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APPROXIMATIONS TO OPTIMAL NON-LINEAR FILTERS

by

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ABSTRACT

Let the signal and noise processes be given as solutions to non-linear stochastic differential equations. The optimal filter for the problem, derived elsewhere, is usually infinite dimensional. Several methods of obtaining possibly useful finite dimensional approximations are considered here, and some of the special problems of simulation are discussed. The numerical results indicate a number of useful features of the approximating filters and suggest methods of improvement. The paper is concerned with problems where the noise and non-linear effects are much too large for the use of 'linearization' methods, (which at least for the simulated problem, was useless).

1. INTRODUCTION. This paper is concerned with the problem of finite-dimensional approximations of optimal filters for a large class of non-linear, large-noise and continuous time problems. The truly optimal filters, as given in [1],[2] are infinite dimensional except for a few special cases (e.g.; the usual Gaussian-linear [3] problem and certain problems where the state space of the signal process has only finitely many points [4],[5]). A physical realization of the exact non-linear filter is impossible. The paper discusses an inter esting approach to a finite approximation, which seems to preserve some of the important qualitative properties of the optimal filter. Using some qualitative properties of the optimal filter, the procedure gives canonical forms for the approximations. A certain amount of experimentation and guesswork is required to select a specific filter.

The approximations are hard to evaluate theoretically, but versions are compared numerically. Although a numerical comparison of statistical filters does not always provide enough information to determine the 'better' one,it may point out some of the possible sources of difficulty, methods of improvement, or desirable properties.

The filters, as described herein, are probably far from ideal. It must be emphasized that in the problems considered the interaction between the non-linearities and the noise is sufficiently important for 'linearization' to be useless (or so it appears from the simulations). The filters can undoubtly be improved, but this appears to be the first work in which a systematic attack is attempted

for the class of signal and noise processes which are considered. As shown by the numerical results, the filters do seem to 'work', and it is indeed unfortunate that there are no other methods with which to compare the results. Given any specific filtering problem, the method is to experimentally (theoretically, where possible) compare several filters suggested by the procedure of the paper.

Since the development requires some published results, which, regrettably, are not sufficiently well known, backgroud material is given in Sections 2 and 3. The optimal filter is discussed in Section 4, and the approximations, etc., in Section 5 and on. It should be understood that the work is exploratory, hoping to help expose and clarify some of the specific and difficult problems in the realization of non-linear continuous time filters.

Notation. Throughout the paper time indices are omitted when no confusion should arise, and the same symbols are used for the approximations to conditional moments, and for the conditional moments themselves.

2. THE SYSTEM MODEL. The importance of state variable representation in control theory, suggests that the model for the signal process include at least special cases of the form

$$\dot{x} = f(x,\alpha) \tag{1a}$$

where α_t is a Markov process, say, a Gaussian process with

correlation $\exp-\beta|t|$. Such an α_t is the unique solution to the stochastic differential (Itô) equation

$$d\alpha = -\beta\alpha dt + dz \tag{1b}$$

where z_t is a Wiener process ($\dot{z}_t$ is formal Gaussian 'white noise').

The exact model is the stochastic differential equation

$$dx = f(x)dt + v(x)dz \tag{2}$$

where the components of z_t are <u>independent</u> Wiener processes and $E(z_t-z_s)(z_t-z_s)' = I|t-s|$. Equation (2) is meant to imply that x_t is a process satisfying

$$x_t-x_o = \int_o^t f(x_s)ds + \int_o^t v(x_s)dz_s \tag{2a}$$

where the latter (stochastic) integral, is given a precise interpertation by Itô [6],[7],[8]. The model (2) is extremely versatile; it is the subject of a large literature, and under broad conditions, the solutions are Markov processes. For more detail see [6],[7],[8], or the introduction to [9].

To anticipate questions in the sequel concerning the computation of the sample paths of equations such as (2), the following

brief and formal discussion of some known results is included. Define $\delta z_t = z_{t+\Delta} - z_t$ for fixed $\Delta > 0$. Define the process q_t^Δ which equals $q_{n\Delta}$ in the t interval $[n\Delta, (n+1)\Delta)$ where $q_o = x_o$ and

$$(3) \qquad q_{(n+1)\Delta} = q_{n\Delta} + f(q_{n\Delta})\Delta + v(q_{n\Delta})(z_{(n+1)\Delta} - z_{n\Delta}).$$

$q_t^\Delta \to x_t$ (in mean square) as $\Delta \to 0$. <u>If</u> v <u>depends on</u> x, then the limit (as $\Delta \to 0$) of r_t^Δ where, analogous to (3),

$$(4) \qquad r_{(n+1)\Delta} = r_{n\Delta} + f(r_{n\Delta})\,\Delta + v(r_{n\Delta})(z_{(n+1/2)\Delta} - z_{(n-1/2)\Delta})$$

is not necessarily the limit of q_t^Δ. The basic reason is that, <u>loosely speaking</u>, $\delta z_t = O(\sqrt{\Delta})$ $(E\delta z_t \delta z_t' = O(\Delta))$ and, hence $x_t = O(\sqrt{\Delta})$. In (3), $v(q_{n\Delta})$ is independent of its coefficient $\delta z_{n\Delta}$; but in (4), $v(r_{n\Delta})$ is not necessarily independent of its coefficient. In fact the expectation of the last terms of (3) and (4) are zero and $O(\Delta)$, respectively. These $O(\Delta)$ terms may add to give a significant difference between the limits of (3) and (4) (as $\Delta \to 0$). (See the excellent work in [10],[11],[12],[13],[14] for more details*.) (3) and (4) have the same limits (in mean square) for coefficients

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* Although only [10],[11] are rigorous, [12] and [13] are quite plausable.

given by (1).

Suppose now that the signal process is the scalar process y_t^n, where

$$(5) \qquad \dot{y}^n = f(y^n) + v(y^n)\xi^n$$

where ξ_t^n is a Gaussian process with a relatively flat spectrum in the set $(-n,n)$ and where $\int^t \xi_s^n ds \xrightarrow{n} z_t$ (in a suitable sense). (In this case, ξ_t is continuous, and y_t^n is the limit of either sequence (as $\Delta \to 0$) (3) or (4), where $\int^{n\Delta} \xi_s^n ds$ replaces $z_{n\Delta}$.) As $n \to \infty$, y_t^n does not necessarily converge to x_t if v depends on x. However, (see [10],[11],[12],[13], for details) there is another stochastic equation

$$(6) \qquad d\tilde{x} = (f(\tilde{x}) + v(\tilde{x})v_x(\tilde{x})/2)dt + v(\tilde{x})dz$$

so that $y_t^n \to \tilde{x}_t$. The term $vv_x/2$ in (6) accounts for the effects of the $o(\Delta)$ terms previously mentioned.

On the other hand, suppose that an approximation to a sample path of the stochastic differential equation (7) is desired

$$(7) \qquad dx = f(x)dt + B(x)dz$$

but, instead of z_t, only a wide (but finite) bandwidth Gaussian function ξ_s is available, where $\int^t \xi_s ds \approx z_t$. Then, the above

argument implies that the solution of

$$(8) \qquad \dot{y} = f(y) - B_y B/2 + B\xi$$

approximates the solution of (7), in a suitable statistical sense.

Now define the function q_t with values $(z_{(n+1)\Delta} - z_{n\Delta})/\Delta$ in the t interval $[n\Delta, (n+1)\Delta)$. Then the solution of the ordinary equation

$$(9) \qquad \dot{y} = f(y) - B_y B/2 + Bq$$

is an approximation to (7) and $E(x_t - y_t)^2 = o(\Delta)$ [13].

Now let z_t be a vector valued Wiener process and $\int^t \xi ds$ a band-width 'n' approximation to z_t; the components of each are supposed independent. Then [13] the solution of the ordinary equation

$$(10) \qquad \dot{y}_i = f_i(y) - \frac{1}{2} \sum_{j,k} B_{kj} \partial B_{ij}/\partial y_k + \sum_j B_{ij} \xi_j \ , \ i = 1, \ldots$$

is an approximation $(E(y_t - x_t)'(y_t - x_t) = O(1/n)$, [13]) to the solution of the stochastic equation

$$(11) \qquad dx_i = f_i(y)dt + \sum_j B_{ij} dz_j \ , \quad i = 1, \ldots \ .$$

For more detail, see the cited references. These results will be helpful in describing the numerical techniques required in the simulation of the approximate filters.

3. THE OBSERVATION MODEL. While the well known linear, continuous time, Kalman-Bucy [3] filter theory can handle a more practical class of signal processes than can the Wiener linear filter theory, the observation noise for the former is restricted to white Gaussian*,** (not necessarily stationary), while the observation noise for the latter** must only be stationary Gaussian. The restriction on the observation noise seems to be, at present, an unavoidable aspect of the state variable formulation. It is supposed here that the observation is the vector process given by

$$
\begin{aligned}
dy &= g(x,t)dt + \sigma dw \\
y_t &= \int_0^t g(x_s,s)ds + \int_0^t \sigma_s dw_s ,
\end{aligned}
\tag{12}
$$

where σ does not depend on x, $\sigma\sigma' = \Sigma$, a positive definite matrix at each t, and w_t is a vector of independent Wiener process; $E(w_t-w_s)(w_t-w_s)' = I|t-s|$. Formally $\dot{w}$ is white Gaussian noise.

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* 'whiteness' is always necessary for the exact optimal filter to be realizable with causal elements.

** Gaussianess is required for the filter output to be a version of the conditional mean, as opposed to being merely the best linear estimator in the mean square sense.

For simplicity, suppose that w_t is independent of z_t. As suggested by the remarks in Section 2, and in the section on computation, the filter may be modified to provide an approximation when $\dot{w}$ in (12) is replaced by a wide band Gaussian process.

4. THE OPTIMUM FILTER. Some previous results will be described briefly and formally, Write $E^t h(x_t)$ for the conditional expectation $E[h(x_t)|y_s, s \leq t]$ and, supposing that there is a conditional density, write $P(x,t)dx = P(x_t \text{ is in } [x,x+dx)|y_s, s \leq t)$. Write $\{v_{ij}\} = v'v = V$ and

$$L = \sum_i f_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} v_{ij} \frac{\partial^2}{\partial x_i \partial x_j} .$$

Then

$$(13) \qquad d(E^t h(x_t)) = E^t Lh(x_t)dt + (dy - E^t g dt)' \Sigma^{-1} (E^t gh - E^t g E^t h)$$

$$(14) \qquad dP(x,t) = L^* P(x,t)dt + P(x,t)(dy - E^t g dt)' \Sigma^{-1} (g - E^t g)$$

where $dy = gdt + \sigma dw$ and L^* is the formal adjoint of L;

$$L^* P = - \sum_i (f_i P)_{x_i} + \frac{1}{2} \sum_{i,j} (v_{ij} P)_{x_i x_j} .$$

Note that $dP = L^* P dt$ is Kolmogorov's forward equation for the

evolution of the density of (2). Under reasonable conditions on h, f, Σ, g and V, (13) is rigorously derived in [2]. (14) is formally derived in [1]. (13) implies that (a version of) the conditional expectation $E^t h$ satisfies a stochastic differential equation. (14) is a 'partial' stochastic differential equation*. The usual equations for the linear-Gaussian filter are a special case of (13).

Equation (15), derived from (13) and (14) in Appendix 1, will be useful. The version for scalar x and y is written, since it is to be used only for illustrative purposes. Until mentioned otherwise, define $m = m_t = E^t x_t$ and $m_{it} = m_i = E^t(x_t - m_t)^i$; then

$$dm = E^t f(x)dt + (dy - E^t g dt)(E^t gx - E^t g \cdot m)/\sigma^2$$

$$\begin{aligned} dm_i = {} & [-i m_{i-1} E^t f(x) + E^t L(x-m)^i]dt + \\ & + \frac{1}{\sigma^2}\left[\frac{i(i-1)}{2} m_{i-2}(E^t gx - E^t g \cdot m)^2 - i(E^t gx - E^t gm)E^t(g - E^t g)(x-m)^{i-1}\right]dt \\ & + \frac{(dy - E^t g dt)}{\sigma^2}\left[-i m_{i-1}(E^t gx - E^t gm) + E^t(g - E^t g)(x-m)^i\right]. \end{aligned} \tag{15}$$

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*The work [2] appears to be the first rigorous treatment of the continuous time non-linear filter. The work [1] was independent of the prior work [15], whose results are not consistent with the Itô calculus. See [12] for a recent discussion of this point. For other results, precise in a state space with only a finite number of points, and formal otherwise, see [4],[5],[12],[15],[16]. The mathematical work [17] discusses a related problem.

The operator L acts on x only.

At least formally (and rigorously under specified conditions [2]), both (14) and (13) represent the truly optimal non-linear filter. The sets (13) and (15)(for the scalar or vector case), seem easier to work with and we will concentrate on them.

5. THE PROBLEM OF PHYSICAL REALIZATION OF THE OPTIMAL FILTER.

Until mentioned otherwise, we let the system and observation be scalar. Also, suppose that the sample paths of x_t, given by $dx = f(x)dt+v(x)dz$, are computable (or suitably approximatable) if the sample paths (or suitable approximations) of z_t were available. (The latter supposition will be treated subsequently.) Let $g = x$, and $\Sigma = \sigma^2$. Then (15) becomes

$$dm = E^t f(x)dt + (dy-mdt)m_2/\sigma^2 \tag{16}$$

$$dm_2 = [2E^t(x-m)f(x)+E^t v^2-m_2^2/\sigma^2]dt+(dy-mdt)m_3/\sigma^2. \tag{17}$$

If f is linear, $P(x,0)$ Gaussian, and v^2 does not depend on x, then it is easily verified that $m_{3t} \equiv 0$ and (16),(17) reduce to the usual Kalman-Bucy filter for the scalar case with uncorrelated system and observation noise. Then, the only dependent variables in (16) and (17) are m and m_2, and the two equations can be solved for m_t and m_{2t}.

In general, (16) and (17) involve $E^t f(x)$, $E^t(x-m)f(x)$,

m_3 and $E^t v^2$. If f and v are polynomials, then the right sides of (16) and (17) contain $m_3, \ldots$ and, hence, cannot be solved for m_t and m_{2t}, unless $m_{3t}, \ldots$, are known. In the polynomial case, (15) becomes(still $g = x$ for simplicity)

$$(18) \qquad dm_i = F_i(m,\ldots,m_{i+N})dt + K_i(m,\ldots,m_{i+N})dt/\sigma^2 + (dy-mdt)G_i(m,\ldots,m_{i+N})/\sigma^2$$

where F_i depends on the dynamics of x_t, and the other terms are due to the observations; i.e., in general, the moments are 'given' by a <u>coupled</u> infinite system of stochastic differential equations. A further difficulty with (18) is that (unless $|x_t| \leq 1$ w.p.1.) $m_{it} \to \infty$ as $i \to \infty$. In any case, (18) cannot (unless it reduces to a finite problem, as in the linear-Gaussian case) be realized by a finite system.

The approximation of the first n equations of (18) by an n-dimensional system (in n variables, say $m, m_2, \ldots, m_n$, and the observation term $(dy-mdt)$), which may be termed the 'closure' problem*, is the subject of the sequel**.

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* The term is taken from certain branches of physics. Typically, a partial differential equation, such as Boltzman's equation, is given, and an infinite set of coupled differential equations for the moments are derived. Various schemes have been proposed for the truncation of the infinite system, while having the solution of the resulting finite system preserve some desirable property of the true moments. See [18],[19],[20]. Unfortunately, the techniques in these references shed little light on the filtering problem.

**There are a variety of expansions, based on either orthogonal systems -- or complete systems of functions and a Galerkin-like procedure. These appear to require far too large an approximate filter to be of use.

6. LINEARIZATION AND ITS SHORTCOMINGS. It appears that previously suggested approaches to the closure problem for various non-linear filters involve various forms of power series expansions and truncations [21],[22],[23]. Essentially, this is equivalent to setting $m_i \equiv 0$, for i greater than some n. A typical version of the procedure follows (for the scalar case of the last section and v^2 independent of x). Write

$$f(x) = f(m+(x-m)) \simeq f(m)+(x-m)f'(m)+\frac{1}{2}(x-m)^2 f''(m) \tag{19}$$

and $E^t f(x) \simeq f(m)+m_2 f''(m)/2$ (with or without the last term) and $E^t(x-m)f(x) \simeq m_2 f'(m)$, $m_3 = m_4 = \ldots = 0$. Then

$$\begin{aligned} dm &\simeq f(m)dt+m_2 f''(m)dt/2+(dy-mdt)m_2/\sigma^2 \\ dm_2 &\simeq (2m_2 f'(m)+v^2-m_2^2/\sigma^2)dt \ , \end{aligned} \tag{20}$$

the usual result of linearization (with or without the f'' term).

When f'' is 'small', $v^2 = 0$ and m_2 not large, then the error process x_t-m_t (due to (20)) may possibly converge to zero. Shortcomings of (20) are evident for truly non-linear problems; (19) is not

even a Taylor series with remainder (then the last term would be $(x-m)^2 f''(\xi)/2$, where $\xi \in [m,x]$ and a random variable), and it is easy to construct meaningful non-linear problems where (20) is useless (the example of Section 13).

A comparison of the structure of (16) and (20) is interesting. The properties of (20) are very sensitive to the properties of f at the current estimate m. If f is a function, such as $(1+\epsilon \sin bx)x$ for large b and small ϵ, then (for not sufficiently small m_2) (20) will exhibit 'large' undesirable oscillations*. In this case f'' is not negligable, and perhaps no formal power series and truncation will be useful. (20) is just too sensitive to f'' and f' to be of use except when f'' is small over a sufficiently large region. The linearization method is even more questionable if m is not sufficiently close to the conditional mode, or if the variance of the estimate is large, in which case the sensitivity of the filter to the local structure of f should be small.

Now examine (16). The differentiations in (20) are replaced by a local <u>smoothing or averaging of the dynamics about the</u>

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*The local structure of f in the example may not be typical, nevertheless, the concern here is with the general problems of a non-linear theory, which would, hopefully, be useful on problems which are not 'nearly' linear.

estimate. The terms

$$(21) \qquad f''(m)m_2/2 \quad , \quad m_2 f'(m)$$

are replaced by

$$(22) \qquad \int(f(x)-f(m))u(dx,t) \quad , \int(x-m)f(x)u(dx,t)$$

where $u(\cdot,t)$ is the conditional measure of x_t ($u(dx,t) = P(x,t)dx$ if a density exists). The smoothing operation (22) is opposed to the unsmoothing operation (21).

This distinction between smoothing (or averaging of dynamics), and unsmoothing is one of the salient distinctions between the exact and the 'linearized' procedures. The approximations of the sequel are devised to retain this property.

7. APPROXIMATION TO THE OPTIMAL FILTER 1: ASSUMED CONDITIONAL PROBABILITY DENSITY. Suppose the form (18) temporarily. Generally, a finite system approximation to the system (18) requires some sort of substitution $W_i(m,m_2,\ldots,m_n)$ for m_{n+i}, $i = 1,\ldots$. (Linearization sets $W_i = 0$.) Motivated by the previous remarks concerning the smoothing effect of the E^t operation in (18), several types of substitutions will now be suggested.

First pick an n parameter probability density $\tilde{P}(x,m,m_2,\ldots,m_n)$.

Then, suppose, arbitrarily, that at time t, $\tilde{P}$ is the conditional density of x_t. Then the terms $E^t g$, $E^t f$, $m_{n+1}, \ldots$ can be computed immediately from $\tilde{P}$, and $m_{n+i} = W_i\ (m, m_2, \ldots, m_n)$ for a known W_i. Under this supposition, either (13) and (15), (16) or (18) becomes a system of n equations in n unknowns, and can be solved. <u>Note that E^t is replaced by a local averaging operation.</u>

Set $n = 2$, consider equations for m and m_2 only, and let $\tilde{P}(x,m,m_2) = (2\pi m_2)^{-1/2} \exp -(x-m)^2/2m_2$. Then odd moments m_{2n+1} are replaced by zero, m_4 by $3m_2^2$, m_6 by $15m_2^3$, etc.

The choice of $\tilde{P}$ is, of course, important, and a generally satisfactory algorithm for choosing it in any given problem is not yet available. Let $\tilde{E}^t$ correspond to integration with respect to $\tilde{P}$. $\tilde{P}$ should retain the important qualitative properties of the true P; for example, considering its method of use here, the qualitative effects of $\tilde{E}^t g$, etc., on the approximations, should be similar to those of $E^t g$, etc., on the true estimates. Actually, a primary concern was with $\tilde{P}$ forms that are easy to compute with, while still providing useful filters.

From another point of view, linearization is equivalent to the replacement of P(in E^t) by a function $\hat{P}$ which <u>is not a probability density</u>. While this may work in some 'nearly linear' cases, it could often be disastrous.

Appendix 2 contains some further discussion of the relation between the meaning of the estimates, and the form of $\tilde{P}$.

EXAMPLE. The general principle is illustrated by the trival case $\dot{x} = \cos x$, $g = x$ and where $\tilde{P}$ is the uniform density, $\tilde{P} = 1/2A$ in $[m+A, m-A]$ and 0 outside, where $A = \sqrt{3m_2}$. The exact equations are

$$dm = E^t \cos x dt + (dy - mdt) m_2/\sigma^2$$

$$dm_2 = [2E^t(x-m)\cos x - m_2^2/\sigma^2]dt + (dy - mdt)m_3/\sigma^2.$$

The approximations require $\tilde{E}^t$ replacing E^t and $m_3 = 0$ (by symmetry). Let the true initial m_2 be large, then, under reasonable conditions on $P(x,0)$, the immediate effects of the dynamics, represented by $E^t \cos x$, etc. are small compared to the effects of the observation on the estimates. $\tilde{E}^t$ accomplishes this, and, as the estimate of m_2 decreases, the effect of the dynamics on the estimate of m increases. For 'very small' m_2, the equations reduce to those of linearization.

8. A RESULT ON MOMENT SEQUENCES. The sequence $S_N = \{m, m_2, \ldots, m_N\}$ is said to be a moment sequence if there is a non-decreasing function $\alpha(t)$, with $\alpha(-\infty) = 0$, $\alpha(\infty) = 1$ and

$$m = \int x d\alpha(x) = 0$$

$$\int x^i d\alpha(x) = m_i .$$

Set $m = 0$ here. If $m \neq 0$, α is shifted to the right m units. Clearly, an aim of Section 7 is to replace m_{n+i} by functions W_i so that $m_2, \ldots, m_n, W_1, \ldots$ is a moment sequence. Given S_N, or S_∞, the question of the existence of such an $\alpha(x)$ is the classical Hamburger moment problem [24],[25]. Necessary and sufficient conditions on S_N or S_∞, are [24]

$$\sum_{i,j} m_{i+j} \xi_i \overline{\xi_j} > 0 \tag{23}$$

for all ξ with positive norms ($\overline{\xi_j}$ is the complex conjugate of ξ_j). Alternatively, the positive definitness of (23) is implied if the principal minors of (24) are positive

$$\begin{bmatrix} 1 & 0 & m_2 & \cdots \\ 0 & m_2 & m_3 & \cdots \\ m_2 & m_3 & m_4 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \tag{24}$$

$$m_2 > 0, \quad m_4 > m_2^2, \quad m_6 > m_4^2/m_2, \ \ldots .$$

Replacing $>$ by $\geq$ in (23) yields an α which changes at possibly only a finite number of points.

9. APPROXIMATION TO THE OPTIMAL FILTER 2: MOMENT SEQUENCES. If $m_2, \ldots, m_n$ are given, then (24) gives conditions on $m_{n+1}, \ldots, m_N$

so that S_N is a moment sequence: (24) gives all allowable substitutions for $m_{i+1},\ldots,m_N$.

Except for a few special cases, it is hard to find useful forms $\tilde{P}$ which depend on only n moments. (24) characterizes all such $\tilde{P}$, and each moment sequence corresponds to some local smoothing. A method of choosing the proper sequences $m_{n+1},\ldots,$ is discussed in connection with the numerical example. Generally several members of a suitable family of sequences are compared by means of simulation, i.e., the method yields a family of filters which have the smoothing property, and one is selected experimentally.

10. <u>COMPUTATIONAL METHOD.</u> Let the filter, approximating the optimal filter (15) or (18), be

$$
\begin{aligned}
dm_i &= F_i(m,\ldots,m_n)dt+K_i(m,\ldots,m_n)dt/\sigma^2 \\
&\qquad + (dy-E(m,\ldots,m_n)dt)G_i(m,\ldots,m_n)/\sigma^2 \\
dm &= F_1dt+K_1dt/\sigma^2+(dy-Edt)G/\sigma^2 \quad ,
\end{aligned}
\tag{25}
$$

where $E = E(m,\ldots,m_n)$ is the approximation of $E^t g$, etc.

Since there does not appear to be a finite device which will generate either an exact Wiener process, or solve (25) exactly if such a process were available, two approaches to the numerical realization of the filter (25) are outlined. The first approach

supposes a type of discrete time filter; the second is essentially equivalent to the supposition that w_t is replaced by a suitable wide band process.

APPROACH 1. Suppose that observations are taken at the discrete times $t = n\Delta$, $n=1,\ldots,$ where Δ is small, and suppose that the observation δy can be approximated by the expression $\sigma\delta w_{n\Delta} + g(x_{n\Delta})\Delta$, or by $\int_{n\Delta}^{(n+1)\Delta} (\sigma_s dw_s + g(x_s)ds)$. Divide the computation of (25) into 2 parts. In the time between observations, the dynamics alone are involved and $dm_i = F_i dt$, $dm = F_1 dt$. To compute $m_{i,n\Delta+0} - m_{i,n\Delta-0}$, the change in the moment due to the observation at $n\Delta$, use the discrete approximation to (25)

$$(26) \qquad m_{i,n\Delta+0} - m_{i,n\Delta-0} = \frac{K_i\Delta}{\sigma^2} + (\delta y_{(n-1)\Delta} - E\,\Delta)G_i/\sigma^2$$

where the arguments of K_i, E and G_i are evaluated at $n\Delta-0$. If σ^2 depends on s, replace σ^2 by $\frac{1}{\Delta}\int_{(n-1)\Delta}^{n\Delta}\sigma^2(s)ds$ in (26). The procedure is consistent with the interpretation of (25) based on the discussion of the limit properties of (3), and will converge to the correct solutions of (25) (in some statistical sense) as $\Delta \to 0$.

APPROACH 2. Suppose, temporarily, that $g(x_t)$ is available, and that an approximation to the sample paths of (25) is desired when the values of $w_{n\Delta}$, $n = 1,\ldots,$ are available. Write (25) as

$$(27) \qquad dm_i = F_i dt + K_i dt/\sigma^2 + (g(x_t)dt - Edt + \sigma dw)G_i/\sigma^2$$

and similarly for dm. By those results of [10],[11],[12],[13] which are presented in Section 2 (here $B = [G_1/\sigma,\ldots,G_n/\sigma]'$), the ordinary equation (28) is an approximation to (27). q_t is equal to $[w_{(n+1)\Delta} - w_{n\Delta}]/\Delta$ in the interval $[n\Delta, (n+1)\Delta)$.

$$(28) \qquad \dot{m}_i = F_i + K_i/\sigma^2 - \frac{1}{2\sigma^2}\sum_k G_k \frac{\partial G_i}{\partial m_k} + (g - E + \sigma q)G_i/\sigma^2.$$

Finally, (28) is equivalent to the supposition that the observation is really $\dot{y}_t = g(x_t) + \sigma q_t$ for a wide band Gaussian process q_t. Approach 2 appears to be advantageous from the computational point of view and (the vector version) is used in the simulation.

More sophisticated, and hopefully better, numerical methods are being studied.

11. <u>A FILTER FOR THE VAN DER POL EQUATION</u>. The system of the simulation is the Van der Pol oscillater

$$(29) \qquad \begin{aligned} \dot{x}_1 &= x_2 \qquad\qquad \epsilon = 3 \\ \dot{x}_2 &= -x_1 + \epsilon x_2(1 - x_1^2) \\ dy &= x_1 dt + \sigma dw\ . \end{aligned}$$

<u>Define</u> $p_1 = E^t x_1$, $p_2 = E^t x_2$, $p_{ij} = E^t(x_i - p_i)(x_j - p_j)$, $p_{ijk\ell} = E^t(x_i - p_i)(x_j - p_j)(x_k - p_k)(x_\ell - p_\ell)$. For simplicity of design of the

filter, set <u>the estimates of</u> p_{ijk} <u>equal to zero</u>. Then, by (13),

$$
(30)\qquad
\begin{aligned}
dp_1 &= p_2 dt + (dy - p_1 dt) p_{11}/\sigma^2 \\
dp_2 &= (-p_1 + \epsilon p_2) dt - \epsilon(2p_{12}p_1 + p_{11}p_2 + p_1^2 p_2) dt + (dy - p_1 dt) p_{12}/\sigma^2
\end{aligned}
$$

$$
(31)\qquad
\begin{aligned}
\dot{p}_{11} &= -p_{11}^2/\sigma^2 + 2p_{12} \\
\dot{p}_{12} &= -p_{11}p_{12}/\sigma^2 + p_{22} - p_{11} - \epsilon[-p_{12} + p_{1112} + p_{12}p_1^2 + 2p_{11}p_1p_2] \\
\dot{p}_{22} &= -p_{12}^2/\sigma^2 - 2p_{12} - 2\epsilon[-p_{22} + p_{1122} + p_1^2 p_{22} + 2p_1p_2p_{12}]
\end{aligned}
$$

If p_{ijk} were not set equal to zero, then (31) would contain an observation term. (30) and (31) represent the filter and a substitution for p_{1112} and p_{1122} is required. (Note that, if a Gaussian $\tilde{P}$ were used, then $p_{1122} = p_{11}p_{22} + 2p_{12}^2$ and $p_{1112} = 3p_{11}p_{12}$.) Since (31) does not depend on the observation noise, and the coefficients of the noise in (30) depend on p_{ij}, then the vector version of (28) (see also (10),(11)) is exactly (30) and (31) with a division by dt and $\dot{y}$ replaced by $g + \sigma q$; i.e., no correction terns are needed if the observation noise is 'wide band'.

12. <u>TWO DIMENSIONAL MOMENT SEQUENCES</u>. The apparant lack of a general two dimensional analog of (23) and (24) suggests the use of

the limited class of two dimensional moment sequences which can be derived from (23),(24). We shoose a special subclass. Let y_1, y_2 be independent random variables with corresponding moment sequences $S_{y_1} = (u, u_1, \ldots)$ and $S_{y_2} = (v, v_2, \ldots)$. (23) and (24) hold for S_{y_1} and S_{y_2}. Define the distribution, and moment sequences of x by the orthogonal transformaion

$$(32) \qquad x = Ay = \begin{bmatrix} a & \sqrt{1-a^2} \\ -\sqrt{1-a^2} & a \end{bmatrix} y \quad , \quad |a| \leq 1 \quad .$$

Since it is supposed that $p_{ijk} = 0$, we may let $u_{2n+1} = v_{2n+1} = 0$, $n = 0, 1, \ldots$. From (32)

$$(33) \qquad \begin{aligned} p_{12} &= (v_2 - u_2) a \sqrt{1-a^2} \\ p_{11} &= a^2 u_2 + (1-a^2) v_2 \\ p_{22} &= (1-a^2) u_2 + a^2 v_2 \end{aligned}$$

$$(34) \qquad \begin{aligned} p_{1112} &= a^3 \sqrt{1-a^2}\, u_4 + 3a\sqrt{1-a^2}(2a^2-1) u_2 v_2 - a(1-a^2)^{3/2} v_4 \\ p_{1122} &= a^2(1-a^2) u_4 + [a^4 - 4a^2(1-a^2) + (1-a^2)^2] u_2 v_2 + a^2(1-a^2) v_4 \end{aligned}$$

To compute $p_{ijk\ell}$, for the filter (30),(31), make the admissible substitutions $u_4 = bu_2^2, v_4^2 = cv_2^2$, where $b \geq 1$, $c \geq 1$, determine the filter and are to be chosen. Via (33) and (34), p_{ij} and $p_{ijk\ell}$ are specified by 3 quantities, u_2, v_2 and a. Conversely,

given all p_{ij}, the u_2, v_2 and a can be computed, then u_4, v_4 can be calculated and, finally, p_{1112} and p_{1122} obtained.

Define $T = p_{11}+p_{22} = u_2+v_2$, $\Delta = p_{11}p_{22}-p_{12}^2 = u_2v_2$.

Then

$$u_2 = T + \sqrt{T^2-4\Delta}/2$$

$$v_2 = T-u_2$$

$$a^2 = (p_{11}-v_2)/(u_2-v_2)$$

$$\text{sign } a = \text{sign } (v_2-u_2)a\sqrt{1-a^2} = \text{sign } p_{12} .$$

Note that, if $u_2 = v_2$, then a is not important and $p_{11} = p_{22} = u_2 = v_2$, $p_{12} = 0$.

13. <u>NUMERICAL DATA. A REMARK ON THE METHOD</u>. The methods described in Sections 7 and 9 do not account for the initial density $P(x,0)$, which may be part of the problem data. Consistency with the given $P(x,0)$ requires that $\tilde{P}(x,0) = P(x,0)$, or that the chosen moment sequence coincides with that of $P(x,0)$ at $t = 0$. For the moment method, this implies that the b, c of Section 12 should be functions of time, whose initial values are determined by $P(x,0)$, and whose later values are chosen for quality of the filter. The procedure taken here is not consistent with these statements. The goal of the numerical work is the study of general qualitative properties

of the filters as they appear in the several computer runs, rather than to do a statistically correct evaluation of a filter.

The values of b, c are fixed for the entire run. The results stand by themselves, but if desired, it can be supposed that either $P(x,0)$ is consistent with the choice of b, c, or take the following 'practical' rationalization: Data on $P(x,0)$ may not be reliably known. Although the filters of Section 4 are optimal for a specific Baysian problem formulation, the Baysian approach can be considered as merely a suggestive device used to obtain a family of possibly useful filters for some non-Baysian problem, and the initial data adjusted for convenience of the filter. In any case, the filters must be checked under a variety of conditions.

DYNAMICS OF THE SIGNAL. The signal system paths are plotted in Figures 1,2 and the limit cycle in Figure 6. Roughly speaking, each half cycle has 2 parts; a nearly linear part in which the velocity is nearly constant at about .26, and a non-linear part in which the velocity changes by a large amount in a relatively short time. The non-linearity of the system, as reflected in these large velocity changes, helps explain some of the salient points of the data. Errors in the estimates change rapidly during this period*.

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*For example, the pair of initial conditions of the Van der Pol equation $(x_1 = 1.59,\ x_2 = -.32)$, $(x_1 = 1.7, x_2 = -.28)$ are mapped into the pair $(x_1 = .63, x_2 = -1.42), (x_1 = -.38, x_2 = 4.24)$, resp., at $t = 2$.

A 'linearized' filter was simulated by dropping the 4th moments in (31)*, but the estimates of p_i, p_{ij} were, in all cases, extremely unstable and were completely useless within a fraction of a unit of time.

DATA. All the runs for the conditions $p_{11}(0) = 5$, $p_{12}(0) = 0$, $p_{22}(0) = 20$, $\sigma^2 = 4$, $b = c = 3$ (corresponding to a Gaussian density) were rather similar and will be described in detail**. Note the relatively large noise (virtually masking the signal) and the large initial values of p_{ii}.

Refer to Figures 1-3. The large values of b and c, and corresponding large initial value of p_{1122}, caused p_{22} to decrease extremely rapidly at first. p_2 decreased rapidly to about 0.4 and remained there for about 4 units of time. Initially, the filter is essentially tracking a moving point which, it supposes, moves with a constant velocity. By the start of the second half cycle, the estimates are rather good, and the magnitude of the errors are consistent with the values of the p_{ij}. The initial velocity estimate of 0.4 is consistent with the estimate that the initial position is about 2 and is slowly decreasing. The signal point spends most time in the nearly linear region (where $|x_2| \approx .26$).

---

*This is equivalent to expanding the optimal filter and retaining only the first and second moments.

**This sequence of runs was generally the best.

By $t = 0.4$, the filter has decided that one of these nearly linear regions is a much more likely location of x_t than the other; the large rate of decrease in p_{22} is due in part to the fact that x_2 changes relatively little in the nearly linear region, and the initial estimate $p_2 \approx 0.4$ is very likely an effect of the averaging of the dynamics (a desirable effect). The averaging at this time should (it seems) suppress large changes in the velocity (p_2), yet follow the moving x_1 and catch up with it in the nearly linear regions. To do this, the required velocity changes (in p_2) must be spread out in time. See Figure 2.

The shift to the right, of the first peak of p_2, is possibly due to an initial conservatism; i.e., initially, p_{11} is large and the filter 'requires' more evidence concerning the change of the 'nearly linear' region in which x_1 is located before moving p_1 rapidly to a different region. This effect decreases in later cycles (provided that the estimates p_t are close to x_t)*.

When the estimate $|p_2|$ increases rapidly, the errors are presumed to increase, and the variances p_{ij} changes accordingly. When p_t changes rapidly, the observations contain more information on the values of x_t, since the observation component $x_1 - p_1$ is

---

*For smaller b,c the first peak of p_{2t} usually occurs to the left of that in Figure 2. As b,c decrease to 1, the corresponding density degenerates to one concentrated at 4 points (2 on y_1 axis, 2 on y_2 axis), and it appears that, for the same initial variance, the filter supposes that more information concerning the true location of x_t is available in the observations.

larger. The variances $|p_{ij}|$ all increase, during the periods of rapid increase in p_t, to take advantage of the added information in the observation (recall that the observation effects on p_i are proportional to p_{1i}). The depression at 8.2 in the p_{22} curve of Figure 3 is due to the fact that p_{2t} is close to a local maximum at this time (see also Figure 6, for the same run), and the rate of change of the estimate of velocity is near a local minimum; hence, the 'magnification' of estimate errors is near a local minimum (i.e., loosely speaking, two points on the phase space path of the estimate which are close at $t = 7.5$, are further apart at $t = 8$, are closer at $t = 8.3$, and spread again at $t = 8.6$, etc.). The two peaks of m_{22} in the interval $[7.5, 8.5]$ appear to be desirable (and intuitively expected in the optimum filter).

For t greater than about 0.5, and in all runs, $p_{12}^2 \approx p_{11}p_{22}$ (correlation coefficient ≈ 1). This is not completely understood, but is one of the most important features of the data. While not expected in the optimal filter, it does serve useful purposes. $p_{12}^2 \approx p_{11}p_{22}$ means

$$\left(\int (x_{1t}-p_{1t})(x_{2t}-p_{2t})P(x,t)dx\right)^2$$

$$\approx \int (x_{1t}-p_{1t})^2 P(x,t)dx \int (x_{2t}-p_{2t})^2 P(x,t)dx$$

which, implies that $(x_{1t}-p_{1t}) = k(x_{2t}-p_{2t})$; in particular, the

filter supposes that the probability is concentrated on a line (in phase space) through p_t with slope $(p_{22}/p_{11})^{1/2}$ sign p_{12}.

Figures 4,5 give data on another run for large time, and the same b,c and initial p_{ij} values. This run was not quite as successful as that of Figures 1,2,3,6, but illustrates some important properties of the filter. In Figure 4 there are plotted several pairs of corresponding x_t, p_t points of one cycle. The letters identify values of x_t, p_t at the same time. The arrows indicate the general direction (the direction of points on the graph is in the same quadrant as the true direction) of the effects of the bias term $p_{1i}(x_1-p_1)$ in the observation at the indicated times. The regions of positivity and negativity of p_{12} are also indicated. Although not plotted, the p_{ij} pattern of variation was not too much different from the pattern in the last part of Figure 3, so that the observations still do have a corrective effect. In Figure 5, parts of two cycles of the same run are plotted. The estimates on the last cycle are better, and the path of the estimate on the last cycle is closer to the limit cycle.

Putting Figures 4,5,6 together, the following picture emerges. The filter is initially conservative and averages the dynamics considerably. The path (in phase space) of the estimate (as a trend) is an outward spiral tending slowly to the limit cycle of the oscillator. At the nearly linear regions, the estimates are good, and degenerate when the velocity $|p_2|$ or $|x_2|$ increase rapidly. The observation bias $p_{ij}(x_1-p_1)$ serves as a corrective force. The change in the velocity p_2 is smaller than the change

in x_2, since when p_2 tends to change rapidly, the increased p_{ij} and consequent averaging of the dynamics holds the rate of change down. This conservatism decreases as time increases. Since $p_{11} > 0$, the instantaneous effect of the non-noise part of the observation, is to reduce the error (x_1-p_1).

Figure 4 illustrates the desirability of the change in sign of p_{12} at certain points, so that the bias term $(x_1-p_1)p_{12}$ has a corrective effect. Consider points d,e,f. At all three points, $x_1 > p_1$, but between e and f, the sign of (x_2-p_2) changes, and it <u>seems</u> reasonable that p_{12} should also change sign in this region, which it does.

The system is rather non-linear, and it is hard to compare the results to any absolute standard. (Also, there is no other filter, suitable for this problem, known to the author.) The asymptotic properties are unclear, although it is suspected that the estimates for the described runs would converge to the true values. Several runs were less successful, in particular for $\sigma^2 = 4$ and smaller values of b,c (for larger σ^2, the data suggest that $b = c = 3$ is not as good as some smaller values), but the general features described above were retained, except that the sign of p_{12} differed a little more often from that which would allow $p_{12}(x_1-p_1)$ to instantaneously decrease the error in the velocity estimate.

A number of specific directions for further investigation are suggested. The degeneration of the correlation $p_{12}^2/p_{11}p_{22}$ to

1 implies that only one of the y_i (Section 12) was non-zero, and suggests that the method of Section 12 be repeated for y_i with independent but skew distributions. The specification of non-zero 3rd moments for the y_i would allow the introduction of two independent 3rd moments into the filter, with the corresponding extra filter equations. In the event that the degeneracy of the correlation still obtained, the skewness would seem to provide for a more natural averaging of the dynamics. Also, the x_i (in Section 12) could be suitable non-linear functions of the y_i. An analysis of the qualitative asymptotic properties of the filter equations (29),(30),(31) could be attempted, but this seems rather hard. It is desirable to simulate another non-linear problem, and to obtain more data on the example of the paper in order to improve the understanding of the effects of various types of averaging assumptions. The relation between the quality of the filter and the sign of p_{12} must be further clarified for the example of the paper. A careful study of the evolution of the true moments, when there are no observations, should yield some useful guides. In fact, some computations of these moments have suggested that the properties of the covariance estimates are generally correct -- except for the possibility that at their 'low' points, they are too small. Finally, it would be quite helpful to have independent estimates of the properties of the optimal filter (e.g., useful bounds on the variance of the optimal p_i).

APPENDIX 1. DERIVATION OF (15). For brevity, a formal derivation is presented and is based on Itô's calculus for differentials of functions of solutions of stochastic differential equations. Let $dx = f(x)dt + u(x)dz$. Then, the stochastic differential of a suitable function $g(x)$ is

$$dg = g'_x dx + \frac{1}{2} \sum g_{x_i x_j} dx_i dx_j \tag{A-1}$$

where $dx_i dx_j$ is written for $E[dx_i dx_j | x_t] = \sum_k u_{ij} u_{kj} dt$ ($u = \{u_{ij}\}$). Then, w.p.1. with the Itô interpretation of the integral $\int^t f dz_s$, [7],[8] (see these references and also [6] for the definition of the stochastic integral $\int^t f dz_s$), Itô's Lemma yields

$$g(x_t) - g(x_o) = \int_o^t dg_s \ .$$

Define $m = E^t x_j$ and $m_i = E^t (x_j - m)^i$; j is fixed. Then, via a formal application of Itô's Lemma,

$$\begin{aligned} dm_i &= d \int (x_j - m)^i P(x,t) dx \\ &= -dm \int i(x_j - m)^{i-1} P(x,t) dx \\ &\quad + \frac{(dm)^2 i(i-1)}{2} \int (x_j - m)^{i-2} P(x,t) dx \\ &\quad + \int (x_j - m)^i dP(x,t) dx - i \int (x_j - m)^{i-1} dm dP(x,t) dx \ , \end{aligned} \tag{A-2}$$

where $(dm)^2$ and $dmdP$ are to be replaced by their expectations conditioned on m_t, $P(x,t)$. Recall that (13), (14),

$$\text{(A-3)}\quad \begin{aligned} dm &= dE^t x_j = E^t f_j(x)dt + (dy-E^t g dt)' \Sigma^{-1}(E^t g x_j - E^t g m) \\ dP &= L^*P dt + P(dy-E^t g dt)' \Sigma^{-1}(g-E^t g) \end{aligned}$$

and that, formally, for suitable $q(x)$,

$$\int q(x)L^*P(x,t)dx = \int (Lq(x))P(x,t)dx.$$

Substituting (A-3) into (A-2) gives

$$\text{(A-4)}\quad \begin{aligned} dm_i = dt\{&-im_{i-1}E^t f_j(x) + \frac{i(i-1)}{2} m_{i-2}(E^t g x_j - E^t g m)'\Sigma^{-1}(E^t g x_j - E^t g m) \\ &+ E^t L(x_j-m)^i - i(E^t g x_j - Egm)'\Sigma^{-1} E^t(g-E^t g)(x_j-m)^{i-1}\} \\ &+ (dy-E^t g dt)'\Sigma^{-1}\{-im_{i-1}(E^t g x_j - Egm) + E^t(g-E^t g)(x_j-m)^i\} . \end{aligned}$$

APPENDIX 2. DISCUSSION OF THE SIGNIFICANCE OF A SELECTED $\tilde{P}$.

A loose and formal description of an alternative and suggestive view of the significance of $\tilde{P}$ will be given for a scalar problem with no observations. The observation terms can be added to the procedure. The limits on all integrals are $\pm\infty$. x is a scalar and subscripts x, t denote differentation. Let

$$\text{(B-1)} \qquad P_t = L^*P \qquad\qquad L = (L^*)^* \quad , \; P(x,0) \text{ given}$$

where L is given in Section 4. Suppose that all $\varphi(x)$ of the sequel are such that the operations have meaning, and that $\tilde{P}(x,t)$ and $P(x,t)$ are such that all terms in (B-2) go to zero as $|x| \to \infty$.

$$\text{(B-2)} \qquad \begin{aligned} &P(x,t)f(x)\varphi(x) \\ &(P(x,t)v^2(x))_x\varphi(x) \\ &(P(x,t)v^2(x))\varphi_x(x) \; . \end{aligned}$$

Then if $P(x,t)$ is the solution to (B-1), for any $\varphi(x)$, (B-3) holds.

$$\text{(B-3)} \qquad \int P_t(x,t)\varphi(x)dx = \int (L^*P(x,t))\varphi(x)dx = \int P(x,t)L\varphi(x)dx$$

(B-3) implies that

$$\frac{d}{dt} E^t\varphi(x) = E^t L\varphi(x) \; .$$

Let $\{\varphi_i(x)\}$ be a complete (in a suitable sense) family.

Then under suitable conditions on $P(x,0)$ and L, the procedure of Galerkin [26], [27], can be used to obtain a sequence $P^n(x,t)$ converging to $P(x,t)$: Let $P^n(x,t) = \sum_1^n a_i^n(t)\varphi_i(x)$ and choose the $a_i^n(t)$ so that

$$\text{(B-4)} \qquad \int[P_t^n(x,t)-L^*P^n(x,t)]\varphi_i(x)dx = 0, \quad i = 1,\ldots,n .$$

(B-4) gives a set of linear differential equations for the $a_i^n(t)$, and the initial conditions of this set are determined by $P(x,0)$. A large n may be required for $P^n(x,t)$ to be 'close' to $P(x,t)$ but, in any case the 'error' $P_t^n-L^*P^n$ is orthogonal to $\varphi_1(x),\ldots,\varphi_n(x)$.

Now write $\tilde{P}$ for any approximation to P. At least formally, $\tilde{P} = \tilde{P}(x,m,r_2,\ldots,r_n)$ may be non-linearly parameterized where $m = \int x\tilde{P}(x,t)dx$ and r_i are other parameters (there are no exact results for the non-linear parameterization). Equations for $\dot{m}$ and $\dot{r}_i$ are obtained by imposing the condition that the error $\tilde{P}_t-L^*\tilde{P}$ be orthogonal to $\varphi_1(x),\ldots,\varphi_n(x)$, where $\varphi_1(x) = x$. Then

$$\text{(B-5)} \qquad 0 = \int[\tilde{P}_t-L^*\tilde{P}]\varphi_i dx = \int\tilde{P}_t\varphi_i dx - \int\tilde{P}(L\varphi)dx$$

or

$$\frac{d}{dt}\tilde{E}^t\varphi_i = \tilde{E}^t\varphi_i , \quad E^t\varphi_i = \int\tilde{P}\,\varphi_i dx .$$

If $\varphi_1 = x$ and $r_i = \tilde{E}^t\varphi_i$, $i = 2,\ldots,n$, then (B-5) gives equations

for the $\dot{r}_i$. Otherwise, the $\dot{r}_i$ are obtained from

$$\int \tilde{P}_t \varphi_i dx = \sum_j \dot{r}_j \int \tilde{P}_{r_j} \varphi_i dx + \dot{m} \int \tilde{P}_m \varphi_i dx \ .$$

For any given form $\tilde{P}(x,m,r_2,\ldots,r_n)$ there are many sets of $\{\varphi_i\}$, $\varphi_1 = x$, which can be used, and the equations for the r_i depend on the $\{\varphi_i\}$. For example, consider

$$\dot{x} = x^2 \quad , \ r_1 = m$$

$$\tilde{P} = \tilde{P}(x,r_1,r_2) = (2\pi r_2)^{-1/2} \exp-(x-r_1)^2/2r_2 \ .$$

Let $\varphi_1 = x$, $\varphi_2 = x^2$, then

$$\dot{r}_1 = r_2 + r_1^2 = \tilde{E}^t Lx$$

(B-7)

$$\dot{r}_2 = 4r_1 r_2 \ .$$

Here the error is orthogonal to x, x^2 and $(x-r_1)^2$. Now, let $\varphi_2 = x^4$, then

$$\dot{r}_1 = r_2 + r_1^2$$

(B-8)

$$\dot{r}_2 = 4r_1 r_2 [1 + \frac{r_2}{r_1^2 + r_2}]$$

Owing to the use of x^4 in lieu of x^2, the computation of (B-8) weighs the error at larger values of $|x|$ more heavily. The use of $\varphi_2 = (x-r_1)^4$ gives (B-7).

The derivation with the inclusion of observation terms, is similar. The author has not been successful at exploiting the possibilities offered by the various choices of $\{\varphi_i\}$. In the text $r_i = m_i$, $i \geq 2$ and the observation terms are included. The derived equations for the dm_i and dm are those which would be obtained by the above procedure (with observations) for $\varphi_1 = x$, $\varphi_i = x^i$, $i \geq 2$. The estimates m_t and m_{it} are such that the function $\tilde{P}_t - L^*\tilde{P}_t$ is orthogonal to x^i, $i = 1,\ldots,n$, for each t, where $\tilde{P} = \tilde{P}(x,m_t,\ldots,m_{nt})$ (the t subscript on m, m_i, denotes time.)

CONCLUSION

A class of finite dimensional approximations to the optimal filter has been discussed. An analysis would be difficult, but the procedure is suggestive and the numerical results indicate that some of the approximations have rather desirable properties. The paper is, in a sense, exploratory. Since there do not appear to be alternative filters for the non-linear problems of concern here, comparison of our results is impossible. Linearization appears to be useless for our problems ('linearized' filters 'blew' up). The problems of realizing useful non-linear filters for the class of signal and noise processes considered here are difficult, but the numerical results indicate that the methods (and problems) isolated here merit much further study.

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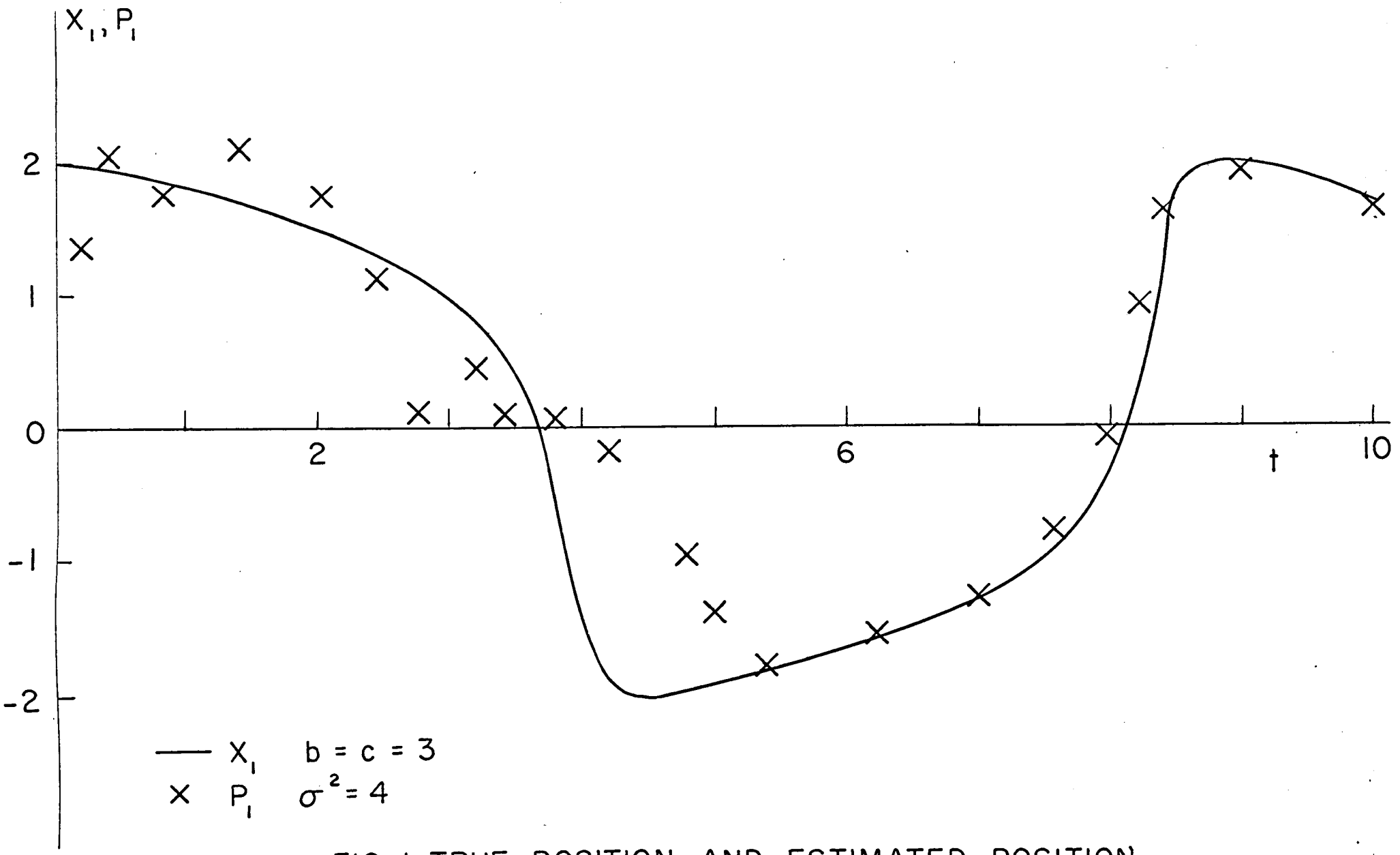


FIG. 1 TRUE POSITION AND ESTIMATED POSITION

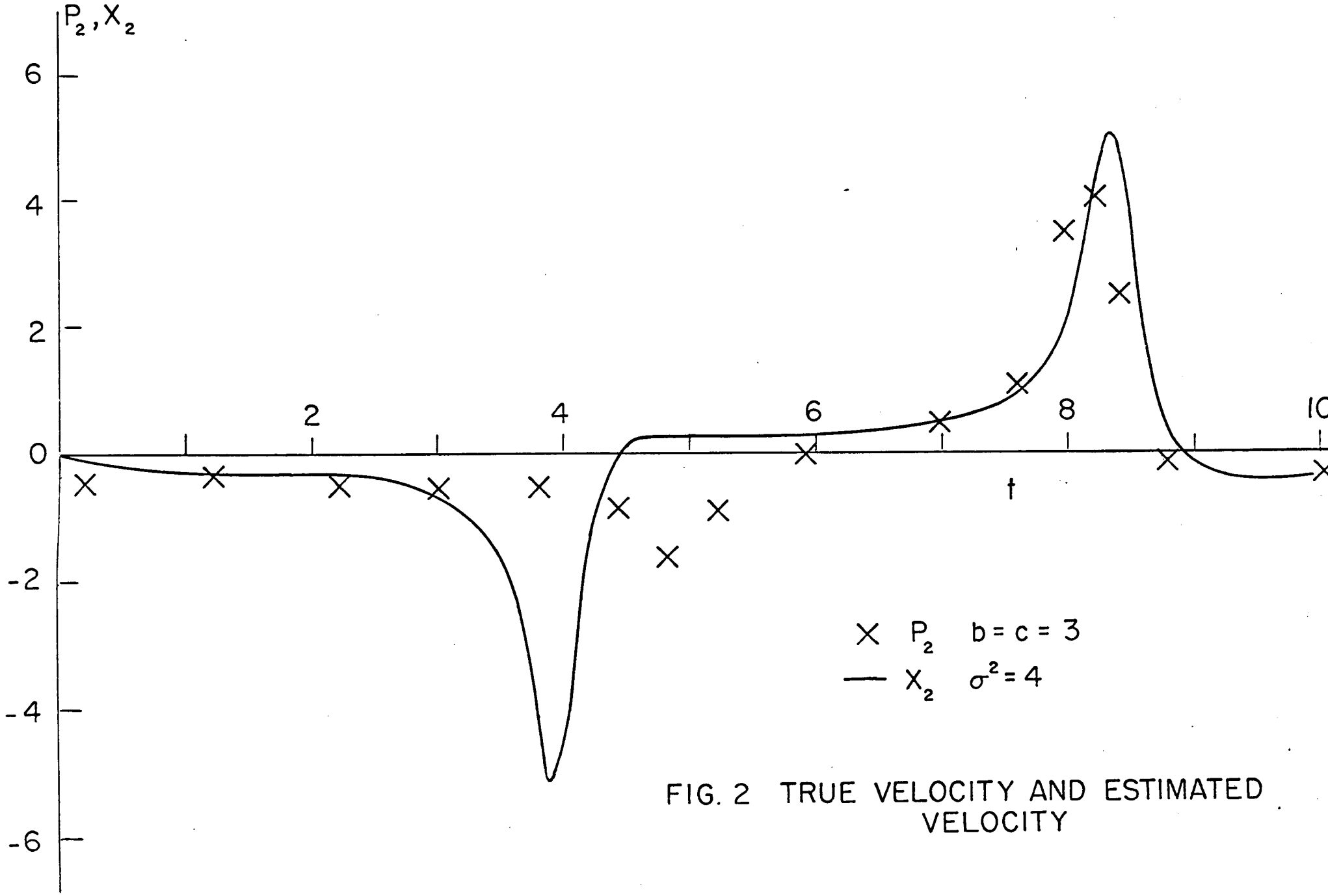
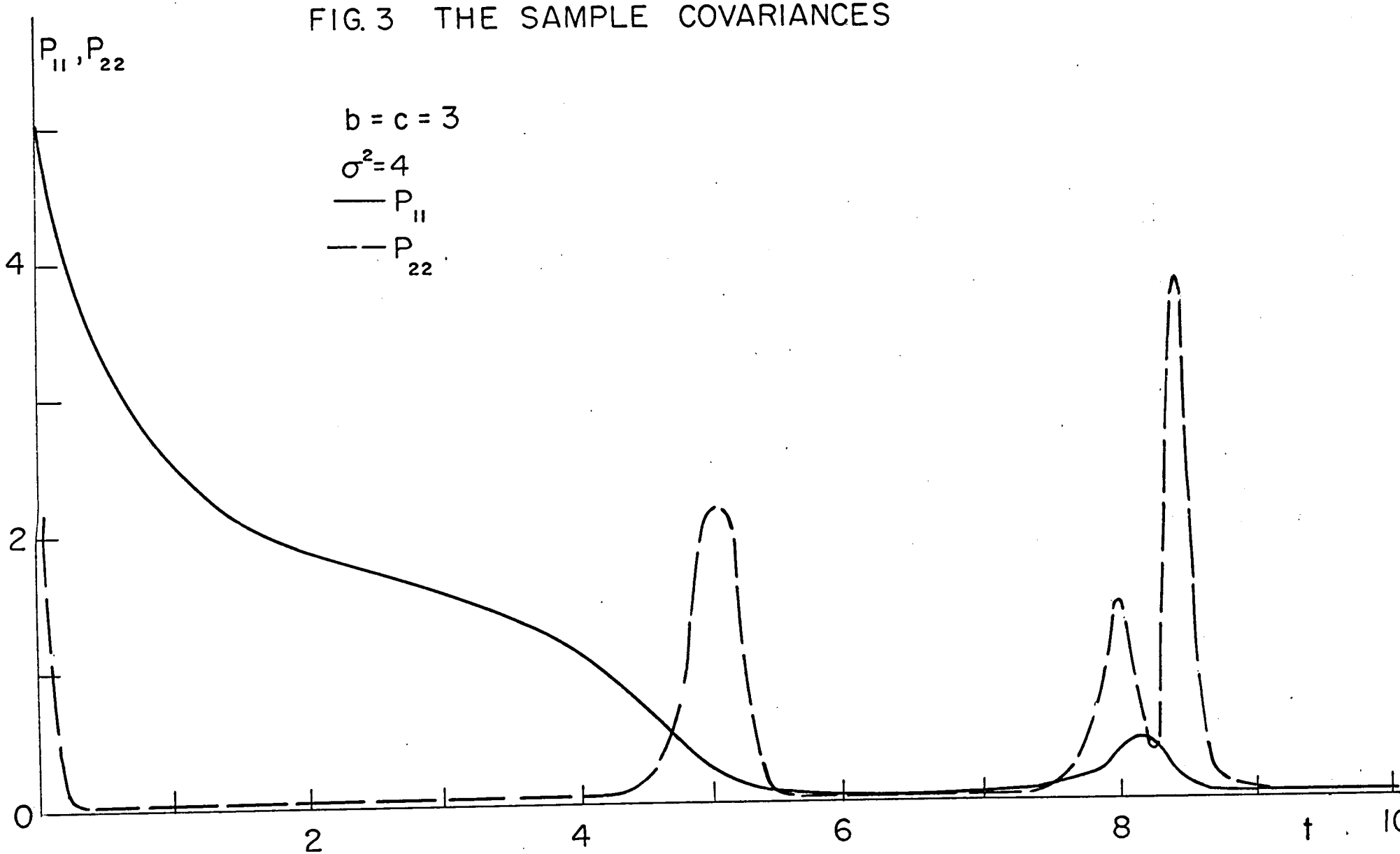


FIG. 2 TRUE VELOCITY AND ESTIMATED VELOCITY

FIG. 3 THE SAMPLE COVARIANCES



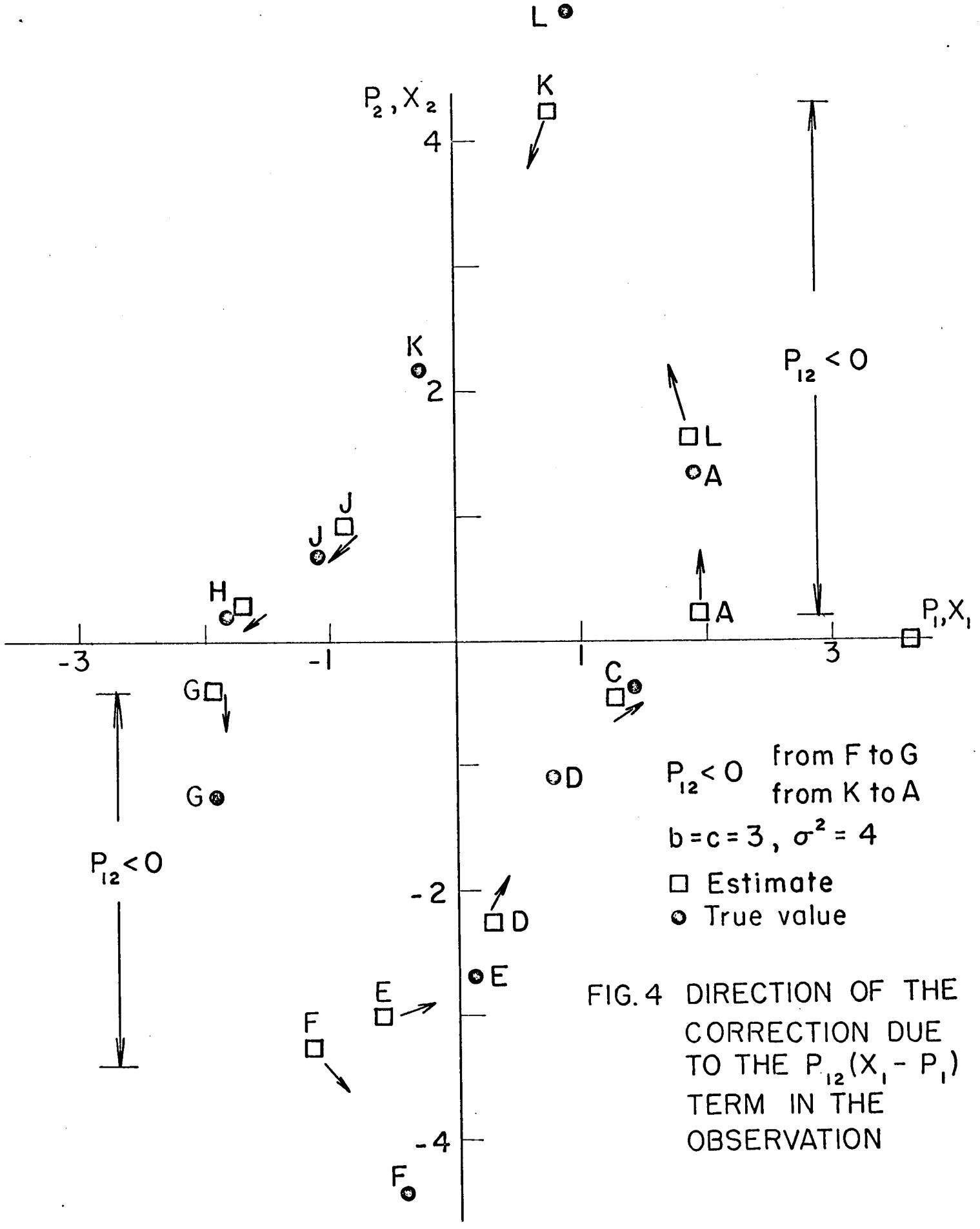


FIG. 4 DIRECTION OF THE CORRECTION DUE TO THE $P_{12}(X_1 - P_1)$ TERM IN THE OBSERVATION

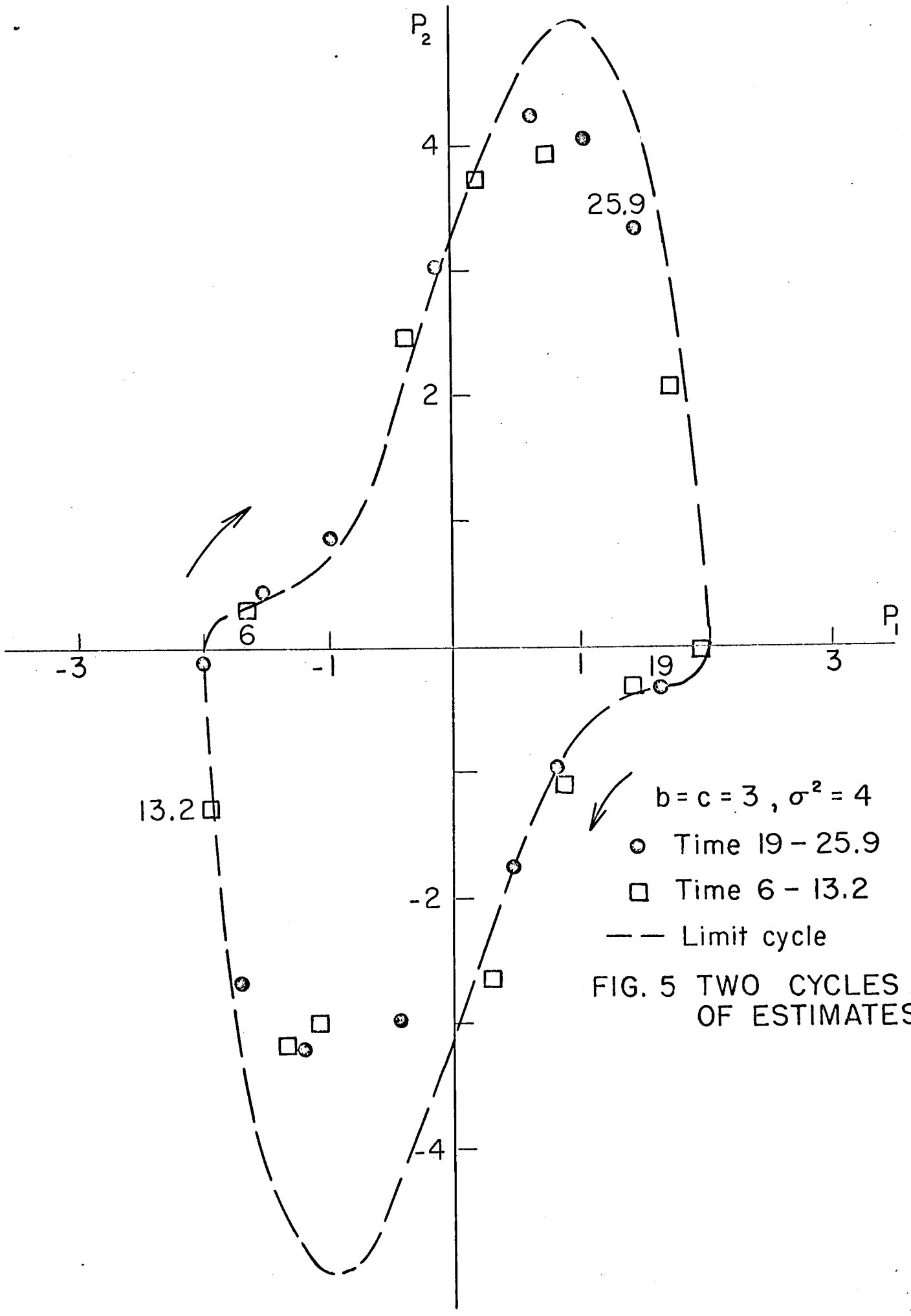


FIG. 5 TWO CYCLES OF ESTIMATES

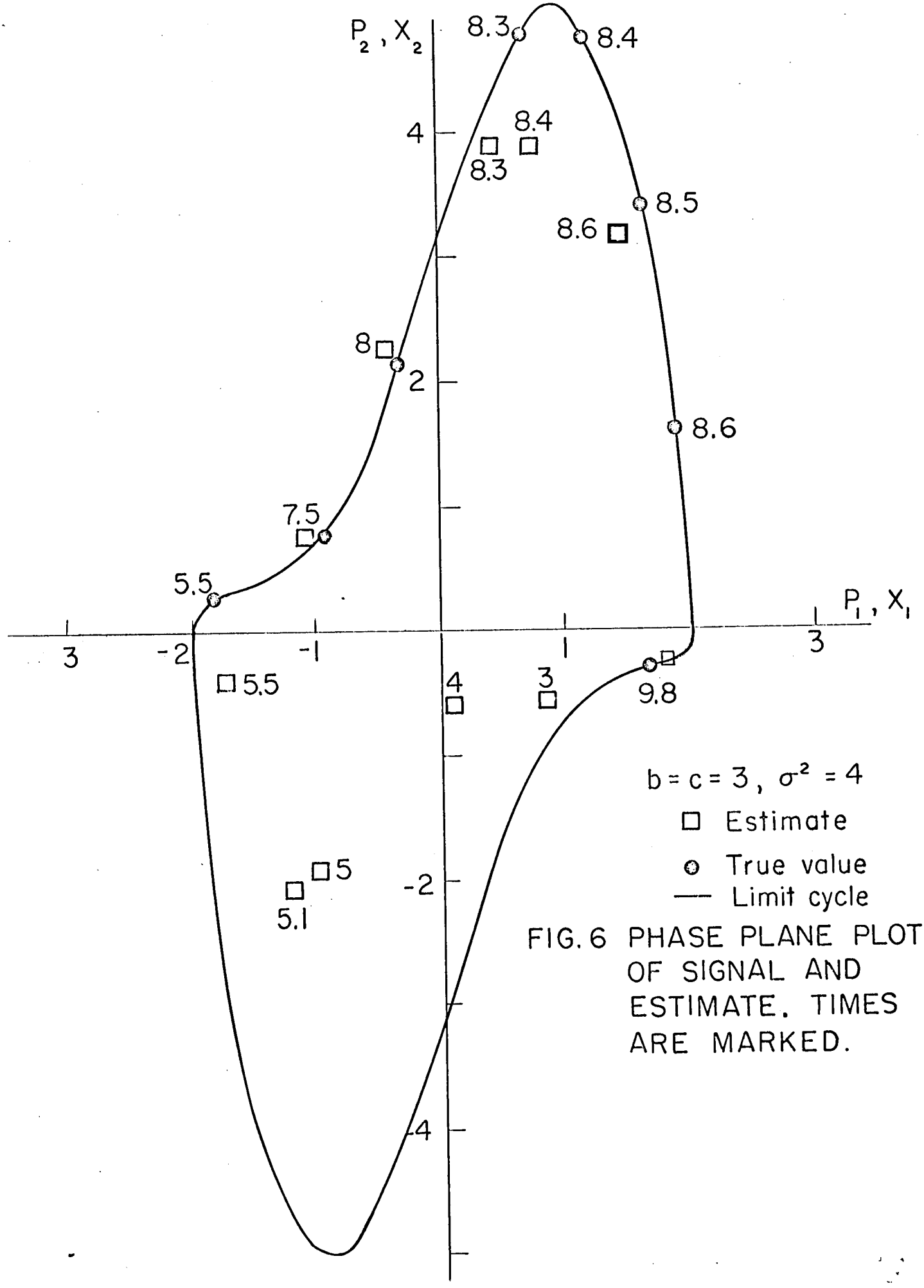


FIG. 6 PHASE PLANE PLOT OF SIGNAL AND ESTIMATE. TIMES ARE MARKED.

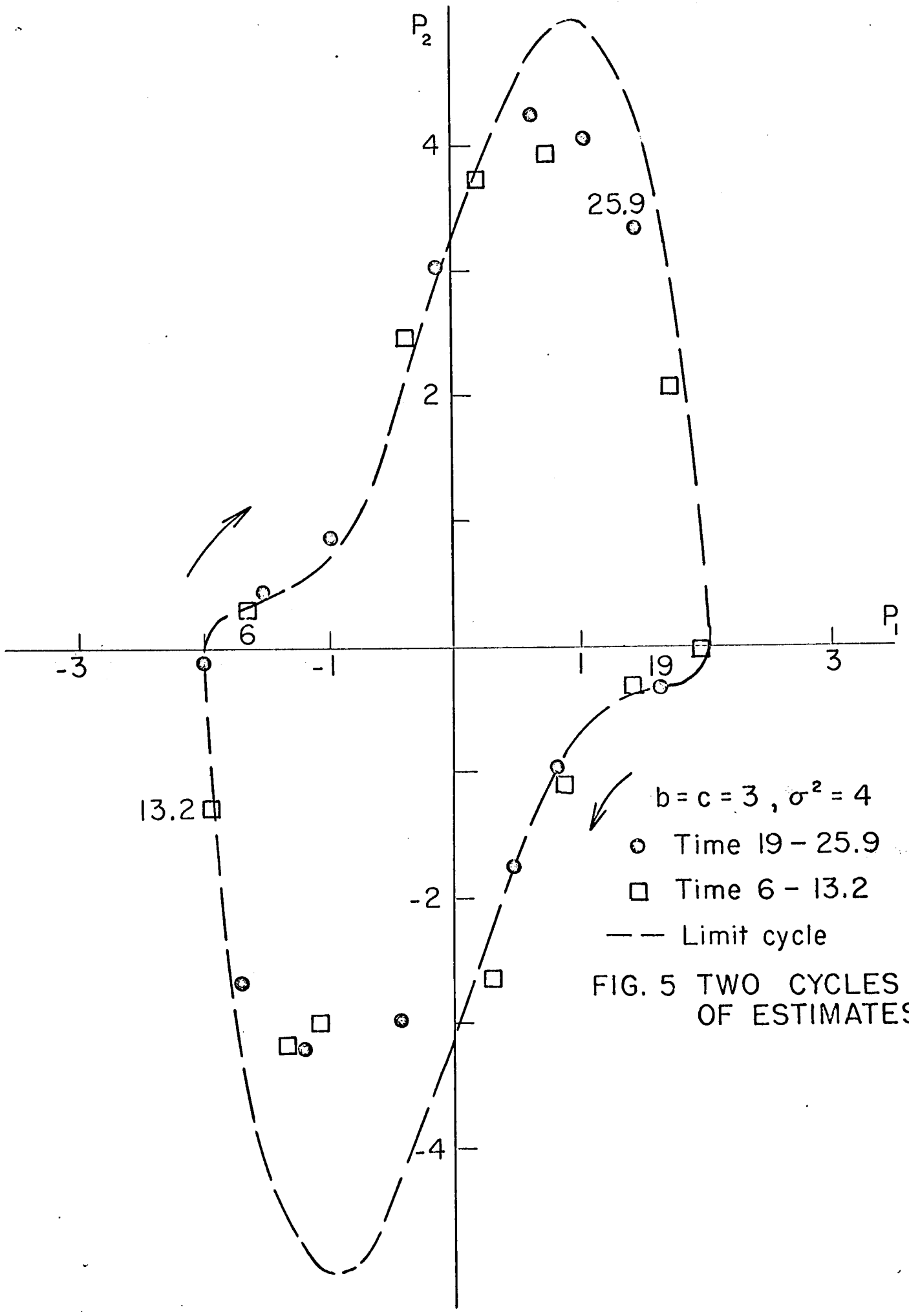


FIG. 5 TWO CYCLES OF ESTIMATES