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ON DYNAMICS OF TWO CABLE-CONNECTED SPACE STATIONS

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ABSTRACT

Presented is an analysis of the dynamic problems of two cable-connected space stations rotating about an axis normal to their orbital plane to provide artificial gravity. The dynamics of cable-connected stations in which the cable tension is zero (non-spinning case) are not treated herein. Differential equations of vibration of the elastic cable and the angular movements of the stations are derived. These motions are coupled through the nonhomogeneous boundary conditions of the cable. This mathematical difficulty is resolved by using the concept of concentrated fictitious masses. The cable equation is solved by using Galerkin's approach for both free and forced oscillations. A general nth order determinantal frequency equation of free vibration of the system is obtained. The responses of the space stations to applied time-varying moments are presented in analytical forms.

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TECHNICAL AND SCIENTIFIC STAFF
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DEFINITION OF SYMBOLS

Symbol	Definition
c	center of rotation to space station
EI	bending rigidity of cable
I_1, I_2, I_3	principal moment of inertias of space station
ℓ	distance of cable connection to c.g. of space station
L	length of cable
m	mass of space station
$M^*(t)$	applied moment
$p(t)$, $s(t)$	generalized coordinates
$r_i(x)$	eigen-function of a free-free beam
R	center of rotation to c.g. of space station
t	time variable
$u(x,t)$	displacement relative to moving coordinates
v, w	components of u relative to moving coordinates
x, y, z	rotating coordinates
X, Y, Z	inertial coordinates
x_1, x_2, x_3	body-fixed principal axes
α_n	constant in eigen-function of beam
$\beta_n L$	eigen-value of free-free beam
γ_n	eigen-value of matrix $(A^{-1}M)$
λ_n	eigen-value of matrix Q
ρ	mass density per unit length of cable

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
ψ, θ, Φ	Eulerian angles of space station
ψ, θ, ϕ	column matrices of the Eulerian angles

Barred symbols refer to counter-weight of space station.

Subscripts i, n denote the ith and nth mode of vibrations of cable.

The dot above a symbol denotes the time derivative.

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SUMMARY

Presented is an analysis of the dynamic problems of two cable-connected space stations rotating about an axis normal to their orbital plane to provide artificial gravity. The dynamics of cable-connected stations in which the cable tension is zero (non-spinning case) are not treated herein. Differential equations of vibration of the elastic cable and the angular movements of the stations are derived. These motions are coupled through the nonhomogeneous boundary conditions of the cable. This mathematical difficulty is resolved by using the concept of concentrated fictitious masses. The cable equation is solved by using Galerkin's approach for both free and forced oscillations. A general nth order determinantal frequency equation of free vibration of the system is obtained. The responses of the space stations to applied time-varying moments are presented in analytical forms.

I. INTRODUCTION

The dynamic problems of cable-connected space stations rotating in the orbital plane have been discussed in much of the published literature. Reference 1 presents a detailed analysis of many types of arrangements of the stations, as well as a bibliography on this subject. The configuration of the space stations treated in this report is two arbitrarily shaped space vehicles which are connected by a long elastic cable. The system is considered rotating about an axis normal to the orbital plane with constant angular velocity. The dynamic problems concerned here are (1) the dynamic stability criteria, (2) the natural frequencies of free vibrations and (3) the dynamic responses of the stations to applied moments.

In a near-zero earth-gravitational field, the dynamic behavior of the space stations in free flight is predominated by the response of the connecting cable. Therefore, it is essential to develop the dynamic equation of the rotating cable coupled with the free angular movements of the space stations. This concept leads to an approach entirely different from that used in reference 1.

The equations of motions of the space stations and of the elastic cable are formulated independently, but the two sets of variables are linked kinematically at the points of connection. This enables us to eliminate the angular variables of the stations. By using concentrated fictitious masses to represent the dynamic coupling, the problem is reduced to the solution of the cable equation with nonhomogeneous mass distribution but homogeneous boundary conditions.

II. ANALYSIS

A. Description of the Coordinate Systems

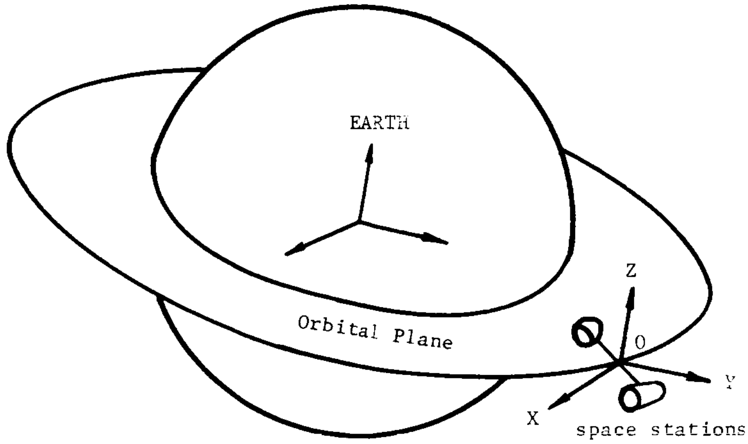
The formulation of the dynamic problems of space vehicles can be greatly simplified by carefully choosing the coordinates and the variables. To achieve this, several coordinate systems are required (see figure 1).

(1) The inertial coordinates XYZ: Let the X and Y axes be in the orbital plane of the space stations and the center of mass of the system be the origin. This coordinate system has fixed orientation with respect to earth but with moving origin.

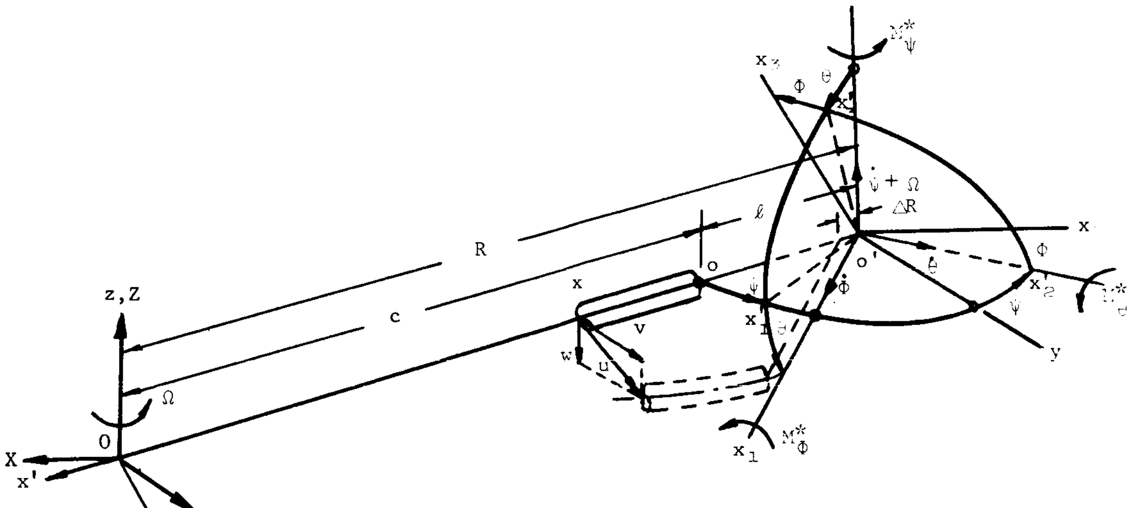
(2) The rotating axes x'yz and xyz: The z-axis is coincidental with Z about which the coordinates rotate with a constant angular velocity Ω. Another set of rotating axes, xyz, with their origin at one end of the cable will be used in dealing with the cable equations.

(3) The principal axes $x_1x_2x_3$ and the Euler angles: Let the center of mass of the space station be the origin of the principal axes and let the x_3-axis of the undisturbed station be parallel to Z. The cable is attached to the space station at a distance ℓ on x_1-axis. The orientation of the disturbed station can be completely defined by Euler's angles. First, as shown in figure 1b, the station rotates about the axis normal to the orbital plane with angular displacement ψ. This moves the other two principal axes to x_1' and x_2'. Since the two stations are rotating as a unit about the Z-axis with angular velocity Ω, the first Euler angle is $\psi + \Omega t$. The second rotation, which is about the x_2'-axis with angle θ, shifts x_1' to its final direction x_1 and the other axis to x_3'. The last rotation Φ about x_1-axis gives the final orientations x_2 and x_3.

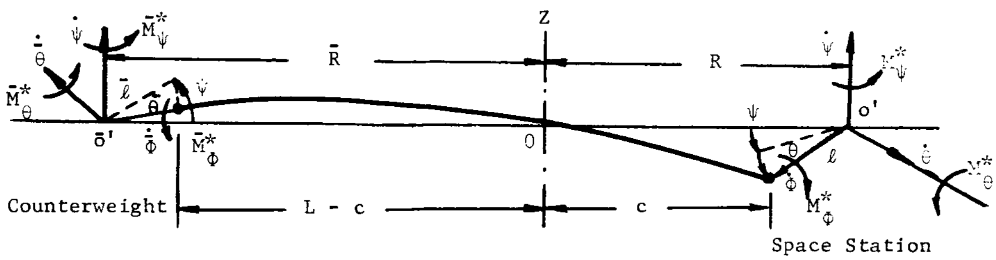
(4) Symbols for the counterweight: The barred symbols are used for the counterpart of the space station which will be referred to as the counterweight.



(a) The Inertial Axes



(b) The Rotating Axes and Euler Angles



(c) The Layout of the System

Figure 1. The Coordinate Systems

B. Formulation of the Moment Equations of the Space Stations

Let us consider that the masses of the space stations are much greater than the mass of the connecting cable and that the system is free of external forces. Also, we may restrict the disturbed motions of the system as follows:

(1) The motion of the space stations is in the plane of rotation and along the line connecting the two c.g.'s, and

(2) the orbital motion of the center of mass of the system is not affected by the disturbance.

We shall now derive the equations of motion of the space stations and the connecting cable separately.

Under the assumption that the cable cannot be extended, the radial velocity of the space station is small in comparison with its absolute angular velocities ω_1, ω_2 and ω_3 about the principal axes. Hence, the rotational kinetic energy can be used as the kinetic energy of the body:

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right), \tag{1}$$

where I_1, I_2 and I_3 are the principal moments of inertia. Upon substitution of the relationships of the absolute angular velocities with the Euler angles,

$$\begin{aligned}
\omega_1 &= \dot{\Phi} - (\Omega + \dot{\psi}) \sin\theta \\
\omega_2 &= \dot{\theta}\cos\Phi + (\Omega + \dot{\psi})\cos\theta\sin\Phi \\
\omega_3 &= (\Omega + \dot{\psi})\cos\theta\cos\Phi - \dot{\theta}\sin\Phi,
\end{aligned} \tag{2}$$

we have

$$T = \frac{1}{2} I_1[\dot{\Phi} - (\Omega + \dot{\psi})\sin\theta]^2 + \frac{1}{2} I_2[\dot{\theta}\cos\Phi + (\Omega + \dot{\psi})\cos\theta\sin\Phi]^2$$

$$+ \frac{1}{2} I_3[(\Omega + \dot{\psi})\cos\theta\cos\Phi - \dot{\theta}\sin\Phi]^2. \tag{3}$$

Applying the Lagrangian equation of motion,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i,$$

results in, after neglecting the nonlinear terms,* three moment equations:

$$M_\psi = I_3\ddot{\psi} + \frac{\partial V}{\partial \psi}$$

$$M_\theta = I_2\ddot{\theta} + (I_3 - I_1)\ \Omega^2\theta + (I_1 + I_2 - I_3)\ \Omega\dot{\Phi} + \frac{\partial V}{\partial \theta} \tag{4a}$$

$$M_\Phi = I_1\ddot{\Phi} + (I_3 - I_2)\ \Omega^2\Phi - (I_1 + I_2 - I_3)\ \Omega\dot{\theta} + \frac{\partial V}{\partial \Phi}\ ,$$

where M is a moment due to all external forces about the axis through the center of mass. Three identical equations can be written for the counterweight:

$$M_{\bar{\psi}} = \bar{I}_3\ddot{\bar{\psi}} + \frac{\partial \bar{V}}{\partial \bar{\psi}}$$

$$M_{\bar{\theta}} = \bar{I}_2\ddot{\bar{\theta}} + (\bar{I}_3 - \bar{I}_1)\ \Omega^2\bar{\theta} + (\bar{I}_1 + \bar{I}_2 - \bar{I}_3)\ \Omega\dot{\bar{\Phi}} + \frac{\partial \bar{V}}{\partial \bar{\theta}} \tag{4b}$$

$$M_{\bar{\Phi}} = \bar{I}_1\ddot{\bar{\Phi}} + (\bar{I}_3 - \bar{I}_2)\ \Omega^2\bar{\Phi} - (\bar{I}_1 + \bar{I}_2 - \bar{I}_3)\ \Omega\dot{\bar{\theta}} + \frac{\partial \bar{V}}{\partial \bar{\Phi}}\ .$$

The potential energies due to the artificial gravity for the two bodies are

$$V = mR\Omega^2(-\Delta R) \quad \text{and} \quad \bar{V} = \bar{m}\bar{R}\Omega^2(-\Delta\bar{R}), \tag{5}$$

---

*These terms become critical only when the spinning rate is very small.

where

$$-\Delta R = \frac{1}{2}\int_{o}^{c}\left(\frac{\partial u}{\partial x}\right)^2 dx + \ell(1 - \cos\theta) + \ell(1 - \cos\psi)$$

$$-\Delta\bar{R} = \frac{1}{2}\int_{c}^{L}\left(\frac{\partial u}{\partial x}\right)^2 dx + \bar{\ell}(1 - \cos\bar{\theta}) + \bar{\ell}(1 - \cos\bar{\psi})$$

in which $u = u(x,t)$ is the deflection of the cable relative to the rotating axes. Notice that the potential energy of the cable is small and can be disregarded.

C. The Vibratory Motions of the Connecting Cable

For the convenience of using the tabulated data and the integration formulas of eigen-functions given by references 4 and 5, the origin of the rotating coordinates is transferred to one end of the cable, and the new coordinates are denoted by xyz and their unit vectors by i,j and k. Let $M(x,t)$ be the bending moment due to centrifugal load, $N(x,t)$ the axial force in the cable, and $\vec{u}$ the position vector of an element from the center of rotation,

$$\vec{u}(x,t) = (c - x)i + v(x,t)j + w(x,t)k.$$

The equation of vibration can be expressed by either

$$EID_x^4\,\vec{u} + D_x^2M(x,t) + \rho(D_t^2\,\vec{u})_a = 0 \qquad (6a)$$

or

$$EID_x^4\,\vec{u} - D_x(ND_x\,\vec{u}) + \rho(D_t^2\,\vec{u})_a = 0, \qquad (6b)$$

where D_x and D_t denote partial differential operators with respect to x and t, respectively. The last term in equation (6) denotes the absolute acceleration excluding the acceleration of the origin. By the principle of differentiation of a vector,

$$(D_t^2 \vec{u})_a = D_t^2 \vec{u} + (\Omega k) \times (\Omega k) \times \vec{u} + 2(\Omega k) \times (D_t \vec{u}) + (\dot{\Omega} k) \times \vec{u}$$

$$= - [2\Omega D_t v + (c - x) \Omega^2] i + (D_t^2 v - \Omega^2 v) j + D_t^2 w k. \quad (7)$$

The formulation of the middle term of equation (6) is given in appendix A. Now, the following equations of motions are obtained:

$$EID_x^4 v + [\frac{1}{2} \rho(x^2 - 2cx) - mR] \Omega^2 D_x^2 v + \rho\Omega^2 (x - c) D_x v + \rho(D_t^2 v - \Omega^2 v) = 0 \quad (8a)$$

$$EID_x^4 w + [\frac{1}{2} \rho(x^2 - 2cx) - mR] \Omega^2 D_x^2 w + \rho\Omega^2 (x - c) D_x w + \rho D_t^2 w = 0. \quad (8b)$$

There is no known exact solution to the above equations. However, an approximate solution can be obtained by using the well-known Galerkin's approach. Fictitious masses are used to provide conditions in which it is applicable.

D. The Boundary Conditions of the Cable

1. The Kinematic Boundary Conditions

It can be seen from figure 1b that the angular displacements of the space stations and the displacements of the cable are related as follows:

$$\begin{aligned} v(0,t) &= \ell\psi(t) \qquad & w(0,t) &= \ell\theta(t) \\ -v(L,t) &= \bar{\ell}\bar{\psi}(t) \qquad & w(L,t) &= \bar{\ell}\bar{\theta}(t). \end{aligned} \quad (9)$$

As a result of these important relationships, it enables us to combine the three sets of equations into a single set.

2. The Dynamic Couplings

The external moments in equation (4) come from two sources. One source is the applied moments from the control rockets on board the space stations as denoted by M^*, and the other is the elastic shearing forces at the ends of the cable. Conversely, the disturbed motions of the stations act upon the ends of the cable as shearing forces. The action of the stations on the cable can be considered as concentrated masses attached to the ends of the cable with magnitudes satisfying the following dynamic conditions:

$$M_{\psi} = \ell m_v (D_t^2 v(0,t))_a + M_{\psi}^* \qquad -M_{\bar{\psi}} = \bar{\bar{\ell}} \bar{\bar{m}}_v (D_t^2 v(L,t))_a - M_{\bar{\psi}}^* \tag{10a}$$

$$M_{\theta} = \ell m_w (D_t^2 w(0,t))_a + M_{\theta}^* \qquad M_{\bar{\theta}} = \bar{\bar{\ell}} \bar{\bar{m}}_w (D_t^2 w(L,t)_a + M_{\bar{\theta}}^* , \tag{10b}$$

where m_v and m_w are the fictitious masses with respect to the motions of v and w. Let us assume that the ends of the cable are free to rotate with respect to the stations; thus,

$$M_{\Phi} = M_{\Phi}^* \quad \text{and} \quad M_{\bar{\Phi}} = M_{\bar{\Phi}}^* . \tag{10c}$$

By substituting the right side of equation (4) into equation (10) and making use of equation (9), we now reduce the problem to free vibrations of a rotating cable with fictitious concentrated masses attached to the ends.

E. The Modified Eigen-Functions

Let us consider that the fictitious masses are distributed uniformly over a small interval ϵ at both ends of the cable. Consequently, the constant ρ of the last term in equation (8) should be substituted by the mass density functions $\rho_v(\epsilon,x)$ and $\rho_w(\epsilon,x)$ with respect to the motions v and w. These functions are defined as

	$0 < x < \epsilon$	$\epsilon < x < L - \epsilon$	$L - \epsilon < x < L$
$\rho_v(\epsilon,x)$	$\rho + m_v/\epsilon$	ρ	$\rho + \bar{m}_v/\epsilon$
$\rho_w(\epsilon,x)$	$\rho + m_w/\epsilon$	ρ	$\rho + \bar{m}_w/\epsilon$

We now represent the two equations of equation (8) by an equation of a single variable u(x,t) in the form

$$EID_x^4u + m\Omega^2 \left\{\left[\frac{\rho}{m}\left(\frac{1}{2}x^2 - cx\right) - R\right] D_x^2u + \frac{\rho}{m}(x - c)\, D_x u\right\} + \rho_u(\epsilon,x)(D_t^2u)_a = 0. \tag{11}$$

1. Eigen-Functions of a Uniform Free-Free Cable

The eigen-functions of a uniform, free-free cable which satisfy the differential equation,

$$\frac{d^4r_i}{dx^4} - \beta_i^4 r_i = 0, \quad i = 1,2 \ \ldots$$

and the orthogonality condition

$$\int_0^L \rho r_n(x)\, r_i(x)\, dx = \rho L\delta_{ni}, \quad (\delta_{ni} \text{ is Kronecker delta})$$

are

$r_1(x) = 1$ rigid body mode, symmetric

$r_2(x) = d(c - x)$ rigid-body mode, anti-symmetric about $x = c$

$d = (c^2 - cL + L^2/3)^{-1/2}$

$$r_i(x) = \cosh \beta_i x + \cos \beta_i x - \alpha_i(\sinh \beta_i x + \sin \beta_i x), \quad i \geq 3.$$

Notice that the ith mode given above is the (i - 2)th mode of references 4 and 5.

2. The Modified Eigen-Functions Adapted to Equation (11)

Let $r_i(\epsilon,x)$ be the eigen-function of a free-free cable with a mass density function $\rho_u(\epsilon,x)$ of which ϵ serves as a small parameter. In the limiting case, we have

$$\lim_{\epsilon \to 0} r_i(\epsilon,x) = r_i(x), \quad i = 1,2, \ \ldots \ .$$

Furthermore, the orthogonality condition for the modified eigen-functions is

$$\int_0^L \rho_u(\epsilon,x)\; r_n(\epsilon,x)\; r_i(\epsilon,x)\; dx = 0, \quad n \neq i. \tag{12a}$$

In dealing with the integration

$$\int_0^L \rho_u(\epsilon,x)\; r_n(\epsilon,x)\; f(x)\; dx = \int_0^L \rho r_n(\epsilon,x)\; f(x)\; dx$$

$$+ \int_0^{\epsilon} (m_u/\epsilon)\; r_n(\epsilon,x)\; f(x)\; dx + \int_{L-\epsilon}^{L} (\bar{m}_u/\epsilon)\; r_n(\epsilon,x)\; f(x)\; dx,$$

we may apply the mean-value theorem to the last two integrals of the above and obtain

$$\lim_{\epsilon \to 0} \int_0^L \rho_u(\epsilon,x)\; r_n(\epsilon,x)\; f(x)\; dx = \int_0^L \rho r_n(x)\; f(x)\; dx + m_u r_n(0)\; f(0)$$

$$+ \bar{m}_u r_n(L)\; f(L). \tag{12b}$$

Notice that

$$r_1(0) = 1 \qquad r_1(L) = 1$$

$$r_2(0) = dc \qquad r_2(L) = -d(L - c)$$

$$r_n(0) = 2 \qquad r_n(L) = -(-1)^{n_2} \qquad n = 3,4, \ldots .$$

F. Approximate Solution by Galerkin's Approach

The general approach to the approximate solution of equation (8) is to assume the solution in the form

$$
\begin{aligned}
v(x,t) &= R\sum_{i=1}^{\infty} r_i(\epsilon,x)\, p_i(t) \\
w(x,t) &= R\sum_{i=1}^{\infty} r_i(\epsilon,x)\, s_i(t).
\end{aligned}
\tag{13}
$$

In these expressions, $r_i(\epsilon,x)$ is the modified eigen-function which satisfies the given boundary conditions; $p_i(t)$ and $s_i(t)$ are the variables to be determined. By the same token, we may write

$$
\begin{aligned}
\psi(t) &= \sum_{i=1}^{\infty} \psi_i(t) \qquad & \theta(t) &= \sum_{i=1}^{\infty} \theta_i(t) \qquad & \Phi(t) &= \sum_{i=1}^{\infty} \phi_i(t) \\
\bar{\psi}(t) &= \sum_{i=1}^{\infty} \bar{\psi}_i(t) \qquad & \bar{\theta}(t) &= \sum_{i=1}^{\infty} \bar{\theta}_i(t) \qquad & \bar{\Phi}(t) &= \sum_{i=1}^{\infty} \bar{\phi}_i(t).
\end{aligned}
\tag{14}
$$

In accordance with equation (9), we have the following kinematic couplings between the variables:

$$
\begin{aligned}
Rr_i(0)\, p_i(t) &= \ell\psi_i(t) \qquad & Rr_i(0)\, s_i(t) &= \ell\theta_i(t) \\
Rr_i(L)\, p_i(t) &= -\bar{\ell}\bar{\psi}_i(t) \qquad & Rr_i(L)\, s_i(t) &= \bar{\ell}\bar{\theta}_i(t).
\end{aligned}
\tag{15}
$$

By making use of equations (4) and (13) to (15), the dynamic boundary conditions given by equation (10) become

$$\ell R m_v r_i(0)\ (\ddot{p}_i)_a = (I_3R/\ell)\ r_i(0)\ \ddot{p}_i + \frac{\ell}{r_i(0)}\frac{\partial V}{R\partial p_i} - M^*_{\psi_i} \tag{16a}$$

$$\begin{aligned}\ell R m_w r_i(0)\ (\ddot{s}_i)_a = (I_2R/\ell)\ r_i(0)\ \ddot{s}_i + (I_3 - I_1)(R/\ell)\ r_i(0)\ \Omega^2 s_i \\ + (I_1 + I_2 - I_3)\ \Omega\dot{\phi}_i + \frac{\ell}{r_i(0)}\frac{\partial V}{R\partial s_i} - M^*_{\theta_i}\end{aligned} \tag{16b}$$

$$\bar{\ell} R\bar{m}_v r_i(L)(\ddot{p}_i)_a = -(\bar{I}_3R/\bar{\ell})\ r_i(L)\ \ddot{p}_i - \frac{\bar{\ell}}{r_i(L)R}\frac{\partial \bar{V}}{\partial p_i} + M^*_{\bar{\psi}_i} \tag{16c}$$

$$\begin{aligned}\bar{\ell} R\bar{m}_w r_i(L)(\ddot{s}_i)_a = (\bar{I}_2R/\bar{\ell})\ r_i(L)\ \ddot{s}_i + (\bar{I}_3 - \bar{I}_1)(R/\bar{\ell})\ r_i(L)\ \Omega^2 s_i \\ + (\bar{I}_1 + \bar{I}_2 - \bar{I}_3)\ \Omega\dot{\bar{\phi}}_i + \frac{\bar{\ell}}{r_i(L)R}\frac{\partial \bar{V}}{\partial s_i} - M^*_{\bar{\theta}_i}\ ,\end{aligned} \tag{16d}$$

where $M^*_{\psi_i}$, $M^*_{\bar{\psi}_i}$, $M^*_{\theta_i}$ and $M^*_{\bar{\theta}_i}$ are the components of M^*_{ψ}, $M^*_{\bar{\psi}}$, M^*_{θ} and $M^*_{\bar{\theta}}$, respectively. For the purpose of combining the treatment of the two variables v and w, we use the following matrix notations:

$$(q_i) = \begin{bmatrix} p_i \\ s_i \end{bmatrix} \qquad (\ddot{q}_i)_a = \begin{bmatrix} p_i - \Omega^2 p_i \\ s_i \end{bmatrix} \qquad (u_i) = \begin{bmatrix} v_i \\ w_i \end{bmatrix} = r_i(\epsilon,x)(q_i)$$

$$(m_i) = \begin{bmatrix} m_{v_i} & 0 \\ 0 & m_{w_i} \end{bmatrix} \quad (\bar{m}_i) = \begin{bmatrix} \bar{m}_{v_i} & 0 \\ 0 & \bar{m}_{w_i} \end{bmatrix} \qquad (\rho(\epsilon,x)) = \begin{bmatrix} \rho_v(\epsilon,x) & 0 \\ 0 & \rho_w(\epsilon,x) \end{bmatrix}. \tag{17}$$

Substituting from equation (13) into equation (8) and making use of equation (17) results in

$$\sum_{i=1}^{\infty} \left\{ EI\beta_i^4 r_i(\epsilon,x) + m\Omega^2 \left[\left\{ \frac{\rho}{m} \left(\frac{1}{2}x^2 - cx\right) - R \right\} D_x^2 r_i(\epsilon,x) \right.\right.$$

$$\left.\left. + \frac{\rho}{m}(x - c)\, D_x r_i(\epsilon,x) \right] (q_i) + r_i(\epsilon,x) \Big(\rho(\epsilon,x)\Big) (\ddot{q}_i)_a \right\} = 0. \qquad (18)$$

By integrating the product of $r_n(\epsilon,x)$ and equation (18) from $x = 0$ to $x = L$ and taking the limiting case that ϵ approaches to zero, we obtain

$$EI\beta_n^4 L(q_n) + m\Omega^2 \sum_{i=1}^{\infty} f_{ni}(q_i) + \rho L(\ddot{q}_n)_a + r_n^2(0)(m_n)(\ddot{q}_n)_a + r_n^2(L)(\bar{m}_n)(\ddot{q}_n)_a = 0. \qquad (19)$$

Notice that use has been made of the orthogonality condition given by equation (11) and the integration formula given by equation (12). The constant f_{ni} in equation (19) is defined as

$$f_{ni} = \int_0^L r_n(x) \left\{ \left[\frac{\rho}{m}\left(\frac{1}{2}x^2 - cx\right) - R \right] D_x^2 r_i(x) + \frac{\rho}{m}(x - c)\, D_x r_i(x) \right\} dx, \qquad (20)$$

which gives

$$f_{11} = f_{12} = f_{21} = f_{n1} = f_{n2} = 0 \qquad f_{22} = \frac{\rho L}{m}$$

$$f_{1n} = (-1)^n \frac{\rho L}{m} \alpha_n \beta_n L \left(\frac{2c}{L} - 1\right) - 2\alpha_n \beta_n L[1 - (-1)^n] \frac{R}{L} \qquad (n \geq 2)$$

$$f_{2n} = (-1)^n \frac{\rho L d}{m} \left\{ \alpha_n \beta_n L(2c^2/L - 3c + L) + 2c - L \right\}$$

$$- 2Rd \left[\left\{ [1 - (-1)^n] \frac{c}{L} + (-1)^n \right\} \alpha_n \beta_n L - 2[1 + (-1)^n] \right]$$

$$f_{ni} = \frac{(-1)^{n+i} \frac{2\rho L}{m} (L - 2c)(\alpha_i\beta_n - \alpha_n\beta_i) - 4R[1 + (-1)^{n+i}](\alpha_n\beta_n - \alpha_i\beta_i)}{\beta_n^4 - \beta_i^4} \beta_i^4$$

$$(n, i \geq 2)$$

$$f_{nn} = \frac{\rho L}{2m}\left[\frac{5}{2} + (1 - 2c\,\alpha_n/L)\,\beta_n L + (c\alpha_n^2/L - \frac{1}{3})\,\beta_n^2 L^2\right] + R(\alpha_n\beta_n L - 2)\,\alpha_n\beta_n .$$

The last two terms of equation (19) are recognized as the dynamic couplings given by equation (16). The partial derivatives in equation (16) are now determined from equation (5) by using equations (10) and (14). The results are written as follows:

$$\frac{1}{R}\frac{\partial V}{\partial p_n} = mR\Omega^2 \sum_{i=1} g_{ni} p_i \qquad \frac{1}{R}\frac{\partial \bar{V}}{\partial p_n} = \bar{m}\bar{R}\Omega^2 \sum_{i=1} \bar{g}_{ni} p_i$$

$$\frac{1}{R}\frac{\partial V}{\partial s_n} = mR\Omega^2 \sum_{i=1} g_{ni} s_i \qquad \frac{1}{R}\frac{\partial \bar{V}}{\partial s_n} = \bar{m}\bar{R}\Omega^2 \sum_{i=1} \bar{g}_{ni} s_i \tag{21}$$

$$\frac{\partial V}{\partial \phi_n} = 0 \qquad \frac{\partial \bar{V}}{\partial \bar{\phi}_n} = 0 .$$

The formulations of g_{ni} and $\bar{g}_{ni}$ are shown in appendix B.

We now have a set of four equations of motion, two of which are obtained from equation (19) by separating the variables and two of which are the third equation of equations (4a) and (4b).

$$a_n(\ddot{p}_n/\Omega^2) + \frac{\rho L}{m}[\kappa(\beta_n L)^4 - 1]\,p_n + \sum_{i=1}^{\infty} c_{ni} p_i = r_n(0)\,\hat{M}^*_{\psi_n} - r_n(L)\,\hat{M}^*_{\psi_n} \tag{22}$$

$$b_n(\ddot{s}_n/\Omega^2) + \left[\frac{\rho L}{m}\,\kappa(\beta_n L)^4 + \frac{I_3 - I_1}{m\ell^2}\,r_n^2(0) + \frac{\bar{I}_3 - \bar{I}_1}{m\bar{\ell}^2}\,r_n^2(L)\right] s_n + \sum_{i=1}^{\infty} c_{ni}s_i$$

$$+ ar_n(0)\ (\dot{\phi}_n/\Omega) + \bar{a}r_n(L)(\dot{\bar{\phi}}_n/\Omega) = r_n(0)\ \hat{M}^*_{\theta_n} + r_n(L)\ \hat{M}^*_{\bar{\theta}_n} \qquad (23a)$$

$$(\ddot{\phi}_n/\Omega^2) + \delta\phi_n - br_n(0)(\dot{s}_n/\Omega) = \hat{M}^*_{\phi_n} \qquad (23b)$$

$$(\ddot{\bar{\phi}}_n/\Omega^2) + \bar{\delta}\phi_n - \bar{b}r_n(L)(\dot{s}_n/\Omega) = \hat{M}^*_{\bar{\phi}_n}\ , \qquad (23c)$$

where

$$a_n = \rho L/m + r_n^2(0)\ I_3/m\ell^2 + r_n^2(L)\ \bar{I}_3/m\bar{\ell}^2$$

$$b_n = \rho L/m + r_n^2(0)\ I_2/m\ell^2 + r_n^2(L)\ \bar{I}_2/m\bar{\ell}^2$$

$$c_{ni} = f_{ni} + g_{ni} + (\bar{m}\bar{R}/mR)\bar{g}_{ni} \qquad \kappa = \frac{EI}{\rho L^4\Omega^2}$$

$$a = (I_1 + I_2 - I_3)/mR\ell \qquad \delta = (I_3 - I_2)/I_1 \qquad b = (1 + \delta)\ R/\ell$$

$$\bar{a} = (\bar{I}_1 + \bar{I}_2 - \bar{I}_3)/mR\bar{\ell} \qquad \bar{\delta} = (\bar{I}_3 - \bar{I}_2)/\bar{I}_1 \qquad \bar{b} = (1 + \bar{\delta})\ R/\bar{\ell}$$

$$\hat{M}^*_{\psi_n} = \frac{1}{mR\Omega^2\ell}\,M^*_{\psi_n} \qquad \hat{M}^*_{\theta_n} = \frac{1}{mR\Omega^2\ell}\,M^*_{\theta_n} \qquad \hat{M}^*_{\phi_n} = \frac{1}{I_1\Omega^2}\,M^*_{\phi_n}$$

$$\hat{M}^*_{\bar{\psi}_n} = \frac{1}{mR\Omega^2\bar{\ell}}\,M^*_{\bar{\psi}_n} \qquad \hat{M}^*_{\bar{\theta}_n} = \frac{1}{mR\Omega^2\bar{\ell}}\,M^*_{\bar{\theta}_n} \qquad \hat{M}^*_{\bar{\phi}_n} = \frac{1}{\bar{I}_1\Omega^2}\,M^*_{\bar{\phi}_n}\ .$$

Notice that equations (22) and (23) represent the equations of motions of two infinite-degrees-of-freedom systems of which the former is in the variable p_n alone while the latter has three variables coupled together.

In studying the dynamic stability or computing the frequencies of vibrations of a many-degrees-of-freedom system, it is often convenient to have the equations of motions in matrix forms. To this end, let us take a finite number of modes for each variable, say, N, we obtain readily from equations (22) and (23)

$$\frac{1}{\Omega^2} A\ddot{p} + Mp = E(t) \tag{24}$$

$$\frac{1}{\Omega^2} B \begin{bmatrix} \ddot{s} \\ \ddot{\phi} \\ \ddot{\bar{\phi}} \end{bmatrix} + \frac{1}{\Omega} C \begin{bmatrix} \dot{s} \\ \dot{\phi} \\ \dot{\bar{\phi}} \end{bmatrix} + K \begin{bmatrix} s \\ \phi \\ \bar{\phi} \end{bmatrix} = F(t), \tag{25}$$

where

$$\underset{N \times 1}{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix} \qquad \underset{N \times 1}{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix} \qquad \underset{N \times 1}{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} \qquad \underset{N \times 1}{\bar{\phi}} = \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \vdots \\ \bar{\phi}_N \end{bmatrix}$$

$\underset{N \times N}{U}$ = Identity Matrix $\qquad$ $D(\;)$ = Diagonal Matrix

$$\underset{N \times N}{A} = D(a_n) \qquad \underset{N \times N}{B_1} = D(b_n) \qquad \underset{N \times N}{R_o} = D(r_n(0)) \qquad \underset{N \times N}{R_L} = D(r_n(L))$$

$$\underset{N \times N}{M} = [c_{ni}] + \frac{\rho L}{m} [D(\beta_n^4 L^4) - U]$$

$$\underset{3N \times 3N}{B} = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{bmatrix} \qquad \underset{3N \times 3N}{C} = \begin{bmatrix} 0 & aR_o & \bar{a}R_L \\ -bR_o & 0 & 0 \\ -\bar{b}R_L & 0 & 0 \end{bmatrix} \qquad \underset{3N \times 3N}{K} = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{bmatrix}$$

$$\underset{N \times N}{K_1} = [c_{ni}] + \frac{I_3 - I_1}{m\ell^2} R_o^2 + \frac{\bar{I}_3 - \bar{I}_1}{m\bar{\ell}^2} R_L^2 + \frac{\rho L}{m} \kappa D(\beta_n^4 L^4) \qquad \underset{N \times N}{0} = \text{Null Matrix}$$

$$\underset{N \times 1}{E(t)} = \hat{M}_\psi^* R_o(e) + \hat{M}_\psi^{\bar{*}} R_L(\bar{e})$$

$$\underset{3N \times 1}{F(t)} = \begin{bmatrix} \hat{M}_\theta^* R_o(f) + \hat{M}_\theta^{\bar{*}} R_L(\bar{f}) \\ \hat{M}_\Phi^*(g) \\ \hat{M}_\Phi^{\bar{*}}(\bar{g}) \end{bmatrix}.$$

In the expressions of $E(t)$ and $F(t)$, we have introduced the load distribution vectors, (e), $(\bar{e})$, (f), $(\bar{f})$, (g), and $(\bar{g})$, such that

$$M_\psi^*(e) = \begin{bmatrix} M_{\psi_1}^* \\ M_{\psi_2}^* \\ \vdots \\ M_{\psi_N}^* \end{bmatrix} \qquad M_\theta^*(f) = \begin{bmatrix} M_{\theta_1}^* \\ M_{\theta_2}^* \\ \vdots \\ M_{\theta_N}^{\bar{*}} \end{bmatrix} \qquad M_\Phi^*(g) = \begin{bmatrix} M_{\phi_1}^* \\ M_{\phi_2}^* \\ \vdots \\ M_{\phi_N}^* \end{bmatrix}$$

$$M_\psi^{\bar{*}}(\bar{e}) = \begin{bmatrix} M_{\psi_1}^{\bar{*}} \\ M_{\psi_2}^{\bar{*}} \\ \vdots \\ M_{\psi_N}^{\bar{*}} \end{bmatrix} \qquad M_\theta^{\bar{*}}(\bar{f}) = \begin{bmatrix} M_{\theta_1}^{\bar{*}} \\ M_{\theta_2}^{\bar{*}} \\ \vdots \\ M_{\theta_N}^{\bar{*}} \end{bmatrix} \qquad M_\phi^{\bar{*}}(\bar{g}) = \begin{bmatrix} M_{\phi_1}^{\bar{*}} \\ M_{\phi_2}^{\bar{*}} \\ \vdots \\ M_{\phi_N}^{\bar{*}} \end{bmatrix}. \tag{26}$$

These vectors remain undetermined at present time.

By transformation of variables, equation (25) can be reduced to the first order matrix differential equation [7],

$$\frac{1}{\Omega}\dot{y} - Qy = G(t), \tag{27}$$

where

$$\underset{6N \times 1}{y} = \begin{bmatrix} \dot{s} \\ \dot{\phi} \\ \dot{\bar{\phi}} \\ s \\ \phi \\ \bar{\phi} \end{bmatrix} \qquad \underset{6N \times 6N}{Q} = \begin{bmatrix} Q_1 & Q_2 \\ U & 0 \end{bmatrix} \qquad \underset{6N \times 1}{G(t)} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

$$\underset{3N \times 3N}{Q_1} = -B^{-1}C = \begin{bmatrix} 0 & -aB_2R_o & -\bar{a}B_2R_L \\ bR_o & 0 & 0 \\ \bar{b}R_L & 0 & 0 \end{bmatrix} \qquad \underset{3N \times 3N}{Q_2} = -B^{-1}K = \begin{bmatrix} -B_2K_1 & 0 & 0 \\ 0 & -\delta U & 0 \\ 0 & 0 & -\bar{\delta}U \end{bmatrix}$$

$$B_2 = B_1^{-1} = D(1/b_n).$$

G. Solution of the Homogeneous Differential Equations

Let the complementary solution of equations (24) and (27) be, respectively,

$$p = Pe^{j\gamma\Omega t} \qquad j = \sqrt{-1} \tag{28a}$$

$$y = Se^{\lambda\Omega t}. \tag{28b}$$

Substitution of the assumed solution into the respective reduced differential equations yields

$$(M - \gamma^2 A)P = 0 \tag{29a}$$

$$(\lambda U - Q)S = 0 \tag{29b}$$

and the characteristic equations

$$|M - \gamma^2 A| = 0 \quad \text{(Nth degree polynomial of } \gamma^2\text{)} \tag{30a}$$

$$|\lambda U - Q| = 0 \quad \text{(6Nth degree polynomial of } \lambda\text{).} \tag{30b}$$

Since the sub-matrices of matrix Q are either null or diagonal matrices except one, B_2K_1, the 6Nth order determinant given by equation (30b) can be readily reduced to an Nth order determinant,

$$|(\lambda^2 + \delta)(\lambda^2 + \bar{\delta})(\lambda^2 B_1 + K_1) + \lambda^2(\lambda^2 + \bar{\delta})\, abR_o^2 + \lambda^2(\lambda^2 + \delta)\bar{a}\bar{b}R_L^2| = 0. \tag{31}$$

This gives a 3Nth degree polynomial of λ^2. For a symmetric system, all the barred symbols are equal to the unbarred symbols and $R_o^2 = R_L^2$; the determinant of equation (13) degenerates to an Nth order determinant,

$$|(\lambda^2 + \delta)(\lambda^2 B_1 + K_1) + 2ab\lambda^2 R_o^2| = 0, \tag{32}$$

which is a 2Nth degree polynomial of λ^2. This can be seen from the relationship

$$\phi_n = (-1)^n\, \bar{\phi}_n \qquad n = 1,2, \ldots \tag{33}$$

obtained by eliminating the variable $\dot{s}_n$ from equations (23b) and (23c). Hence, N variables of $\bar{\phi}_n$ can be eliminated from the set of 3N variables of a symmetric system. In equation (31) the constants a, $\bar{a}$, b, $\bar{b}$, δ, $\bar{\delta}$ and the matrices B_1, K_1 are directly related to the geometry and moments of inertia of the system and bending stiffness of the cable. The matrices R_o^2, R_L^2 and $D(\beta_n^4 L^4)$ can be written out immediately:

$$R_o^2 = \begin{bmatrix} 1 & & & & & & \\ & d^2c^2 & & & & & \\ & & 4 & & & & \\ & & & 4 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & 4 \end{bmatrix} \qquad R_L^2 = \begin{bmatrix} 1 & & & & & & \\ & d^2(L - c)^2 & & & & & \\ & & 4 & & & & \\ & & & 4 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & 4 \end{bmatrix}$$

$$D(\beta_n^4 L^4) = \begin{bmatrix} 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & 500.6 & & & & & & & \\ & & & 3803 & & & & & & \\ & & & & 14618 & & & & & \\ & & & & & 39944 & & & & \\ & & & & & & 89135 & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & \cdot \end{bmatrix} \qquad \beta_n L \simeq \frac{1}{2}(2n - 3)\pi \qquad n > 7$$

$$R_o^2 = R_L^2 = \begin{bmatrix} 1 & & & & & & & \\ & 3 & & & & & & \\ & & 4 & & & & & \\ & & & 4 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & 4 \end{bmatrix} \qquad \text{for a symmetric system .}$$

H. Dynamic Stability and Free Vibrations of the System

1. Dynamic Stability Criteria

From the solution of the homogeneous differential equations, the dynamic stability of a system which is free of external moments and internal damping force can be readily given as

(1) $\gamma_i^2 > 0$ $i = 1, 2, \ldots, N$ for motions in the plane of rotation

(2) $\lambda^2 < 0$ $i = 1, 2, \ldots, 3N$ for motions out of the plane of rotation.

2. Free Vibrations of the System

If a system is dynamically stable, its disturbed motions are vibratory and can be written in the forms:

$$p = \sum_{n=1}^{N} (k_{n1} \cos \gamma_n \Omega t + k_{n2} \sin \gamma_n \Omega t) P_n \tag{34}$$

$$y = \sum_{n=1}^{3N} \mathrm{Re} \left\{ S_n e^{\lambda_n \Omega t} \right\} \quad (\lambda_n\text{'s are positive, pure imaginary}) \tag{35}$$

where Re { } means the real part of { }. In the above equations, P_n and S_n denote the eigen-vectors which are the solution vectors of

$$(M - \gamma_n^2 A)\, P_n = 0 \qquad n = 1, 2, \ldots, N \tag{36}$$

$$(\lambda_n U - Q)\, S_n = 0 \qquad n = 1, 2, \ldots, 3N, \tag{37}$$

respectively. The arbitrary constants k_{n1} and k_{n2} and the arbitrary magnitudes of the eigen-vectors S_n can be determined for a given set of initial deformation and velocity of the cable as functions of x by using eigen-function expansion method. However, this is of little practical interest to us.

I. Special Case - Free Vibrations of Two Identical Stations Connected by a Massless, Flexible Cable

Before we proceed to determine the responses of the system to external moments, let us apply the results of the previous section to a simple case. If the connecting cable is massless and flexible, it remains straight under centrifugal force. This is to say that the disturbed motions of the space stations produce only the two rigid-body modes of the cable. For free vibrations of a symmetric system connected by a massless and flexible cable, we substitute the following in equation (32),

$$R_o^2 = R_L^2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad B_1 = \frac{I_2}{\ell^2}\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}, \qquad 2abR_o^2 = \left(\frac{I_1 + I_2 - I_3}{\ell}\right)^2 \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$$K_1 = \frac{1}{\ell^2}\begin{bmatrix} 2J_1 & 0 \\ 0 & 6J_2 \end{bmatrix} + \frac{I_3 - I_1}{\ell^2}\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}, \qquad J_1 = mR\ell \qquad J_2 = 2mR^2\ell/L,$$

and obtain the characteristic equation of the motions out of the plane of rotation

$$\begin{vmatrix} 2f_1(\lambda) & 0 \\ 0 & 6f_2(\lambda) \end{vmatrix} = 0, \tag{38}$$

where

$$f_n(\lambda) = I_1I_2\lambda^4 + [I_3(I_3 - I_2 - I_1) + 2I_1I_2 + I_1J_n]\lambda^2 + (I_3 - I_2)(I_3 - I_1 + J_n),$$

$$n = 1,\ 2.$$

This gives two values of λ for each mode from the equations

$$f_1(\lambda) = 0 \quad \text{and} \quad f_2(\lambda) = 0.$$

Inserting

$$A = \frac{I_3}{\ell^2}\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \qquad M = \frac{1}{\ell^2}\begin{bmatrix} 2J_1 & 0 \\ 0 & 6J_2 \end{bmatrix}$$

into equation (30a) yields

$$\gamma_n = \sqrt{J_n/I_3}, \qquad n = 1,\ 2,$$

for the motions in the plane of rotation.

J. Responses of the System to External Moments

Let us consider a system which is dynamically stable; i.e., the eigenvalues γ_n ($n = 1, 2, \ldots, N$) are real and positive and the eigenvalues λ_n ($n = 1, 2, \ldots, 3N$) are pure imaginary. We now define the following:

$\underset{N \times N}{P} = (P_1 P_2 \ldots P_N)$ modal matrix of matrix $(A^{-1}M)$ or a row matrix of the eigen-vectors of equation (29a)

$\underset{6N \times 6N}{S}$ = modal matrix of Q

$$\underset{N \times N}{\Gamma^2} = D(\gamma_n^2) = P^{-1}(A^{-1}M)P \tag{39}$$

$$\underset{6N \times 6N}{\Lambda} = \begin{bmatrix} j\Lambda_1 & 0 \\ 0 & -j\Lambda_1 \end{bmatrix} = S^{-1}QS \qquad \Lambda_1 = D(\lambda_n)$$

$$\underset{N \times N}{\sin \Gamma\Omega t} = D(\sin \gamma_n \Omega t) \qquad \underset{6N \times 6N}{e^{\Lambda\Omega t}} = \begin{bmatrix} D(e^{\lambda_n \Omega t}) & 0 \\ 0 & D(e^{-\lambda_n \Omega t}) \end{bmatrix}.$$

By making the transformation of variables

$$p = Pq \tag{40}$$

and

$$y = Sz, \tag{41}$$

in equations (24) and (27), respectively, we obtain

$$\ddot{q} + (\Omega\Gamma)^2 q = P^{-1}A^{-1}E(t)\ \Omega^2 \tag{42}$$

and

$$\dot{z} - (\Omega\Lambda)z = \Omega S^{-1}G(t). \tag{43}$$

It follows immediately that

$$p = Pq = P\Gamma^{-1} \int_0^t \sin \Gamma\Omega(t - \tau)\ P^{-1}A^{-1}E(\tau)\ d\tau \tag{44}$$

$$y = Sz = S \int_0^t e^{\Lambda\Omega(t-\tau)}\ S^{-1}G(\tau)\ d\tau. \tag{45}$$

In the above, formulations of the solutions E(t) and G(t), as mentioned earlier, are yet to be determined.

Before we proceed to determine the moment distribution vectors defined in equation (26), some considerations about the applied moments are necessary. To confine ourselves to the assumptions made earlier for the derivation of the equations of motion, we require that (1) the resultant moment in the plane of rotation is small such that the change of the rotational velocity Ω is negligible and (2) the resultant moment normal to the plane of rotation is also small so that the motions of the station normal to the orbital plane can be disregarded. Based on the above considerations and on the assumption that the moment of inertia of the system $(mR^2 + \bar{m}\bar{R}^2)$ is much greater than the principal moments of inertia of the stations, we may assume that the disturbed motions of the stations due to the applied moments do not have appreciable effect on the motions of the whole system.

In other words, the applied moments which are used for the purposes of navigation, changing orbital plane, spinning up or down of the system, etc., are excluded. Although there is no means by which to determine the exact components of the applied moments on each generalized coordinate, it seems plausible to determine the loads distribution in a static equilibrium state and use it as an approximation in the solution of dynamic responses.

Static Loads Distribution

Let ψ and $\bar{\psi}$ be the angular displacements of the stations and u(x) be the deflection of the cable produced by the application of a unit moment M_ψ^*. The work of the unit moment is equal to the sum of the change of potential energy of the stations and the strain energy of the cable.

$$1 \cdot \psi = \Delta V + \Delta\bar{V} + \Delta U. \tag{46}$$

Notice that the kinetic energy of the system is omitted under the assumption that the system is in static equilibrium configuration. We now assume that

$$u(x) = R \sum_{i=1}^{N} c_i r_i(x), \tag{47}$$

of which the constants c_i's are to be determined from the condition

$$\epsilon = \psi - (\Delta V + \Delta\bar{V} + \Delta U), \tag{48}$$

is a minimum; i.e.,

$$\frac{\partial \epsilon}{\partial c_n} = 0 \qquad n = 1, 2, \ldots, N. \tag{49}$$

Some of the terms in equations (48) and (49) can be written using results already obtained:

$$\frac{\partial \Delta V}{\partial c_n} = mR^2\Omega^2 \sum_{i=1}^{N} g_{ni} c_i \qquad \frac{\partial \Delta \bar{V}}{\partial c_n} = \bar{m}\bar{R}^2\Omega^2 \sum \bar{g}_{ni} c_i \tag{50}$$

$$\psi = \sum_{n=1}^{N} \frac{R}{\ell} c_n r_n(0) \qquad \frac{\partial \psi}{\partial c_n} = \frac{R}{\ell} r_n(0) \tag{51}$$

$$\Delta U = \frac{1}{2EI} \int_o^L M_x^2 \, dx \qquad \frac{\partial \Delta U}{\partial c_n} = \frac{1}{EI} \int_o^L M_x \frac{\partial M_x}{\partial c_n} dx = \frac{1}{EI} \sum_{i=1}^{N} h_{ni} c_i, \tag{52}$$

where

$$M_x = \sum_{n=1}^{N} c_n \int_o^x \rho\Omega^2(c - \xi)[r_n(x) - r_n(\xi)] \, d\xi - mR\Omega^2[r_n(x) - r_n(0)],$$

as given by equation (A-1). It is tedious to perform the integration given in equation (52); however, with the aid of reference 5 a closed form of h_{ni} can be obtained. This work is omitted here.

Substituting from equations (50) to (52) into (49) results in a system of equations in c_i

$$(mR^2\Omega^2 g_{ni} + \bar{m}\bar{R}R\Omega^2\bar{g}_{ni} + \frac{1}{EI} h_{ni})c_i = \frac{R}{\ell} r_n(0) \qquad n = 1, 2, \ldots, N \tag{53}$$

from which c_i can be solved. We now replace the static deflection $r_n(x)$ by $r_n(x)\ p_n(t)$ to represent the dynamic deflection, and rewrite the work expression as follows:

$$M_\psi^* \psi(t) = M_\psi^* \sum_{n=1}^{N} \frac{R}{\ell} c_n r_n(0)\ p_n(t) = \sum_{n=1}^{N} M_{\psi_n}^* \psi_n(t).$$

Since $\psi_n(t) = \frac{R}{\ell} r_n(0)\ p_n(t)$, the above expression gives

$$M_{\psi_n}^* = M_\psi^* c_n. \tag{54a}$$

Similarly,

$$M^*_{\theta_n} = M^*_\theta \, c_n \tag{54b}$$

$$M^{\bar{*}}_{\psi_n} = M^{\bar{*}}_\psi \, \bar{c}_n \qquad M^{\bar{*}}_{\theta_n} = M^{\bar{*}}_\theta \, \bar{c}_n \tag{54c}$$

where $\bar{c}_n = \dfrac{\ell r_n(L)}{\bar{\ell} r_n(0)} c_n$. This relationship is obtained by comparing equation (53) with the equation which generates $\bar{c}_n$,

$$(mR^2\Omega^2 g_{ni} + \bar{m}\bar{R}R\Omega^2\bar{g}_{ni} + \frac{1}{EI} h_{ni}) \bar{c}_i = \frac{R}{\bar{\ell}} r_n(L). \tag{53a}$$

However, the proposed static approach cannot be applied to the pitch moments, since the rotsional vibration of the cable is not considered. Therefore, the response to these moments cannot be treated at the present time.

We now return to equations (25) and (26) and rewrite the load functions in the forms:

$$\underset{N \times 1}{E(t)} = (M^*_\psi(t) \; R_o + M^{\bar{*}}_\psi(t) \; R_L) \; (c_n) \tag{55}$$

$$\underset{3N \times 1}{F(t)} = \begin{bmatrix} (M^*_\theta(t)R_o + M^{\bar{*}}_\theta(t)R_L)(c_n) \\ 0 \\ 0 \end{bmatrix}. \tag{56}$$

where $\bar{R}_o = \frac{\ell}{\bar{\ell}} R_o^{-1} R_L$.

K. Discussion and Recommendations

A great deal of computer time can be saved in computing equation (45) by using partitioned matrices to take advantage of the fact that the eigen-values and eigen-vectors of matrix Q are in conjugate pairs. For this purpose, let us partition the matrices as follows:

$$\underset{6N \times 6N}{S} = \left[\begin{array}{c|c} j\Omega\Sigma\Lambda_1 & -j\Omega\Sigma\Lambda_1 \\ \hline \Sigma & \tilde{\Sigma} \end{array}\right] \qquad \underset{3N \times 3N}{\Sigma} = \begin{bmatrix} \sigma \\ \mu \\ \nu \end{bmatrix} \tag{57}$$

$$\underset{6N \times 6N}{S^{-1}} = \left[\begin{array}{c|c} \Pi_1 & \Pi \\ \hline \tilde{\Pi}_1 & \tilde{\Pi} \end{array}\right] \qquad \underset{3N \times 3N}{\Pi} = (\xi, \; \eta, \; \zeta) \tag{58}$$

where "$\sim$" above a letter denotes its conjugate; σ, μ, ν are N x 3N sub-matrices; and ξ, η, ζ are 3N x N sub-matrices. It is easy to prove that matrix Π can be computed directly from the equation

$$\underset{3N \times 3N}{\Pi} = (\Sigma\Lambda_1 + \tilde{\Sigma}\Lambda_1 \tilde{\Sigma}^{-1}\Sigma)^{-1} \tilde{\Sigma}\Lambda_1 \tilde{\Sigma}^{-1}. \tag{59}$$

Now, equation (45) becomes

$$\begin{aligned}
\underset{N \times 1}{s} &= 2\mathrm{Re}\,\{\sigma H(t)\} & \underset{N \times 1}{\dot{s}} &= 2\mathrm{Re}\,\{j\sigma\Lambda_1\, H(t)\} \\
\underset{N \times 1}{\phi} &= 2\mathrm{Re}\,\{\mu H(t) & \underset{N \times 1}{\dot{\phi}} &= 2\mathrm{Re}\,\{j\mu\Lambda_1\, H(t)\} \\
\underset{N \times 1}{\bar{\phi}} &= 2\mathrm{Re}\,\{\nu H(t)\} & \underset{N \times 1}{\dot{\bar{\phi}}} &= 2\mathrm{Re}\,\{j\nu\Lambda_1\, H(t)\},
\end{aligned} \tag{60}$$

where Re denotes the "real part of" and

$$H(t) = \int_o^t e^{j\Lambda_1\Omega(t-\tau)}\ \xi(M_\theta^*(\tau)R_o + M_\theta^{\bar{*}}(\tau)\bar{R}_o)\ (c_n)\ d\tau. \tag{61}$$

From equations (44) and (52),

$$p = P\Gamma^{-1} \int_o^t \sin\,\Omega\Gamma(t - \tau)P^{-1}A^{-1}(M_\psi^*(\tau)R_o + M_\psi^{\bar{*}}(\tau)\bar{R}_o)(c_n)\ d\tau \tag{62}$$

and

$$\dot{p} = \Omega P \int_o^t \cos\,\Omega\Gamma(t - \tau)\ P^{-1}A^{-1}(M_\psi^*(\tau)R_o + M_\psi^{\bar{*}}(\tau)\bar{R}_o)(c_n)\ d\tau.$$

Finally, the Euler angles and their time derivatives of the space stations are

$$
\begin{aligned}
\psi &= \frac{1}{\ell}\, i R_o p & \theta &= \frac{1}{\ell}\, i R_o s & \Phi &= i\phi \\
\bar{\psi} &= \frac{1}{\bar{\ell}}\, i R_L p & \bar{\theta} &= \frac{1}{\bar{\ell}}\, i R_L s & \bar{\Phi} &= i\bar{\phi} \\
\dot{\psi} &= \frac{1}{\ell}\, i R_o p & \dot{\theta} &= \frac{1}{\ell}\, i R_o s & \dot{\Phi} &= i\dot{\phi} \\
\dot{\bar{\psi}} &= \frac{1}{\bar{\ell}}\, i R_L p & \dot{\bar{\theta}} &= \frac{1}{\bar{\ell}}\, i R_L s & \dot{\bar{\Phi}} &= i\dot{\bar{\phi}}
\end{aligned}
\tag{63}
$$

where $i = (1\ 1\ \ldots\ 1)$, a $1 \times n$ row matrix.

In conclusion, the following problems are suggested for further study:

(1) Refine the approach used for the determination of the load distribution factor.

(2) Obtain solutions on other types of loadings such as reel-out and reel-in of the cable, spin-up and spin-down of the stations, movements of the astronauts, etc.

(3) Take into consideration the torsional stiffness of the cable and other end conditions of the cable.

(4) Study other configurations of the stations, such as two stations connected to a massive hub.

APPENDIX A

Determination of M(x,t) and N(x,t) and Their Derivatives

As observed from Figure A-1, the bending moment at x due to the centrifugal force is

$$M(x,t) = -\int_0^x \rho\Omega^2(c - \xi)[u(x,t) - u(\xi,t)]\, d\xi - mR\Omega^2[u(x,t) - u(0,t)] \quad x< c$$

$$\bar{M}(x,t) = -\int_x^L \rho\Omega^2(\xi - c)[u(x,t) - u(\xi,t)]\, d\xi - \bar{m}\bar{R}\Omega^2[u(x,t) - u(L,t)] \quad c< x <L. \tag{A-1}$$

Since x = c is the center of rotation,

$$mR + \frac{1}{2}\rho c^2 = \bar{m}\bar{R} + \frac{1}{2}\rho(L - c)^2. \tag{A-2}$$

It can be proved readily that

$$\frac{\partial^2}{\partial x^2} M(x,t) = \frac{\partial^2}{\partial x^2}\bar{M}(x,t)$$

$$= \frac{1}{2}\rho\Omega^2\left[(x^2 - 2cx)\frac{\partial^2 u}{\partial x^2} + 2(x - c)\frac{\partial u}{\partial x}\right] - mR\Omega^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L. \tag{A-3}$$

The axial force in the cable is

$$N(x,t) = \int_0^x \rho\Omega^2(c - \xi)\, d\xi + mR\Omega^2, \quad x < c$$

$$\bar{N}(x,t) = \int_x^L \rho\Omega^2(\xi - c)\, d\xi + \bar{m}\bar{R}\Omega^2, \quad x > c. \tag{A-4}$$

It is seen that

$$-\frac{\partial}{\partial x}\left(N\,\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\bar{N}\,\frac{\partial u}{\partial x}\right) = \frac{\partial^2}{\partial x^2}\,M(x,t). \qquad \text{(A-5)}$$

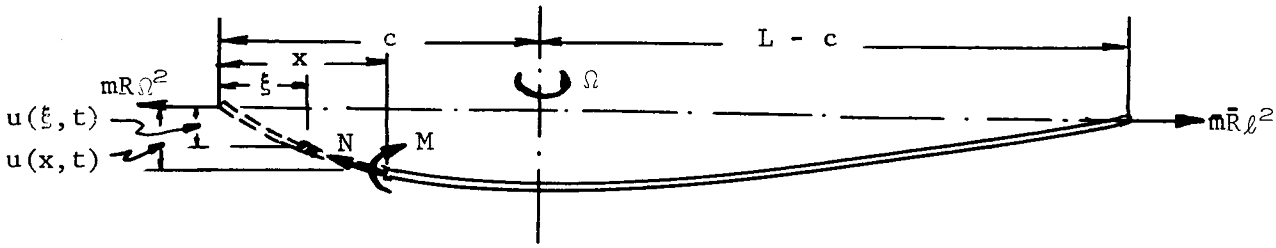


Figure A-1. Displacement of Cable

APPENDIX B

The Potential Energy

Substituting equations (13), (14) and (15) into equation (5) results in

$$\frac{-\Delta R(q)}{R^2} = \frac{1}{2} \sum_{n,i=1}^{N}\sum^{N} \left[\int_{o}^{c} \left(\frac{dr_n}{dx}\right)\left(\frac{dr_i}{dx}\right) dx + \frac{1}{\ell} r_n(0)\, r_i(0) \right] q_n q_i$$

$$= \frac{1}{2} d^2 c q_2^2 - d \sum_{i=3}^{N} [r_i(c) - 2]\, q_2 q_i + \frac{1}{8} \sum_{n=3}^{N} [a_{nn}(c) + 12\alpha_n \beta_n]\, q_n^2$$

$$+ \frac{1}{2} \sum_{n,i=3}^{N}\sum^{N} [a_{ni}(c) + 4(\beta_n^4 \alpha_i \beta_i - \beta_i^4 \alpha_n \beta_n)](\beta_n^4 - \beta_i^4)^{-1}\, q_n q_i$$

$$+ \frac{1}{2} \sum_{n,i=1}^{N}\sum^{N} \frac{1}{\ell} r_n(0)\, r_i(0)\, q_n q_i . \qquad \text{(B-1)}$$

$$\frac{-\Delta \bar{R}(q)}{R^2} = \frac{1}{2} \sum_{n,i=1}^{N}\sum^{N} \left[\int_{c}^{L} \left(\frac{dr_n}{dx}\right)\left(\frac{dr_i}{dx}\right) dx + \frac{1}{\bar{\ell}} r_n(L)\, r_i(L) \right] q_n q_i$$

$$= \frac{1}{2} d^2 (L - c) q_2^2 + d \sum_{i=1}^{N} [(-1)^i\, 2 + r_i(c)]\, q_2 q_i$$

$$+ \frac{1}{8} \sum_{n-1}^{N} [4\alpha_n \beta_n (3\alpha_n \beta_n + 1) - a_{nn}(c)]\, q_n^2$$

$$+\frac{1}{2}\sum_{n,i=3}^{N}\sum^{N}[(-1)^{n+i}4(\beta_n^4\alpha_i\beta_i - \beta_i^4\alpha_n\beta_n) - a_{ni}(c)]\, q_n q_i$$

$$+\frac{1}{2}\sum_{n,i=1}^{N}\sum^{N}\frac{1}{\ell}\, r_n(L)\, r_i(L)\, q_n q_i, \qquad \text{(B-2)}$$

where

$$a_{ni}(x) = \beta_n^4 r_n \frac{dr_i}{dx} - \beta_i^4 r_i \frac{dr_n}{dx} - \frac{d^2r_i}{dx^2}\frac{d^3r_n}{dx^3} + \frac{d^2r_n}{dx^2}\frac{d^3r_i}{dx^3} \qquad n \neq i$$

$$a_{nn}(x) = 3r_n \frac{dr_n}{dx} + x\left(\frac{dr_n}{dx}\right)^2 - 2xr_n\frac{d^2r_n}{dx^2} - \beta_n^{-4}\frac{d^2r_n}{dx^2}\frac{d^3r_n}{dx^3} + (x/\beta_n^4)\left(\frac{d^3r_n}{dx^3}\right)^2 .$$

Let us denote

$$\frac{1}{R}\frac{\partial V}{\partial q_n} = mR\Omega^2\sum_{i=1}^{N} g_{ni}q_i \qquad \frac{1}{R}\frac{\partial \bar{V}}{\partial q_n} = \bar{m}\bar{R}\Omega^2\sum_{i=1}^{N}\bar{g}_{ni}q_i,$$

where the formulas of g_{ni} and $\bar{g}_{ni}$ are tabulated as follows:

$g_{in} = g_{ni}$

$g_{11} = R/\ell \qquad g_{12} = dcR/\ell \qquad g_{1n} = 2R/\ell \quad (n > 2)$

$g_{22} = Rd^2c(1 + c/\ell) \qquad g_{2n} = Rd[2c/\ell + 2 - r_n(c)]$

$(n > 2)$

$$g_{ni} = \left[\frac{a_{ni}(c)+4(\beta_n^4\alpha_i\beta_i - \beta_i^4\alpha_n\beta_n)}{\beta_n^4 - \beta_i^4} + \frac{4}{\ell}\right]R \qquad g_{nn} = \left[\frac{1}{4}a_{nn}(c)+3\alpha_n\beta_n + \frac{4}{\ell}\right]R.$$

$$\bar{g}_{in} = \bar{g}_{ni}$$

$$\bar{g}_{11} = R/\bar{\ell} \qquad \bar{g}_{12} = -d(L-c)R/\bar{\ell} \qquad \bar{g}_{1n} = -(-1)^n 2R/\bar{\ell} \qquad (n > 2)$$

$$\bar{g}_{22} = Rd^2(L-c)\left(1 + \frac{L-c}{\bar{\ell}}\right) \qquad \bar{g}_{2n} = Rd[(-1)^n 2(L-c)/\bar{\ell} + (-1)^n 2 + r_n(c)]$$

$$\bar{g}_{ni} = \left[\frac{4(\beta_n^4\alpha_i\beta_i - \beta_i^4\alpha_n\beta_n)(-1)^{n+i} - a_{ni}(c)}{\beta_n^4 - \beta_i^4} + (-1)^{n+i}\, 4/\bar{\ell}\right] R$$

$$\bar{g}_{nn} = [\alpha_n\beta_n(3\alpha_n\beta_n + 1) - \frac{1}{4}\, a_{nn}(c) + 4/\bar{\ell}]\ R.$$

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ON DYNAMICS OF TWO CABLE-CONNECTED SPACE STATIONS

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