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RECURSION FORMULAS FOR THE COEFFICIENTS OF THE f AND g SERIES

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If the xy-plane coincides with the plane of motion and if the positive x-axis is along the position vector $\bar{r}_0$ at time t_0, the two-body equations can be written in the form

$$x\dot{y} - y\dot{x} = \dot{y}_0 r_0 , \tag{1}$$

$$x\dot{x} + y\dot{y} = \dot{r}_0 x + \left(\dot{y}_0 - \mu/\dot{y}_0 r_0\right) y , \tag{2}$$

$$x(t_0) = x_0 = r_0 , \qquad y(t_0) = y_0 = 0 ,$$

where μ is a constant and r is the magnitude of $\bar{r}$. Zero subscripts designate values at time t_0, and a dot over a symbol indicates a time derivative.

Equation 1 expresses the constancy of angular momentum and Equation 2 is obtained by multiplying through by r the equation

$$\begin{aligned} \dot{r} &= (\mu e \sin\theta)/h \\ &= \left[\left(\mu e \sin\theta_0\right)/h\right] \cos\left(\theta - \theta_0\right) + \left[\left(\mu e \cos\theta_0\right)/h\right] \sin\left(\theta - \theta_0\right) , \end{aligned}$$

where θ is the true anomaly, e the eccentricity, and h the magnitude of the angular momentum.

In the reference frame described above, the equation

$$\bar{r} = f\bar{r}_0 + g\dot{\bar{r}}_0 \tag{3}$$

reduces to

$$x = f r_0 + g\dot{r}_0 , \tag{4}$$

$$y = g\dot{y}_0 . \tag{5}$$

Equation 2 is not valid when $\dot{y}_0 = 0$, i.e., for the rectilinear cases. Substituting Equation 5, however, into Equations 1 and 2,

$$x\dot{g} - g\dot{x} = r_0 \ , \tag{6}$$

$$x\dot{x} + bg\dot{g} = \dot{r}_0 x + cg \ , \tag{7}$$

$$b = \dot{y}_0^2 = v_0^2 - \dot{r}_0^2 \ , \qquad c = b - \mu/r_0 \ , \qquad v_0 = \left|\dot{\vec{r}}_0\right| \ ,$$

$$x_0 = r_0 \ , \qquad g_0 = 0 \ . \tag{8}$$

Substitution of

$$x = \sum_{i=0}^{\infty} a_i \left(t - t_0\right)^i \ , \qquad g = \sum_{i=0}^{\infty} b_i \left(t - t_0\right)^i \tag{9}$$

into Equations 6 and 7 with $a_0 = r_0$, $a_1 = \dot{r}_0$, $b_0 = 0$, and $b_1 = 1$ gives, for $i \geq 1$,

$$r_0 \left(i + 1\right) a_{i+1} = a_1 a_i + cb_i - \sum_{j=0}^{i-1} \left(j + 1\right) \left(a_{j+1} a_{i-j} + bb_{j+1} b_{i-j}\right) \ , \tag{10}$$

$$r_0 \left(i + 1\right) b_{i+1} = \sum_{j=0}^{i-1} \left(j + 1\right) \left(a_{j+1} b_{i-j} - b_{j+1} a_{i-j}\right) \ . \tag{11}$$

Taking advantage of symmetry in the summations, Equations 10 and 11 become, for $i \geq 2$,

$$r_0 a_{i+1} = \left(a_1 a_i + cb_i\right)\Big/\left(i + 1\right) - \sum_{j=0}^{k} a_{j+1} a_{i-j} - b \sum_{j=0}^{k} b_{j+1} b_{i-j} - q\left(a_{k+2}^2 + bb_{k+2}^2\right) , \tag{12}$$

$$r_0 (i+1) b_{i+1} = \sum_{j=0}^{k} \left[(i-j) - (j+1) \right] \left(b_{j+1} a_{i-j} - a_{j+1} b_{i-j} \right) , \quad (13)$$

$$k = \text{integral part of } (\tfrac{1}{2})(i-2),$$

$$q = \text{fractional part of } (\tfrac{1}{2})(i-2),$$

$$a_2 = -\mu/2r_0^2 , \ b_2 = 0$$

Having $x(t)$ and $g(t)$, f can be computed from Equation 4 and $\bar{r}(t)$ from Equation 3. Inversion of Equations 6 and 7 yields

$$r^2 \dot{x} = \left(\dot{r}_0 x + cg \right) x - br_0 g , \quad (14)$$

$$r^2 \dot{g} = r_0 x + \left(\dot{r}_0 x + cg \right) g , \quad (15)$$

where

$$r^2 = x^2 + bg^2$$

Then $\dot{f}(t)$ and $\dot{\bar{r}}(t)$ are obtained from

$$\dot{f} = \left(\dot{x} - \dot{r}_0 \dot{g} \right) / r_0 ,$$

$$\dot{\bar{r}} = \dot{f} \bar{r}_0 + \dot{g} \dot{\bar{r}}_0 .$$

For purposes of numerical control it may be advisable to let

$$t - t_0 = h\tau , \qquad h = t_1 - t_0 ,$$

where $\left[t_0 , t_1 \right]$ is a time interval of interest. The corresponding interval for τ is $[0, 1]$. Equations 12 and 13 still hold provided $a_0 = r_0$, $b_0 = b_2 = 0$,

$a_1 = \dot{r}_0 h, \; b_1 = h, \; a_2 = -\mu h^2/2r_0{}^2, \; c = (b - \mu/r_0)\, h.$
Equations 9 become

$$x = \sum_{i=0}^{\infty} a_i \tau^i , \qquad g = \sum_{i=0}^{\infty} b_i \tau^i ,$$

which offer a further advantage when $t = t_1$, i.e., when $\tau = 1$.

Formulas for the radii of convergence of the f and g series are given by Moulton.[1]

REFERENCES

1. Moulton, F. R., "The True Radii of Convergence of the Expressions for the Ratios of the Triangles When Developed as Power Series in the Time Intervals," The Astronomical Journal 23, 93-102 (1903).