COMPUTING THE PSEUDO-INVERSE

BY

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ABSTRACT

An orthogonalization algorithm for producing the pseudo-inverse of a matrix is described, and a FORTRAN program which realizes the algorithm is given in detail.
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E. R. Lancaster, under whose supervision this paper was written, was particularly helpful in the development of certain theoretical aspects and supplied many perceptive suggestions on overall organization. G. H. Wyatt's programming skill was instrumental in the debugging phase of the programming effort.
To every matrix $A$ there corresponds a unique matrix $A^+$ with the following properties:

\begin{align*}
AA^+A &= A \quad (1) \\
A^+AA^+ &= A^+ \quad (2) \\
(A^+A)^T &= A^+A \quad (3) \\
(AA^+)^T &= AA^+ \quad (4)
\end{align*}

Penrose [1], one of the originators of this concept, called $A^+$ the generalized inverse of $A$, and equations (1) through (4) are often called Penrose's Lemmas. Recent usage applies generalized inverse to any matrix satisfying (1), (1) and (2), or (1), (2), and (3), referring to the unique $A^+$ as the pseudo-inverse of $A$. Other definitions of $A^+$ have been given (e.g. Albert [2], Ben-Israel [3]) but the most common is that given above.

For simplicity's sake, the rest of this paper considers only real matrices, although most results hold for complex matrices as well. The pseudo-inverse provides a way to handle the ubiquitous matrix-vector equation

\[ Ax = y. \quad (5) \]

If $A$ is square and non-singular, $A^+$ is $A^{-1}$ and the vector $A^+y$ solves the equation. The particular advantage of the pseudo-inverse appears when $A$ is singular or non-square, since $A^+y$ then is the minimal vector for this equation; that is, if $M$ is the set of all vectors $x_0$ such that

\[ \| Ax_0 - y \| \leq \| Ax - y \| \quad (6) \]

for all $x$, then $A^+y \in M$ and

\[ \| A^+y \| = \min_{x_0 \in M} \left\{ \| Ax_0 \| \right\} \quad (7) \]

Here we use the standard Euclidean norm.
A theorem which dates back to the time of Gauss (Newhouse [41]) states, in effect, that if \( x_0 \in M \), then \( x_0 \) is a solution of

\[
A^T A x = A^T y .
\]

This type of system, often called a set of normal equations, is found repeatedly in least squares problems. (See, e.g., Rao [51]). Since \( A^T y \in M \), the application of \( A^T \) in these circumstances is evident.

The same theorem also states that if \( x_0 \in M \), then \( x_0 \) is a projection of \( y \) onto the column space of \( A \). Newhouse later gives a theorem which proves condition (7), that \( A^T y \) is the "shortest" of these projections, giving rise to Greville's assertion [6] that \( A^T y \) is the best solution to equation (5) in the least squares sense.

Naturally, the theoretical existence of such a useful mathematical object makes a method for its computation very desirable. Most of the methods suggested, however, require that the product \( A^T A \) be formed and that Gaussian elimination (or one of its variants such as pivotal condensation or sweep out) be performed on it. Should we be faced with an ill-conditioned matrix, it is entirely possible that numerical difficulties will prevent any significant computation using such methods. For example, consider the matrix

\[
H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
5 & 6 & 7 & 8 \\
1 & 1 & 1 & 1 \\
6 & 7 & 8 & 9 \\
1 & 1 & 1 & 1 \\
7 & 8 & 9 & 10 \\
1 & 1 & 1 & 1 \\
8 & 9 & 10 & 11
\end{bmatrix}
\]

The Hilbert matrix is notoriously ill-conditioned with respect to Gaussian elimination. The upper left-hand \( 4 \times 4 \) corner of it has a condition number \( \lambda_{\text{max}} / \lambda_{\text{min}} \) given by Marcus (Ref. [7]) as 15,514, so our \( 4 \times 4 \) segment of it would certainly be suspect. Fox (Ref. [8]) shows that our suspicions are justified, giving the Gaussian elimination process for \( H \), in which the steady decrease in magnitude of the pivots leads to very unreliable quantities. More to our point, he demonstrates that Gauss elimination fails completely when applied
to $H^T H$. We should realize that a bad but workable problem can become pathologically unmanageable if such a product is formed, and, as a general rule, avoid such approaches.

The method of Rust, Burrus, and Schneeberger (Ref. [9]) was used to compute the pseudo-inverse because it does conform to this general rule. Briefly, it can be characterized as follows: if the $m \times n$ matrix $A$ is in the form $[R|S]$, where the $k$ linearly independent columns form the submatrix $R$ and the linearly dependent columns form the submatrix $S$, make up the $n \times n$ identity matrix and write, symbolically,

$$
\begin{bmatrix}
R & S \\
I_k & 0 \\
0 & I_{n-k}
\end{bmatrix}
$$

Then perform the Gram-Schmidt (G.S.) orthogonalization process on $[R|S]$, and apply these elementary column operations to the lower submatrix to get

$$
\begin{bmatrix}
Q & 0 \\
Z & -U \\
0 & I_{n-k}
\end{bmatrix}
$$

Next, perform the G.S. process on the submatrix $[U \quad I_{n-k}]$ to produce

$$
\begin{bmatrix}
Q & 0 \\
Z & -UP \\
0 & P
\end{bmatrix}
$$

form the matrix

$$
\begin{bmatrix}
Q^T \\
(UP)^T Z Q^T
\end{bmatrix}
$$
and, finally,

\[ A^+ = [R|S]^+ = \begin{bmatrix} Z & -UP \\ 0 & P \end{bmatrix} \begin{bmatrix} QT \\ (UP)^TZQ^T \end{bmatrix} \]

A complete derivation is given in Ref. [9] and a few auxiliary notes are given in Appendix A.

Of course, not every matrix will be in the convenient $[R|S]$ form, but if we can determine which columns of $A$ are dependent we can certainly permute columns to produce it; then $[R|S]^+$ is found and by the authority of Theorems I and II, Appendix A, the rows of $[R|S]^+$ are likewise permuted to get $A^+$. Since the G.S. process not only orthogonalizes the independent columns of $A$ but also makes the dependent ones zero, we can use it to find the dependent columns.

Now we have a straightforward way to proceed:

1. Use G.S. to find the dependent columns.
2. Permute to get $[R|S]$.
4. Permute to get $A^+$.

The reader will have noticed that the G.S. process is used in step (1) and again in step (3). We could save some computation time if we combined the two steps and performed the G.S. process only once. A closer examination of the process reveals that we can, under certain conditions, make this combination.

Our program uses a modified Gram–Schmidt process which is more accurate than the classic textbook version. A recursive algorithm describing our version is:

1. Orthogonalize $c_j$, the next column of $A$:

\[ b_j = c_j - \sum_{i=1}^{j-1} \left( \frac{c_i \cdot b_i^T}{b_i^T \cdot b_i} \right) b_i^T \]
(2) Is $b_j \approx 0$? If so, zero it out and go to step (1). If not, do step (3).

(3) Re-orthogonalize $b_j$:

$$b_j' = b_j - \sum_{i=1}^{j-1} \left( \frac{b_j \cdot b_i'}{b_i' \cdot b_i'} \right) b_i' ,$$

and go to step (1).

The initial condition is $b_1' = c_1$. After we run out of columns, we normalize each one and we have an orthonormal matrix $A_1$.

If we want to duplicate these elementary column operations on another matrix $D$, we could save the numbers

$$\left( \frac{c_j \cdot b_i'}{b_i' \cdot b_i'} \right) , \quad \left( \frac{b_j \cdot b_i'}{b_i' \cdot b_i'} \right) , \quad \text{and} \quad (b_j' \cdot b_j')^{1/2}$$

and then go through the algorithm again, this time letting $c_j$ be the columns of $D$. More precisely, we might save these numbers in an $n \times n$ matrix $S$, defined as

$$S_{ji} = \frac{c_j \cdot b_i'}{b_i' \cdot b_i'} \quad (j > i) ,$$

$$S_{ij} = \frac{b_j \cdot b_i'}{b_i' \cdot b_i'} \quad (j > i) ,$$

$$S_{jj} = (b_j' \cdot b_j')^{1/2} \quad (1 \leq j \leq n) .$$

As an example, let $A$ be

$$\begin{bmatrix}
1 & 1 & 3 & 6 \\
2 & 2 & 6 & 7 \\
3 & 3 & 9 & 8
\end{bmatrix}$$
Using eight-digit arithmetic and rounding the final answers to three digits, we have

\[
A_1 = \begin{bmatrix}
.267 & 0 & 0 & .873 \\
.535 & 0 & 0 & .218 \\
.802 & 0 & 0 & -.436
\end{bmatrix}
\]

\[
S = \begin{bmatrix}
3.74 & 0 & 0 & .213 \times 10^{-7} \\
1.00 & 0 & 0 & 0 \\
3.00 & 0 & 0 & 0 \\
3.14 & 0 & 0 & 3.27
\end{bmatrix}
\]

Once we have done the G.S. process on \( A \), we have done it for all column permutations of \( A \) which do not disturb the relative order of the independent columns. If \( P \) is a permutation matrix such that \( AP = [R|S] \), where \( R \) is the matrix of independent columns of \( A \) in their original relative order, the orthonormal matrix \( [Q|0] \) produced by the G.S. process on \( [R|S] \) will be \( AP \). In our example, suppose

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

Then

\[
[R|S] = AP = \begin{bmatrix}
1 & 6 & 1 & 3 \\
2 & 7 & 2 & 6 \\
3 & 8 & 3 & 9
\end{bmatrix},
\]

and

\[
[Q|0] = A_1 P = \begin{bmatrix}
.267 & .873 & 0 & 0 \\
.535 & .218 & 0 & 0 \\
.802 & -.436 & 0 & 0
\end{bmatrix}
\]

We can also produce a new \( S \) matrix (\( F \)) by permutations. Referring back to the definition of \( S \), one can see that a particular column \( c_j \) has its initial
orthogonalization coefficients \( (c_i \cdot b_j')/(b_i' \cdot b_i) \) on the jth row and below the diagonal, and its secondary coefficients \( (b_j \cdot b_i')/(b_i' \cdot b_i) \) fall on the jth column and above the diagonal. Once \( c_j \) is converted into \( b_j' \), all initial coefficients having \( b_j' \) as a factor fall on the jth column below the diagonal, and all such secondary coefficients fall on the jth row above the diagonal. When \( b_j' \) is normalized, its "length" falls on the jth diagonal element. Moving \( c_j \) to a new position therefore means that we must move the jth row and column of S to corresponding positions, or

\[
F = P^T S P
\]

In our example,

\[
F = \begin{bmatrix}
3.74 & 213 \times 10^{-7} & 0 & 0 \\
3.14 & 3.27 & 0 & 0 \\
1.00 & 0 & 0 & 0 \\
3.00 & 0 & 0 & 0
\end{bmatrix}
\]

If we go through the algorithm with \( c_i \) taken as the columns of a matrix D and the numbers \( (c_i \cdot b_j')/(b_i' \cdot b_i) \) and \( (b_j \cdot b_i')/(b_i' \cdot b_i) \) taken as \( f_{ji} \) and \( f_{ij} \) respectively, we have applied the elementary column operations of the G.S. process on \( R \mid S \) to D.

Now we have the desired result: once the G.S. process on A is complete, it is not necessary to do it again on \( R \mid S \) to derive its effects; merely execute the indicated permutations on \( A_\perp \) and S and we have all the necessary matrices. Using this result, the procedure (1) through (4) on page 4 can be rewritten:

1. Use G.S. on A; save the G.S. coefficients in S and save \( A_\perp \). Note which columns are dependent.

2. Permute \( A_\perp \) to get \( [Q \mid 0] \); permute S to get F.

3. Use the entries of F to operate on

\[
\begin{bmatrix}
I_k & 0 \\
0 & I_{n-k}
\end{bmatrix}
\]
producing

\[
\begin{bmatrix}
  Z & -U \\
  0 & I_{n-k}
\end{bmatrix}
\]

(4) Proceed as usual to find \([R|S]^+\).

(5) Permute to get \(A^+\).

The program whose flow chart and FORTRAN listing appear in Appendices B and C has been checked with a variety of matrices on the IBM 7094 and appears to run properly. Two particular cautions might be extended, however: first, one will note that a decision on the dependency of any column is made by comparing the "length" of the generated orthogonal column with the "length" of the original column. If the check number \((b_j \cdot b_j)/(c_j \cdot c_j)\) is smaller than a certain tolerance, the column \(b_j\) is made zero. When the check number is very close to the tolerance, any decision made will not be a good one and the resulting perturbations can become serious; for example, the Hilbert matrix gives poor results for this very reason. One might vary the tolerance to suit special cases.

Second, although this program finds the inverse if it exists, there are routines in general use which get better inverses. For example, the SHARE routine MATINV was tested against this program on a sequence of Pei matrices (Ref. [10], Ref. [11]) and consistently got one more accurate digit in the worst cases. The difference is not great but the prospective user should realize that it exists.

Finally, an experienced programmer will see that the FORTRAN realization in Appendix C is not in optimal form. A more streamlined, double-precision version is being prepared for the IBM 360 as of this writing. The author would appreciate hearing of mistakes in, or improvements upon, the original.
APPENDIX A
(Supplementary notes for Ref. [9])

Theorem I

If $P$ is a permutation matrix (possibly a product of elementary permutation matrices) and $A^+$ is the pseudo-inverse of $A$, then

$$(AP)^+ = PTA^+.$$  

Proof: We need only verify that Penrose's Lemmas hold. Noting that $PP^T = P^TP = I$, we have

(a) $$(AP)(P^TA^+)(AP) = AP$$

(b) $P^TA^+(AP)P^TA^+ = P^TA^+$$

(c) $$[(AP)(P^TA^+)]^T = (AA^+)^T = AA^+ = (AP)(P^TA^+)$$

(d) $$[(P^TA^+)(AP)]^T = \left[ P^T(A^+A)P \right]^T$$

$$= P^T(A^+A)^TP = P^T(A^+A)P$$

$$= (P^TA^+)(AP).$$

Theorem II

If $P$ is a permutation matrix and the operation $AP$ effects a column permutation of $A$, then $P^TA$ effects that same permutation on the rows of $A$.

Proof: Suppose one of the effects of $AP$ is to change column $i$ to the $j$th place. Then $P_{ij} = 1, P^T_{ji} = 1,$ and $P^TA$ changes row $i$ to the $j$th place.

We use this result to get $A^+$ from a row permutation of $[R|S]^+$ — that same permutation of columns which transformed $A$ into $[R|S]$.

The paper states (p. 383, right column) that the G.S. process turns a dependent vector into the zero vector. One might check this statement by referring
to Hoffmann and Kunze, p. 230, Theorem III (Ref. [12]). If $a_{k+1}$ is a linear combination of $a_1, \ldots, a_k$ then it is a linear combination of $q_1, \ldots, q_k$ since the vectors $q_i$ span the space of the vectors $a_i$. Furthermore, by the above-mentioned theorem,

$$a_{k+1} = \sum_{i=1}^{k} (a_{k+1}^H q_i) q_i$$

and $c_{k+1} = 0$.

On p. 384, left column we are to note that $I_{n-k}$ remains unchanged. Suppose we are operating on column $k+p$ ($p > 0$) of the matrix $[R|S]$. We have

$$c_{k+p} = a_{k+p} - \sum_{i=1}^{k+p-1} (a_{k+p}^H q_i) q_i$$

$$= a_{k+p} - \sum_{i=1}^{k} (a_{k+p}^H q_i) q_i - \sum_{i=k+1}^{k+p-1} (a_{k+p}^H q_i) q_i .$$

But each $q_i$, $k+1 \leq i \leq k+p-1$, has been zeroed out already, since they came from vectors dependent upon $a_1, \ldots, a_k$, so the above is

$$c_{k+p} = a_{k+p} - \sum_{i=1}^{k} (a_{k+p}^H q_i) q_i .$$

Similar column operations on the identity matrix then use only the first $k$ columns, whose lower $n-k$ entries are all zero and cannot contribute to any modification of $I_{n-k}$.
APPENDIX B

INITIALIZATION

BUMP COUNTER IN BY 1

ORTHOGONALIZE Nth COLUMN WITH RESPECT TO PREVIOUS COLUMNS

ZERO THIS COLUMN OUT

YES

RE-ORTHOGONALIZE FOR ACCURACY

NO

PUT ITS NUMBER IN HOLD

IS THIS COLUMN DEPENDENT?

NO

PUT ITS NUMBER IN HOLD

YES

WAS THAT THE LAST COLUMN?

NO

NORMALIZE ALL COLUMNS

YES

ARE ALL COLUMNS INDEPENDENT?

NO

PERMUTE A TO GET [Q D] FORM. PERMUTE FACTOR TO WATCH

FORM Q^-1 X TR

YES

ARE ALL COLUMNS INDEPENDENT?

NO

ORTHOGONALIZE LAST N-K COLUMNS OF A_1 TO GET [C P]

FORM I XP 207

AUGMENT ATR TO GET [Q^T I XP 207] ATR

FORM A X ATR = AXR

PERMUTE AXR TO AGREE WITH ORIGINAL

WRITE RESULTS AND CHECK

STOP
APPENDIX C

DIMENSION A(10,10), FACTOR(10,10), S2(10,10), IHOLD(10), JHOLD(10),
1A1(10,10), ORIG(10,10), PROD1(10,10), PROD2(10,10),
2PROD2(10,10), ATR(10,10), AINV(10,10)
DIMENSION UPRTR(10,10)
DIMENSION AINV1(10,10)
DATA PRA1/5HAINV1/ DATA PRAT2/2FACT, PRS2/1HA, 4FACT, 2MS2/
DATA PRA1/2HA1/ DATA PRAINV/4 HA1NV/ DATA PROPR1/PRF1, PRP12, PRATK/4HUK1G, 4HPROD, 5HPROD1, 5HPROD2,
13HATR/
DATA PRUPTR/4HLPTR/

154 READ(5,1) NROWS, NCOLS
1 FORMAT(215)
2 DO 112 I = 1, NROWS
112 READ(5,2) (A(I,J), J = 1, NCOLS)
2 FORMAT(6E12.8)
3 TOL = (10.**0.5**27)**2
4 DO 110 I = 1, NROWS
5 DO 110 J = 1, NCOLS
110 ORIG(I,J) = A(I,J)
DO 100 I = 1, NCOLS
100 I = 1, NCOLS
JHOLD(I) = 0
DO 102 J = 1, NCOLS
102 A1(I,J) = 0.
100 A1(I,J) = 1.
JHOLD(1) = 1
KK = 1
JJ = 0
II = 0
N = 1
152 NLESS1 = N
N = N + 1
CHECK = DOT(A,N,N,NROWS)
DO 101 I = 1, NLESS1
101 FACTOR(N,I) = DOT(A,N,I,NROWS)/DOT(A,I,I,NROWS)
DO 101 J = 1, NROWS
A(I,J) = A(I,J) - FACTOR(N,I)*A(I,J)
CHECK = DOT(A,N,N,NROWS)/CHECK
IF (CHECK - TOL) 150, 150, 151
150 DO 103 J = 1, NROWS
103  A(J,N) = 0.
    JJ = JJ + 1
    IHOLD(JJ) = N
    GO TO 155

151  DO 104 I = 1,NLESS1
    FACTOR(I,N) = DOT(A,N,1,NROWS)/DOT(A,1,1,NROWS)
    DO 104 J = 1,NROWS

104  A(J,N) = A(J,N) - FACTOR(I,N)*A(J,I)
    KK = KK + 1
    JHOLD(KK) = N

155  IF(N - NCOLS) 152,153,153

153  DO 105 J = 1,NCOLS
    FACTOR(J,J) = SQRT(DOT(A,J,J,NROWS))
    IF(FACTOR(J,J) .EQ. 0.) GO TO 105

    DO 106 K = 1,NROWS

106  A(K,J) = A(K,J)/FACTOR(J,J)

165  CONTINUE
    CALL WRITE(FACTOR,NCOLS,NCOLS,PRFACT)
    CALL WRITE(A,NROWS,NCOLS,PRA)
    IF(KK .EQ. NCOLS) GO TO 156
    DO 120 I = 1,KK
    ISUB = JHOLD(I)
    DO 120 J = 1,NCOLS
        A(J,I) = A(J,ISUB)

120  S2(J,I) = FACTOR(J,ISUB)
    DO 121 I = 1,KK
    ISUB = JHOLD(I)
    DO 121 J = 1,NCOLS
        FACTOR(I,J) = S2(ISUB,J)
    KK = KK + 1
    DO 125 I = KK,NCOLS
        II = II + 1
        ISUB2 = IHOLD(II)
        DO 125 J = 1,NCOLS
            FACTOR(I,J) = S2(ISUB2,J)
    DO 162 I = KK,NCOLS
    DO 162 J = 1,NROWS

162  A(J,I) = 0.
    KK = KK - 1

156  CALL WRITE(A,NROWS,NCOLS,PRA)
    CALL WRITE(FACTOR,NCOLS,NCOLS,PRFACT)
    CALL WRITE(A1,NCOLS,NCOLS,PRA)
    DO 502 I = 2,NCOLS
    ILESS1 = I - 1
    DO 500 J = 1,ILESS1

14
DO 500 K = 1, NROWS
A1(K,J) = A1(K,J) - FACTOR(I,J) * A1(K,J)
DO 500 J = 1, ILESS1
DO 502 K = 1, NROWS
A1(K,J) = A1(K,J) - FACTOR(J,J) * A1(K,J)
DO 501 I = 1, KK
DO 501 J = 1, NROWS
A1(J,J) = A1(J,J) / FACTOR(I,I)
CALL WRITE(A1, NCOLS, NCOLS, PRA1)
CALL TRANSPI(A, NROWS, KK, ATR)
CALL WRITE(ATR, NCOLS, NROWS, PRA1)
IF(KK .EQ. NCOLS) GO TO 157
IST = KK + 1
IF(KK .EQ. (NCOLS-1)) GO TO 158
N = KK + 1
      NLESS1 = N
      N = N + 1
DO 107 I = IST, NLESS1
      FACTOR(N,J) = DOT(A1, N, NCOLS) / DOT(A1, J, J, NCOLS)
DO 107 J = 1, NCOLS
DO 108 I = IST, NLESS1
      FACTOR(I,N) = DOT(A1, N, NCOLS) / DOT(A1, I, I, NCOLS)
DO 108 J = 1, NCOLS
IF(N .LT. NCOLS) GO TO 159
CALL WRITE(A1, NCOLS, NCOLS, PRA1)
      N = N + 1
      N = N + 1
DO 128 I = IST, NCOLS
      FACTOR(I,J) = SORT(DOT(A1, I, J, NCOLS))
DO 128 J = 1, NCOLS
A1(J,J) = A1(J,J) / FACTOR(I,J)
CALL WRITE(A1, NCOLS, NCOLS, PRA1)
CALL MATMPY(A1, KK, KK, ATR, NROWS, PROD1)
CALL WRITE(PROD1, KK, NROWS, PRPR1)
LIM = NCOLS - KK
DO 122 I = 1, LIM
      ISUB1 = KK + I
DO 122 J = 1, NCOLS
UPTR(I,J) = -A1(J,ISUB1)
CALL WRITE(UPTR, LIM, NCOLS, PRUPTR)
CALL MATMPY(UPTR, LIM, KK, PROD1, NROWS, PROD2)
CALL WRITE(PROD2, LIM, NROWS, PRPR2)
DO 123 I = 1, LIM
      ISUB3 = KK + I
DO 123 J = 1, NROWS
123  ATR(ISUB3+J) = PROD2(I,J)
    CALL WRITE(ATR,NCOLS,NROWS,PRATR)
157  CALL MATMPY(AL,NCOLS,NCOLS,ATR,NROWS,AINV).
    CALL WRITE(AINV,NCOLS,NROWS,PRAINV)
    CALL WRITE(ORIG,NROWS,NCOLS,PRORIG)
    DO 126 I = 1, KK
        ISUB4 = JHOLD(I)
    DO 126 J = 1, NROWS
126  AINV(I,ISUB4,J) = AINV(I,J)
    IF (KK .EQ. NCOLS) GO TO 160
    KK = KK + 1
    II = 0
    DO 127 I = KK, NCOLS
        II = II + 1
        ISUB5 = IHOLD(II)
    DO 127 J = 1, NROWS
127  AINV(I,ISUB5,J) = AINV(I,J)
    KK = KK - 1
160  CALL WRITE(AINV1,NCOLS,NROWS,PRAIN1)
    CALL MATMPY(ORIG,NROWS,NCOLS,AINV1,NROWS,PROD)
    CALL WRITE(PROD,NROWS,NCOLS,PRPR)
    CALL MATMPY(PROD,NROWS,NCOLS,ORIG,NCOLS,PROD2)
    CALL WRITE(PROD2,NROWS,NCOLS,PRPR2)
    CALL MATMPY(AINV1,NCOLS,NROWS,PROD,NROWS,PROD2)
    CALL WRITE(PROD2,NCOLS,NCOLS,PRPR2)
    CALL MATMPY(AINV1,NCOLS,NROWS,ORIG,NCOLS,PROD)
    CALL WRITE(PROD,NCOLS,NCOLS,PRPR)
    GO TO 154
END

FUNCTION DOT (A,J,K,NROWS)
DIMENSION A(10,10)
    DOT = 0.
    DO 10 I = 1, NROWS
10   DOT = DOT + A(I,J) * A(I,K)
RETURN
END

SUBROUTINE WRITE(X,NROWS,NCOLS,NAME)
DIMENSION X(10,10)
WRITE(6,3) NAME
3 FORMAT(1HO///7H MATRIX,1A7)
    DO 21 I = 1, NROWS
21   WRITE(6,4) (X(I,J), J = 1, NCOLS)
SUBROUTINE MATMPS(A,NRA,NCA,B,NCA,PROD)
DIMENSION A(10,10),B(10,10),PROD(10,10)
DO 600 I = 1,NRA
    DO 600 J = 1,NCA
        PROD(I,J) = 0
        DO 600 K = 1,NCA
            PROD(I,J) = PROD(I,J) + A(I,K) * B(K,J)
600 CONTINUE
RETURN
END

SUBROUTINE TRANS(A,NRA,NCA,ATR)
DIMENSION A(10,10),ATR(10,10)
DO 601 I = 1,NRA
    DO 601 J = 1,NCA
        ATR(J,I) = A(I,J)
601 CONTINUE
RETURN
END
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