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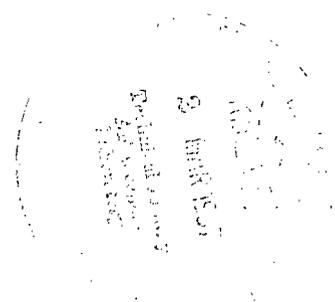


ECCENTRICALLY STIFFENED SHALLOW SHELLS OF DOUBLE CURVATURE

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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By John A. McElman
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SUMMARY

Equilibrium equations and boundary conditions are derived which govern the buckling and vibration of eccentrically stiffened shallow shells of double curvature. The equations are then solved for the case of simple-support boundary conditions and results are presented which illustrate the effects of eccentric stiffening on the dynamic and buckling characteristics of shells of positive and negative Gaussian curvature. Results show that eccentric stiffening can have a significant effect on the natural frequencies and buckling loads of such shells.

INTRODUCTION

Shells of double curvature have become common structural members in aerospace vehicles. The understanding of the effects of stiffening on the behavior of such shells has grown in importance with the increasing need for precision in the design of lightweight structures. The effects of stiffener eccentricities on the buckling and vibration characteristics of circular cylinders and flat plates have been studied analytically in references 1 to 7. A typical stiffened cylinder is shown in figure 1. Experimental results of references 3 and 8 show that for some configurations a circular cylindrical shell stiffened with stringers attached only to its external surface can carry more than twice as much load in axial compression as its internally stiffened counterpart.

In the present study, nonlinear equilibrium equations and boundary conditions are derived for eccentrically stiffened shells of double curvature. The types of doubly curved shells considered are shown in figure 2. These particular shell configurations are chosen because exact closed-form solutions can be obtained which exhibit typical eccentricity effects for shells of double curvature. The derivation is accomplished by defining nonlinear strain-displacement relations for the shell and stiffeners. The potential energy of

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the system is then formulated and the nonlinear equilibrium equations and boundary conditions are obtained by the application of the principle of minimum potential energy. The nonlinear equations are subsequently linearized to obtain equations which govern the small-amplitude vibrations of prestressed eccentrically stiffened shells. Closed-form solutions to these equations are presented for a membranelike prestress state and simple-support boundary conditions. If the natural frequency is assumed to be zero, the linear equations and the solution apply to buckling problems.

Because of the large number of parameters involved, presentation of results of a general nature would be impractical. Results are included, however, for the free vibration of specific shell configurations to demonstrate the effects of eccentric stiffening on vibration problems. Data are also presented for the buckling in axial compression and under hydrostatic pressure for specific shell configurations. These findings illustrate typical effects which exist in eccentrically stiffened shells of double curvature.

SYMBOLS

The units used for the physical quantities defined in this paper are given in both the U.S. Customary Units and the International System of Units (SI) (ref. 9). Factors relating these two systems are presented in appendix A.

A	cross-sectional area of stiffener
a	length of shell
C_1, C_2, C_3, C_4	constants defined in appendix B
D	flexural stiffness of isotropic shell wall, $\frac{Et^3}{12(1 - \mu^2)}$
d	stringer spacing
E	Young's modulus
e	distance from shell middle surface to line on which \bar{N}_x acts
\bar{e}_r, \bar{e}_s	nondimensional eccentricity parameters
f	frequency, $\frac{\omega}{2\pi}$
G	shear modulus

I	moment of inertia of stiffener about an axis through its centroid parallel to the shell middle surface
I_0	moment of inertia of stiffener about middle surface of shell
J	torsional constant for stiffener
l	ring spacing
M	mass per unit area of stiffened shell $\left(\rho_{sh}t + \rho_s \frac{A_s}{d} + \rho_r \frac{A_r}{l}\right)$
M_x, M_y, M_{xy}, M_{yx}	moment resultants
m, n	integers
N_x, N_y, N_{xy}	stress resultants
\bar{N}_x	externally applied compressive load resultant in x-direction
p	external pressure load
R_1	radius of shell equator (fig. 2)
R_2	radius of curvature (fig. 2)
\bar{R}	nondimensional parameter, $\frac{E_r A_r}{E t l}$
\bar{S}	nondimensional parameter, $\frac{E_s A_s}{E t d}$
t	thickness of shell
u, v, w	tangential displacements and normal displacement of shell middle surface in x-, y-, and z-directions, respectively
$\bar{u}, \bar{v}, \bar{w}$	displacement amplitudes
x, y, z	rectangular Cartesian coordinates
Z	curvature parameter, $\frac{a^2}{R_1 t} (1 - \mu^2)^{1/2}$

\bar{z} distance from middle surface of shell to centroid of stiffener
 α, β wavelength parameters
 Γ defined by equation (33)
 $\epsilon_x, \epsilon_y, \gamma_{xy}$ normal and shearing strains at shell middle surface
 $\epsilon_{xT}, \epsilon_{yT}, \gamma_{xyT}$ total normal and shearing strains
 $\Lambda, \Lambda_o, \Lambda_r, \Lambda_s, \Lambda_{rs}$ eccentricity parameters defined by equations (34)
 $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ defined by equations (B10)
 μ Poisson's ratio
 ν curvature parameter, $\frac{R_1}{R_2}$
 Π potential energy
 ρ mass density
 ω circular frequency

Subscripts:

A prestress state
B small changes away from prestress state
i inertial
L load
r stiffening in y-direction (rings)
s stiffening in x-direction (stringers)

sh shell

T total

A subscript preceded by a comma indicates partial differentiation with respect to the subscript.

ANALYSIS

In this section, the nonlinear equilibrium equations and boundary conditions for the shallow shells of double curvature illustrated in figure 2 are derived. Solutions are obtained for the case of simple-support boundary condition.

Energy Formulation

Strain-displacement relations.- The strain-displacement relations for the shells of double curvature are derived in reference 10 and are as follows:

$$\left. \begin{aligned} \epsilon_{xT} &= u_{,x} + \frac{1}{2} w_{,x}^2 \pm \frac{w}{R_2} - zw_{,xx} \\ \epsilon_{yT} &= v_{,y} + \frac{1}{2} w_{,y}^2 + \frac{w}{R_1} - zw_{,yy} \\ \gamma_{xyT} &= u_{,y} + v_{,x} + w_{,x}w_{,y} - 2zw_{,xy} \end{aligned} \right\} \quad (1)$$

where u and v can be identified as the tangential displacements of the middle surface of the shallow shell and w can be regarded as the normal displacement. In equations (1) and in subsequent equations, a double sign is used; the upper sign applies to shells with positive Gaussian curvature and the lower sign applies to shells with negative Gaussian curvature.

Strain energy of isotropic shell.- The strain energy of the unstiffened thin-wall isotropic shell is

$$\Pi_{sh} = \frac{E}{2(1-\mu^2)} \int_{-t/2}^{t/2} \int_0^{2\pi R_1} \int_0^a \left(\epsilon_{xT}^2 + \epsilon_{yT}^2 + 2\mu \epsilon_{xT} \epsilon_{yT} + \frac{1-\mu}{2} \gamma_{xyT}^2 \right) dx dy dz \quad (2)$$

Substitution of equations (1) into equation (2) and integration with respect to z yields the following expression for shell strain energy:

$$\begin{aligned} \Pi_{sh} = & \frac{Et}{2(1-\mu^2)} \int_0^{2\pi R_1} \int_0^a \left(\epsilon_x^2 + \epsilon_y^2 + 2\mu\epsilon_x\epsilon_y + \frac{1-\mu}{2} \gamma_{xy}^2 \right) dx dy \\ & + \frac{D}{2} \int_0^{2\pi R_1} \int_0^a \left[w_{,xx}^2 + w_{,yy}^2 + 2\mu w_{,xx}w_{,yy} + 2(1-\mu)w_{,xy}^2 \right] dx dy \end{aligned} \quad (3)$$

where

$$\epsilon_x = \epsilon_{xT}|_{z=0}, \quad \epsilon_y = \epsilon_{yT}|_{z=0}, \quad \gamma_{xy} = \gamma_{xyT}|_{z=0} \quad (4)$$

and $D = \frac{Et^3}{12(1-\mu^2)}$ is the flexural stiffness of the shell wall.

Strain energy of stiffeners.- The strain energy of the stiffeners is derived by using the fact that the displacements in the shell and stiffeners are equal at the point of attachment. Integration with respect to the thickness coordinate yields terms whose signs depend upon whether the stiffeners are attached to the inner or the outer surface of the shell. The stiffeners are assumed to be in a state of uniaxial stress, and stiffener twisting is accounted for in an approximate manner. For configurations in which both rings and stringers are attached to the same surface of the shell, the effects of joints in the stiffener framework are ignored.

The total strain energy of K stringers in the x -direction is written as

$$\Pi_S = \sum_{k=1}^K \left(\int_0^a \int_{A_S} \frac{E_S}{2} \epsilon_{xT}^2 dA_S dx + \frac{G_S J_S}{2} \int_0^a w_{,xy}^2 dx \right)_k \quad (5)$$

where the first term within the parentheses is the strain energy of bending and extension in the stiffener, and the second term is the strain energy of twisting of the stiffener. This latter term results from assuming that the stiffener twists in a fashion such that its angle of twist is equal to the local angle of twist of the shell. The quantity dA_S is an element of the cross-sectional area of the stiffener and the quantity $G_S J_S$ is the twisting stiffness of the stringer section. After substitution from equations (1) and (4), the first term inside the parentheses can be written as

$$\int_0^a \frac{E_S}{2} \left(\epsilon_x^2 \int_{A_S} dA_S - 2\epsilon_x w_{,xx} \int_{A_S} z dA_S + w_{,xx}^2 \int_{A_S} z^2 dA_S \right) dx$$

Inspection of these terms reveals that the first integral inside the parentheses is the area A_S of the stiffener cross section, the second integral is the first moment of the area $\bar{z}_S A_S$ where \bar{z}_S is the distance from the middle surface of the isotropic shell ($z = 0$) to

the centroid of the stiffener cross section, and the third integral is the moment of inertia of the stringer I_{OS} about $z = 0$. Note that the centroid distance \bar{z}_S is positive for external stiffeners and negative for internal stiffeners. If the stiffener spacing d at the equator is sufficiently small, the effects of the stiffeners can be averaged or "smeared out," and the finite sum in equation (5) can be replaced by an integral. Equation (5) then becomes

$$\Pi_S = \frac{1}{d} \int_0^{2\pi R_1} \int_0^a \left[\frac{E_S}{2} \left(A_S \epsilon_x^2 - 2\bar{z}_S A_S \epsilon_x w_{,xx} + I_{OS} w_{,xx}^2 \right) + \frac{G_S J_S}{2} w_{,xy}^2 \right] dx dy \quad (6)$$

The strain energy of stiffening in the y -direction can be derived in a similar manner and is written as follows:

$$\Pi_R = \frac{1}{l} \int_0^{2\pi R_1} \int_0^a \left[\frac{E_R}{2} \left(A_R \epsilon_y^2 - 2\bar{z}_R A_R \epsilon_y w_{,yy} + I_{OR} w_{,yy}^2 \right) + \frac{G_R J_R}{2} w_{,xy}^2 \right] dx dy \quad (7)$$

where l is the ring spacing and the subscript r is used to denote ring properties.

Potential energy of applied loads.- The only applied loads to be considered in this analysis are an external pressure p and an externally applied load resultant \bar{N}_x (positive in compression). The potential energy associated with these loads is

$$\Pi_L = \int_0^{2\pi R_1} \int_0^a p w dx dy + \int_0^{2\pi R_1} \bar{N}_x (u - e w_{,x}) \Big|_0^a dy \quad (8)$$

The quantity e is the distance from the middle surface of the isotropic shell ($z = 0$) to the line on which \bar{N}_x acts.

Potential energy of inertia loading.- If the stiffened shell is undergoing simple harmonic motion of circular frequency ω (inplane inertia neglected), and if u , v , and w are amplitudes of such motion, the potential energy of inertia loading at maximum deflection is

$$\Pi_\omega = - \frac{1}{2} \int_0^{2\pi R_1} \int_0^a M \omega^2 w^2 dx dy \quad (9)$$

where $M = \rho_{sh} t + \rho_s \frac{A_S}{d} + \rho_r \frac{A_R}{l}$ is the smeared-out mass per unit of surface area of the stiffened shell. The quantities ρ_{sh} , ρ_s , and ρ_r are the mass densities of the shell, x -direction stiffeners, and y -direction stiffeners, respectively.

Nonlinear Equilibrium Equations and Boundary Conditions

The total potential energy Π_T is the sum of the energies given by equations (3), (6), (7), (8), and (9):

$$\Pi_T = \Pi_{sh} + \Pi_S + \Pi_R + \Pi_L + \Pi_\omega \quad (10)$$

Equation (10) may be written in terms of stress and moment resultants as

$$\begin{aligned} \Pi_T = & \frac{1}{2} \int_0^{2\pi R_1} \int_0^a \left(N_x \epsilon_x + N_y \epsilon_y + N_{xy} \gamma_{xy} - M_x w_{,xx} - M_y w_{,yy} + M_{xy} w_{,xy} - M_{yx} w_{,xy} \right) dx dy \\ & + \int_0^{2\pi R_1} \int_0^a \left(pw - \frac{1}{2} M \omega^2 w^2 \right) dx dy + \int_0^{2\pi R_1} \bar{N}_x (u - ew_{,x}) \Big|_0^a dy \end{aligned} \quad (11)$$

where the stress and moment resultants are defined as follows

$$\left. \begin{aligned} N_x &= \frac{Et}{1 - \mu^2} \left[u_{,x} \pm \frac{w}{R_2} + \frac{1}{2} w_{,x}^2 + \mu \left(v_{,y} + \frac{w}{R_1} + \frac{1}{2} w_{,y}^2 \right) \right] \\ &+ \frac{E_S A_S}{d} \left(u_{,x} \pm \frac{w}{R_2} + \frac{1}{2} w_{,x}^2 - \bar{z}_S w_{,xx} \right) \\ N_y &= \frac{Et}{1 - \mu^2} \left[v_{,y} + \frac{w}{R_1} + \frac{1}{2} w_{,y}^2 + \mu \left(u_{,x} \pm \frac{w}{R_2} + \frac{1}{2} w_{,x}^2 \right) \right] \\ &+ \frac{E_R A_R}{l} \left(v_{,y} + \frac{w}{R_1} + \frac{1}{2} w_{,y}^2 - \bar{z}_R w_{,yy} \right) \\ N_{xy} &= Gt (u_{,y} + v_{,x} + w_{,x} w_{,y}) \\ M_x &= - \left[D (w_{,xx} + \mu w_{,yy}) + \frac{E_S I_S}{d} w_{,xx} - \frac{\bar{z}_S E_S A_S}{d} \left(u_{,x} \pm \frac{w}{R_2} + \frac{1}{2} w_{,x}^2 - \bar{z}_S w_{,xx} \right) \right] \\ M_y &= - \left[D (w_{,yy} + \mu w_{,xx}) + \frac{E_R I_R}{l} w_{,yy} - \frac{\bar{z}_R E_R A_R}{l} \left(v_{,y} + \frac{w}{R_1} + \frac{1}{2} w_{,y}^2 - \bar{z}_R w_{,yy} \right) \right] \\ M_{xy} &= \left(\frac{Gt^3}{6} + \frac{G_S J_S}{d} \right) w_{,xy} \\ M_{yx} &= - \left(\frac{Gt^3}{6} + \frac{G_R J_R}{l} \right) w_{,xy} \end{aligned} \right\} \quad (12)$$

and

$$I_S = I_{OS} - \bar{z}_S^2 A_S$$

$$I_R = I_{OR} - \bar{z}_R^2 A_R$$

The stress resultants N_x and N_y are positive in tension. The nonlinear equilibrium equations and boundary conditions are obtained from equation (11) by application of the principle of minimum potential energy ($\delta\Pi = 0$) and the fundamental lemma of the calculus of variations. The equations so obtained are

$$\left. \begin{aligned}
 N_{x,x} + N_{xy,y} &= 0 \\
 N_{y,y} + N_{xy,x} &= 0 \\
 -M_{x,xx} + M_{xy,xy} - M_{yx,xy} - M_{y,yy} \pm \frac{N_x}{R_2} + \frac{N_y}{R_1} \\
 - N_x w_{,xx} - N_y w_{,yy} - 2N_{xy} w_{,xy} - M\omega^2 w + p &= 0
 \end{aligned} \right\} \quad (13)$$

A set of boundary conditions to be satisfied at each end of the shell ($x = 0, a$) are

$$M_{x,x} - (M_{xy,y} - M_{yx,y}) + N_x w_{,x} + N_{xy} w_{,y} = 0 \quad (14a)$$

$$\text{or } w = 0 \quad (14b)$$

$$M_x + \bar{N}_x e = 0 \quad (15a)$$

$$\text{or } w_{,x} = 0 \quad (15b)$$

$$N_x + \bar{N}_x = 0 \quad (16a)$$

$$\text{or } u = 0 \quad (16b)$$

$$N_{xy} = 0 \quad (17a)$$

$$\text{or } v = 0 \quad (17b)$$

The natural boundary conditions are given by the expressions in equations (14a), (15a), (16a), and (17a) and the geometric boundary conditions are given by equations (14b), (15b), (16b), and (17b). The condition in equation (14a) requires that a shear resultant comparable to the Kirchhoff shear vanish and hence is a free-edge boundary condition. The three natural boundary conditions in equations (15a), (16a), and (17a) correspond to conditions in which the edge moment resultant, the extensional stress resultant, and the shear stress resultant, respectively, vanish.

Homogeneous Equations Governing a Prestressed Vibrating Shell

In this section, the nonlinear equilibrium equations (eqs. (13)) are used to obtain linear equations which govern the small-amplitude vibration of a prestressed eccentrically

stiffened shallow shell of double curvature. The deformations u , v , and w associated with the vibration of a prestressed shell are divided into two parts as follows:

$$u = u_A + u_B, \quad v = v_A + v_B, \quad w = w_A + w_B \quad (18)$$

The first part, denoted by the subscript A , is assumed to be an axisymmetric static prestress deformation which occurs prior to the excitation of one of the natural frequencies. The second part, denoted by the subscript B , is a small additional deformation which occurs as a result of the excitation. Since equations (13) are equilibrium equations for the system, the displacements denoted by subscript A as well as the sum of the two displacements denoted by subscripts A and B must satisfy these equations. After substitution of the axisymmetric (subscript A) displacements, equations (13) become

$$\left. \begin{aligned} N_{xA,x} &= 0 \\ N_{xyA,x} &= 0 \\ -M_{xA,xx} \pm \frac{N_{xA}}{R_2} + \frac{N_{yA}}{R_1} - N_{xA}w_{A,xx} + p &= 0 \end{aligned} \right\} \quad (19)$$

where now

$$\left. \begin{aligned} N_{xA} &= \frac{Et}{1-\mu^2} \left[u_{A,x} \pm \frac{w_A}{R_2} + \frac{1}{2} w_{A,x}^2 + \mu \left(\frac{w_A}{R_1} \right) + \frac{E_S A_S}{d} \left(u_{A,x} \pm \frac{w_A}{R_2} + \frac{1}{2} w_{A,x}^2 - \bar{z}_S w_{A,xx} \right) \right] \\ N_{yA} &= \frac{Et}{1-\mu^2} \left[\frac{w_A}{R_1} + \mu \left(u_{A,x} \pm \frac{w_A}{R_2} + \frac{1}{2} w_{A,x}^2 \right) + \frac{E_r A_r}{l} \left(\frac{w_A}{R_1} \right) \right] \\ N_{xyA} &= Gt v_{A,x} \\ M_{xA} &= - \left[D w_{A,xx} + \frac{E_S I_S}{d} w_{A,xx} - \frac{\bar{z}_S E_S A_S}{d} \left(u_{A,x} \pm \frac{w_A}{R_2} + \frac{1}{2} w_{A,x}^2 - \bar{z}_S w_{A,xx} \right) \right] \end{aligned} \right\} \quad (20)$$

A set of appropriate boundary conditions is found from equations (14) to (17) to be

$$\left. \begin{aligned} M_{xA,x} + N_{xA} w_{A,x} &= 0 \quad \text{or} \quad w_A = 0 \\ M_{xA} + \bar{N}_x e &= 0 \quad \text{or} \quad w_{A,x} = 0 \\ N_{xA} + \bar{N}_x &= 0 \quad \text{or} \quad u_A = 0 \\ N_{xyA} &= 0 \quad \text{or} \quad v_A = 0 \end{aligned} \right\} \quad (21)$$

A solution to equations (19) satisfying the conditions (21) completely describes the pre-stressed state.

The equilibrium equations governing the additional deformations (subscript B) are obtained by substituting equations (18) into equations (13). If only linear terms in the additional deformations are retained and equations (19) are considered, equations (13) become

$$\left. \begin{aligned} N_{xB,x} + N_{xyB,y} &= 0 \\ N_{yB,y} + N_{xyB,x} &= 0 \\ -M_{xB,xx} + M_{xyB,xy} - M_{yxB,xy} - M_{yB,yy} \pm \frac{N_{xB}}{R_2} + \frac{N_{yB}}{R_1} \\ &- N_{xA}w_{B,xx} - N_{yA}w_{B,yy} - N_{xB}w_{A,xx} - M\omega^2w_B = 0 \end{aligned} \right\} \quad (22)$$

and the boundary conditions become

$$\left. \begin{aligned} M_{xB,x} - (M_{xyB,y} - M_{yxB,y}) + N_{xA}w_{B,x} + N_{xB}w_{A,x} &= 0 \quad \text{or} \quad w_B = 0 \\ M_{xB} = 0 \quad \text{or} \quad w_{B,x} &= 0 \\ N_{xB} = 0 \quad \text{or} \quad u_B = 0 \\ N_{xyB} = 0 \quad \text{or} \quad v_B = 0 \end{aligned} \right\} \quad (23)$$

where

$$\begin{aligned} N_{xB} = \frac{Et}{1 - \mu^2} &\left[u_{B,x} \pm \frac{w_B}{R_2} + w_{A,x}w_{B,x} + \mu \left(v_{B,y} + \frac{w_B}{R_1} \right) \right] \\ &+ \frac{E_s A_s}{d} \left(u_{B,x} \pm \frac{w_B}{R_2} + w_{A,x}w_{B,x} - \bar{z}_s w_{B,xx} \right) \end{aligned} \quad (24a)$$

$$N_{yB} = \frac{Et}{1 - \mu^2} \left[v_{B,y} + \frac{w_B}{R_1} + \mu \left(u_{B,x} \pm \frac{w_B}{R_2} + w_{A,x}w_{B,x} \right) \right] + \frac{E_r A_r}{l} \left(v_{B,y} + \frac{w_B}{R_1} - \bar{z}_r w_{B,yy} \right) \quad (24b)$$

$$N_{xyB} = Gt \left(u_{B,y} + v_{B,x} + w_{A,x}w_{B,y} \right) \quad (24c)$$

$$M_{xB} = - \left[D(w_{B,xx} + \mu w_{B,yy}) + \frac{E_S I_S}{d} w_{B,xx} - \frac{\bar{z}_S E_S A_S}{d} \left(u_{B,x} \pm \frac{w_B}{R_2} + w_{A,x} w_{B,x} - \bar{z}_S w_{B,xx} \right) \right] \quad (24d)$$

$$M_{yB} = - \left[D(w_{B,yy} + \mu w_{B,xx}) + \frac{E_r I_r}{l} w_{B,yy} - \frac{\bar{z}_r E_r A_r}{l} \left(v_{B,y} + \frac{w_B}{R_1} - \bar{z}_r w_{B,yy} \right) \right] \quad (24e)$$

$$M_{xyB} = \left(\frac{Gt^3}{6} + \frac{G_S J_S}{d} \right) w_{B,xy} \quad (24f)$$

$$M_{yxB} = - \left(\frac{Gt^3}{6} + \frac{G_r J_r}{l} \right) w_{B,xy} \quad (24g)$$

The homogeneous equations (22) and the homogeneous boundary conditions (23) represent an eigenvalue problem which governs the natural frequencies of a prestressed eccentrically stiffened shell of double curvature. The coefficients in the equations are determined by considering solutions to the axisymmetric prestress problem described by equations (19) and the associated boundary conditions (eqs. (21)).

Solution for Prestressed Vibrating Shell

Equations (22) have variable coefficients and would be quite difficult to solve in most instances. If, however, an assumption analogous to that made in classical buckling theory is made (i.e., that the lateral prestress deformation w_A is constant prior to the excitation), the solution is greatly simplified. The implications of this assumption are given in reference 6, and the exact solution to equations (19) and (20) is presented in appendix B.

For w_A constant and no applied shear, the prestress equations are

$$\left. \begin{aligned} N_{xA} &= \text{Constant} \\ N_{xyA} &= \text{Constant} = 0 \\ \pm \frac{N_{xA}}{R_2} + \frac{N_{yA}}{R_1} + p &= 0 \end{aligned} \right\} \quad (25)$$

From the boundary conditions (eqs. 21))

$$N_{xA} = -\bar{N}_x \quad (26)$$

Substituting equation (26) into the third of equations (25) yields

$$N_{yA} = -(pR_1 + \bar{N}_x \nu) \quad (27)$$

where $\nu = \frac{R_1}{R_2}$ is a curvature parameter.

Also for w_A constant, equations (22) become

$$\left. \begin{aligned} N_{xB,x} + N_{xyB,y} &= 0 \\ N_{yB,y} + N_{xyB,x} &= 0 \\ -M_{xB,xx} + M_{xyB,xy} - M_{yxB,xy} - M_{yB,yy} \pm \frac{N_{xB}}{R_2} + \frac{N_{yB}}{R_1} \\ &- N_{xA} w_{B,xx} - N_{yA} w_{B,yy} - M\omega^2 w_B = 0 \end{aligned} \right\} \quad (28)$$

where now

$$N_{xB} = \frac{Et}{1 - \mu^2} \left[u_{B,x} \pm \frac{w_B}{R_2} + \mu \left(v_{B,y} + \frac{w_B}{R_1} \right) \right] + \frac{E_s A_s}{d} \left(u_{B,x} \pm \frac{w_B}{R_2} - \bar{z}_s w_{B,xx} \right) \quad (29a)$$

$$N_{yB} = \frac{Et}{1 - \mu^2} \left[v_{B,y} + \frac{w_B}{R_1} + \mu \left(u_{B,x} \pm \frac{w_B}{R_2} \right) \right] + \frac{E_r A_r}{l} \left(v_{B,y} + \frac{w_B}{R_1} - \bar{z}_r w_{B,yy} \right) \quad (29b)$$

$$N_{xyB} = Gt(u_{B,y} + v_{B,x}) \quad (29c)$$

$$M_{xB} = - \left[D(w_{B,xx} + \mu w_{B,yy}) + \frac{E_s I_s}{d} w_{B,xx} - \frac{\bar{z}_s E_s A_s}{d} \left(u_{B,x} \pm \frac{w_B}{R_2} - \bar{z}_s w_{B,xx} \right) \right] \quad (29d)$$

$$M_{yB} = - \left[D(w_{B,yy} + \mu w_{B,xx}) + \frac{E_r I_r}{l} w_{B,yy} - \frac{\bar{z}_r E_r A_r}{l} \left(v_{B,y} + \frac{w_B}{R_1} - \bar{z}_r w_{B,yy} \right) \right] \quad (29e)$$

$$M_{xyB} = \left(\frac{Gt^3}{6} + \frac{G_S J_S}{d} \right) w_{B,xy} \quad (29f)$$

$$M_{yxB} = - \left(\frac{Gt^3}{6} + \frac{G_R J_R}{l} \right) w_{B,xy} \quad (29g)$$

If the origin of the coordinate system is taken at one edge of the shell, the simple-support boundary conditions to be satisfied are

$$\left. \begin{aligned} w_{B(0,y)} = M_{xB(0,y)} = v_{B(0,y)} = N_{xB(0,y)} = 0 \\ w_{B(a,y)} = M_{xB(a,y)} = v_{B(a,y)} = N_{xB(a,y)} = 0 \end{aligned} \right\} \quad (30)$$

Expressions for the displacements u_B , v_B , and w_B which satisfy these boundary conditions are given as

$$\left. \begin{aligned} u_B &= \bar{u} \cos \frac{m\pi x}{a} \cos \frac{ny}{R_1} \\ v_B &= \bar{v} \sin \frac{m\pi x}{a} \sin \frac{ny}{R_1} \\ w_B &= \bar{w} \sin \frac{m\pi x}{a} \cos \frac{ny}{R_1} \end{aligned} \right\} \quad (31)$$

where m is the number of axial half waves and n is the number of circumferential full waves. After substitution of equations (31) into equations (28) and nondimensionalization, the following equation is obtained:

$$\begin{bmatrix} [A_{11}] & [\bar{e}_s \bar{s} \alpha^2 (1 - \mu^2) + \mu \pm \nu \pm \bar{s} \nu (1 - \mu^2)] & [\bar{e}_r \bar{R} (1 - \mu^2) n^2 \pm \mu \nu + 1 + (1 - \mu^2) \bar{R}] \\ [\bar{e}_s \bar{s} \alpha^2 (1 - \mu^2) + \mu \pm \nu \pm \bar{s} \nu (1 - \mu^2)] & -[1 + \bar{s} (1 - \mu^2) + \frac{1 - \mu}{2} \beta^2] & -[\frac{1 + \mu}{2}] \\ [\bar{e}_r \bar{R} (1 - \mu^2) n^2 \pm \mu \nu + 1 + (1 - \mu^2) \bar{R}] & -[\frac{1 + \mu}{2}] & -[1 + \bar{R} (1 - \mu^2) + \frac{1 - \mu}{2 \beta^2}] \end{bmatrix} \begin{bmatrix} \bar{w} \\ \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (32)$$

where

$$\begin{aligned}
A_{11} = & -\frac{D\alpha^4(1-\mu^2)(1+\beta^2)^2}{EtR_1^2} - \frac{N_{yA}n^2(1-\mu^2)}{Et} - \frac{N_{xA}\alpha^2(1-\mu^2)}{Et} \\
& - \left(\frac{G_S J_S}{d} + \frac{G_R J_R}{l}\right) \frac{\alpha^2 n^2 (1-\mu^2)}{EtR_1^2} - \frac{E_R I_R (1-\mu^2) n^4}{l EtR_1^2} - \frac{E_S I_S (1-\mu^2) \alpha^4}{Et d R_1^2} - \bar{e}_S^2 \alpha^4 (1-\mu^2) \bar{S} \\
& - \bar{e}_R^2 n^4 (1-\mu^2) \bar{R} - 2\bar{e}_R n^2 (1-\mu^2) \bar{R} \mp 2\alpha^2 \nu \bar{e}_S (1-\mu^2) \bar{S} - (1 \pm 2\mu\nu + \nu^2) \\
& - \bar{R}(1-\mu^2) - \bar{S}(1-\mu^2)\nu^2 + \frac{M\omega^2 R_1^2 (1-\mu^2)}{Et}
\end{aligned}$$

and the following nondimensional parameters have been defined:

$$\bar{e}_S = \frac{\bar{z}_S}{R_1}, \quad \bar{e}_R = \frac{\bar{z}_R}{R_1}, \quad \alpha = \frac{m\pi R_1}{a}, \quad \beta = \frac{na}{m\pi R_1}, \quad \bar{R} = \frac{E_R A_R}{Et l}, \quad \bar{S} = \frac{E_S A_S}{Et d}$$

For a nontrivial solution to exist, the determinant of the coefficients of \bar{u} , \bar{v} , and \bar{w} must equal zero. After some manipulation, the nondimensional characteristic equation becomes

$$\begin{aligned}
\frac{M\omega^2 a^4}{\pi^4 D} - (N_{yA}\beta^2 + N_{xA}) \left(\frac{m^2 a^2}{D\pi^2}\right) = \Gamma = m^4 (1+\beta^2)^2 + m^4 \left[\frac{E_S I_S}{Dd} + \left(\frac{G_S J_S}{dD} + \frac{G_R J_R}{lD}\right) \beta^2 + \frac{E_R I_R}{Dl} \beta^4 \right] \\
+ \frac{12Z^2}{\pi^4} \left(\Lambda_O + \frac{\bar{S}\Lambda_S + \bar{R}\Lambda_R + \bar{R}\bar{S}\Lambda_{RS}}{\Lambda} \right) \quad (33)
\end{aligned}$$

where

$$\Lambda_O = (1 \pm \nu\beta^2)^2 \quad (34a)$$

$$\Lambda_S = 1 \pm \nu\beta^2 (2 \pm \nu\beta^2) + 2\bar{e}_S \alpha^2 (\beta^2 - \mu) (1 \pm \nu\beta^2) + \bar{e}_S^2 \alpha^4 (1 + \beta^2)^2 \quad (34b)$$

$$\Lambda_R = 1 \pm \nu\beta^2 (2 \pm \nu\beta^2) + 2\bar{e}_R n^2 (1 - \mu\beta^2) (1 \pm \nu\beta^2) + \bar{e}_R^2 n^4 (1 + \beta^2)^2 \quad (34c)$$

$$\Lambda_{rs} = \bar{e}_s^2 n^2 \alpha^2 \left[(1 - \mu^2) \beta^2 + 2(1 + \mu) \right] + \bar{e}_r^2 n^4 \left[(1 - \mu^2) + 2\beta^2(1 + \mu) \right] + 2\bar{e}_r \bar{e}_s n^4 (1 + \mu)^2 + 2(1 - \mu^2) n^2 (1 \pm \nu \beta^2) \bar{e}_r + 2(1 - \mu^2) n^2 (1 \pm \nu \beta^2) \bar{e}_s + (1 - \mu^2) \left[1 \pm \nu \beta^2 (2 \pm \nu \beta^2) \right] \quad (34d)$$

$$\Lambda = (\beta^2 + 1)^2 + (\bar{R} + \bar{S})(1 + \mu) 2\beta^2 + (1 - \mu^2) \left[\bar{S} + \bar{R}\beta^4 + 2\beta^2 \bar{R}\bar{S}(1 + \mu) \right] \quad (34e)$$

and the nondimensional parameter $Z^2 = \frac{a^4(1 - \mu^2)}{R_1^2 t^2}$ has been defined.

Equation (33) is a closed-form expression which gives the natural frequencies for a prestressed eccentrically stiffened shell of double curvature. If the natural frequency ω is set equal to zero in equation (33), a stability equation results which may be minimized to obtain buckling loads or buckling coefficients for a variety of loadings.

In equation (33), the effect of eccentric stiffening is reflected by the terms containing the quantities \bar{e}_r and \bar{e}_s . These quantities are positive for stiffening attached to the external surface of the shell and negative for stiffening attached to the internal surface. The term $(\beta^2 - \mu)$ in equation (34b) and the term $(1 - \mu\beta^2)$ in equation (34c) may also change sign, depending upon the shell geometry and its deflected shape. It should also be noted that every term that is first degree in \bar{e}_r or \bar{e}_s is modified by the term $(1 \pm \nu\beta^2)$, an indication that the type and magnitude of the shell curvature can greatly influence the effects exhibited by eccentric stiffening.

RESULTS AND DISCUSSION

The results presented include the vibration of shells in the absence of prestress, the buckling of shells in axial compression, and the buckling of shells under a hydrostatic pressure loading with $\bar{N}_x = \frac{pR_1}{2}$. The prestress load resultants for axial compression are:

$$\left. \begin{aligned} N_{xA} &= -\bar{N}_x \\ N_{yA} &= \pm \bar{N}_x \nu \end{aligned} \right\} \quad (35)$$

and the prestress load resultants for hydrostatic pressure are:

$$\left. \begin{aligned} N_{xA} &= -p \frac{R_1}{2} \\ N_{yA} &= -pR_1 \left(1 \mp \frac{\nu}{2} \right) \end{aligned} \right\} \quad (36)$$

The following characteristic equations for each of these cases are found from equation (33) by using equations (35) and (36):

$$\text{Vibration (no prestress): } \frac{M\omega^2 a^4}{\pi^4 D} = \Gamma \quad (37)$$

$$\text{Buckling (axial compression): } \frac{\bar{N}_x a^2}{D\pi^2} = \frac{\Gamma}{m(1 \mp \nu\beta^2)} \quad (38)$$

$$\text{Buckling (hydrostatic pressure): } \frac{pR_1 a^2}{D\pi^2} = \frac{\Gamma}{\left(\frac{1}{2} \mp \frac{\nu\beta^2}{2} + \beta^2\right)m^2} \quad (39)$$

Numerical results obtained from these equations are discussed in the following sections.

Vibration

All the vibration results presented for shells of double curvature were obtained from equation (37) for a shell whose stiffener configuration is shown in figure 3. The natural frequencies in the absence of prestress for a stringer-stiffened shell of positive Gaussian curvature are shown in figure 4. The natural frequencies for external stiffening are in general higher than those for internal stiffening, the difference being about 35 percent for $m = 2$ and $n = 8$. A crossover does occur at low values of n due to the fact that the term $(\beta^2 - \mu)$ has changed sign in equation (34b). The corresponding frequencies for a cylinder ($\nu = 0$) are shown in figure 1 of reference 7. The shell of positive Gaussian curvature is stiffer and as a result, it exhibits higher frequencies than the corresponding cylinder.

Figure 5 shows the natural frequencies in the absence of prestress for a stringer stiffened shell of negative Gaussian curvature ($\nu = -0.25$). The curves for external and internal stiffeners cross twice for each m . The first crossing occurs when the term $(\beta^2 - \mu)$ changes sign, whereas the second crossing occurs when the term $(1 - \nu\beta^2)$ changes sign. This trend makes it difficult to determine which type of stiffening yields the highest frequency for a given m and n . The figure does illustrate, however, that stiffening eccentricities can significantly alter the natural frequencies of a stiffened shell of this type. External stiffening yields a natural frequency that is approximately 50 percent higher than that for internal stiffening for $m = 2$ and $n = 4$, and internal stiffening gives approximately a 50-percent higher frequency at $m = 2$ and $n = 8$. The analysis of the shell of negative Gaussian curvature is thus more complicated than that of either the corresponding shell of positive Gaussian curvature or the cylinder (ref. 7). It is necessary to note that the results depend on the shell geometry since β is a

function of the shell length and radius as well as mode shape and they also depend on the ratio of principal curvatures. The crossings in the figure can always be predicted, however, by setting the terms $(\beta^2 - \mu)$ and $(1 - \nu\beta^2)$ equal to zero.

Figures 6 and 7 show the natural frequencies with no prestress of ring stiffened shells with $\nu = 0.25$ and $\nu = -0.25$, respectively. As in the case of ring stiffened cylinders, the shells of double curvature considered herein exhibit relatively small eccentricity effects when stiffened by rings alone, the maximum eccentricity effect being about 10 percent. The points where the curves cross in the figure can be predicted by considering the terms $(1 - \mu\beta^2)$ for figure 6 and $(1 - \mu\beta^2)$ and $(1 - \nu\beta^2)$ for figure 7.

Buckling

The buckling results obtained from the present analysis are for a shell with a stiffener configuration like that shown in figure 3 loaded either by end loads or hydrostatic pressure. Tables I and II give the pressures and the compressive end loads for buckling of shells of positive ($\nu = 0.25$) and negative ($\nu = -0.25$) Gaussian curvatures, respectively. The first column of results in the tables gives the critical axial load resultants for a shell loaded in axial compression, and the second column of results gives the critical values of pressure for a shell loaded by hydrostatic pressure. The effects of eccentric stiffening are evident from a consideration of the tables; a few important facts should be mentioned, however. The eccentricity effects are not as large for the shell loaded in axial compression as they are for the corresponding cylinder (ref. 6). It is also evident that stringers are more effective stiffeners than rings under axial compressive loadings for the shell of positive Gaussian curvature but rings are more effective stiffeners than stringers under this type of loading for a shell of negative Gaussian curvature. Rings are more effective than stringers under hydrostatic pressure loading for both types of shells, as is true for cylinders (ref. 6).

The data of table I show that external stringers are more effective (approximately 20 percent) than internal stringers under axial compressive loading for a shell of positive Gaussian curvature with no ring stiffeners. However, internal stringers are more effective under this loading for a shell of negative Gaussian curvature, as shown by the data of table II.

Examination of equations (35) and (33) shows that a shell of positive Gaussian curvature can buckle with an applied tensile edge load because of the buildup of compressive hoop stresses. Table III gives the tensile buckling loads for such a shell ($\nu = 0.25$). It is evident from the table that rings provide the most effective type of stiffening for this case with internal rings giving a 28-percent higher buckling load than external rings. Combining both types of stiffening does not greatly increase the buckling load and, in fact, table III shows that external rings and internal stringers give a lower buckling load than internal rings alone.

CONCLUDING REMARKS

An analysis is made of the buckling and vibration of eccentrically stiffened shallow shells of double curvature. An expression is presented relating the natural frequencies to prestress terms and to a variety of nondimensional shell and stiffening parameters. This expression may be used to determine the vibration and buckling characteristics of particular shell-stiffener configurations or to perform parametric studies to optimize the stiffening configuration for a particular application.

All the results presented herein illustrate the complicated behavior of eccentrically stiffened shells. It is impossible to make generalizations regarding the dynamic or buckling behavior of such shells. The eccentricity effects depend on the type of loading, the configuration, and the physical properties of both the shell and the stiffening. As a result, each particular shell-stiffening configuration must be thoroughly analyzed to determine these effects.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., August 30, 1966,
124-11-06-04-23.

APPENDIX A

CONVERSION OF U.S. CUSTOMARY UNITS TO SI UNITS

The International System of Units (SI) was adopted by the Eleventh General Conference on Weights and Measures held in Paris, October 1960 in Resolution No. 12 (ref. 9). Conversion factors for the units used herein are given in the following table:

Physical quantity	U.S. Customary Unit	Conversion factor (*)	SI Unit
Force	lbf	4.448	newtons (N)
Frequency	cps	1	hertz (Hz)
Length	in.	0.0254	meters (m)
Stress and pressure	lbf/in ²	6.895×10^3	newtons/meter ² (N/m ²)
Unit loading	lbf/in.	175.1	newtons/meter (N/m)

*Multiply value given in U.S. Customary Unit by conversion factor to obtain equivalent value in SI Unit.

Prefixes to indicate multiple of units are as follows:

Prefix	Multiple
centi (c)	10^{-2}
mega (M)	10^6
giga (G)	10^9

APPENDIX B

PRESTRESS DEFORMATIONS

The solutions presented in this report are based on the assumption of constant lateral displacement w_A prior to buckling. The implications of this assumption for a cylindrical shell are discussed in reference 6. The discussion in reference 6 applies directly to the shells of double curvature except that the added curvature can possibly increase the importance of considering exact prestress deformations rather than the simple assumption of constant lateral displacement. Consideration of exact prestress deformations would have necessitated an approximate solution to equations (22) for buckling and vibrating. An exact prestress solution can be obtained, however, and is presented as follows.

The equations which govern axisymmetric prestress deformations are

$$N_{xA,x} = 0 \quad (B1)$$

$$N_{xyA,x} = 0 \quad (B2)$$

$$-M_{xA,xx} \pm \frac{N_{xA}}{R_2} + \frac{N_{yA}}{R_1} - N_{xA} w_{A,xx} + p = 0 \quad (B3)$$

where

$$N_{xA} = \frac{Et}{1 - \mu^2} (\epsilon_{xA} + \mu \epsilon_{yA}) + \frac{E_S A_S}{d} (\epsilon_{xA} - \bar{z}_S w_{A,xx}) \quad (B4)$$

$$N_{yA} = \frac{Et}{1 - \mu^2} (\epsilon_{yA} + \mu \epsilon_{xA}) + \frac{E_r A_r}{l} (\epsilon_{yA}) \quad (B5)$$

$$M_{xA} = - \left[D w_{A,xx} + \frac{E_S I_S}{d} w_{A,xx} - \frac{\bar{z}_S E_S A_S}{d} (\epsilon_{xA} - \bar{z}_S w_{A,xx}) \right] \quad (B6)$$

Equation (B1) implies that $N_{xA} = \text{Constant}$. This constant is denoted by $-\bar{N}_x$, where \bar{N}_x is an applied compressive end load. In addition, equation (B2) implies that $N_{xyA} = \text{Constant} = 0$ with no applied shear load. For axisymmetric deformations, $\epsilon_{yA} = \frac{w_A}{R_1}$; thus equation (B4) can be solved for ϵ_{xA} as follows:

$$\epsilon_{xA} = \frac{\frac{E_S A_S \bar{z}_S}{d} w_{A,xx} - \frac{Et\mu}{1 - \mu^2} \frac{w_A}{R_1} - \bar{N}_x}{\frac{Et}{1 - \mu^2} + \frac{E_S A_S}{d}} \quad (B7)$$

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Substitution of equation (B7) into equations (B4) to (B6), and subsequent substitution of equations (B4) to (B6) into equation (B3) yields

$$w_{A,xxxx} + \frac{C_2}{C_1} w_{A,xx} + \frac{C_3}{C_1} w_A + \frac{C_4}{C_1} = 0 \quad (B8)$$

where

$$C_1 = \left(D + \frac{E_S I_S}{d} \right) \left[1 + \bar{S}(1 - \mu^2) \right] + \frac{\bar{z}_S^2 E_S A_S}{d}$$

$$C_2 = \frac{2\bar{z}_S \bar{S} E t \mu}{R_1} + \bar{N}_x \left[1 + \bar{S}(1 - \mu^2) \right]$$

$$C_3 = \frac{E t}{R_1^2} \left[1 + \bar{R} + \bar{S} + \bar{R}\bar{S}(1 - \mu^2) \right]$$

$$C_4 = \left(p \mp \frac{\bar{N}_x}{R_2} \right) \left[1 + \bar{S}(1 - \mu^2) \right] - \frac{\bar{N}_x \mu}{R_1}$$

A solution to equation (B8) is of the following form:

$$w_A = K_1 e^{\lambda_1 x} + K_2 e^{\lambda_2 x} + K_3 e^{\lambda_3 x} + K_4 e^{\lambda_4 x} - \frac{C_4}{C_3} \quad (B9)$$

where the K 's are constants to be determined from the boundary conditions and the λ 's are defined as follows:

$$\lambda_1 = \sqrt{\frac{-\frac{C_2}{C_1} + \sqrt{\left(\frac{C_2}{C_1}\right)^2 - 4\left(\frac{C_3}{C_1}\right)}}{2}} \quad (B10a)$$

$$\lambda_2 = \sqrt{\frac{-\frac{C_2}{C_1} - \sqrt{\left(\frac{C_2}{C_1}\right)^2 - 4\left(\frac{C_3}{C_1}\right)}}{2}} \quad (B10b)$$

$$\lambda_3 = -\sqrt{\frac{-\frac{C_2}{C_1} + \sqrt{\left(\frac{C_2}{C_1}\right)^2 - 4\left(\frac{C_3}{C_1}\right)}}{2}} \quad (B10c)$$

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$$\lambda_4 = -\sqrt{\frac{-\frac{C_2}{C_1} - \sqrt{\left(\frac{C_2}{C_1}\right)^2 - 4\left(\frac{C_3}{C_1}\right)}}{2}} \quad (\text{B10d})$$

Equation (B9) can be used in conjunction with equations (22) to determine the effect of prestress deformations on the buckling and vibrating characteristics of doubly curved shallow shells.

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TABLE I.- BUCKLING RESULTS FOR SHELL OF POSITIVE GAUSSIAN CURVATURE

Shell properties	
ν	= 0.25
$l = d$	= 1.0 in. (2.54 cm)
$E = E_S = E_R$	= 10.5×10^6 psi (72.4 GN/m ²)
a	= 23.75 in. (60.3 cm)
R_1	= 9.55 in. (24.3 cm)
μ	= 0.3

Configuration	Axial compressive load, \bar{N}_x		Hydrostatic pressure load, p	
	lbf/in.	MN/m	psi	MN/m ²
No stringers Rings external	744	0.130	156	1.076
No stringers Rings internal	717	0.126	150	1.034
No rings Stringers external	7,913	1.385	24.5	0.169
No rings Stringers internal	6,664	1.166	22.1	0.152
Rings external Stringers internal	10,144	1.776	374	2.580
Rings external Stringers external	11,778	2.063	413	2.849
Rings internal Stringers internal	7,928	1.389	415	2.860
Rings internal Stringers external	9,183	1.607	462	3.187

TABLE II.- BUCKLING RESULTS FOR SHELL OF NEGATIVE GAUSSIAN CURVATURE

Shell properties	
ν	= -0.25
$l = d$	= 1.0 in. (2.54 cm)
$E = E_S = E_R$	= 10.5×10^6 cm (72.4 GN/m ²)
a	= 23.75 in. (60.3 cm)
R_1	= 9.55 in. (24.3 cm)
μ	= 0.3

Configuration	Axial compressive load, \bar{N}_x		Hydrostatic pressure load, p	
	lbf/in.	MN/m	psi	MN/m ²
No stringers Rings external	744	0.130	69	0.476
No stringers Rings internal	697	0.122	63	0.434
No rings Stringers external	282	0.049	14.5	0.100
No rings Stringers internal	340	0.060	18.5	0.128
Rings external Stringers internal	2,040	0.354	78	0.538
Rings external Stringers external	1,913	0.335	70	0.483
Rings internal Stringers internal	1,770	0.310	84	0.579
Rings internal Stringers external	1,842	0.322	68	0.469

TABLE III.- BUCKLING RESULTS FOR SHELL OF POSITIVE GAUSSIAN CURVATURE
LOADED IN TENSION

Shell properties	
$\nu = 0.25$	
$l = d = 1.0 \text{ in. (2.54 cm)}$	
$E = E_S = E_T = 10.5 \times 10^6 \text{ cm (72.4 GN/m}^2)$	
$a = 23.75 \text{ in. (60.3 cm)}$	
$R_1 = 9.55 \text{ in. (24.3 cm)}$	
$\mu = 0.3$	

Configuration	Axial tensile load, \bar{N}_x	
	lbf/in.	MN/m
No stringers Rings external	13,441	2.354
No stringers Rings internal	17,200	3.010
No rings Stringers external	833	0.146
No rings Stringers internal	751	0.132
Rings external Stringers internal	16,617	2.910
Rings external Stringers external	17,788	3.115
Rings internal Stringers internal	19,408	3.400
Rings internal Stringers external	20,396	3.570

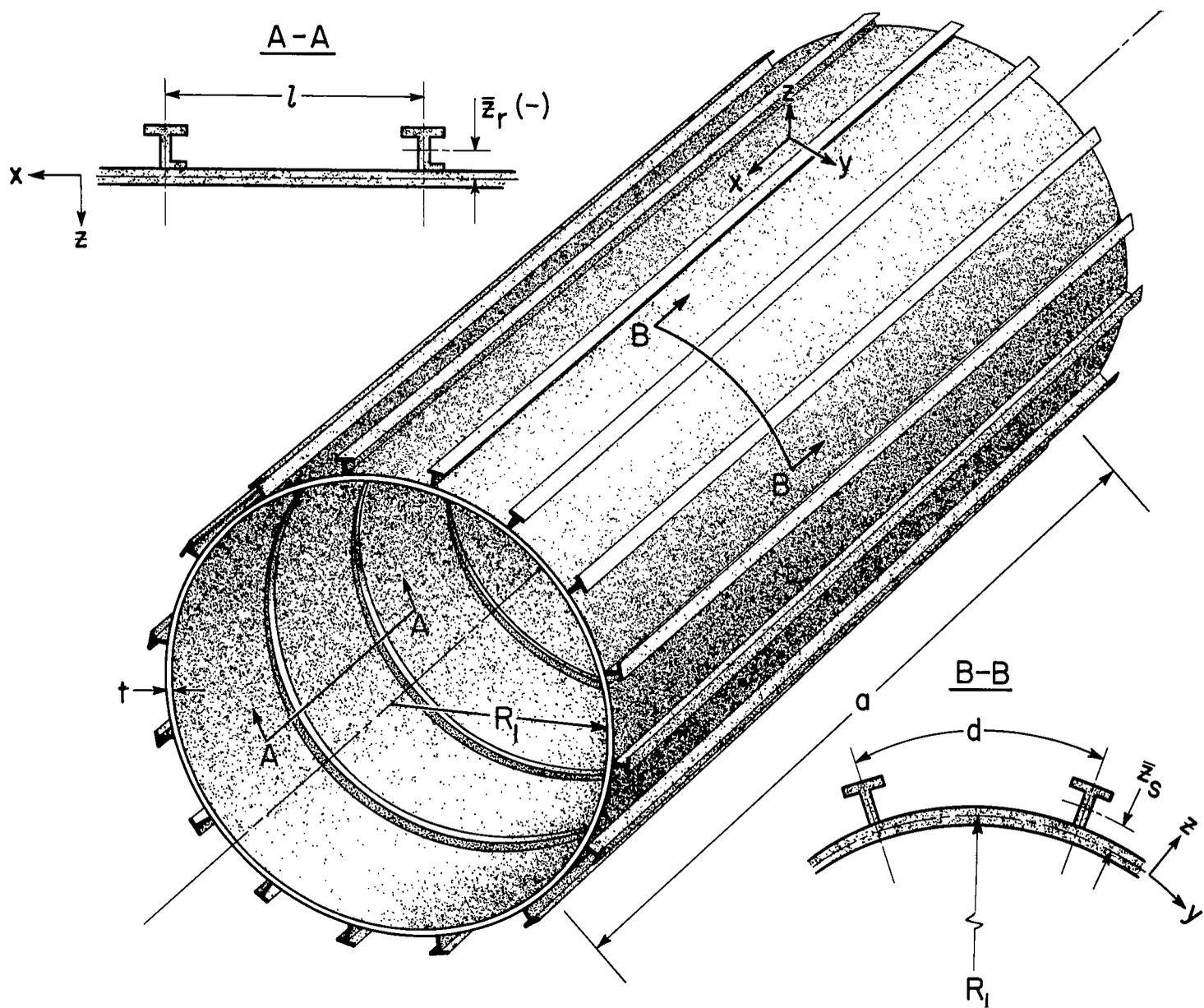


Figure 1.- Geometry of eccentrically stiffened cylinder.

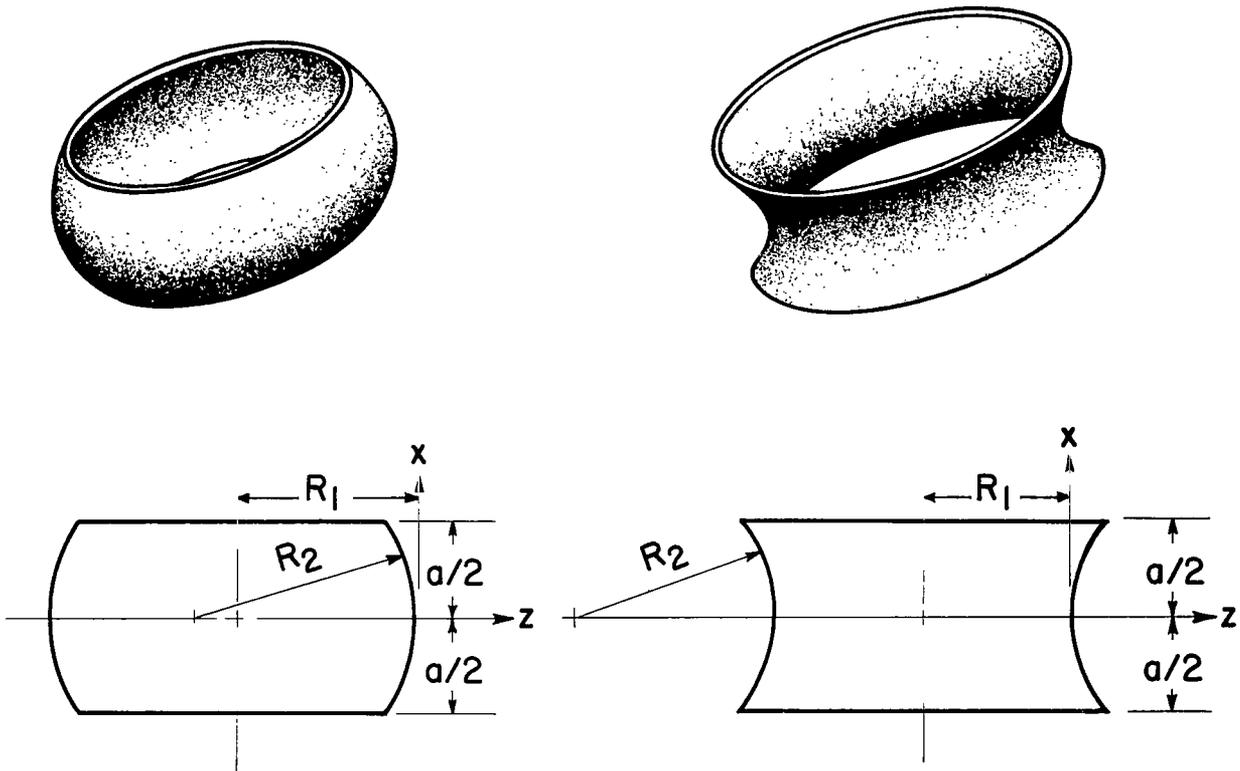


Figure 2.- Geometry of shells of double curvature.

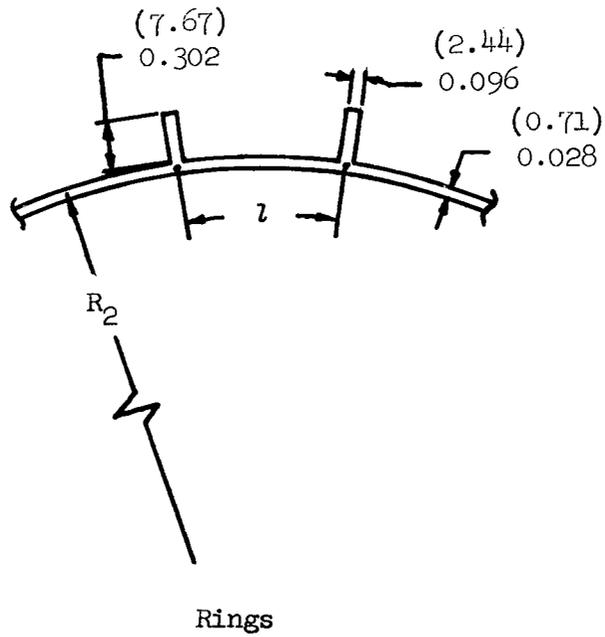
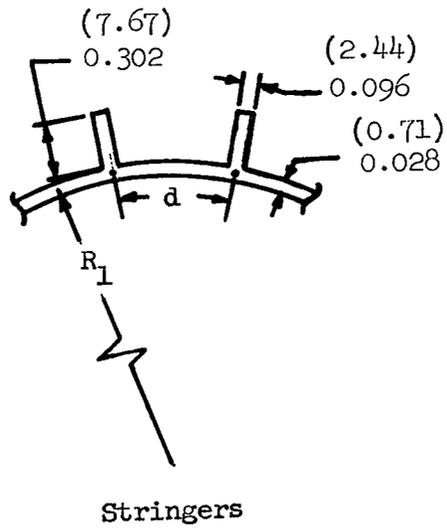


Figure 3.- Stiffening configuration. All dimensions are in inches (millimeters).

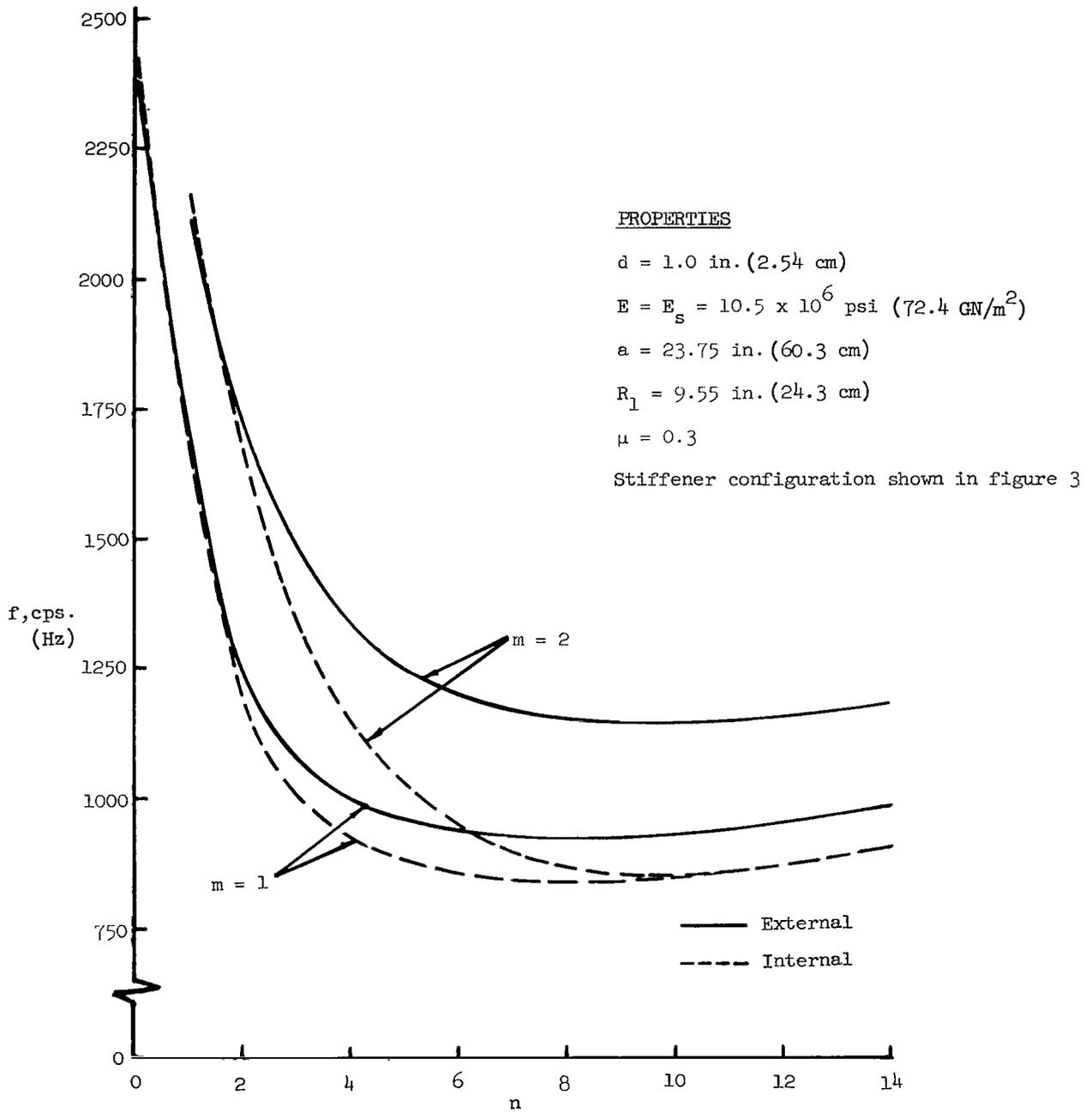


Figure 4.- Natural frequencies of stringer-stiffened shell of positive Gaussian curvature ($\nu = 0.25$).

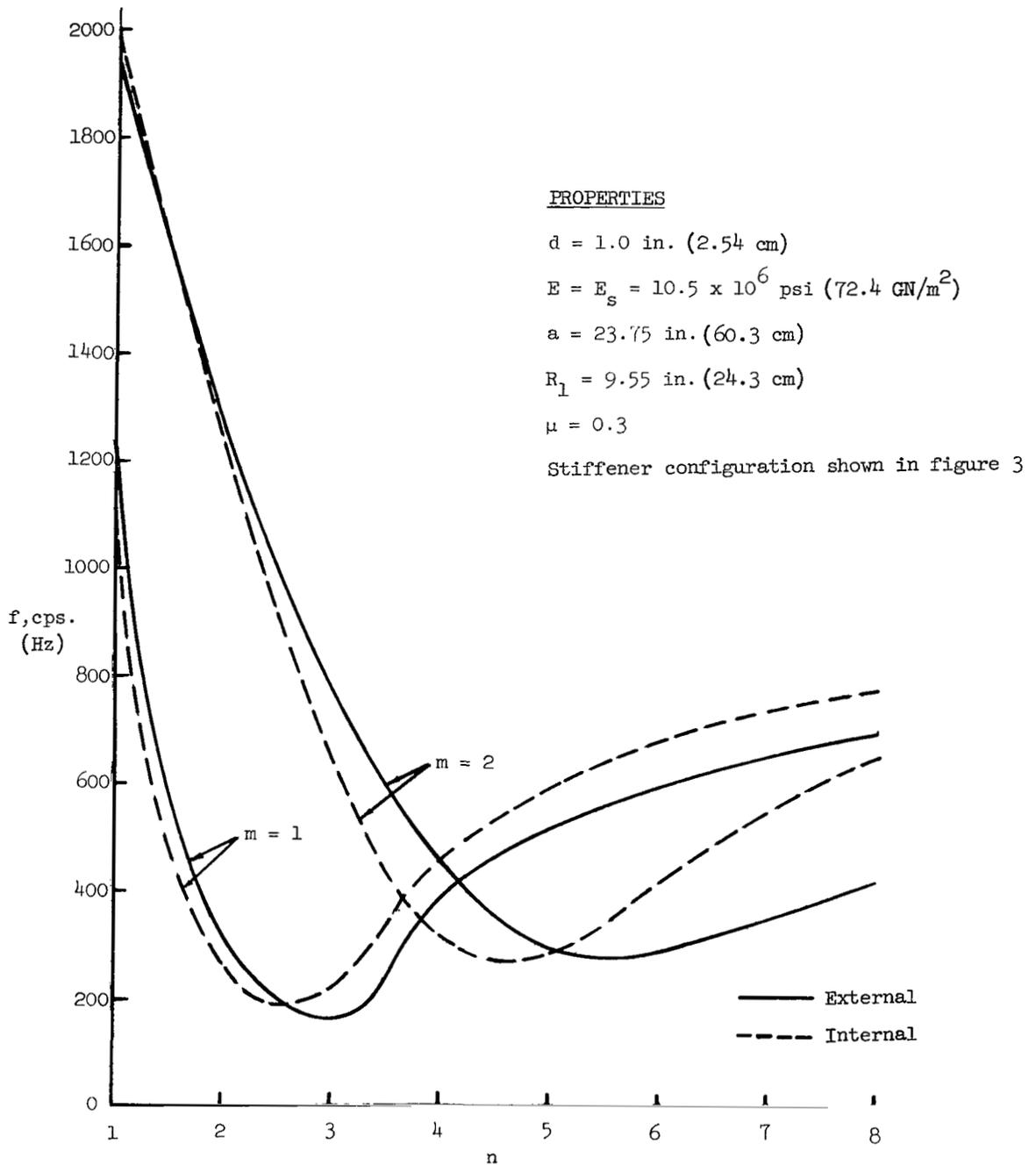


Figure 5.- Natural frequencies of stringer-stiffened shell of negative Gaussian curvature ($\nu = -0.25$).

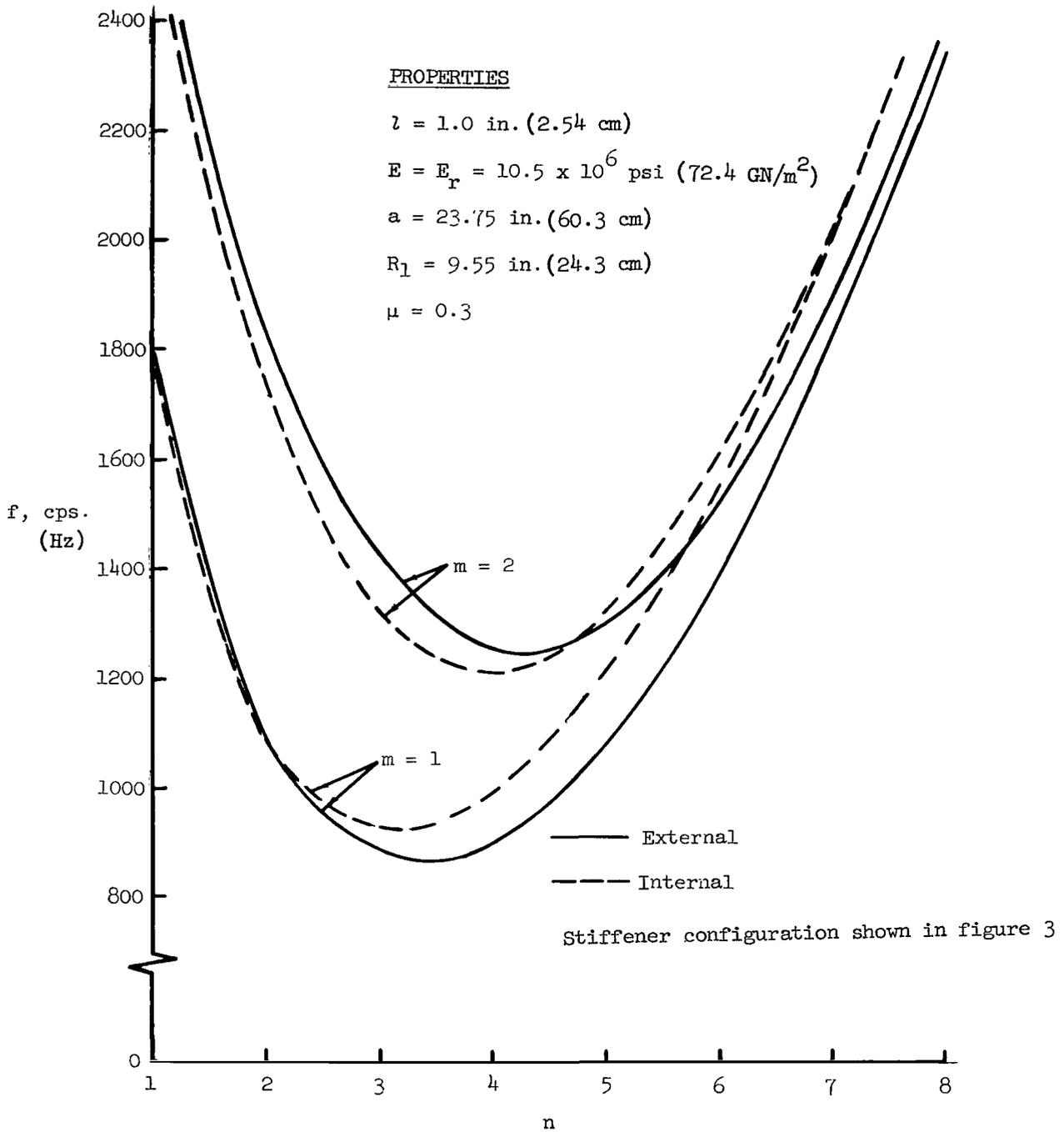


Figure 6.- Natural frequencies of ring-stiffened shell of positive Gaussian curvature ($\nu = 0.25$).

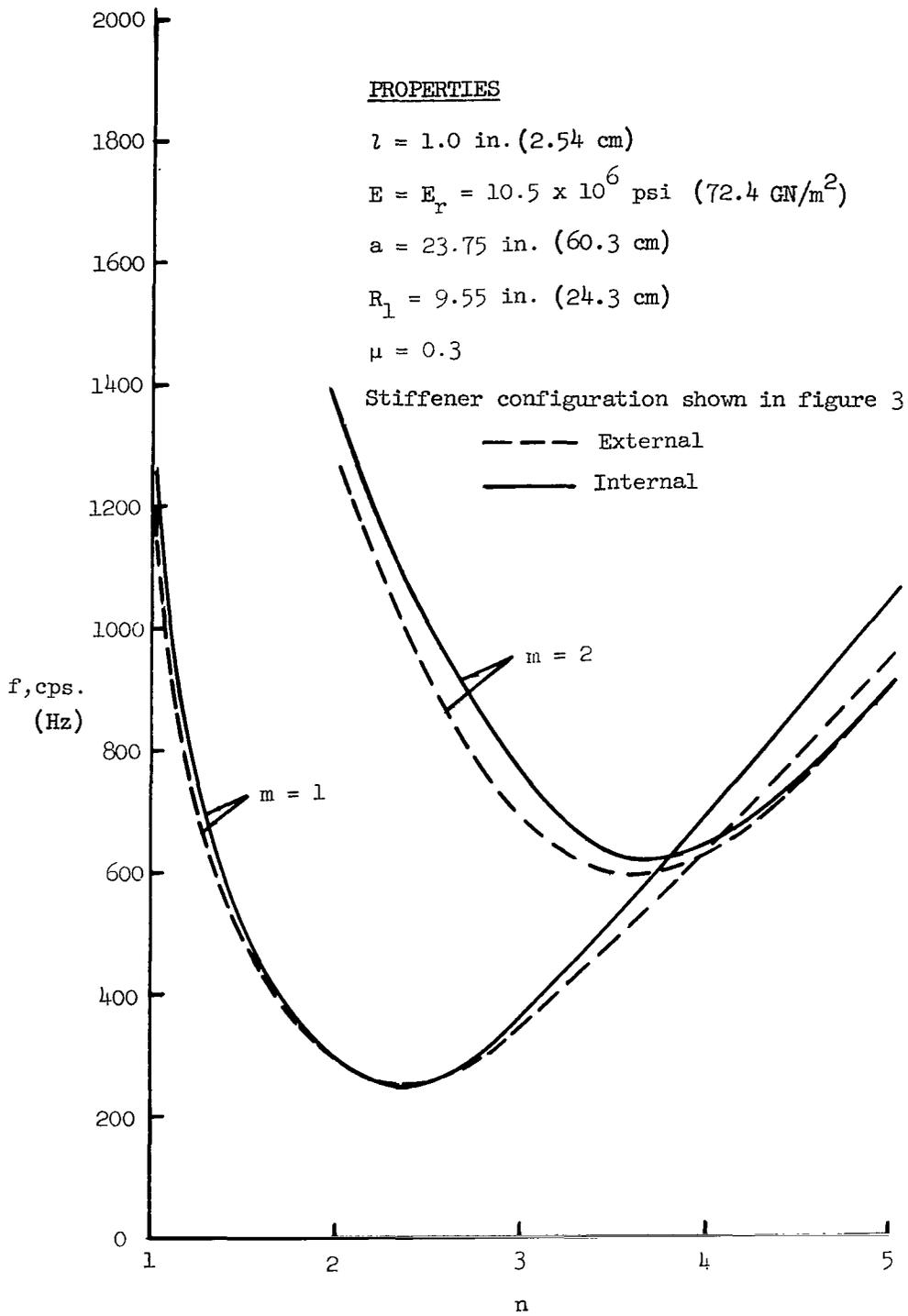


Figure 7.- Natural frequencies of ring-stiffened shell of negative Gaussian curvature ($\nu = -0.25$).

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