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A NEW LIE SERIES METHOD FOR THE NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS, WITH AN APPLICATION TO THE RESTRICTED PROBLEM OF THREE BODIES

by Siegfried Filippi

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Huntsville, Ala.

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TABLE OF CONTENTS

A NEW LIE SERIES METHOD FOR THE NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS, WITH AN APPLICATION TO THE RESTRICTED PROBLEM OF THREE BODIES[1]

By

Siegfried Filippi[2]

SUMMARY

In this paper a new method for the numerical solution of problems of initial value in ordinary differential equations, originally developed by W. Gröbner, will be described. The first step will be to describe the most important theorems of W. Gröbner's Lie series with reference to the generalized Lie series of Filatov. Then follows a detailed explanation of the new method of Lie series illustrated by the restricted problem of three bodies. For this a number of comparing calculations are made using the power series method and the Runge-Kutta - Fehlberg method. This specific method of Lie series compared with W. Gröbner's method distinguishes itself by being a numerical method of any high truncation error. A simple and efficient automatic step-size control can also be realized.

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1. The decisive suggestion to use the method of power series as a basis for the new Lie series method was made by Dr. Erwin Fehlberg of the Computation Laboratory, George C. Marshall Space Flight Center (see also Fehlberg and Filippi [1]). The work was financially supported by means of the NASA Contract NAS 8-11209 to the General Electric Company. The numerical computations were performed on an IBM 7094, Model II computer at the George C. Marshall Space Flight Center.

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I. INTRODUCTION

During the last few years the method of Lie series developed by W. Gröbner [2] was often applied to the numerical solution of problems of initial value in ordinary differential equations [3, 4, 5]. In this form the method of Lie series could in no way compete with other methods for the numerical solution of ordinary differential equations. This was mainly because of the difficulties in finding at first a satisfactory analytical initial approximation for the problem considered, and secondly because the perturbation integrals to be computed in using quadrature formulas required a very considerable computational effort. Furthermore the method of Lie series as originally developed had a relatively low order of the truncation error, i.e., the highest order of the truncation error was $0\,(h^7)$ [4]. And it was impossible to enlarge the order of the truncation error by adding further terms of Lie series, because the appropriate operator expressions $D^{\nu} z_i$ for $\nu = 4, 5, \ldots$ became much too long, and in comparison with the solution the effort was by far too much. The method of Lie series described in this paper (see also [1]) eliminates all these difficulties. Now a solution of problems of initial value of <u>any high truncation error</u> can be realized by means of a numerical method; also included is an efficient automatic step-size control. Only by this treatment did the new method turn out to be a very effective numerical method.

In the first part of the paper the most important theorems of the method of Lie series will partly be stated and partly proved. Then follows the new method as applied to the restricted problem of three bodies. And finally a number of comparing calculations are made with reference to the method of power series and the one of Runge-Kutta-Fehlberg.

II. GRÖBNER'S METHOD OF LIE SERIES

A. General

In 1960 W. Gröbner developed a new method for the numerical solution of problems of initial value in ordinary differential equations, by using the Lie series. During the last few years Gröbner's method sometimes was applied to the numerical solution of multibody problems.

In the first part of our paper the most important theoretical basic theorems of the method of Lie series will partly be explained and partly be proved.

We introduce a linear differential operator

$$D = f_1(z)\frac{\partial}{\partial z_1} + f_2(z)\frac{\partial}{\partial z_2} + \ldots + f_n(z)\frac{\partial}{\partial z_n} \tag{1}$$

with the holomorphic functions $f_k(z)$ of the complex variables $z_1, z_2, \ldots, z_n$. We call a function $f(z)$ of these variables holomorphic at one point (e.g., at $z_1 = z_2 = \ldots = z_n = 0$) if it can be developed as a regular absolutely convergent power series.

If all $f_k(z)$ and $f(z)$ in the neighborhood of this point are holomorphic, then the operator, equation (1), can be applied to $f(z)$

$$Df(z) = f_1(z)\frac{\partial f(z)}{\partial z_1} + f_2(z)\frac{\partial f(z)}{\partial z_2} + \ldots + f_n(z)\frac{\partial f(z)}{\partial z_n} \tag{2}$$

and we obtain, according to well known theorems of function theory, a function that is again holomorphic at the same point. By repeated use of the operator (1) to $f(z)$

$$D^k f(z) = D\,[D^{k-1} f(z)] \qquad (k = 0, 1, 2, \ldots) \tag{3}$$

we always get a function being holomorphic at the same point.

Using operator (1) we define a Lie series by

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k f(z) = e^{tD} f(z) \tag{4}$$

where $e^{tD} f(z)$ is a symbolic way of writing series (4).

In equation (4), t stands for a new complex variable, independent of the variables $z_1, z_2, \ldots, z_n$. Every term in equation (4) is a well-defined holomorphic function of these $n + 1$ complex variables.

If it can be shown that a number $T > 0$ exists so that series (4) is absolutely convergent for $|t| < T$, then we can say that series (4) is a holomorphic function of the variables $z_1\ z_2, \ldots, z_n, t$.

B. Proof of Convergence

Using Cauchy's method of majorants [2] we are going to prove now the convergence of the Lie series, i.e., we shall prove the following theorem:

Theorem 1

If G is the common holomorphic region of $f(z)$ and of all $f_k(z)$, then at every point of G there is a number $T > 0$ so that the Lie series

$$e^{tD} f(z) = \sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f(z) \tag{5}$$

assuming

$$D = f_1(z) \frac{\partial}{\partial z_1} + f_2(z) \frac{\partial}{\partial z_2} + \ldots + f_n(z) \frac{\partial}{\partial z_n}$$

is absolutely convergent for all $|t| < T$.

The proof of this theorem, using the Cauchy majorant criterion, is given by the following three steps:

(a) Determination of a suitable convergent majorant for $f(z)$ and for operator D, so that $f(z) \leq g(y)$ and $|D| \leq \Delta$

(b) Proof that $|D^k f(z)| \leq \Delta^k G(y)$ follows from (a)

(c) Determination of a number $T > 0$ for which there is absolute convergence for all $|t| < T$ in series (5)

Step a

If P is a point within G, then the function $f(z)$, assumed to be holomorphic in G, can be developed into a power series convergent at P. If by an appropriate coordinate transformation, P is moved to the origin $z_1 = z_2 = \ldots = z_n = 0$, then the power series expansion is

$$f(z) = \sum_{i_1 \ldots i_n}^{0 \ldots \infty} a_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n} \tag{6}$$

with $|z_j| \leq \rho_j > 0$. Power series (6) is absolutely convergent in a certain region of the origin. The bounds ρ_j are chosen so that convergence of series (6) for $|z_j| \leq \rho_j$ is certain. Generally, it is very difficult to state the whole domain of convergence of equation (6). But it is sufficient to find only one domain of convergence of equation (6).

The convergence of

$$\sum_{i_1 \ldots i_n}^{0 \ldots \infty} |a_{i_1 \ldots i_n}| \rho_i^{i_1} \ldots \rho_n^{i_n} \tag{7}$$

follows from equation (6). Since the terms in a convergent series are bounded

$$|a_{i_1 \ldots i_n}| \rho_i^{i_1} \ldots \rho_n^{i_n} \leq M$$

or

$$|a_{i_1 \ldots i_n}| \leq \frac{M}{\rho_i^{i_1} \ldots \rho_n^{i_n}} = b_{i_1 \ldots i_n} \tag{8}$$

we get as the majorant for f (z)

$$|f(z)| \leq g(y_i) = \sum_{i_1 \ldots i_n}^{0 \ldots \infty} b_{i_1 \ldots i_n} y_i^{i_1} \ldots y_n^{i_n} \tag{9}$$

with $|z_j| \leq y_i < \rho_j$. It follows from equation (7) that

$$g(y_i) = \sum_{i_1 \ldots i_n}^{0 \ldots \infty} \frac{M}{\rho_i^{i_1} \ldots \rho_n^{i_n}} y_1^{i_1} \ldots y_n^{i_n} = M \sum_{i_1 \ldots i_n}^{0 \ldots \infty} \frac{y_1^{i_1} \ldots y_n^{i_n}}{\rho_1^{i_1} \ldots \rho_n^{i_n}}$$

$$= \frac{M}{\left(1 - \frac{y_1}{\rho_1}\right)\left(1 - \frac{y_2}{\rho_2}\right) \ldots \left(1 - \frac{y_n}{\rho_n}\right)} \tag{10}$$

because $y_i / \rho_j < 1$.

Since an absolutely convergent series within its domain of convergence can be rearranged arbitrarily, we arrange equation (10) in the following way

$$\sum_{i_1 \ldots i_n}^{o \ldots \infty} b_{i_1 \ldots i_n} y_i^{i_1} \ldots y_n^{i_n}$$

$$= \sum_i \sum_{i_1 + i_2 + \ldots + i_n = i} b_{i_1 \ldots i_n} y_i^{i_1} \ldots y_n^{i_n} \tag{11}$$

the summation $i_1 + i_2 + \ldots + i_n = i$ being extended over all possible combinations of indices. If we then replace y_j and/or ρ_j with y and/or ρ so that $y = \max (y_j)$ and/or $\rho = \min (\rho_j)$, we get for the function $g(y_i)$ with n real variables y_i function $G(y)$ with one real variable y. So we have

$$|f(z)| \leq G(y) = \sum_{i=o}^{\infty} b_i y^i . \tag{12}$$

It follows from

$$b_i \geq \sum_{i_1 + \ldots + i_n = i} |a_{i_1 \ldots i_n}| \qquad (i = 0, 1, 2, \ldots) \tag{12a}$$

and

$$|z_j| \leq y_j < y < \rho \leq \rho_j$$

that

$$\frac{y_i}{\rho_i} \leq \frac{y}{\rho} < 1 \text{ and } \frac{1}{1 - \frac{y_j}{\rho_i}} = \frac{1}{1 - \frac{y}{\rho}}$$

We get

$$|f(z)| \leq g(y_j) \leq G(y) = \frac{M}{\left(1 - \frac{y}{\rho}\right)^n} \tag{13}$$

Since an absolutely convergent series can be differentiated term by term as often as desired within the domain of convergence, and the majorizing operation can be transferred to the derivatives, we get

$$\frac{\partial f(z)}{\partial z_k} \leq \frac{\partial g(y_k)}{\partial y_k} \leq \frac{\partial G(y)}{\partial y}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{14}$$

$$\frac{\partial^n f(z)}{\partial z_k^{\,n}} \leq \frac{\partial^n g(y_k)}{\partial y_k^{\,n}} \leq \frac{\partial^n G(y)}{\partial y^n} \, .$$

In the same way as for $f(z)$ we find a majorant for the functions $f_k(z)$

$$|f_k(z)| \leq \frac{N_k}{\left(1 - \frac{y}{\rho}\right)^n} \tag{15}$$

We assume that the ρ_i are equal for all estimates; this can, of course, always be achieved.

We therefore get as majorizing operator

$$|D| \leq \Delta = \frac{N}{\left(1 - \frac{y}{\rho}\right)^n} \frac{d}{dy} \quad . \quad (N = k \max \{N_k\}) \tag{16}$$

Step b

It follows from

$$|f(z)| \leq G(y) \text{ and } |D| \leq \Delta$$

that

$$|D^k f(z)| \leq \Delta^k G(y) \quad . \tag{17}$$

Since the functions $f_k(z)$ are holomorphic, they can be expanded into a power series

$$f_k(z) = \sum_{j_1 \ldots j_n}^{o \ldots \infty} c^{(k)}_{j_1 \ldots j_n} \quad z_1^{j_1} \ldots z_n^{j_n} \quad . \tag{18}$$

We also develop the majorizing operator in a power series

$$f(y) \frac{d}{dy} = \sum_{j=o}^{\infty} \gamma_j y^j \frac{d}{dy} \quad . \tag{19}$$

With $f(y) \geq |f_k(z)|$, it follows that

$$\gamma_j = \sum_{j_1 + \ldots + j_n = j} c^{(k)}_{j_1 \ldots j_n} \quad . \; (j = 0,\ 1,\ 2,\ \ldots) \tag{20}$$

Assuming this, it now follows that

$$Df(z) = \sum c^{(k)}_{j_1 \ldots j_n} \quad z_1^{j_1} \ldots z_n^{j_n} \frac{d}{dz_k} \sum a_{i_1 \ldots i_n} \; z_1^{i_1} \ldots z_n^{i_n}$$

$$= \sum c^{(k)}_{j_1 \ldots j_n} \; a_{i_1 \ldots i_n} \; i_k \; z_1^{i_1 + j_1} \ldots z_k^{i_k + j_k - 1} \ldots z_n^{i_n + j_n} \tag{21}$$

and

$$\Delta G\,(y) \quad = \sum i \gamma_j b_i y^{i + j - 1} \quad . \tag{22}$$

With equations (12a), (20), and $i_k \leq i \; (i_1 + i_2 + \ldots + i_n = i)$, these relations hold:

1. $|z_j| \leq y$

2. $\displaystyle\sum_{k=1}^{n} \sum_{i_1 + \ldots + i_n = i} \sum_{j_1 + \ldots + j_n = j} |c^{(k)}_{j_1 \ldots j_n} \; a_{i_1 \ldots i_n} \; i| \leq \gamma_j b_i^{\,i}$,

i.e., $\Delta G(y)$ is a true majorant for $Df(z)$ because the coefficients of the series 2.are always smaller than the corresponding coefficients of series (22). Thus

$$|Df(z)| \leq \Delta G(y) \quad . \tag{23}$$

But since $Df(z)$ and/or $\Delta G(y)$, as was shown above, fulfill the same conditions as $f(z)$ and/or $G(y)$, we get

$$\begin{aligned} &D[Df(z)] \leq \Delta [\Delta G(y)] \\ &\cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ &D[D^{k-1} f(z)] \leq \Delta [\Delta^{k-1} G(y)] \end{aligned} \tag{24}$$

Since the sum of the majorants is majorant for the sum of the corresponding minorants, we get

$$\left| \sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f(z) \right| \leq \sum_{k=o}^{\infty} \frac{|t|^k}{k!} \Delta^k G(y) \quad . \tag{25}$$

Step c.

To determine the convergence radius of the majorant Lie series (25), we first calculate from

$$\Delta = \frac{N}{\left(1-\frac{y}{\rho}\right)^n} \frac{d}{dy}, \qquad G(y) = \frac{M}{\left(1-\frac{y}{\rho}\right)^n}$$

and

$$\Delta G(y) = \frac{N}{\left(1-\frac{y}{\rho}\right)^n} \frac{Mn}{\rho\left(1-\frac{y}{\rho}\right)^{n+1}} = \frac{Mi!}{\left(1-\frac{y}{\rho}\right)^n} \binom{-\frac{n}{n+1}}{i} \left(\frac{-N(n+1)}{\rho\left(1-\frac{y}{\rho}\right)^{n+1}}\right)^i$$

$$\Delta^k G(y) = \frac{Mk!}{\left(1-\frac{y}{\rho}\right)^n} \binom{-\frac{n}{n+1}}{k} \left(\frac{-N(n+1)}{\rho\left(1-\frac{y}{\rho}\right)^{n+1}}\right)^k \quad . \tag{26}$$

The proof of equations (26) is obtained by mathematical induction. The validity of equations (26) for $k = 1$ has already been demonstrated above.

From equations (25) and (26) it now follows that

$$\sum_{k=o}^{\infty} \frac{t^k}{k!} \Delta^k G(y) = \frac{M}{\left(1 - \frac{y}{\rho}\right)^n} \sum_{k=o}^{\infty} \binom{-\frac{n}{n+1}}{k} \left(\frac{-N(n+1)t}{\rho \left(1 - \frac{y}{\rho}\right)^{n+1}} \right)^k . \quad (27)$$

Series (27), and also the Lie series (25), converge for

$$\left| \frac{-N(n+1)t}{\rho \left(1 - \frac{y}{\rho}\right)^{n+1}} \right| < 1 ,$$

i.e., for

$$|t| < \frac{\rho}{(n+1)N} \left(1 - \frac{y}{\rho}\right)^{n+1} = T . \quad (28)$$

Thus N depends on operator D, while ρ depends on $f(z)$ as well as on D, and y can be freely selected between 0 and ρ . At the expansion point $P(y = 0)$ the series is absolutely convergent for

$$|t| < T^* = \frac{\rho}{(n+1)N} . \quad (29)$$

By comparison with the binomial series

$$\sum_{k=o}^{\infty} \binom{-\ell}{k} x^k = (1 + x)^{-k} \quad (30)$$

we obtain an evaluation for the sum of the Lie series by taking as upper bound the summation value for the majorant

$$\left| \sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f(z) \right| \leq \frac{M}{\left(1 - \frac{y}{\rho}\right)^n} \left(1 + \frac{-N(n+1)}{\rho\left(1 - \frac{y}{\rho}\right)^{n+1}} \right)^{-\frac{n}{n+1}} \quad (31)$$

Estimate of Remainder

To estimate the error of the Lie series

$$\sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f(z) = e^{tD} f(z) \tag{32}$$

when it is truncated after the m-th term, we get from equation (27)

$$\frac{M}{\left(1-\frac{y}{\rho}\right)^n} \sum_{k=m}^{\infty} \binom{-\frac{n}{n+1}}{k} \left(\frac{-N(n+1)\,|t|}{\rho\left(1-\frac{y}{\rho}\right)^{n+1}}\right)^k$$

$$< \frac{M}{\left(1-\frac{y}{\rho}\right)^n} \sum_{k=m}^{\infty} \left(\frac{N(n+1)\,|t|}{\rho\left(1-\frac{y}{\rho}\right)^{n+1}}\right)^k$$

$$= \frac{M}{\left(1-\frac{y}{\rho}\right)^n} \sum_{k=m}^{\infty} \left(\frac{|t|}{T}\right)^{k+m} = \frac{M}{\left(1-\frac{y}{\rho}\right)^n \left(1-\frac{|t|}{T}\right)} \left(\frac{|t|}{T}\right)^m \tag{33}$$

since

$$\frac{|t|}{T} < 1 \quad .$$

NOTE: If the $f_k(z)$ are polynomials, the estimate of the remainder (33) can be further improved.

C. The Lie Series for the Solution of Ordinary Differential Equations

The following important theorem [10] can be derived from the analyses in II-B:

Theorem 2

If G is a finite, closed region in z-space where all $f_k(z)$ and $f(z)$ are holomorphic, then there exists a positive number T so that Lie series (4) in II-A is absolutely and uniformly convergent in the entire region G for $|t| \leq T$ and is a holomorphic function of the $n+1$ complex variables $z_1, z_2, \ldots, z_n, t$.

Lie series (4) can be differentiated term by term as often as desired within G and $|t| \leq T$ with reference to the variables $z_1, z_2, \ldots, z_n, t$. Thus

$$\frac{\partial}{\partial t}\left[\sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f(z)\right] = \sum_{k=o}^{\infty} \frac{t^k}{k!} D^{k+1} f(z) \tag{34}$$

and

$$\frac{\partial}{\partial z_\ell}\left[\sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f(z)\right] = \sum_{k=o}^{\infty} \frac{t^k}{k!} \frac{\partial}{\partial z_\ell}[D^k f(z)] \quad . \tag{35}$$

NOTE: In the special case $f(z) = Z_i$ we get the Lie series

$$Z_i = e^{tD} z_i \quad . \qquad (i = 1, 2, \ldots, n) \tag{36}$$

In a certain region of the variables $z_1, z_2, \ldots z_n$ in which all $f_k(z)$ of operator (1) in II-A are holomorphic, Lie series (36) represents, according to Theorem 2, for $|t| \leq T$, holomorphic functions of $z_1, z_2, \ldots, z_n, t$. For $t = 0$ the functions Z_i in equation (36) take the initial values z_i

$$(Z_i)_{t=o} = z_i \quad . \qquad (i = 1, 2, \ldots, n) \tag{37}$$

The following important properties can be demonstrated [2] for the Lie series defined by (2) in II-A:

(a) For the sums and products of Lie series formed with the same operator D:

$$e^{tD}[c_1 g_1(z) + c_2 g_2(z) + \dots + c_m g_m(z)]$$

$$= c_1 e^{tD} g_1(z) + c_2 e^{tD} g_2(z) + \dots + c_m e^{tD} g_m(z) \tag{38}$$

where the c_i are constant, and

$$e^{tD}[g_1(z) g_2(z) \dots g_m(z)] = [e^{tD} g_1(z)]\,[e^{tD} g_2(z)] \dots [e^{tD} g_m(z)]\,. \tag{39}$$

(b) Theorem of permutability:

If $F(z)$ is an arbitrary holomorphic function near $z_1, z_2, \dots, z_n$ and its power series expansion also converges at the point $\{z_1, z_2, \dots, z_n\}$, then the functional symbol F and the symbol e^{tD} for the Lie series

$$F(Z) = \sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f(z)$$

or (40)

$$F(e^{tD} z) = e^{tD} F(z)$$

can alternate.

Also, according to equation (34), the special Lie series in (36) can be differentiated term by term beyond t in the region $|t| < T$. Thus we have

$$\dot{Z}_i = \frac{\partial Z_i}{\partial t} = \sum_{k=1}^{\infty} \frac{k t^{k-1}}{k(k-1)!} D^k z_i = \sum_{k=o}^{\infty} \frac{t^k}{k!} D^{k+1} z_i$$

$$= \sum_{k=o}^{\infty} \frac{t^k}{k!} D^k (D z_i) = \sum_{k=o}^{\infty} \frac{t^k}{k!} D^k f_i(z) = f_i(Z) \quad . \tag{40a}$$

$(i = 1, 2, \dots, n)$

The functions $f_i(z)$ are holomorphic in the neighborhood of the initial point $z_1, z_2, \ldots, z_n$ and the independent variable t is not explicit.

According to (40a) the functions Z_i in (36) satisfy an autonomous system of first-order differential equations. From this we immediately obtain the theorem:

Theorem 3

If we have an autonomous system of first-order differential equations of the form

$$\dot{Z}_i = f_i(Z_1, Z_2, \ldots, Z_n) \qquad (i = 1, 2, \ldots, n) \tag{41}$$

with the initial conditions

$$(Z_i)_{t=0} = z_i \qquad (i = 1, 2, \ldots, n) \tag{42}$$

and in whose neighborhood the functions $f_i(z)$ are holomorphic, then the Lie series

$$Z_i = e^{tD} z_i \tag{43}$$

where

$$D = f_1(z) \frac{\partial}{\partial z_1} + f_2(z) \frac{\partial}{\partial z_2} + \ldots + f_n(z) \frac{\partial}{\partial z_n}$$

represent the desired solutions of this system of differential equations.

Thus the z_i should be considered primarily as variables to which operator D is applied. After all the differentiations, prescribed by the operator $D^\nu z_i$, have been performed, the z_i can be replaced by the initial values.

Supplement

A nonautonomous system of first-order differential equations

$$\dot{Z}_i = f_i(Z_1, Z_2, \ldots, Z_n, t) \qquad (i = 1, 2, \ldots, n) \tag{44}$$

can always, by the substitution

$$t = Z_{n+1} \tag{45}$$

be converted into the autonomous system

$$\dot{Z}_i = f_i(Z_1, Z_2, \ldots, Z_n, Z_{n+1}) \cdot (i=1, 2, \ldots, n, n+1) \tag{46}$$

It follows from equation (45) that

$$\frac{dZ_{n+1}}{dt} = \frac{dt}{dt} = 1 \equiv f_{n+1}$$

and $(Z_{n+1})_{t=t_0} = t_0$.

In general, Lie series (43) converge very slowly and are therefore not usable in this form in a numerical computation. This disadvantage can be eliminated by rearranging (43). To achieve this, operator D is split into

$$D = D_1 + D_2 \tag{47}$$

where

$$D_1 = \sum_{i=1}^{n} \varphi_i (Z_1, Z_2, \ldots, Z_n) \frac{\partial}{\partial Z_i}$$

and

$$D_2 = \sum_{i=1}^{n} [f_i(Z_1, Z_2, \ldots, Z_n) - \varphi_i(Z_1, Z_2, \ldots, Z_n)] \frac{\partial}{\partial Z_i} \cdot \tag{48}$$

Thus near the initial point $|f_i - \varphi_i| < |\varphi_i|$ and the functions belonging to D_1

$$Z_{ia}(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} [D_1^k Z_i]_{Z^{(0)}} \quad (i = 1, 2, \ldots, n) \tag{49}$$

should be regular and expressed by known functions in closed form for all finite values of t.

The subscript $Z^{(0)}$ following the brackets in equation (49) indicates that the variables $Z_1, Z_2, \ldots, Z_n$ are to be replaced by the initial values (42) only after all the differentiations prescribed by D^k have been performed.

By splitting (47) the desired solutions $Z_i(t)$ can be written in the following form (see Gröbner [2] for details)

$$Z_i(t) = Z_{ia}(t) + \sum_{m=o}^{\infty} \int_{t_0}^{t} \frac{(t-\tau)^m}{m!} [D_2 D^m Z_i]_{Z_{ia}(\tau)} d\tau .$$

$$(i = 1, 2, \ldots, n) \qquad (50)$$

NOTE: If the term $\partial/\partial t$ is added to operator D, the splitting of operator D becomes considerably more flexible with respect to possible choices of the functions φ_1. By splitting D into

$$D_1 = \sum_{i=1}^{n} \varphi_i (Z_1, Z_2, \ldots, Z_n, t) \frac{\partial}{\partial Z_i} + \frac{\partial}{\partial t} \qquad (51)$$

and

$$D_2 = \sum_{i=1}^{n} [f_i(Z_1, Z_2, \ldots, Z_n) - \varphi_i(Z_1, Z_2, \ldots, Z_n, t)] \frac{\partial}{\partial Z_i} \qquad (52)$$

we can choose any desired functions of t -- polynomials, trigonometric sums, m-terms of the power series expansions, etc. --as functions φ_k.

D. Generalized Lie Series

To extend the scope of Gröbner's Lie series, Filatov [6] introduced "generalized Lie series": Under the assumptions that we have $\varphi_m(t, z_1 z_2, \ldots, z_n)$ $(m = 0, 1, 2, \ldots, n)$ holomorphic functions of the complex variables $t, z_1, z_2, \ldots, z_n$ in a region G of the $(n + 1)$-dimensional space $R_{n+1}(t, z_1, z_2, \ldots, z_n)$ containing the origin of the coordinate system. Furthermore, the function $f(t, z_1 z_2, \ldots, z_n)$ may also be a holomorphic function in the same region G.

We now introduce operator W

$$W = \varphi_0(t, z_1 z_2, \ldots, z_n) \frac{\partial}{\partial t} + \sum_{k=1}^{n} \varphi_k(t, z_1, z_2, \ldots, z_n) \frac{\partial}{\partial z_k} \quad (53)$$

for which these relations apply when operator W is repeatedly applied to function f:

$$W^0 \equiv f \quad \text{and} \quad W^n f = W(W^{n-1} f). \quad (54)$$

We can now, just as in equation (4), define the "generalized Lie series"

$$\sum_{k=o}^{\infty} \frac{t^k}{k!} W^k f(t, z_1, z_2, \ldots, z_n) = e^{tW} f. \quad (55)$$

From equation (55) we again get Gröbner's Lie series, for the special case $\varphi_0 = 0$ and for the functions $\varphi_k (k = 1, 2, \ldots, n)$ and f which are independent of t.

Absolute and uniform convergence in a finite and closed region G of R_{n+1}-space can also be demonstrated for these "generalized Lie series" [6].

III. APPLICATION TO THE RESTRICTED PROBLEM OF THREE BODIES

A. General

Since the Lie series for a given initial value problem generally converge very slowly, it is usually necessary to split operator D into two operators $D = D_1 + D_2$ and use formula (50) for the calculation. But in doing this it is necessary that the functions belonging to D_1 [see equation (49)] represent a good approximation of the desired solution, and have to be regular for all finite values of t, and can be expressed by known functions in closed form. Until now, a low-order polynomial, second or third-order, in $(t - t_0)$ with the coefficients $D^\nu z_i$ has been used at each initial point.

The operators $D^\nu z_i$ for $\nu = 0, 1, 2, \ldots$ had to be computed in this way, but generally this is possible only up to $\nu = 2, 3$, or at most 4, because the operator terms soon become too long. A good initial approximation was rarely achieved and the highest order of the error term for the Lie series method was very low [4].

Since the Lie series for a problem in differential equations, for reasons of the uniqueness of the solution, is simply the Taylor series for the solution function, the power series method can obviously be used to get a good initial approximation for the method of Lie series [2]. Thus it becomes possible to expand the method of Lie series into a numerical method of any desired order of accuracy. By adding perturbation integrals to formula (50), the order of accuracy of the method of Lie series can be raised by a full h-power over the power series method by every further perturbation integral. The perturbation integrals are therefore analogous to the k-values of the Runge-Kutta-Fehlberg method.

B. The Restricted Problem of Three Bodies

For the numerical solution of the restricted problem of three bodies in a rotating coordinate system the problem of initial value [7, 8, 4]

$$
\begin{aligned}
\dot{x} &= \vartheta_1 \\
\dot{y} &= \vartheta_2 \\
\dot{\vartheta}_1 &= x + 2\vartheta_2 - \mu' \frac{x+\mu}{[(x+\mu)^2 + y^2]^{3/2}} - \mu \frac{x-\mu'}{[(x-\mu')^2+y^2]^{3/2}} = \vartheta_3 \\
\dot{\vartheta}_2 &= y - 2\vartheta_1 - \mu' \frac{y}{[(x+\mu)^2+y^2]^{3/2}} - \mu \frac{y}{[(x-\mu')^2+y^2]^{3/2}} = \vartheta_4
\end{aligned}
\tag{56}
$$

with the initial conditions

$$
\begin{aligned}
x(0) &= x_0, & \dot{x}(0) &= \dot{x}_0 \\
y(0) &= y_0, & \dot{y}(0) &= \dot{y}_0
\end{aligned}
\tag{56a}
$$

will be integrated step by step. Additionally, $\mu = 1/82.45$ and $\mu' = 1 - \mu$.

With the abbreviations

$$
\begin{aligned}
\frac{1}{\mu'}\,[(x+\mu)^2 + y^2]^{3/2} &= N_1 \\
\frac{1}{\mu}\,[(x-\mu')^2 + y^2]^{3/2} &= N_2
\end{aligned}
\tag{57}
$$

equations (56) become

$$
\begin{aligned}
\dot{x} &= \vartheta_1 \\
\dot{y} &= \vartheta_2 \\
\dot{\vartheta}_1 = \vartheta_3 &= x + 2\vartheta_2 - \frac{1}{N_1}(x+\mu) - \frac{1}{N_2}(x - \mu') \\
\dot{\vartheta}_2 = \vartheta_4 &= y - 2\vartheta_1 - \frac{y}{N_1} - \frac{y}{N_2} \quad .
\end{aligned}
\tag{58}
$$

Operator D [see equations (43) or (51)] for equations (58) is

$$
D = \frac{\partial}{\partial t} + \vartheta_1 \frac{\partial}{\partial x} + \vartheta_2 \frac{\partial}{\partial y} + \vartheta_3 \frac{\partial}{\partial \vartheta_1} + \vartheta_4 \frac{\partial}{\partial \vartheta_2} \quad . \tag{59}
$$

We split operator D in equation (59) into

$$
D_1 = \vartheta_1 \frac{\partial}{\partial x} + \vartheta_2 \frac{\partial}{\partial y} + \varphi_3 \frac{\partial}{\partial \vartheta_1} + \varphi_4 \frac{\partial}{\partial \vartheta_2} + \frac{\partial}{\partial t}
$$

and

$$
D_2 = (\vartheta_3 - \varphi_3) \frac{\partial}{\partial \vartheta_1} + (\vartheta_4 - \varphi_4) \frac{\partial}{\partial \vartheta_2} \tag{60}
$$

where $D = D_1 + D_2$, and φ_3 and φ_4 are arbitrary functions that will later be chosen in a more convenient form. Using equation (50), the solutions of equations (58) generally take the form

$$
f(t, z) = g(t, z) + \sum_{k=o}^{\infty} \int_{o}^{t} \frac{(t-s)^k}{k!} \left[D_2 D^k z\right]_{(z=g)} ds \tag{61}
$$

where $g(t, z) = e^{tD_1} z$ must be a known approximate solution of equations (58).

We now choose φ_3 and φ_4 as follows:

$$
\begin{aligned}
\varphi_3 &= [D^2 x]_0 + \dots + \frac{(t-t_0)^m}{m!} [D^{m+2} x]_0 \\
\varphi_4 &= [D^2 y]_0 + \dots + \frac{(t-t_0)^m}{m!} [D^{m+2} y]_0
\end{aligned}
\tag{62}
$$

where $[D^{\nu}x]_0$ and $[D^{\nu}y]_0$ represent the ν-th total derivatives of x and y with respect to t, taken at the point $t = t_0$, $x = x_0$, $\dot{x} = \dot{x}_0$, $y = y_0$, and $\dot{y} = \dot{y}_0$.

With φ_3 and φ_4 chosen according to (62), the approximate solutions g(t, z) of (62) are

$$
\begin{aligned}
x_a &= [e^{tD_1}x]_0 \\
y_a &= [e^{tD_1}y]_0 \\
\dot{x}_a &= [e^{tD_1}\vartheta_1]_0 \\
\dot{y}_a &= [e^{tD_1}\vartheta_2]_0
\end{aligned}
\qquad (63)
$$

where $[\ ]_0$ means that the expression in brackets, after having performed all the operations prescribed by D_1, is computed at the point $t = t_0$, $x = x_0$, $\dot{x} = \dot{x}_0$, $y = y_0$, and $\dot{y} = \dot{y}_0$. If, for example, we calculate x_a from equations (63) we get, using

$$D_1^0 x = x$$

$$D_1 x = \vartheta_1$$

$$D_1^2 x = D_1(D_1 x) = D_1\vartheta_1 = \varphi_3 = [D^2x]_0 + \ldots + \frac{(t-t_0)^m}{m!}[D^{m+2}x]_0$$

$$D_1^3 x = D_1(D_1^2 x) = D_1\vartheta_3 = \frac{\partial}{\partial t}\left\{ [D^2x]_0 + \ldots + \frac{(t-t_0)^m}{m!}[D^{m+2}x]_0 \right\}$$

$$= [D^3x]_0 + \frac{2(t-t_0)}{2!}[D^4x]_0 + \ldots + m\frac{(t-t_0)^{m-1}}{m!}[D^{m+2}x]_0$$

$$D_1^4 x = D_1(D_1^3 x) = \frac{\partial}{\partial t}\left\{ [D^3x]_0 + \frac{2(t-t_0)}{2!}[D^4x]_0 + \ldots \right\}$$

$$= [D^4x]_0 + \frac{6(t-t_0)}{3!}[D^5x]_0 + \ldots + \frac{m(m-1)(t-t_0)^{m-2}}{m!}[D^{m+2}x]_0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D_1^m x = [D^{m+2}x]_0$$

$$D_1^{m+j} x = 0 \text{ for } j = 1, 2, 3, \ldots$$

under consideration that in $x_a = [e^{tD_1}x]_0$, $D_1^{\nu}x$ has to be taken at the point $t = t_0$, the following equation for the approximate solution x_a:

$$x_a = \sum_{\nu=o}^{\infty} \frac{(t-t_0)^{\nu}}{\nu!} [D_1^{\nu}x]_0 = \sum_{\nu=o}^{m+2} \frac{(t-t_0)^{\nu}}{\nu!} [D^{\nu}x]_0 .$$

In the same way, we can show that

$$\dot{x}_a = \sum_{\nu=o}^{\infty} \frac{(t-t_0)^{\nu}}{\nu!} [D_1^{\nu}\vartheta_1]_0 = \sum_{\nu=0}^{m+2} \frac{(t-t_0)^{\nu}}{\nu!} [D^{\nu}\vartheta_1]_0$$

$$y_a = \sum_{\nu=o}^{\infty} \frac{(t-t_0)^{\nu}}{\nu!} [D_1^{\nu}y]_0 = \sum_{\nu=o}^{m+2} \frac{(t-t_0)^{\nu}}{\nu!} [D^{\nu}y]_0 \tag{64}$$

$$\dot{y}_a = \sum_{\nu=o}^{\infty} \frac{(t-t_0)^{\nu}}{\nu!} [D_1^{\nu}\vartheta_2]_0 = \sum_{\nu=o}^{m+2} \frac{(t-t_0)^{\nu}}{\nu!} [D^{\nu}\vartheta_2]_0 .$$

NOTE: The solution of the system of differential equations

$$\frac{dZ_i}{dt} = \vartheta_i(z_1, z_2, \ldots, z_n) \tag{65}$$

with

$$Z_i(t_0) = z_i^{(0)} \qquad (i = 1, 2, \ldots, n)$$

can, using the formula of Lie series, be written in the following closed form [see equation (43)]:

$$Z_i(t) = \sum_{\nu=o}^{\infty} \frac{(t-t_0)^{\nu}}{\nu!} [D^{\nu}z_i]_{z(0)} . \tag{66}$$

Operator D is here defined by

$$D = \sum_{k=1}^{n} \vartheta_k(z_1, z_2, \ldots, z_n) \frac{\partial}{\partial z_k} . \tag{67}$$

Given equation (65), i.e., given

$$\frac{dZ_i}{dt} = \vartheta_i \quad ,$$

it follows from equation (67) that

$$Dz_i = \vartheta_i = \frac{dZ_i}{dt} \quad . \tag{68}$$

And from equation (68) we get

$$D^2 z_i = D(Dz_i) = D\vartheta_i$$

$$= \sum_{k=1}^{n} \vartheta_k \frac{\partial \vartheta_i}{\partial z_k} = \frac{d^2 Z_i}{dt^2} \tag{69}$$

so that Lie series (66), as is also shown by the uniqueness of the solution, is identical with the Taylor series for $z_i(t)$.

Therefore it is possible to get the desired accuracy for the initial approximations (64) by any choice of m using the method of power series [7] without explicitly calculating the operators $D^{\nu} z_i$. The explicit calculation of the operators $D^{\nu} z_i$ for $\nu = 4, 5, \ldots$ leads to extremely long expressions being of no value to the practical solution [4]. It was mainly for this reason that the method of Lie series could not compete with other methods for the solution of ordinary differential equations. By use of the method of power series, however, these difficulties could be eliminated. Furthermore it is now possible to expand the method of Lie series by making it a numerical method with any high truncation error possible. This was the decisive step in the new method of Lie series first developed by Fehlberg and Filippi [1].

Parallel to the k-values in the Runge-Kutta-Fehlberg method [1, 8, 9, 10], we add perturbation integrals [see equation (61)] to the initial approximations to raise the order of the truncation error. To solve these perturbation integrals, the operators $D_2 D^k z_i$ for $k = 0, 1, 2, \ldots$ must first be computed. Each perturbation integral added to the initial approximation (64) enlarges the order of the truncation error $0(h^m)$ by <u>one</u> full power.

The following equations are the results of our calculations for k = 1, 2, 3, and 4.

(a) $k = 0$: $D_2D^0x = 0$

$D_2D^0y = 0$

(b) $k = 1$: $D_2Dx = \vartheta_3 - \varphi_3$

$D_2Dy = \vartheta_4 - \varphi_4$

(c) $k = 2$: $D_2D^2x = 2(\vartheta_4 - \varphi_4)$

$D_2D^2y = -2(\vartheta_3 - \varphi_3)$

(d) $k = 3$: $D_2D^3x = (\vartheta_3 - \varphi_3)(I) + (\vartheta_4 - \varphi_4)(II)$

$D_2D^3y = (\vartheta_3 - \varphi_3)(II) + (\vartheta_4 - \varphi_4)(III)$

with

$$(I) = -3 - \mu' \frac{y^2 - 2(x+\mu)^2}{[(x+\mu)^2 + y^2]^{5/2}} - \mu \frac{y^2 - 2(x-\mu')^2}{[(x-\mu')^2 + y^2]^{5/2}}$$

$$(II) = \mu' \frac{3y(x+\mu)}{[(x+\mu)^2 + y^2]^{5/2}} + \mu \frac{3y(x-\mu')}{[(x-\mu')^2 + y^2]^{5/2}}$$

$$(III) = -3 - \mu' \frac{(x+\mu)^2 - 2y^2}{[(x+\mu)^2 + y^2]^{5/2}} - \mu \frac{(x-\mu')^2 - 2y^2}{[(x-\mu)^2 + y^2]^{5/2}}$$

(e) $k = 4$: $D_2D^4x = 2(\vartheta_3 - \varphi_3)[\vartheta_1(IV) + \vartheta_2(V)]$

$+ 2(\vartheta_4 - \varphi_4)[\vartheta_1(V) + \vartheta_2(VI) + (I) + (III) + 4]$

$D_2D^4y = 2(\vartheta_4 - \varphi_4)[\vartheta_1(VI) + \vartheta_2(VII)]$

$+ 2(\vartheta_3 - \varphi_3)[\vartheta_1(V) + \vartheta_2(VI) - (I) - (III) - 4]$

with

$$(IV) = \frac{3(x+\mu)\mu'}{[(x+\mu)^2 + y^2]^{7/2}}[3y^2 - 2(x+\mu)^2] + \frac{3(x-\mu')\mu}{[(x-\mu')^2 + y^2]^{7/2}}[3y^2 - 2(x-\mu')^2]$$

$$(\mathrm{V}) = \frac{3y\mu'}{[(x+\mu)^2+y^2]^{7/2}}\,[y^2-4(x+\mu)^2] + \frac{3y\mu}{[(x-\mu')^2+y^2]^{7/2}}\,[y^2-4(x-\mu')^2]$$

$$(\mathrm{VI}) = \frac{3(x+\mu)\mu'}{[(x+\mu)^2+y^2]^{7/2}}\,[(x+\mu)^2-4y^2] + \frac{3(x-\mu')\mu}{[(x-\mu')^2+y^2]^{7/2}}\,[(x-\mu')^2-4y^2]$$

$$(\mathrm{VII}) = \frac{3y\mu'}{[(x+\mu)^2+y^2]^{7/2}}\,[3(x+\mu)^2-2y^2] + \frac{3\mu y}{[(x-\mu')^2+y^2]^{7/2}}\,[3(x-\mu')^2-2y^2]$$

We shall now combine the solution formulas for equations (58), considering also equation (61) with the perturbation integrals for k = 0, 1, 2, 3, and 4:

Solution Formulas

$$x(t) = x_a(t) + \int_{t_0}^{t}(t-\tau)(\vartheta_3-\varphi_3)_a\,d\tau + \int_{t_0}^{t}(t-\tau)^2(\vartheta_4-\varphi_4)_a\,d\tau$$

$$+\frac{1}{6}\int_{t_0}^{t}(t-\tau)^3\,[(\vartheta_3-\varphi_3)(\mathrm{I}) + (\vartheta_4-\varphi_4)(\mathrm{II})]_a\,d\tau$$

$$+\frac{1}{12}\int_{t_0}^{t}(t-\tau)^4\,\{(\vartheta_3-\varphi_3)[\vartheta_1(\mathrm{IV})+\vartheta_2(\mathrm{V})] + (\vartheta_4-\varphi_4)[\vartheta_1(\mathrm{V})+\vartheta_2(\mathrm{VI})$$

$$+(\mathrm{I})+(\mathrm{III})+4]\}_a\,d\tau$$

$$y(t) = y_a(t) + \int_{t_0}^{t}(t-\tau)(\vartheta_4-\varphi_4)_a\,d\tau - \int_{t_0}^{t}(t-\tau)^2(\vartheta_3-\varphi_3)_a\,d\tau$$

$$+\frac{1}{6}\int_{t_0}^{t}(t-\tau)^3\,[(\vartheta_3-\varphi_3)(\mathrm{II}) + (\vartheta_4-\varphi_4)(\mathrm{III})]_a\,d\tau$$

$$+\frac{1}{12}\int_{t_0}^{t}(t-\tau)^4\,\{(\vartheta_4-\varphi_4)[\vartheta_1(\mathrm{VI})+\vartheta_2(\mathrm{VII})]$$

$$+(\vartheta_3-\varphi_3)[\vartheta_1(\mathrm{V})+\vartheta_2(\mathrm{VI})-(\mathrm{I})-(\mathrm{III})-4]\}_a\,d\tau$$

$$\dot{x}(t) = \dot{x}_a(t) + \int_{t_0}^{t}(\vartheta_3-\varphi_3)_a\,d\tau + 2\int_{t_0}^{t}(t-\tau)(\vartheta_4-\varphi_4)_a\,d\tau$$

$$+\frac{1}{2}\int_{t_0}^{t}(t-\tau)^2\,[(\vartheta_3-\varphi_3)(I)+(\vartheta_4-\varphi_4)(II)\,]_a\,d\tau$$

$$+\frac{1}{3}\int_{t_0}^{t}(t-\tau)^3\{(\vartheta_3-\varphi_3)\,[\vartheta_1(IV)+\vartheta_2(V)]$$

$$+(\vartheta_4-\varphi_4)\,[\vartheta_1(V)+\vartheta_2(VI)+(I)+(III)+4]\}_a\,d\tau$$

$$\dot{y}(t)=\dot{y}_a(t)+\int_{t_0}^{t}(\vartheta_4-\varphi_4)_a\,d\tau-2\int_{t_0}^{t}(t-\tau)(\vartheta_3-\varphi_3)_a\,d\tau$$

$$+\frac{1}{2}\int_{t_0}^{t}(t-\tau)^2\,[(\vartheta_3-\varphi_3)(II)+(\vartheta_4-\varphi_4)(III)]_a\,d\tau$$

$$+\frac{1}{3}\int_{t_0}^{t}(t-\tau)^3\{(\vartheta_4-\varphi_4)\,[\vartheta_1(VI)+\vartheta_2(VII)\,]$$

$$+(\vartheta_3-\varphi_3)\,[\vartheta_1(V)+\vartheta_2(VI)-(I)-(III)-4]\}_a\,d\tau$$

Step-by-Step Arithmetical Performance

By use of the solution formulas in the second part of this section, we can immediately perform the step-by-step computation of $x(t)$, $\dot{x}(t)$, $y(t)$, and $\dot{y}(t)$ if we compute the perturbation integrals that depend on m. The m is arbitrarily chosen for computation of the approximate solutions $x_a(t)$, $\dot{x}_a(t)$, $y_a(t)$, and $\dot{y}_a(t)$ via a quadrature formula with an error of the order $0(h^{m+1})$, $0(h^{m+2})$, $0(h^{m+3})$, or $0(h^{m+4})$. But the effort is by far too great [4].

We shall now suggest a more satisfactory method for calculating the perturbation integrals.

Calculation of the First Three Perturbation Integrals

We start as follows:

$$(\vartheta_3 - \varphi_3)_a = (t - t_0)^{m+1} A_{m+1} + (t - t_0)^{m+2} A_{m+2} + (t - t_0)^{m+3} A_{m+3} \quad (70)$$

$$(\vartheta_4 - \varphi_4)_a = (t - t_0)^{m+1} B_{m+1} + (t - t_0)^{m+2} B_{m+2} + (t - t_0)^{m+3} B_{m+3} \;. \quad (71)$$

If we successively introduce $t = t_0 + 2\Delta t$, $t = t_0 + \Delta t$, and $t = t_0 - \Delta t$ into (70), we get

$$A_{m+1} + 2\Delta t A_{m+2} + 4\,(\Delta t)^2 A_{m+3} = \frac{(\vartheta_3 - \varphi_3)_a^{\,t = t_0 + 2\Delta t}}{(2\Delta t)^{m+1}} = \alpha \quad (72)$$

$$A_{m+1} + \Delta t A_{m+2} + (\Delta t)^2 A_{m+3} = \frac{(\vartheta_3 - \varphi_3)_a^{\,t = t_0 + \Delta t}}{(\Delta t)^{m+1}} = \beta \quad (73)$$

$$A_{m+1} - \Delta t A_{m+2} + (\Delta t)^2 A_{m+3} = \frac{(\vartheta_3 - \varphi_3)_a^{\,t = t_0 - \Delta t}}{(-1)^{m+1} (\Delta t)^{m+1}} = \gamma \;. \quad (74)$$

We get three corresponding relations if we successively introduce $t = t_0 + 2\Delta t$, $t = t_0 + \Delta t$, and $t = t_0 - \Delta t$ into equation (71).

If we add equations (73) and (74), we get

$$A_{m+1} + (\Delta t)^2 A_{m+3} = \frac{\beta + \gamma}{2} \;. \quad (75)$$

If we subtract equation (74) from equation (73), we get

$$2\Delta t A_{m+2} = \beta - \gamma \quad (76)$$

or

$$A_{m+2} = \frac{\beta - \gamma}{2\Delta t} \;. \quad (76a)$$

Using equation (76a) we get from equation (72)

$$A_{m+1} + 4(\Delta t)^2 \dot{A}_{m+3} = \alpha - \beta + \gamma \quad . \tag{77}$$

If we subtract equation (75 from (77), we get

$$3(\Delta t)^2 A_{m+3} = \alpha - \frac{3\beta}{2} + \frac{\gamma}{2} \tag{78}$$

or

$$A_{m+3} = \frac{1}{3(\Delta t)^2} \left(\alpha - \frac{3\beta}{2} + \frac{\gamma}{2} \right) . \tag{78a}$$

It follows from equation (75), using equation (78), that

$$A_{m+1} = \frac{\alpha}{3} + \beta + \frac{\gamma}{3} \quad . \tag{79}$$

We then get these expressions for A_{m+1}, A_{m+2}, and A_{m+3}:

$$A_{m+1} = -\frac{1}{(\Delta t)^{m+1}} \cdot \frac{1}{6} \left[\frac{1}{2^m} (\vartheta_3 - \varphi_3)_a^{t=t_0+2\Delta t} - 6(\vartheta_3 - \varphi_3)_a^{t=t_0+\Delta t} + (-1)^m \cdot 2(\vartheta_3 - \varphi_3)_a^{t=t_0-\Delta t} \right] \tag{80}$$

$$A_{m+2} = \frac{1}{(\Delta t)^{m+2}} \cdot \frac{1}{2} \left[(\vartheta_3 - \varphi_3)_a^{t=t_0+\Delta t} + (-1)^m (\vartheta_3 - \varphi_3)_a^{t=t_0-\Delta t} \right] \tag{81}$$

$$A_{m+3} = \frac{1}{(\Delta t)^{m+3}} \cdot \frac{1}{6} \left[\frac{1}{2^m} (\vartheta_3 - \varphi_3)_a^{t=t_0+2\Delta t} - 3(\vartheta_3 - \varphi_3)_a^{t=t_0+\Delta t} + (-1)^{m+1} (\vartheta_3 - \varphi_3)_a^{t=t_0-\Delta t} \right] \quad . \tag{82}$$

Analogous to this we get three expressions for B_{m+1}, B_{m+2}, and B_{m+3}. We simply say B_{m+1} instead of A_{m+1}, etc., in equations (80), (81), and (82), and we change every $(\vartheta_3 - \varphi_3)_a$ to $(\vartheta_4 - \varphi_4)_a$.

If we now introduce the expressions (80), (81), (82), and the corresponding terms B_{m+1}, B_{m+2}, and B_{m+3} to the first three perturbation integrals we get, by means of a term-by-term integration, the following definitive solution formulas:

$$x(t) = x_a(t) + \frac{A_{m+1}(\Delta t)^{m+3}}{(m+2)(m+3)} + \frac{A_{m+2}(\Delta t)^{m+4}}{(m+3)(m+4)} + \boxed{\frac{A_{m+3}(\Delta t)^{m+5}}{(m+4)(m+5)}}$$

$$+ \frac{2B_{m+1}(\Delta t)^{m+4}}{(m+2)(m+3)(m+4)} + \frac{2B_{m+2}(\Delta t)^{m+5}}{(m+3)(m+4)(m+5)} \tag{83}$$

$$+ \frac{[(I)A_{m+1} + (II)B_{m+1}]_a(\Delta t)^{m+5}}{(m+2)(m+3)(m+4)(m+5)}$$

$$\dot{x}(t) = \dot{x}_a(t) + \frac{A_{m+1}(\Delta t)^{m+2}}{m+2} + \frac{A_{m+2}(\Delta t)^{m+3}}{m+3} + \frac{A_{m+3}(\Delta t)^{m+4}}{m+4}$$

$$+ \frac{2B_{m+1}(\Delta t)^{m+3}}{(m+2)(m+3)} + \frac{2B_{m+2}(\Delta t)^{m+4}}{(m+3)(m+4)} \tag{84}$$

$$+ \frac{[(I)A_{m+1} + (II)B_{m+1}]_a(\Delta t)^{m+4}}{(m+2)(m+3)(m+4)}$$

$$y(t) = y_a(t) + \frac{B_{m+1}(\Delta t)^{m+3}}{(m+2)(m+3)} + \frac{B_{m+2}(\Delta t)^{m+4}}{(m+3)(m+4)} + \boxed{\frac{B_{m+3}(\Delta t)^{m+5}}{(m+4)(m+5)}}$$

$$- \frac{2A_{m+1}(\Delta t)^{m+4}}{(m+2)(m+3)(m+4)} - \frac{2A_{m+2}(\Delta t)^{m+5}}{(m+3)(m+4)(m+5)} \tag{85}$$

$$+ \frac{[(II)A_{m+1} + (III)B_{m+1}]_a(\Delta t)^{m+5}}{(m+2)(m+3)(m+4)(m+5)}$$

$$\dot{y}(t) = \dot{y}_a(t) + \frac{B_{m+1}(\Delta t)^{m+2}}{m+2} + \frac{B_{m+2}(\Delta t)^{m+3}}{m+3} + \frac{B_{m+3}(\Delta t)^{m+4}}{m+4}$$

$$- \frac{2A_{m+1}(\Delta t)^{m+3}}{(m+2)(m+3)} - \frac{2A_{m+2}(\Delta t)^{m+4}}{(m+3)(m+4)} \tag{86}$$

$$+ \frac{[(II)A_{m+1} + (III)B_{m+1}]_a (\Delta t)^{m+4}}{(m+2)(m+3)(m+4)}$$

where $(I)_a$, $(II)_a$, $(III)_a$ are abbreviations used in the expression for $k = 3$ in III-B, d (page 23). Index a means that in (I), (II), and (III) x, $\dot{x}$, y, $\dot{y}$ always has to be replaced by x_a, $\dot{x}_a$, y_a, $\dot{y}_a$ [see (64)]. The expressions $A_{m+1}, \ldots,$ B_{m+3} are found in expressions (80), (81), (82), and in three corresponding expressions. The sign $\boxed{}$ indicates that this term is used for automatic step-size control. All other terms with $(\Delta t)^{m+5}$ are to be neglected when solutions (83), . . . , (86) are computed step-by-step.

Computation of the First Four Perturbation Integrals in III-B.

Using the formulation

$$\begin{aligned} (\vartheta_3 - \varphi_3)_a &= (t - t_0)^{m+1} A_{m+1} + \ldots + (t - t_0)^{m+4} A_{m+4} \\ (\vartheta_4 - \varphi_4)_a &= (t - t_0)^{m+1} B_{m+1} + \ldots + (t - t_0)^{m+4} B_{m+4} \end{aligned} \tag{87}$$

and computing as in the third section of III-B, we get as the definitive solution formulas

$$\begin{aligned} x(t) = x_a(t) &+ A_{m+1} \frac{(\Delta t)^{m+3}}{(m+2)(m+3)} + A_{m+2} \frac{(\Delta t)^{m+4}}{(m+3)(m+4)} \\ &+ A_{m+3} \frac{(\Delta t)^{m+5}}{(m+4)(m+5)} + \boxed{A_{m+4} \frac{(\Delta t)^{m+6}}{(m+5)(m+6)}} \\ &+ 2B_{m+1} \frac{(\Delta t)^{m+4}}{(m+2)(m+3)(m+4)} + 2B_{m+2} \frac{(\Delta t)^{m+5}}{(m+3)(m+4)(m+5)} \\ &+ 2B_{m+3} \frac{(\Delta t)^{m+6}}{(m+4)(m+5)(m+6)} \\ &+ \frac{(\Delta t)^{m+5}}{(m+2)(m+3)(m+4)(m+5)} \left[(I)A_{m+1} + (II)B_{m+1}\right]_a \end{aligned}$$

$$+ \frac{(\Delta t)^{m+6}}{(m+3)(m+4)(m+5)(m+6)} \left[(I) A_{m+2} + (II) B_{m+2} \right]_a$$

$$+ 2 \frac{(\Delta t)^{m+6}}{(m+2)(m+3)(m+4)(m+5)(m+6)} \left\{ A_{m+1} \left[\dot{x}_a (IV) + \dot{y}_a (V) \right]_a \right.$$

$$\left. + B_{m+1} \left[\dot{x}_a (V) + \dot{y}_a (VI) + (I) + (III) + 4 \right]_a \right\} \tag{88}$$

$$y(t) = y_a(t) + B_{m+1} \frac{(\Delta t)^{m+3}}{(m+2)(m+3)} + B_{m+2} \frac{(\Delta t)^{m+4}}{(m+3)(m+4)}$$

$$+ B_{m+3} \frac{(\Delta t)^{m+5}}{(m+4)(m+5)} + \boxed{B_{m+4} \frac{(\Delta t)^{m+6}}{(m+5)(m+6)}}$$

$$- 2A_{m+1} \frac{(\Delta t)^{m+4}}{(m+2)(m+3)(m+4)} - 2A_{m+2} \frac{(\Delta t)^{m+5}}{(m+3)(m+4)(m+5)}$$

$$-2A_{m+3} \frac{(\Delta t)^{m+6}}{(m+4)(m+5)(m+6)}$$

$$+ \frac{(\Delta t)^{m+5}}{(m+2)(m+3)(m+4)(m+5)} \left[(I) A_{m+1} + (II) B_{m+1} \right]_a \tag{89}$$

$$+ \frac{(\Delta t)^{m+6}}{(m+3)(m+4)(m+5)(m+6)} \left[(II) A_{m+1} + (III) B_{m+1} \right]_a$$

$$+ 2 \frac{(\Delta t)^{m+6}}{(m+2)(m+3)(m+4)(m+5)(m+6)} \left\{ B_{m+1} \left[\dot{x}_a (VI) + \dot{y}_a (VII) \right]_a \right.$$

$$\left. + A_{m+1} \left[\dot{x}_a (V) + \dot{y}_a (VI) - (I) - (III) - 4 \right]_a \right\}$$

$$\dot{x}(t) = \dot{x}_a(t) + A_{m+1} \frac{(\Delta t)^{m+2}}{m+2} + A_{m+2} \frac{(\Delta t)^{m+3}}{m+3} + A_{m+3} \frac{(\Delta t)^{m+4}}{m+4}$$

$$+ A_{m+4}\frac{(\Delta t)^{m+5}}{m+5} + 2B_{m+1}\frac{(\Delta t)^{m+3}}{(m+2)(m+3)}$$

$$+ 2B_{m+2}\frac{(\Delta t)^{m+4}}{(m+3)(m+4)} + 2B_{m+3}\frac{(\Delta t)^{m+5}}{(m+4)(m+5)}$$

$$+ \frac{(\Delta t)^{m+4}}{(m+2)(m+3)(m+4)}\left[(I)A_{m+1} + (II)B_{m+1}\right]_a$$

$$+ \frac{(\Delta t)^{m+5}}{(m+3)(m+4)(m+5)}\left[(I)A_{m+2} + (II)B_{m+2}\right]_a \qquad (90)$$

$$+ 2\frac{(\Delta t)^{m+5}}{(m+2)(m+3)(m+4)(m+5)}\Big\{A_{m+2}\left[\dot{x}_a(IV) + \dot{y}_a(V)\right]_a$$

$$+ B_{m+1}\left[\dot{x}_a(V) + \dot{y}_a(VI) + (I) + (III) + 4\right]_a\Big\}$$

$$\dot{y}(t) = \dot{y}_a(t) + B_{m+1}\frac{(\Delta t)^{m+2}}{m+2} + B_{m+2}\frac{(\Delta t)^{m+3}}{m+3} + B_{m+3}\frac{(\Delta t)^{m+4}}{m+4}$$

$$+ B_{m+4}\frac{(\Delta t)^{m+5}}{m+5} - 2A_{m+1}\frac{(\Delta t)^{m+3}}{(m+2)(m+3)}$$

$$- 2A_{m+2}\frac{(\Delta t)^{m+4}}{(m+3)(m+4)} - 2A_{m+3}\frac{(\Delta t)^{m+5}}{(m+4)(m+5)}$$

$$(91)$$

$$+ \frac{(\Delta t)^{m+4}}{(m+2)(m+3)(m+4)}\left[(II)A_{m+1} + (III)B_{m+1}\right]_a$$

$$+ \frac{(\Delta t)^{m+5}}{(m+3)(m+4)(m+5)}\left[(II)A_{m+2} + (III)B_{m+2}\right]_a$$

$$+ 2\frac{(\Delta t)^{m+5}}{(m+2)(m+3)(m+4)(m+5)}\Big\{B_{m+1}\left[\dot{x}_a(VI) + \dot{y}_a(VII)\right]_a$$

$$+ A_{m+1}\left[\dot{x}_a(V) + \dot{y}_a(VI) - (I) - (III) - 4\right]_a \Big\}$$

with

$$A_{m+1} = \frac{1}{6}(-\alpha + 4\beta + 4\gamma - \delta)$$

$$A_{m+2} = \frac{1}{12\Delta t}(-\alpha + 8\beta - 8\gamma + \delta)$$

$$A_{m+3} = \frac{1}{6(\Delta t)^2}(\alpha - \beta - \gamma + \delta)$$

$$A_{m+4} = \frac{1}{12(\Delta t)^3}(\alpha - 2\beta + 2\gamma - \delta)$$

where

$$\alpha = \frac{(\vartheta_3 - \varphi_3)_a^{t = t_0 + 2\Delta t}}{(2\Delta t)^{m+1}} \qquad \gamma = \frac{(\vartheta_3 - \varphi_3)_a^{t = t_0 - \Delta t}}{(-\Delta t)^{m+1}}$$

$$\beta = \frac{(\vartheta_3 - \varphi_3)_a^{t = t_0 + \Delta t}}{(\Delta t)^{m+1}} \qquad \delta = \frac{(\vartheta_3 - \varphi_3)_a^{t = t_0 - 2\Delta t}}{(-2\Delta t)^{m+1}}$$

and also corresponding expressions for B_{m+1}, B_{m+2}, B_{m+3}, and B_{m+4}, where $(\vartheta_4 - \varphi_4)_a$ appears in the expressions for α, β, γ, and δ instead of $(\vartheta_3 - \varphi_3)_a$. The expressions (I), (II), ..., (VII) appear in III-B, d and e (page 23) for k = 3 and k = 4. All other expressions correspond with those in the third section of III-B.

Some Numerical Results

The following table (see also Fehlberg and Filippi [1]) illustrates the efficiency of the new Lie-series method using three perturbation integrals and automatic step-size control as described in the third part of III-B. The restricted problem of three bodies [see (56)] was solved for the initial conditions

$$\begin{aligned} x_0 &= 1.20000\ 00000\ 00000 \\ \dot{x}_0 &= 0 \\ y_0 &= -1.04935\ 75098\ 30320 \\ \dot{y}_0 &= 0 \end{aligned} \qquad (92)$$

For comparison, the problem was also computed using other modern methods of numerical integration: the method of power series and two versions of the Runge-Kutta-Fehlberg method [9, 10]. All four methods were applied with a truncation error of the order $0(h^{13})$. All computations were performed on an IBM 7094, Model II computer in double precision (16 digits).

NUMBER OF INTEGRATION STEPS AND COMPUTER RUNNING TIME AFTER 12 REVOLUTIONS (ABOUT ONE YEAR ACTUAL TIME)

Method	Number of Steps	Running Time [min]
PSE *)	5896	2.49
Lie *)	5191	2.05
RKFe 1964*)	4740	1.83
RKFe 1965 *)	3353	1.35

*)
PSE = Power Series Expansion
Lie = New Lie-Series Method with Three Perturbation Integrals
RKFe 1964 = Runge-Kutta-Fehlberg Method 1964
RKFe 1965 = Runge-Kutta-Fehlberg Method 1965

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama, August 29, 1966

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