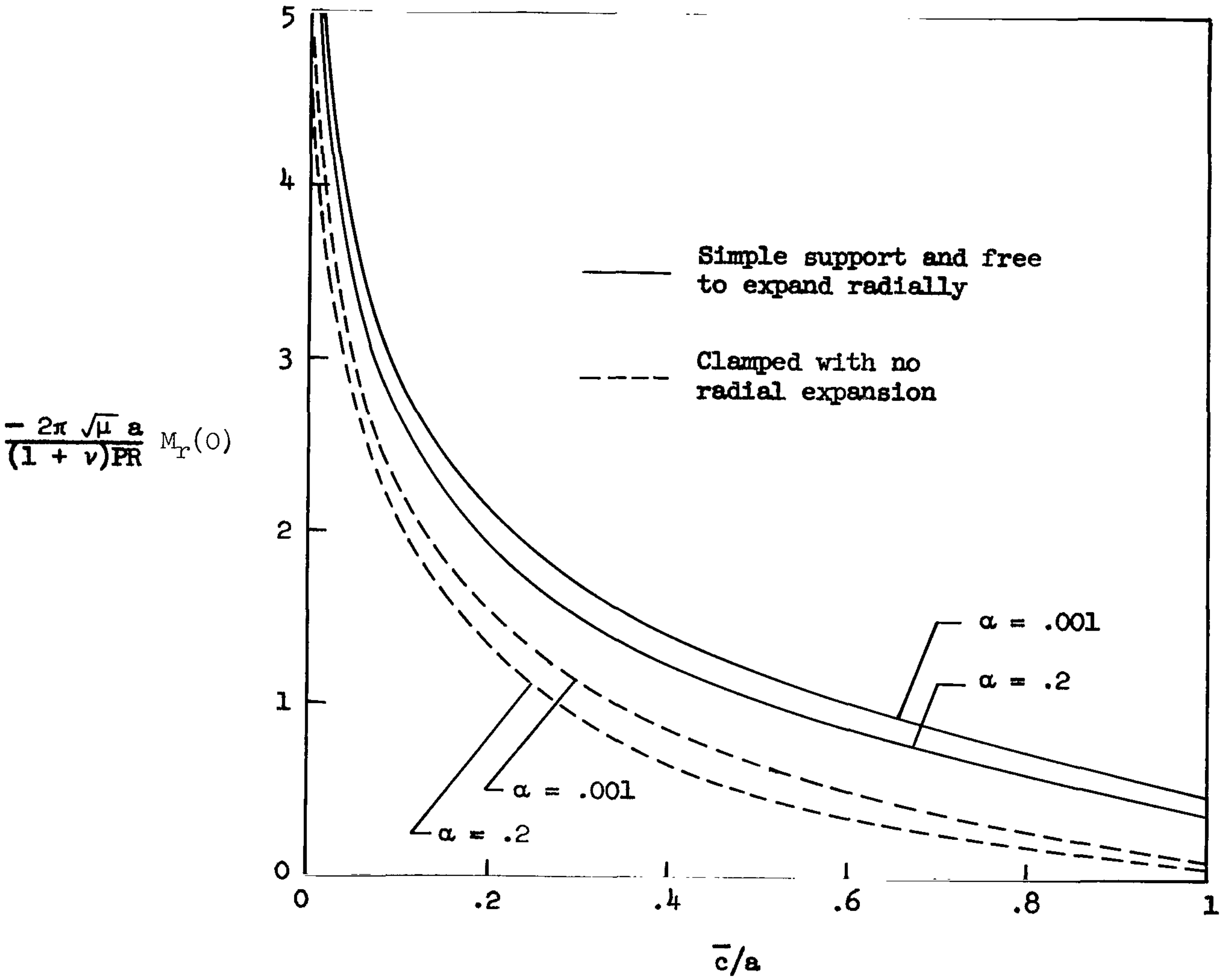


Figure 5.- Meridional bending moment at center of cap for two different edge conditions and two values of α for uniform loading. $a^2/Rh = 2.5$; $\nu = 0.3$.

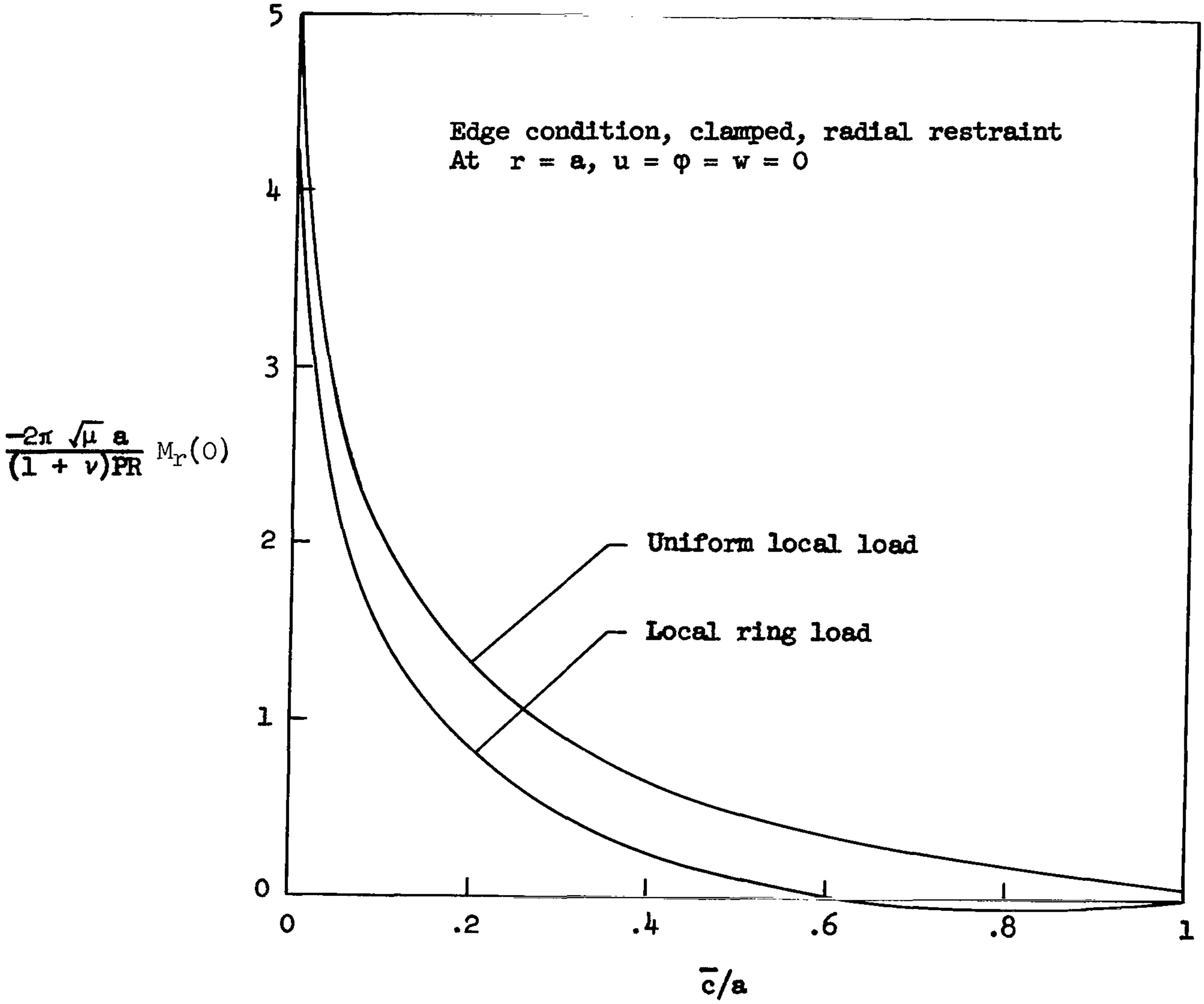


Figure 6.- Comparison of bending moment at center of cap for uniform and local ring loading. $a^2/Rh = 2.5$; $\alpha = 0.2$; $\nu = 0.3$.

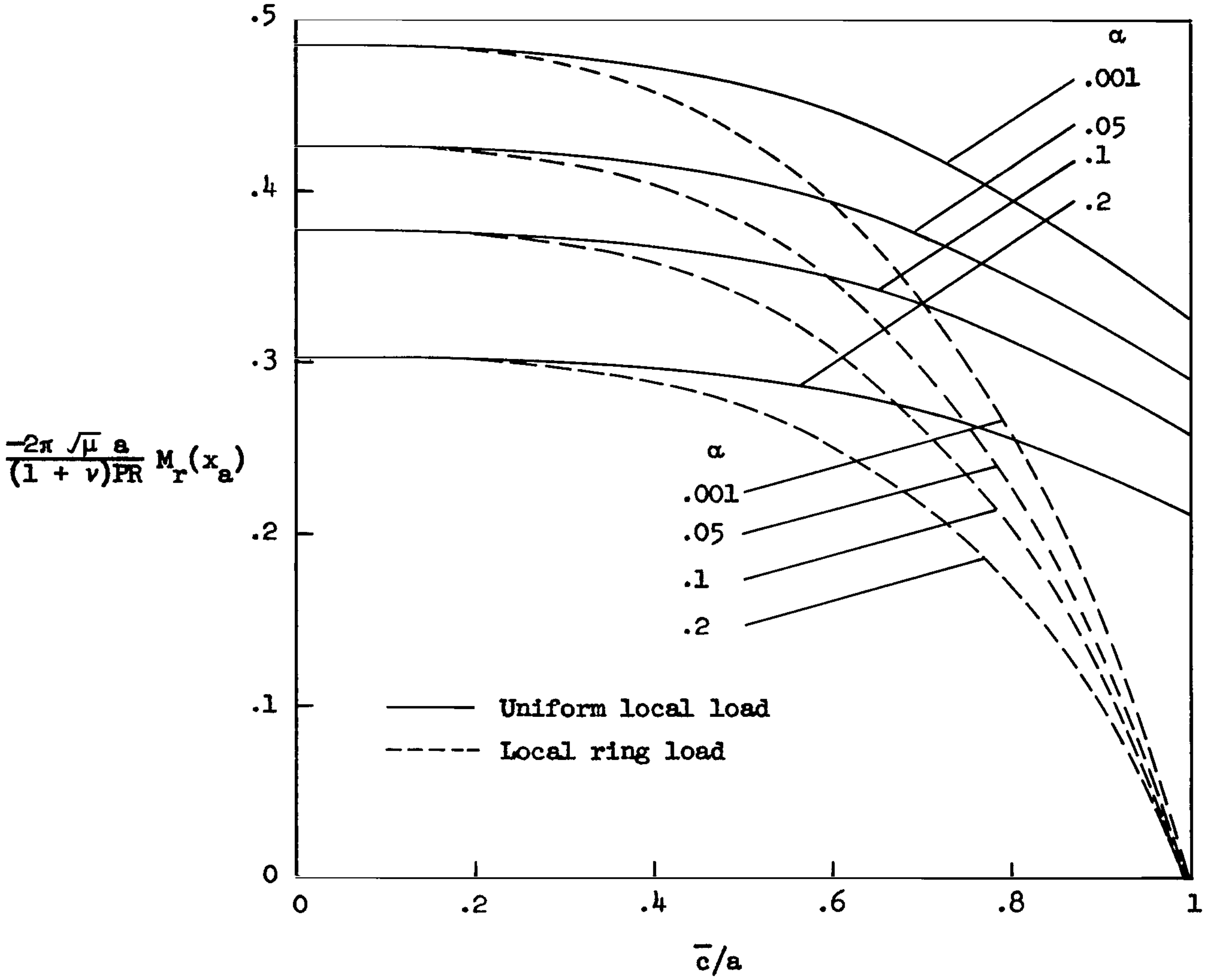


Figure 7.- Meridional bending moment at edge of cap for uniform and local ring loading where the edge is clamped. $a^2/Rh = 2.5$; $\nu = 0.3$.

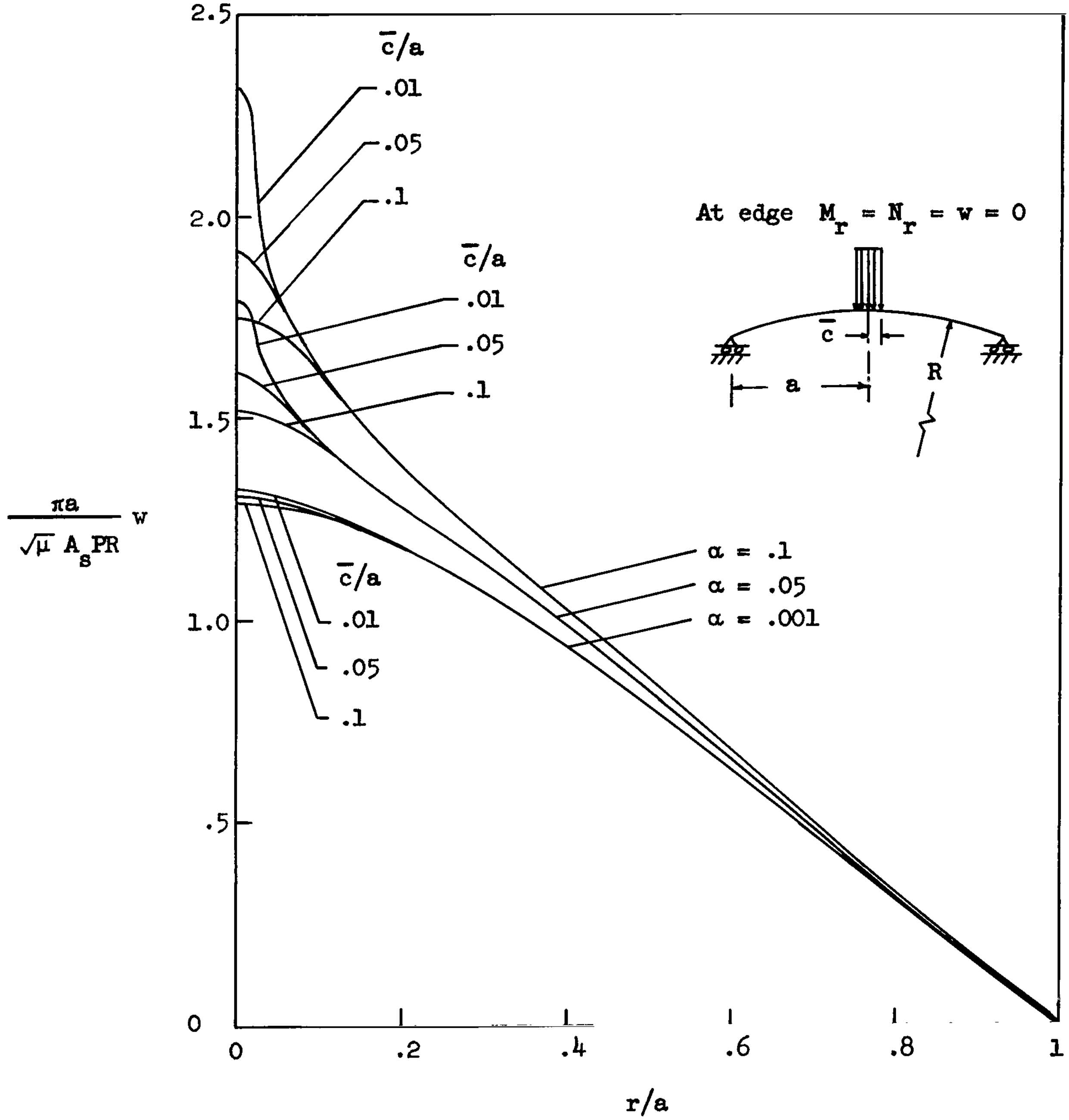


Figure 8.- Displacement w(r/a) caused by load P uniformly distributed over small circular areas defined by three values of $\bar{c}/a$. $a^2/Rh = 2.5$; $\nu = 0.3$.

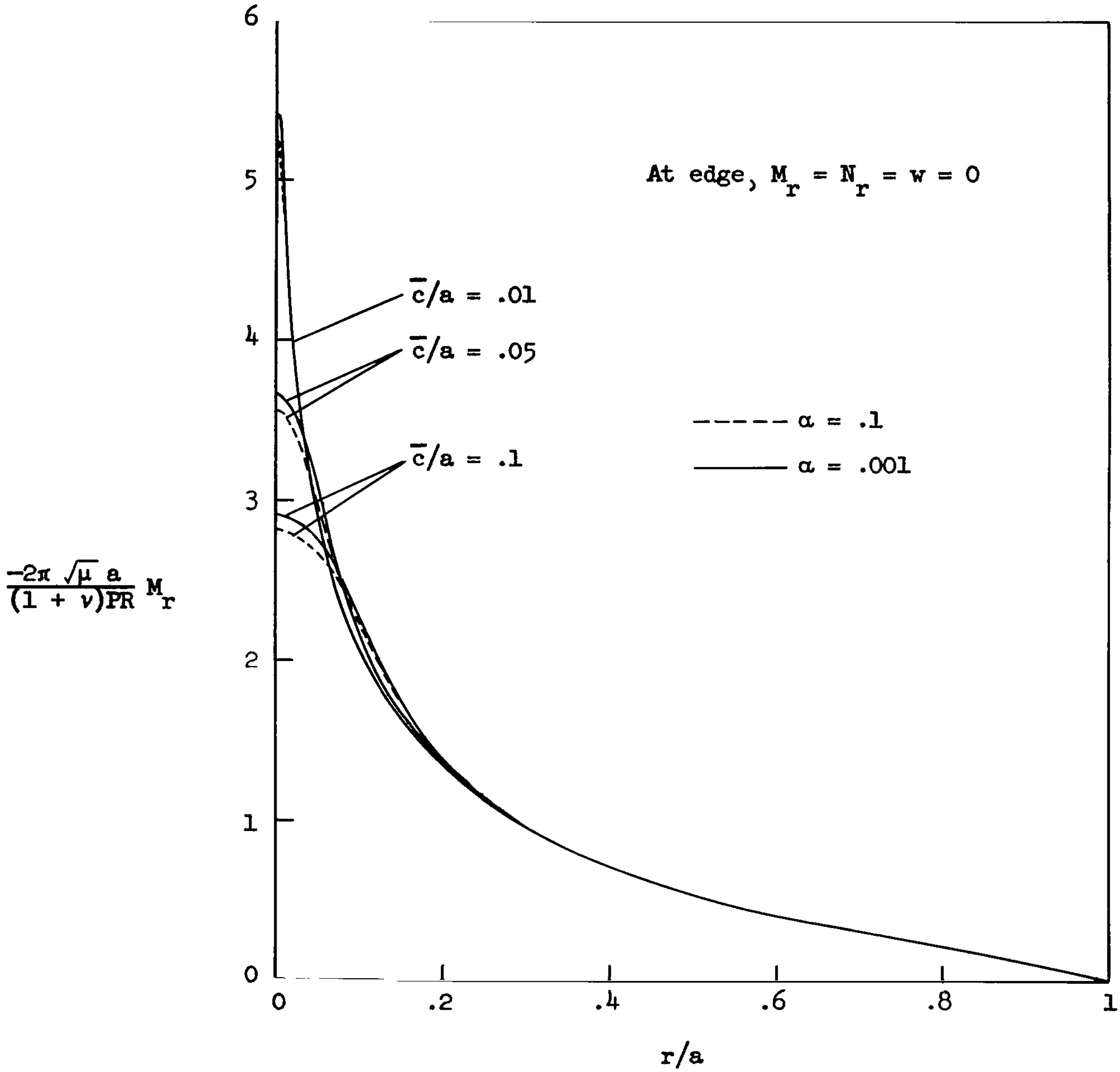


Figure 9.- Meridional bending moment caused by load P uniformly distributed over small circular areas defined by three values of $\bar{c}/a$. $a^2/Rh = 2.5$; $\nu = 0.3$.

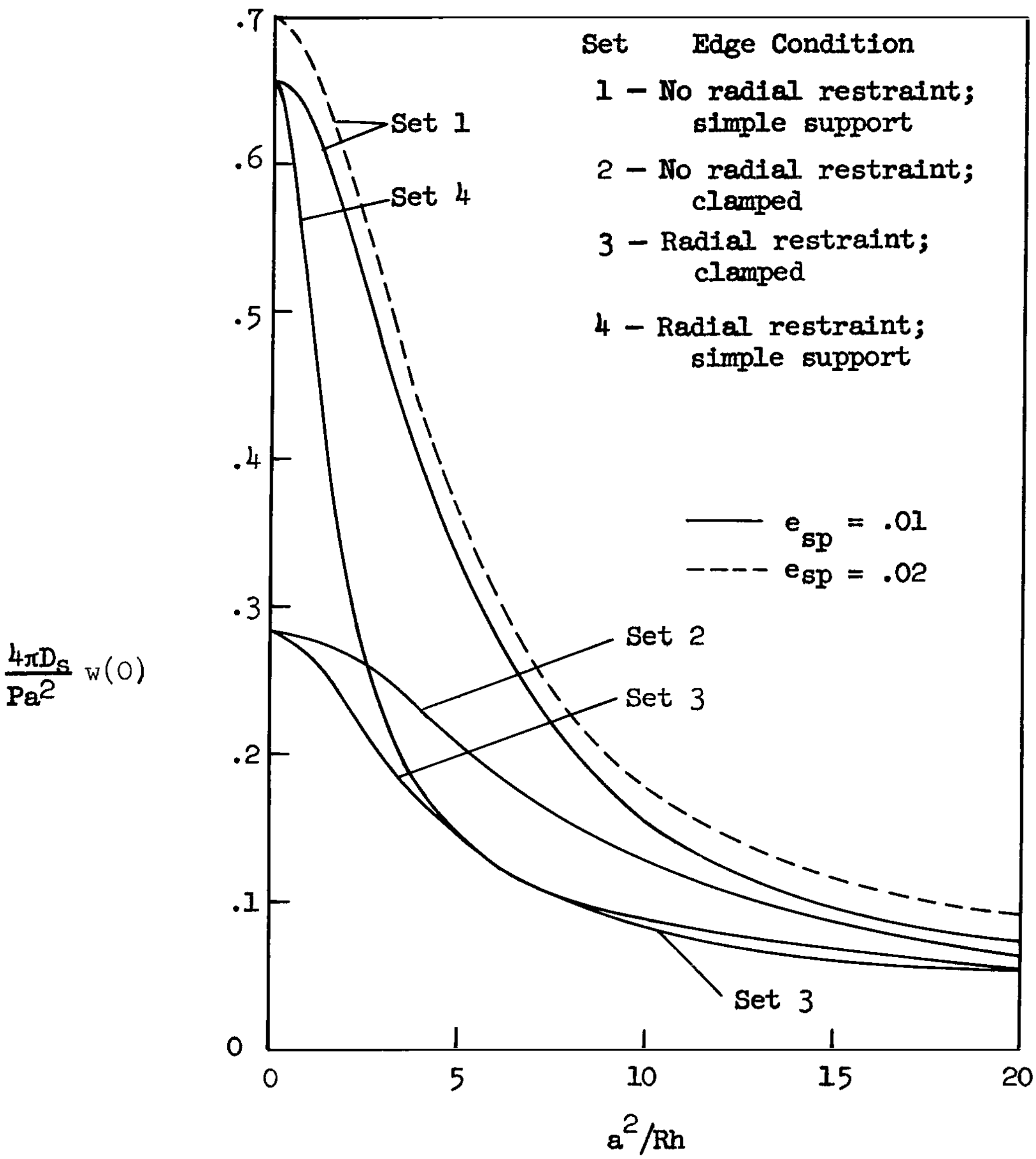


Figure 10.- Center deflection due to uniform local load ($\bar{c}/a = 0.05$) for various edge conditions plotted against curvature parameter a^2/Rh. $\nu = 0.3$.

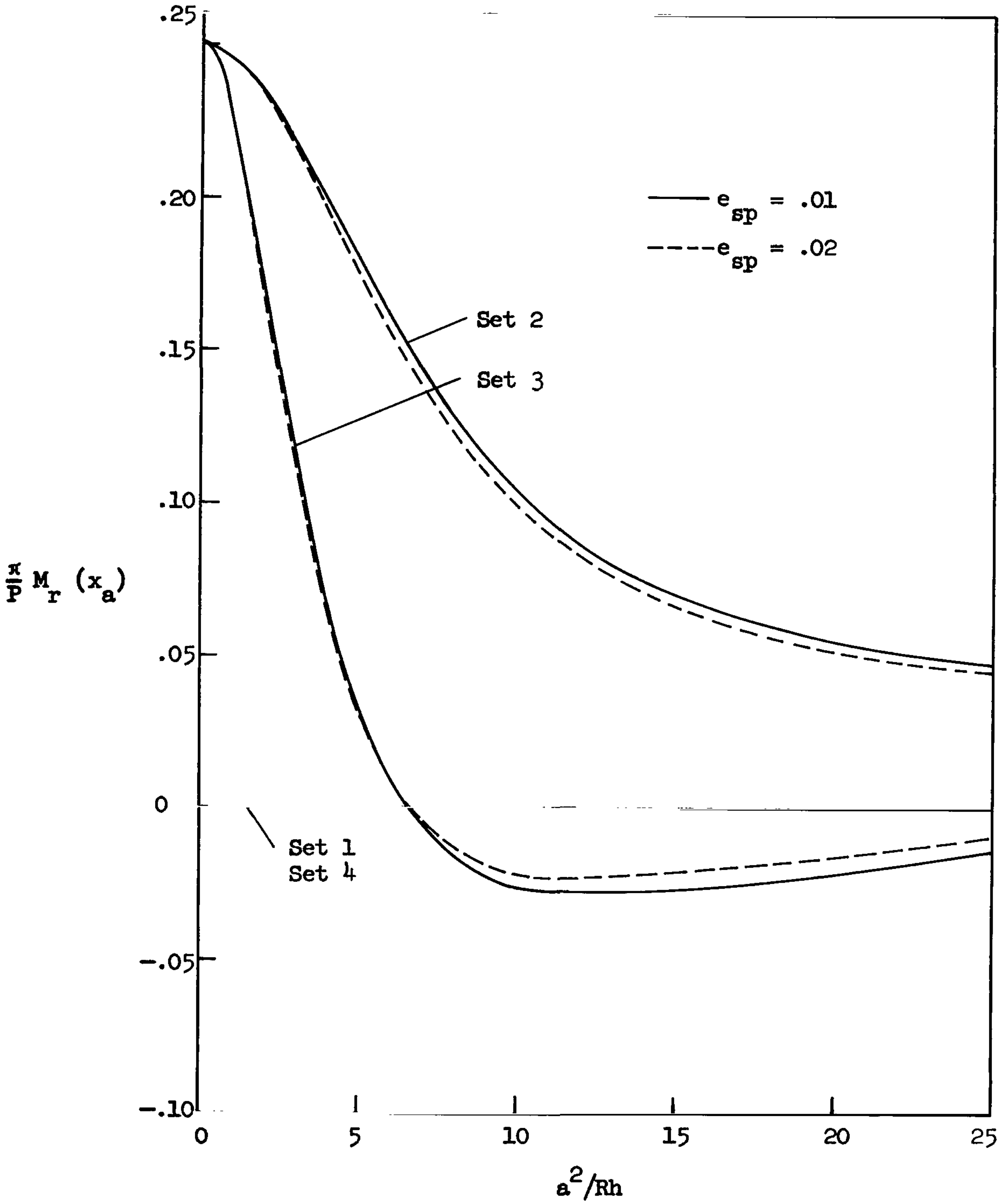


Figure 11.- Meridional bending moment at edge of cap ($r = a$) due to uniform local load ($\bar{c}/a = 0.05$) for various edge conditions. $\nu = 0.3$.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546