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A HOLLOW ELASTIC CYLINDER
BONDED TO A THIN CASING
OF A DIFFERENT MATERIAL

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VIBRATIONS OF A HOLLOW ELASTIC CYLINDER BONDED TO
A THIN CASING OF A DIFFERENT MATERIAL

By Vasant S. Kelkar*

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SUMMARY

Exact solutions are obtained to determine the natural frequencies and mode shapes of a thin cylindrical shell supported by a hollow core of a different material. Materials for both shell and core are assumed to be homogeneous, isotropic, and linearly elastic. A perfect bond is assumed at the junction of the shell and the core. The composite cylinder is free from stresses at its curved boundaries and is supported by a diaphragm at its flat ends. The solutions for the core are based on three-dimensional elasticity theory and for the shell on bending theory. Curves are plotted to show the variation of the frequency with the variation in circumferential and axial wave numbers and in the ratio of inner to outer radii of the core.

INTRODUCTION

Problems relating to vibrations of a thick cylinder bonded to a thin casing of a different material arise during the flight or transportation and handling of solid fuel rocket motors. It is therefore necessary to develop solutions which determine the natural frequencies and mode shapes of vibrations of such composite cylinders. Solid fuel tends to behave like a visco-elastic material; however, an analysis based on the assumption of an elastic core provides useful results.

Solutions for some particular cases have been obtained before as cited below; however, a completely general solution of the problem using three-dimensional elasticity theory for the core and bending theory for the shell is difficult to obtain analytically and has not yet been developed. Such a general solution, useful in its own right, could also be used to check the solutions of approximate methods, such as the finite element method, which could then be used to solve problems of more complicated core geometries. Chu (ref. 1) gave frequency equations for simple axisymmetric axial shear and radial vibrations of composite cylinders, while Achenbach (ref. 2) obtained solutions for torsional oscillations. These solutions are relatively simple since they involve only one displacement component. Problems involving two of the displacement components were also solved; Baltrukonis, Chi, and Gottenberg (ref. 3) and Sann and Shaffer (ref. 4) obtained frequency equations for plane strain vibrations. Neither study considers displacement variations in the

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axial direction which complicate the problem somewhat. Henry and Freudenthal (ref. 5) presented solutions for vibrations of a thin shell with a visco-elastic core of nonablating boundary. Their solutions are for axisymmetric deformations and are based on the use of the correspondence principle.

For the present analysis, we shall consider a composite circular cylinder of length l composed of a thin elastic shell supported by a hollow elastic core of a different material (fig. 1). Materials for both shell and core are assumed to be homogeneous, isotropic, and linearly elastic, and to be perfectly bonded at their junction. The curved surfaces of the composite cylinder are assumed to be free from stresses. The flat ends of the cylinder are assumed to be supported by a diaphragm which prevents displacements in its own plane. The solutions presented are completely general and are based on a three-dimensional elasticity solution for the core and on bending theory solution for the shell. In the analysis which follows, the core and the shell are first considered separately. To solve the problem of the composite cylinder, equilibrium and compatibility conditions are then satisfied at the junction of the shell and the core. The coefficient determinant of the resulting six homogeneous equations yields the frequency equation from which numerical values of the natural frequency are calculated and plotted against variations in axial and circumferential wave numbers for different values of the core thickness ratio. Frequencies of a composite cylinder with an extremely thin shell are compared with those given by Gazis (ref. 6) for a thick cylinder and are found to be in good agreement.

SYMBOLS

a	radius of the middle surface of the shell
b	inside radius of the core
D_c	$\frac{E_c a}{2(1 + \nu_c)}$
D_s	$\frac{E_s t}{1 - \nu_s^2}$
E_c, ν_c	elastic constants for the core material
E_s, ν_s	elastic constants for the shell material
e	$\epsilon_x + \epsilon_\phi + \epsilon_r$
k^2	$\mu^2 p^2 - \frac{\lambda^2}{a^2}$
k_1^2	$\frac{2(1 - \nu_c)}{1 - 2\nu_c} \mu^2 p^2 - \frac{\lambda^2}{a^2}$

\bar{k}^2, \bar{k}_1^2	$-k^2$ and $-k_1^2$, respectively, when k^2 and k_1^2 are less than zero
k_s	$\frac{t^2}{12a^2}$
l	length of cylinder
m	circumferential wave number
n	axial wave number
p	frequency of vibration
p_0	$\frac{1}{\mu a}$
p_x, p_ϕ, p_r	components of applied loading per unit area of shell's middle surface in the $x, \phi,$ and r directions
r, ϕ, x	cylindrical coordinates
t	thickness of the shell
u, v, w	displacements of a point on the middle surface of the shell in the x, ϕ, r directions, respectively
u_c, v_c, w_c	displacements, respectively, in the axial circumferential and radial directions of a point x, ϕ, r in the core
$\bar{u}_c, \bar{v}_c, \bar{w}_c$	displacements of a point on the outer curved surface of the core
u_{mn}, v_{mn}, w_{mn}	amplitudes of displacements $u, v, w,$ respectively
$\alpha_{ij}(r)$	solutions determining the radial variation of stresses $\tau_{rx}, \tau_{r\phi},$ and σ_r
$\delta_{ij}(r)$	solutions determining the radial variation of $u_c, v_c,$ and w_c
$\epsilon_x, \epsilon_\phi, \epsilon_r$ $\gamma_{rx}, \gamma_{x\phi}, \gamma_{r\phi}$	strains at a point in the core
θ	
λ	$\frac{n\pi a}{l}$ for a cylinder of length l

μ^2	$\frac{(1 - 2\nu_c)(1 + \nu_c)}{1 - \nu_c} \frac{\rho_c}{E_c}$
ρ_c	mass density of the material of the core
ρ_s	mass density of the shell material
$\left. \begin{array}{l} \sigma_x, \sigma_\varphi, \sigma_r \\ \tau_{rx}, \tau_{x\varphi}, \tau_{r\varphi} \end{array} \right\}$	stresses at a point in the core
$\bar{\tau}_{rx}, \bar{\tau}_{r\varphi}, \bar{\sigma}_r$	stresses at a point on the outer curved surface of the core

ANALYSIS OF THE CORE

The core is a hollow circular cylinder for which the governing equations for the deformation are (ref. 7):

Equations of motion

$$\left. \begin{aligned} \frac{\partial \tau_{rx}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{x\varphi}}{\partial \varphi} + \frac{\partial \sigma_x}{\partial x} + \frac{1}{r} \tau_{rx} &= \rho_c \frac{\partial^2 u_c}{\partial t^2} \\ \frac{\partial \tau_{r\varphi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{\partial \tau_{x\varphi}}{\partial x} + \frac{2}{r} \tau_{r\varphi} &= \rho_c \frac{\partial^2 v_c}{\partial t^2} \\ \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\partial \tau_{rx}}{\partial x} + \frac{\sigma_r - \sigma_\varphi}{r} &= \rho_c \frac{\partial^2 w_c}{\partial t^2} \end{aligned} \right\} \quad (1)$$

Kinematic relations

$$\left. \begin{aligned} \epsilon_x &= \frac{\partial u_c}{\partial x}, \quad \epsilon_\varphi = \frac{w_c}{r} + \frac{1}{r} \frac{\partial v_c}{\partial \varphi}, \quad \epsilon_r = \frac{\partial w_c}{\partial r} \\ \gamma_{rx} &= \frac{\partial u_c}{\partial r} + \frac{\partial w_c}{\partial x} \\ \gamma_{x\varphi} &= \frac{\partial v_c}{\partial x} + \frac{1}{r} \frac{\partial u_c}{\partial \varphi} \\ \gamma_{r\varphi} &= \frac{1}{r} \frac{\partial w_c}{\partial \varphi} + \frac{\partial v_c}{\partial r} - \frac{v_c}{r} \end{aligned} \right\} \quad (2)$$

Hooke's law

$$\left. \begin{aligned}
 \left. \begin{aligned}
 \sigma_x \\
 \sigma_\varphi \\
 \sigma_r
 \end{aligned} \right\} &= \frac{E_c}{1 + \nu_c} \left\{ \begin{aligned}
 \epsilon_x \\
 \epsilon_\varphi + \frac{\nu_c}{1 - 2\nu_c} e \\
 \epsilon_r
 \end{aligned} \right\} \\
 \\
 \left. \begin{aligned}
 \tau_{x\varphi} \\
 \tau_{r\varphi} \\
 \tau_{rx}
 \end{aligned} \right\} &= \frac{E_c}{2(1 + \nu_c)} \left\{ \begin{aligned}
 \gamma_{x\varphi} \\
 \gamma_{r\varphi} \\
 \gamma_{rx}
 \end{aligned} \right\}
 \end{aligned} \right\} \quad (3)$$

where

$$e = \epsilon_x + \epsilon_\varphi + \epsilon_r = \frac{\partial u_c}{\partial x} + \frac{1}{r} \frac{\partial v_c}{\partial \varphi} + \frac{w_c}{r} + \frac{\partial w_c}{\partial r}$$

Substituting equations (2) into (3) and then introducing the resulting expressions for stresses into equations (1), we get three partial differential equations in u_c , v_c , w_c , and e .

$$\frac{1}{1 - 2\nu_c} \frac{\partial e}{\partial x} + \nabla^2 u_c = \frac{2(1 + \nu_c)}{E_c} \rho_c \frac{\partial^2 u_c}{\partial t^2} \quad (4a)$$

$$\frac{1}{1 - 2\nu_c} \frac{1}{r} \frac{\partial e}{\partial \varphi} + \left(\nabla^2 - \frac{1}{r^2} \right) v_c + \frac{2}{r^2} \frac{\partial w_c}{\partial \varphi} = \frac{2(1 + \nu_c)}{E_c} \rho_c \frac{\partial^2 v_c}{\partial t^2} \quad (4b)$$

$$\frac{1}{1 - 2\nu_c} \frac{\partial e}{\partial r} + \left(\nabla^2 - \frac{1}{r^2} \right) w_c - \frac{2}{r^2} \frac{\partial v_c}{\partial \varphi} = \frac{2(1 + \nu_c)}{E_c} \rho_c \frac{\partial^2 w_c}{\partial t^2} \quad (4c)$$

where ∇^2 is the three-dimensional Laplacian operator

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial x^2}$$

The following operations on these equations

$$\frac{\partial}{\partial r} (4c) + \frac{1}{r} (4c) + \frac{1}{r} \frac{\partial}{\partial \varphi} (4b) + \frac{\partial}{\partial x} (4a)$$

give

$$\nabla^2 e = \mu^2 \frac{\partial^2 e}{\partial t^2} \quad (5)$$

where

$$\mu^2 = \frac{(1 - 2\nu_c)(1 + \nu_c)}{1 - \nu_c} \frac{\rho_c}{E_c}$$

We seek a solution of equation (5) which is periodic in φ with a period 2π and periodic in x with a period $2\pi a/\lambda$:

$$e = E(r) \sin \frac{\lambda x}{a} \cos m \varphi e^{ipt} \quad (6a)$$

where a is a constant which we shall later take as the radius of the middle surface of the shell. This gives

$$\frac{d^2 E}{dr^2} + \frac{1}{r} \frac{dE}{dr} + \left(k^2 - \frac{m^2}{r^2} \right) E = 0 \quad (6b)$$

where

$$k^2 = \mu^2 p^2 - \frac{\lambda^2}{a^2}$$

The solutions of this equation are

$$E(r) = -(1 - 2\nu_c)a(c_1 P_m + c_4 R_m) \quad (7)$$

where c_1 and c_4 are arbitrary constants and where

(1) for $k^2 \geq 0$, that is, $p^2 \geq (1/\mu^2)(\lambda^2/a^2)$

$$P_m = J_m(kr)$$

and

$$R_m = Y_m(kr)$$

where J_m and Y_m are Bessel functions of the first and second kind, respectively.

(2) for $k^2 \leq 0$, that is, $p^2 \leq (1/\mu^2)(\lambda^2/a^2)$

$$P_m = I_m(\bar{k}r)$$

and

$$R_m = K_m(\bar{k}r) , \quad \bar{k}^2 = -k^2$$

where I_m and K_m are modified Bessel functions of the first and second kind, respectively.

Taking

$$u_c = f(r) \cos \frac{\lambda x}{a} \cos m\phi e^{ipt} \quad (8)$$

we get from equation (4a)

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left(k_1^2 - \frac{m^2}{r^2} \right) f = - \frac{1}{1 - 2\nu_c} \frac{\lambda}{a} E \quad (9)$$

where

$$k_1^2 = \frac{2(1 - \nu_c)}{1 - 2\nu_c} \mu^2 p^2 - \frac{\lambda^2}{a^2}$$

The solutions of the homogeneous part of equation (9) are

$$f(r) = c_2 Q_m + c_5 S_m \quad (10)$$

where c_2 and c_5 are arbitrary constants and where

$$(1) \text{ for } k_1^2 \geq 0, \text{ that is, } p^2 \geq \frac{1 - 2\nu_c}{2(1 - \nu_c)} \frac{1}{\mu^2} \frac{\lambda^2}{a^2}$$

$$Q_m = J_m(k_1 r)$$

and

$$S_m = Y_m(k_1 r)$$

$$(2) \text{ for } k_1^2 \leq 0, \text{ that is, } p^2 \leq \frac{1 - 2\nu_c}{2(1 - \nu_c)} \frac{1}{\mu^2} \frac{\lambda^2}{a^2}$$

$$Q_m = I_m(\bar{k}_1 r)$$

and

$$S_m = K_m(\bar{k}_1 r) , \quad \bar{k}_1^2 = -k_1^2$$

When $k_1^2 \geq 0$, k^2 may be negative or not; hence, to the homogeneous solutions of equation (9), we can add two different sets of particular solutions. When $k_1^2 \leq 0$, k^2 must be less than zero; hence, only one particular solution can be added to the homogeneous solutions. All the different solutions thus arising can be divided into three cases as follows:

Case I. $k_1^2 > k^2 \geq 0$, that is, $p^2/p_0^2 \geq \lambda^2 > \lambda^2(1 - 2\nu_c)/2(1 - \nu_c)$

Case II. $k^2 \leq 0 \leq k_1^2$, that is, $\lambda^2 \geq p^2/p_0^2 \geq \lambda^2(1 - 2\nu_c)/2(1 - \nu_c)$

Case III. $0 \geq k_1^2 > k^2$, that is, $p^2/p_0^2 \leq \lambda^2(1 - 2\nu_c)/2(1 - \nu_c) < \lambda^2$

where

$$p_0 = 1/\mu a$$

Having obtained solutions for e and u_c for each of these cases, we can solve for the displacements v_c and w_c from equations (4b) and (4c). This procedure has been illustrated for case I in the appendix. Only the solutions for the displacements obtained in the three cases are listed here:

$$u_c = f(r) \cos \frac{\lambda x}{a} \cos m\varphi e^{ipt} \quad (11a)$$

$$v_c = g(r) \sin \frac{\lambda x}{a} \sin m\varphi e^{ipt} \quad (11b)$$

$$w_c = h(r) \sin \frac{\lambda x}{a} \cos m\varphi e^{ipt} \quad (11c)$$

For a cylinder of length l , $\lambda = n\pi a/l$ ($n = 0, 1, \dots, \infty$). This gives v_c , w_c , and σ_x as zero at the ends $x = 0$ and $x = l$ corresponding to a diaphragm support. For each f , g , and h , three of the six linearly independent solutions are regular and three singular at $r = 0$.

Regular solutions

$$\left. \begin{aligned} f(r) &= \lambda P_m, & Q_m & \text{or } 0 \\ g(r) &= -\frac{m}{r} a P_m, & \frac{\lambda}{a} \frac{1}{k_1^2} \frac{m}{r} Q_m & \text{or } -a \frac{d}{dr} Q_m \\ h(r) &= a \frac{d}{dr} P_m, & -\frac{\lambda}{a} \frac{1}{k_1^2} \frac{d}{dr} Q_m & \text{or } a \frac{m}{r} Q_m \end{aligned} \right\} \quad (12)$$

Singular solutions

$$\left. \begin{aligned} f(r) &= \lambda R_m, & S_m & \text{or } 0 \\ g(r) &= -\frac{m}{r} a R_m, & \frac{\lambda}{a} \frac{1}{k_1^2} \frac{m}{r} S_m & \text{or } -a \frac{d}{dr} S_m \\ h(r) &= a \frac{d}{dr} R_m, & -\frac{\lambda}{a} \frac{1}{k_1^2} \frac{d}{dr} S_m & \text{or } a \frac{m}{r} S_m \end{aligned} \right\} \quad (13)$$

where P_m , Q_m , R_m , and S_m are replaced by the following Bessel functions for the various cases.

Case	P_m	Q_m	R_m	S_m
I	$J_m(kr)$	$J_m(k_1r)$	$Y_m(kr)$	$Y_m(k_1r)$
II	$I_m(\bar{kr})$	$J_m(k_1r)$	$K_m(\bar{kr})$	$Y_m(k_1r)$
III	$I_m(\bar{kr})$	$I_m(\bar{k}_1r)$	$K_m(\bar{kr})$	$K_m(\bar{k}_1r)$

These solutions are the same as those obtained by Gazis (ref. 6) by using scalar and vector potentials. It may be noted that the quantities kr , \bar{kr} , k_1r , and \bar{k}_1r are dimensionless, so also are the displacement solutions (11), (12), and (13). We denote solutions (12) by $\delta_{ij}(r)$ [$i = 1, 2, 3, j = 1, 2, 3$] and (13) by $\delta_{ij}(r)$ [$i = 1, 2, 3, j = 4, 5, 6$]. Then the general solution for displacements can be written:

$$\left. \begin{aligned} u_c &= \sum_{j=1}^6 \delta_{1j}(r) c_j \cos \frac{\lambda x}{a} \cos m \varphi e^{ipt} \\ v_c &= \sum_{j=1}^6 \delta_{2j}(r) c_j \sin \frac{\lambda x}{a} \sin m \varphi e^{ipt} \\ w_c &= \sum_{j=1}^6 \delta_{3j}(r) c_j \sin \frac{\lambda x}{a} \cos m \varphi e^{ipt} \end{aligned} \right\} \quad (14)$$

where c_1, \dots, c_6 are unknown constants. Corresponding expressions for the stresses τ_{rx} , $\tau_{r\varphi}$, and σ_r in the core can be expressed as:

$$\left. \begin{aligned}
\tau_{rx} &= \frac{D_c}{a^2} \sum_{j=1}^6 \alpha_{1j}(r) c_j \cos \frac{\lambda x}{a} \cos m \varphi e^{ipt} \\
\tau_{r\varphi} &= \frac{D_c}{a^2} \sum_{j=1}^6 \alpha_{2j}(r) c_j \sin \frac{\lambda x}{a} \sin m \varphi e^{ipt} \\
\sigma_r &= \frac{D_c}{a^2} \sum_{j=1}^6 \alpha_{3j}(r) c_j \sin \frac{\lambda x}{a} \cos m \varphi e^{ipt}
\end{aligned} \right\} \quad (15)$$

where

$$D_c = \frac{E_c a}{2(1 + \nu_c)}$$

The factors $\alpha_{ij}(r)$ can be calculated for the three cases by substituting equations (12) and (13) into equations (2) and (3). The constants c_1, \dots, c_6 can then be determined from stresses and displacements prescribed at the inner and outer curved surfaces at the core. This step completes the analysis of the core. A few special cases are of interest.

When $m = 0$, we get four linearly independent solutions for each f and h from (12) and (13) corresponding to the axisymmetric deformation problem, while the two nontrivial solutions obtained for g are for a problem with torsional symmetry.¹

When $\lambda = 0$, both k^2 and k_1^2 are non-negative. Hence, only case I is valid. From solutions (12) and (13), we get four linearly independent solutions for g and h^1 for the resulting plane strain problem. The two linearly independent solutions for f correspond to pure axial shear vibrations.

EQUATIONS FOR THE SHELL

The shell has a midsurface radius a and thickness t which is considered to be small compared to a . Using bending theory with Donnell's simplifications and neglecting the effect of rotatory inertia, the following equations of motion of the shell can be written in terms of its midsurface displacements u , v , and w (refs. 8, 9, and 10).

¹It is understood that relevant sines and cosines are interchanged in solutions (14) by a simple coordinate transformation.

$$\left. \begin{aligned}
u'' + \frac{1 - \nu_s}{2} u^{\cdot\cdot} + \frac{1 + \nu_s}{2} v^{\cdot\cdot} + \nu_s w^{\cdot} + p_x \frac{a^2}{D_s} &= \rho_s \frac{a^2}{D_s} t \frac{\partial^2 u}{\partial t^2} \\
\frac{1 + \nu_s}{2} u^{\cdot\cdot} + v^{\cdot\cdot} + \frac{1 - \nu_s}{2} v'' + w^{\cdot} + p_\phi \frac{a^2}{D_s} &= \rho_s \frac{a^2}{D_s} t \frac{\partial^2 v}{\partial t^2} \\
\nu_s u^{\cdot} + v^{\cdot} + w + k_s (w^{iv} + 2w''^{\cdot\cdot} + w^{\cdot\cdot\cdot}) - p_r \frac{a^2}{D_s} &= -\rho_s \frac{a^2}{D_s} t \frac{\partial^2 w}{\partial t^2}
\end{aligned} \right\} \quad (16)$$

where

$$(\)^{\cdot} = \frac{d(\)}{d(x/a)}$$

and

$$(\)^{\cdot\cdot} = \frac{d(\)}{d\phi}$$

and p_x, p_ϕ, p_r are loads acting per unit area of the middle surface of the shell.

ANALYSIS OF THE COMPOSITE CYLINDER

We have a shell of midsurface radius a supported on a core of radius $a(1 - t/2a)$ which we denote by $a\theta$. We shall satisfy equilibrium conditions at the junction of shell and core. Also, for no slip, the displacements of the shell and the core must be compatible at their junction. Displacements and stresses at the outer surface $r = a\theta$ of the core are:

$$\left. \begin{aligned}
\bar{u}_c &= u_c(x, \phi, a\theta) = \sum_{j=1}^6 \delta_{1j}(a\theta) c_j \cos \frac{\lambda x}{a} \cos m \phi e^{ipt} \\
\bar{v}_c &= v_c(x, \phi, a\theta) = \sum_{j=1}^6 \delta_{2j}(a\theta) c_j \sin \frac{\lambda x}{a} \sin m \phi e^{ipt} \\
\bar{w}_c &= w_c(x, \phi, a\theta) = \sum_{j=1}^6 \delta_{3j}(a\theta) c_j \sin \frac{\lambda x}{a} \cos m \phi e^{ipt}
\end{aligned} \right\} \quad (17)$$

$$\left. \begin{aligned}
\bar{\tau}_{rx} = \tau_{rx}(x, \varphi, a\theta) &= \frac{D_c}{a^2} \sum_{j=1}^6 \alpha_{1j}(a\theta) c_j \cos \frac{\lambda x}{a} \cos m \varphi e^{ipt} \\
\bar{\tau}_{r\varphi} = \tau_{r\varphi}(x, \varphi, a\theta) &= \frac{D_c}{a^2} \sum_{j=1}^6 \alpha_{2j}(a\theta) c_j \sin \frac{\lambda x}{a} \sin m \varphi e^{ipt} \\
\bar{\sigma}_r = \sigma_r(x, \varphi, a\theta) &= \frac{D_c}{a^2} \sum_{j=1}^6 \alpha_{3j}(a\theta) c_j \sin \frac{\lambda x}{a} \cos m \varphi e^{ipt}
\end{aligned} \right\} \quad (18)$$

Displacements $\bar{u}_c, \bar{v}_c, \bar{w}_c$ must be the same as those of a corresponding point on the inner surface of the shell. These displacements are related to the midsurface displacements u, v, w of the shell as follows (ref. 10)

$$\left. \begin{aligned}
\bar{u}_c &= u + \frac{t}{2a} w' \\
\bar{v}_c &= \theta v + \frac{t}{2a} w \\
\bar{w}_c &= w
\end{aligned} \right\} \quad (19)$$

We write solutions for u, v, w in the form:

$$\left. \begin{aligned}
u &= u_{mn} \cos \frac{\lambda x}{a} \cos m \varphi e^{ipt} \\
v &= v_{mn} \sin \frac{\lambda x}{a} \sin m \varphi e^{ipt} \\
w &= w_{mn} \sin \frac{\lambda x}{a} \cos m \varphi e^{ipt}
\end{aligned} \right\} \quad (20)$$

wherein we may take $\lambda = m\pi a/l$ in order to satisfy diaphragm support conditions at the ends of a cylinder of finite length l . Substituting equations (20) and (17) into (19) we can express u_{mn}, v_{mn}, w_{mn} as

$$\left. \begin{aligned}
u_{mn} &= \sum_{j=1}^6 \delta_{1j}(a\theta) c_j - \frac{t}{2a} \lambda \sum_{j=1}^6 \delta_{3j}(a\theta) c_j \\
v_{mn} &= \frac{1}{\theta} \sum_{j=1}^6 \delta_{2j}(a\theta) c_j + \frac{t}{2a} \frac{m}{\theta} \sum_{j=1}^6 \delta_{3j}(a\theta) c_j \\
w_{mn} &= \sum_{j=1}^6 \delta_{3j}(a\theta) c_j
\end{aligned} \right\} \quad (21)$$

To satisfy equilibrium at the junction of the shell and the core, consider the shell to be acted upon on its inner surface by loads $-\bar{\tau}_{rx}$, $-\bar{\tau}_{r\phi}$, and $-\bar{\sigma}_r$. The moments caused by these forces about axes through the midsurface of the shell will be neglected since they will be of rather small magnitude. Hence, in equations (16) p_x , p_ϕ , p_r can be directly replaced by $-\bar{\tau}_{rx}$, $-\bar{\tau}_{r\phi}$, and $-\bar{\sigma}_r$, respectively. Moreover, since in the derivation of equations (16) quantities of the order $t/2a$ have been neglected compared to 1, we take $\theta \sim 1$ (ref. 10). When equations (21), (20), and (18) are substituted into equations (16), the following three equations result for the constants c_1, \dots, c_6 :

$$\sum_{j=1}^6 \left[\left(\lambda^2 + \frac{1 - \nu_s}{2} m^2 - \rho_s \frac{a^2 t}{D_s} p^2 \right) \left(\delta_{1j} - \frac{t}{2a} \lambda \delta_{3j} \right) - \frac{1 + \nu_s}{2} \lambda m \left(\delta_{2j} + \frac{t}{2a} m \delta_{3j} \right) - \nu_s \lambda \delta_{3j} + \frac{D_c}{D_s} \alpha_{1j} \right] c_j = 0 \quad (22a)$$

$$\sum_{j=1}^6 \left[- \frac{1 + \nu_s}{2} \lambda m \left(\delta_{1j} - \frac{t}{2a} \lambda \delta_{3j} \right) + \left(m^2 + \frac{1 - \nu_s}{2} \lambda^2 - \rho_s \frac{a^2 t}{D_s} p^2 \right) \left(\delta_{2j} + \frac{t}{2a} m \delta_{3j} \right) + m \delta_{3j} + \frac{D_c}{D_s} \alpha_{2j} \right] c_j = 0 \quad (22b)$$

$$\sum_{j=1}^6 \left\{ -v_s \lambda \left(\delta_{1j} - \frac{t}{2a} \lambda \delta_{3j} \right) + m \left(\delta_{2j} + \frac{t}{2a} m \delta_{3j} \right) \right. \\ \left. + \left[1 + k_s (\lambda^4 + 2\lambda^2 m^2 + m^4) - \rho_s \frac{a^2 t}{D_s} p^2 \right] \delta_{3j} + \frac{D_c}{D_s} \alpha_{3j} \right\} c_j = 0 \quad (22c)$$

where $\delta_{ij} = \delta_{ij}(a)$ and $\alpha_{ij} = \alpha_{ij}(a)$. We prescribe zero stresses at the inner surface $r = b$ of the core. This gives us three additional equations for the constants c_1, \dots, c_6 as follows:

$$\sum_{j=1}^6 \alpha_{1j}(b) c_j = 0 \quad (22d)$$

$$\sum_{j=1}^6 \alpha_{2j}(b) c_j = 0 \quad (22e)$$

$$\sum_{j=1}^6 \alpha_{3j}(b) c_j = 0 \quad (22f)$$

Equations (22) together form six homogeneous equations for the six constants c_1, \dots, c_6 . They will have a nontrivial solution only if the determinant of their coefficients is zero. This gives us the frequency equation - a transcendental equation for p , the roots of which can be calculated. For each frequency p we can calculate from equations (22) the ratios $c_2/c_1, c_3/c_1, \dots, c_6/c_1$. The mode shapes are then obtained from equations (14), (20), and (21). Problems wherein boundary conditions at the inner and outer curved surfaces of the composite cylinder are different from those considered here can also be solved in an analogous manner.

NUMERICAL RESULTS

The results of our analysis are illustrated by numerical values of the frequency calculated for case I which covers a wider range of frequency spectrum than case II or III. Numerical values of frequency in these cases can be calculated in a similar manner. In case I the dimensionless factors $\alpha_{ij}(r)$ have the following form:

$$\begin{aligned}
\alpha_{11}(r) &= 2 \frac{a}{r} m \lambda J_m(kr) - 2ka \lambda J_{m+1}(kr) \\
\alpha_{12}(r) &= \left(1 - \frac{\lambda^2}{k_1^2 a^2}\right) \left[\frac{ma}{r} J_m(k_1 r) - k_1 a J_{m+1}(k_1 r) \right] \\
\alpha_{13}(r) &= m \lambda \frac{a}{r} J_m(k_1 r) \\
\alpha_{21}(r) &= -2(m^2 - m) \frac{a^2}{r^2} J_m(kr) + 2m \frac{ka^2}{r} J_{m+1}(kr) \\
\alpha_{22}(r) &= 2 \frac{(m^2 - m)\lambda}{k_1^2 r^2} J_m(k_1 r) - \frac{2m\lambda}{k_1 r} J_{m+1}(k_1 r) \\
\alpha_{23}(r) &= -2(m^2 - m) \frac{a^2}{r^2} J_m(k_1 r) + k_1^2 a^2 J_m(k_1 r) - 2 \frac{k_1 a^2}{r} J_{m+1}(k_1 r) \\
\alpha_{31}(r) &= 2(m^2 - m) \frac{a^2}{r^2} J_m(kr) + 2k \frac{a^2}{r} J_{m+1}(kr) + (\lambda^2 - k_1^2 a^2) J_m(kr) \\
\alpha_{32}(r) &= -2 \frac{(m^2 - m)\lambda}{k_1^2 r^2} J_m(k_1 r) - \frac{2\lambda}{k_1 r} J_{m+1}(k_1 r) + 2\lambda J_m(k_1 r) \\
\alpha_{33}(r) &= 2(m^2 - m) \frac{a^2}{r^2} J_m(k_1 r) - 2 \frac{mk_1 a^2}{r} J_{m+1}(k_1 r)
\end{aligned} \tag{23}$$

The remaining α_{ij} ($i = 1, 2, 3, j = 4, 5, 6$) are obtained by replacing J_m by Y_m and J_{m+1} by Y_{m+1} in equations (23).

The frequency equation obtained by equating to zero the coefficient determinant of equations (22) was expressed in terms of the dimensionless quantity ka . Numerical work was carried out on an IBM 7094 computer. Values of ν_c and ν_s were fixed at 0.45 and 0.30, respectively. A relatively weak core was considered by taking $E_c/E_s = 10^{-4}$ and $\rho_c/\rho_s = 0.25$. The radius to thickness ratio (a/t) of the shell was taken to be 1000. For various values of λ , m , and a/b , roots ka of the frequency equation were obtained by plotting the value of the determinant against ka . From the definition of k a dimensionless frequency ratio p/p_0 is then given by the relation

$$(p/p_0)^2 = k^2 a^2 + \lambda^2 \tag{24}$$

where

$$p_0 = 1/\mu a$$

In all calculations, the range of values of p/p_0 considered was such that it satisfied the condition $p/p_0 \geq \lambda$ (or $k^2 a^2 \geq 0$) for case I. The results are shown in figures 2 to 6.

For purposes of comparison, numerical values of frequency were also obtained for some cases considered by Gazis (ref. 6) for the vibrations of a thick cylinder. Here, the ratios E_c/E_s , ρ_c/ρ_s , and ν_c/ν_s were taken to be unity, and a/t was assumed to be 10^4 . With these values it was expected that the roots obtained from the frequency equation for our composite cylinder would be very close to those given by Gazis for a thick homogeneous cylinder. Other parameters assumed to correspond to those used by Gazis were:

$$\nu_c = \nu_s = 0.30, \quad a/b = 3.0, \quad \lambda = 1.885, \quad m = 2$$

The value $\lambda = 1.885$ corresponds to the cylinder thickness to axial wave length ratio of 0.2. Values of p/p_c , where

$$p_c = \frac{\pi}{a - b} \left[\frac{E_c}{2\rho_c(1 + \nu_c)} \right]^{1/2}$$

were obtained from the frequency equation and compared with those obtained from Gazis' curves as follows:

p/p_c from frequency equation	---	0.789	1.058	1.415	1.758	2.037	2.257
p/p_c from Gazis' curves	0.425	0.80	1.07	1.43	1.76	2.06	2.26

The first frequency of 0.425 given by Gazis corresponds to a negative value of ka and, hence, is out of the range of case I. Since it will be obtained from case II or III, it is not given in the comparison above. It can be seen that all other roots agree very closely to those given by Gazis with a maximum error of 1.375 percent. Similar agreement was also found in other cases considered for comparison with Gazis' results.

DISCUSSION

For fixed values of the quantities ν_c , ν_s and the ratios ρ_c/ρ_s and a/t , figures 2 to 6 represent the variation of the dimensionless frequency p/p_0 for different values of λ , m , and the radius ratio a/b . Figures 2 to 4 show the variation of p/p_0 against λ for $m = 0, 1, \text{ and } 2$, respectively, and $a/b = 2.0$, whereas figures 3, 5, and 6 show variation of p/p_0 against λ for $a/b = 2, 1.5, \text{ and } 2.5$, respectively. Straight lines $p/p_0 = \lambda$ and 0.3015λ are drawn on each figure to indicate the domains covered by the three cases. However, all the curves are plotted for case I only. Therefore, curves are stopped at the dividing line $p/p_0 = \lambda$. In some cases, curves

are stopped very close to the dividing line and are extended up to the dividing line by dotted lines.

When $m = 0$ (fig. 2), the frequency equation degenerates into two separate equations, one corresponding to the axisymmetric vibrations (shown by solid lines) and the other to pure "torsional" vibrations (shown by dashed lines). Furthermore, when λ is also equal to zero, the axisymmetric vibration frequency equation degenerates further into two equations, one for simple radial vibrations and the other for simple axial shear vibrations. When both m and λ are zero, we expect two of the frequencies to have zero values corresponding to a rigid body axial translation and rotation of the cylinder. Thus, the first two curves in figure 2 go to zero; the dotted curve is for torsional motion and the other is expected to be for predominantly longitudinal motion.

Figures 3 and 4 show the variation of frequency with λ for $m = 1$ and $m = 2$. It seems that for higher values of m the frequency variation with λ is substantially decreased. From figures 3, 5, and 6, one can see the effect on the frequency of changing a/b . This change simulates the different stages during the burning of the solid fuel. It can be seen that the character of the curves remains the same, however; the magnitude of the frequency decreases with increase in a/b ratio. This is as expected since the mass increases as a/b increases; however, the stiffness remains substantially the same as that of the shell.

CONCLUDING REMARKS

Analytical solutions were obtained to determine the natural frequencies and mode shapes of a thin cylindrical shell supported by a hollow core of a different material. The solutions, obtained by using three-dimensional elasticity theory for the core and bending theory for the shell, were in closed form. Roots of the transcendental frequency equation were obtained numerically and curves were plotted to show a typical variation of the frequency with variation in circumferential and axial wave numbers and in the ratio of inner to outer radii of the core. The solutions can be used to obtain the natural frequencies of cylinders with curved surfaces free from stresses and flat ends supported by diaphragms. They can also be used to check frequencies obtained by approximate methods, such as finite elements, which, in turn, can be used to solve problems of complicated core geometries.

Ames Research Center

National Aeronautics and Space Administration

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APPENDIX A

SOLUTIONS FOR u_c , v_c , AND w_c

We shall now illustrate the procedure for obtaining the solutions for displacements u_c , v_c , w_c by considering case I in detail. For this case, the regular solution for $E(r)$ of equation (7) becomes

$$E(r) = -(1 - 2\nu)ac_1J_m(kr) \quad (A1)$$

We shall write only the regular solutions since, for case I, the singular solutions can be obtained merely by replacing J by Y . Substituting equation (A1) into (9), we get

$$\frac{d^2f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left(k_1^2 - \frac{m^2}{r^2}\right) f = c_1\lambda J_m(kr) \quad (A2)$$

The general solution of this equation is given by

$$f(r) = c_2J_m(k_1r) + \frac{c_1}{k_1^2 - k^2} \lambda J_m(kr) \quad (A3)$$

Now from the definition of e

$$\frac{1}{r} \frac{\partial v_c}{\partial \phi} = e - \frac{\partial u_c}{\partial x} - \frac{w_c}{r} - \frac{\partial w_c}{\partial r} \quad (A4)$$

Substituting equation (A4) into (4c), we get

$$\begin{aligned} \frac{\partial^2 w_c}{\partial r^2} + \frac{3}{r} \frac{\partial w_c}{\partial r} + \frac{1}{r^2} \left(w_c + \frac{\partial^2 w_c}{\partial \phi^2} \right) + \frac{\partial^2 w_c}{\partial x^2} + \frac{2}{r} \frac{\partial u_c}{\partial x} - \frac{2}{r} e \\ + \frac{1}{1 - 2\nu_c} \frac{\partial e}{\partial r} = \rho_c \frac{2(1 + \nu_c)}{E_c} \frac{\partial^2 w_c}{\partial t^2} \end{aligned} \quad (A5)$$

Substituting for e , u , and w from equations (6a), (A1), (11), and (A3), we get

$$\frac{d^2h}{dr^2} + \frac{3}{r} \frac{dh}{dr} + \left(k_1^2 - \frac{m^2}{r^2} - \frac{1}{r^2} \right) h = c_1 \left[\frac{2\lambda^2}{a} \frac{(1-2\nu)}{\mu^2 p^2} \frac{1}{r} J_m(kr) - 2(1-2\nu) \frac{a}{r} J_m(kr) + a \frac{d}{dr} J_m(kr) \right] + c_2 \frac{2\lambda}{ar} J_m(kr) \quad (A6)$$

Letting $H = rh$, we can write the left side of equation (A6) as

$$\frac{d^2H}{dr^2} + \frac{1}{r} \frac{dH}{dr} + \left(k_1^2 - \frac{m^2}{r^2} \right) H$$

Equation (A6) can then be solved for H ; hence,

$$h(r) = \frac{H}{r} = c_1 \frac{(1-2\nu)}{\mu^2 p^2} a \frac{d}{dr} J_m(kr) - c_2 \frac{\lambda}{a} \frac{1}{k_1^2} \frac{d}{dr} J_m(k_1 r) + c_3 \frac{m}{r} J_m(k_1 r) \quad (A7)$$

Introducing equation (11b) into (A4) one obtains

$$g(r) = -c_1 \frac{(1-2\nu)a}{\mu^2 p^2} \frac{m}{r} J_m(kr) + c_2 \frac{\lambda}{a} \frac{1}{k_1^2} \frac{m}{r} J_m(k_1 r) - c_3 \frac{d}{dr} J_m(k_1 r) \quad (A8)$$

Equations (A3), (A6), and (A8) form three sets of linearly independent solutions for f , g , and h in case I. Three more sets are obtained by replacing the Bessel functions of the first kind by those of the second kind. Similarly, solutions for the other two cases were also obtained. All these solutions are summarized (within a multiplicative constant) in equations (12) and (13).

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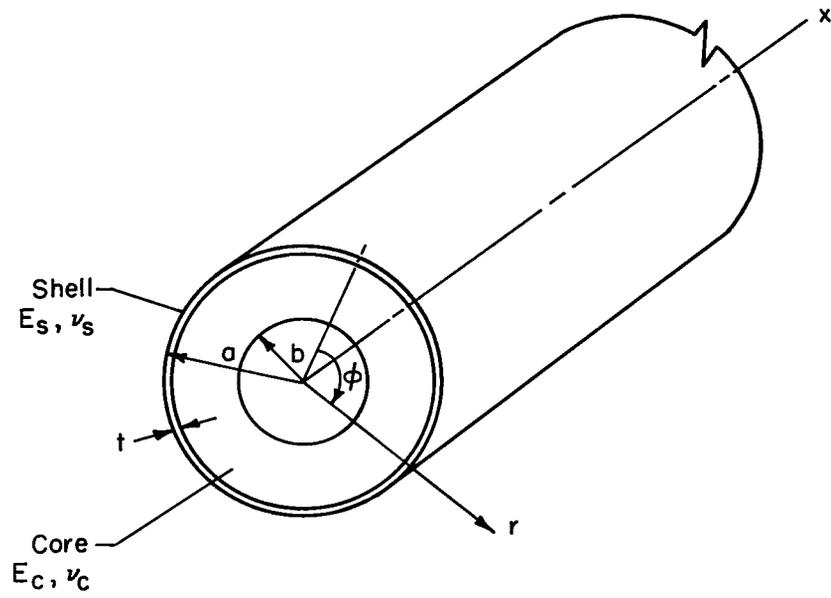


Figure 1.- Cylinder geometry.

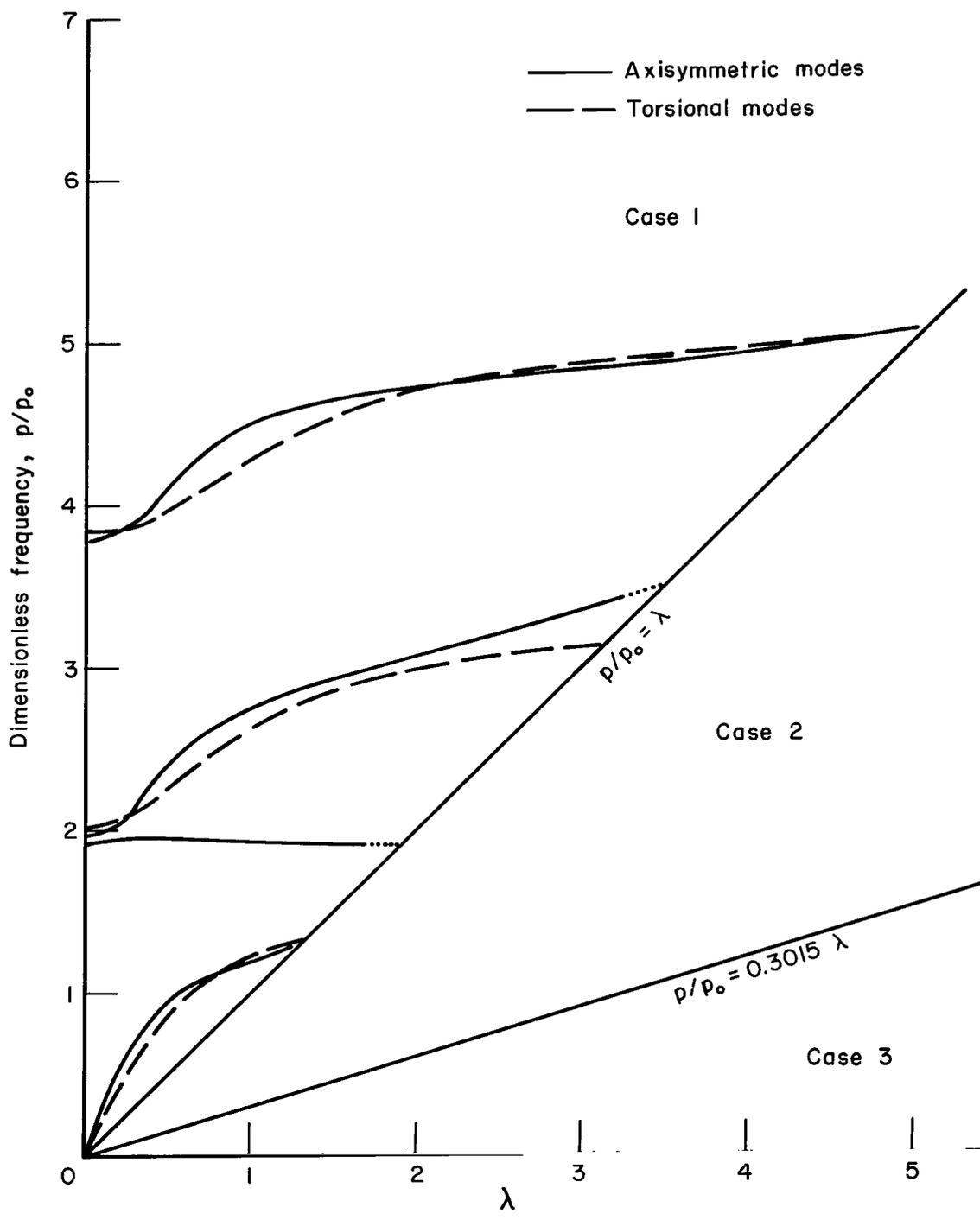


Figure 2.- Frequency versus λ ; $m = 0$, $a/b = 2.0$.

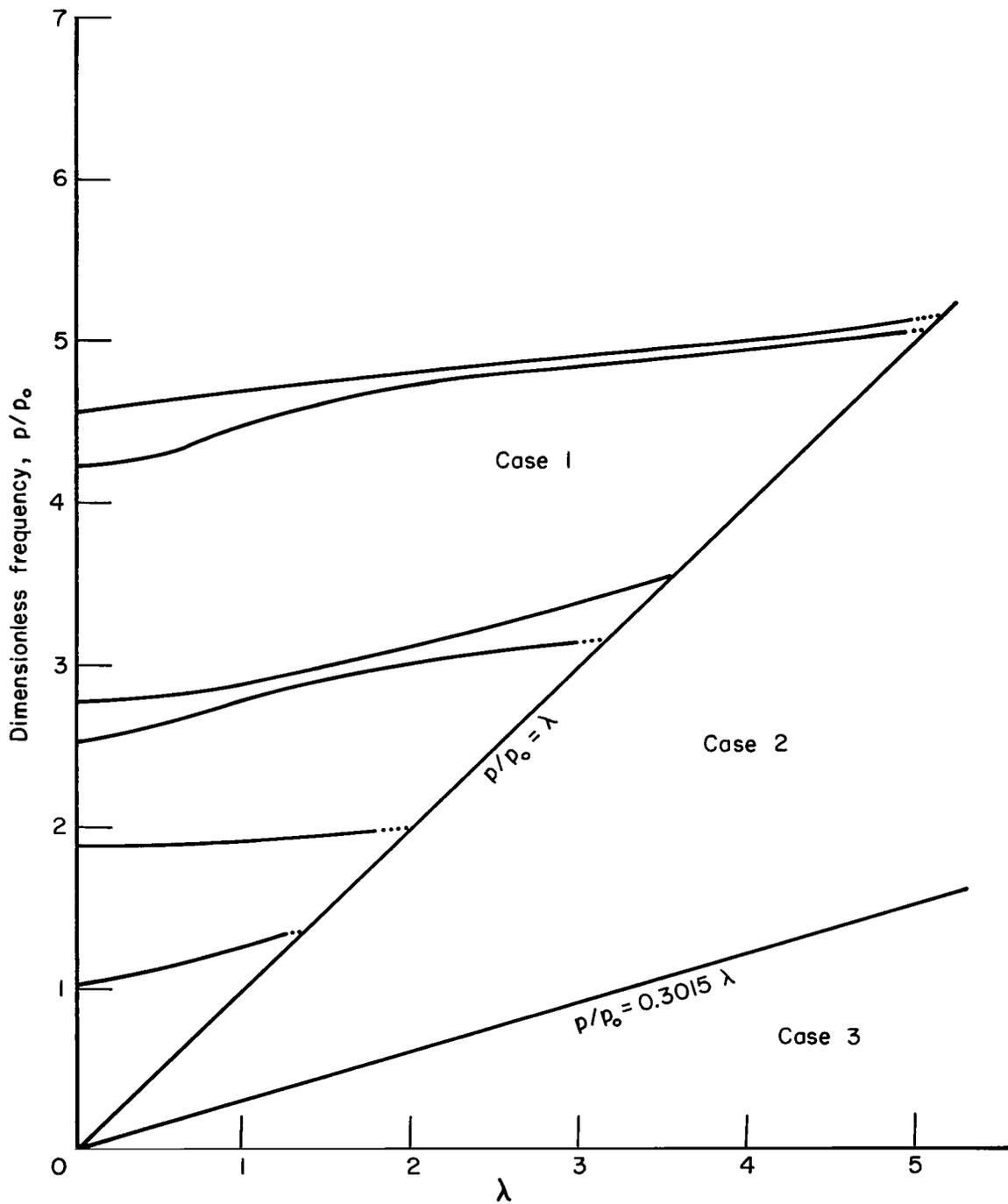


Figure 3.- Frequency versus λ ; $m = 1$, $a/b = 2.0$.

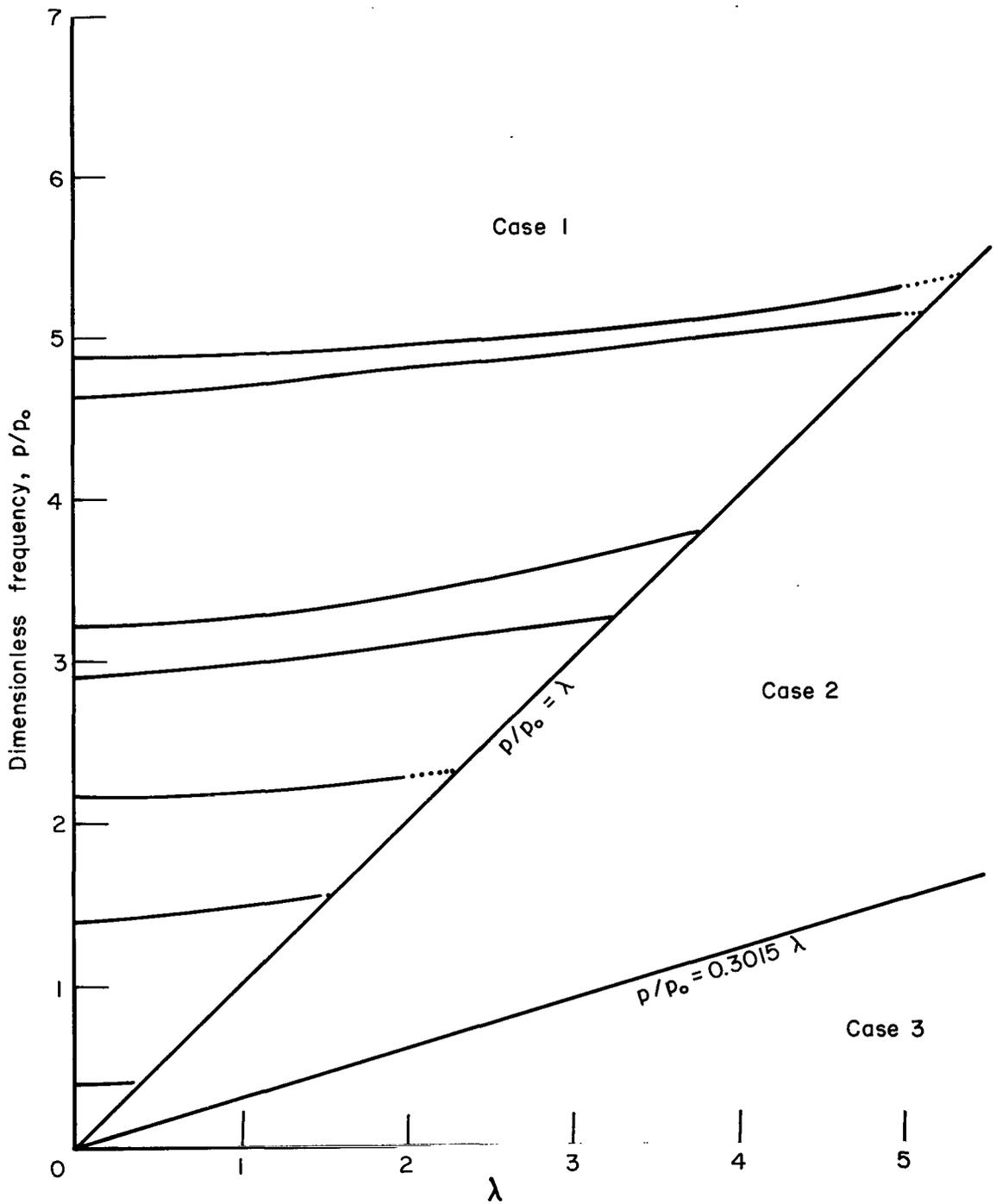


Figure 4.- Frequency versus λ ; $m = 2$, $a/b = 2.0$.

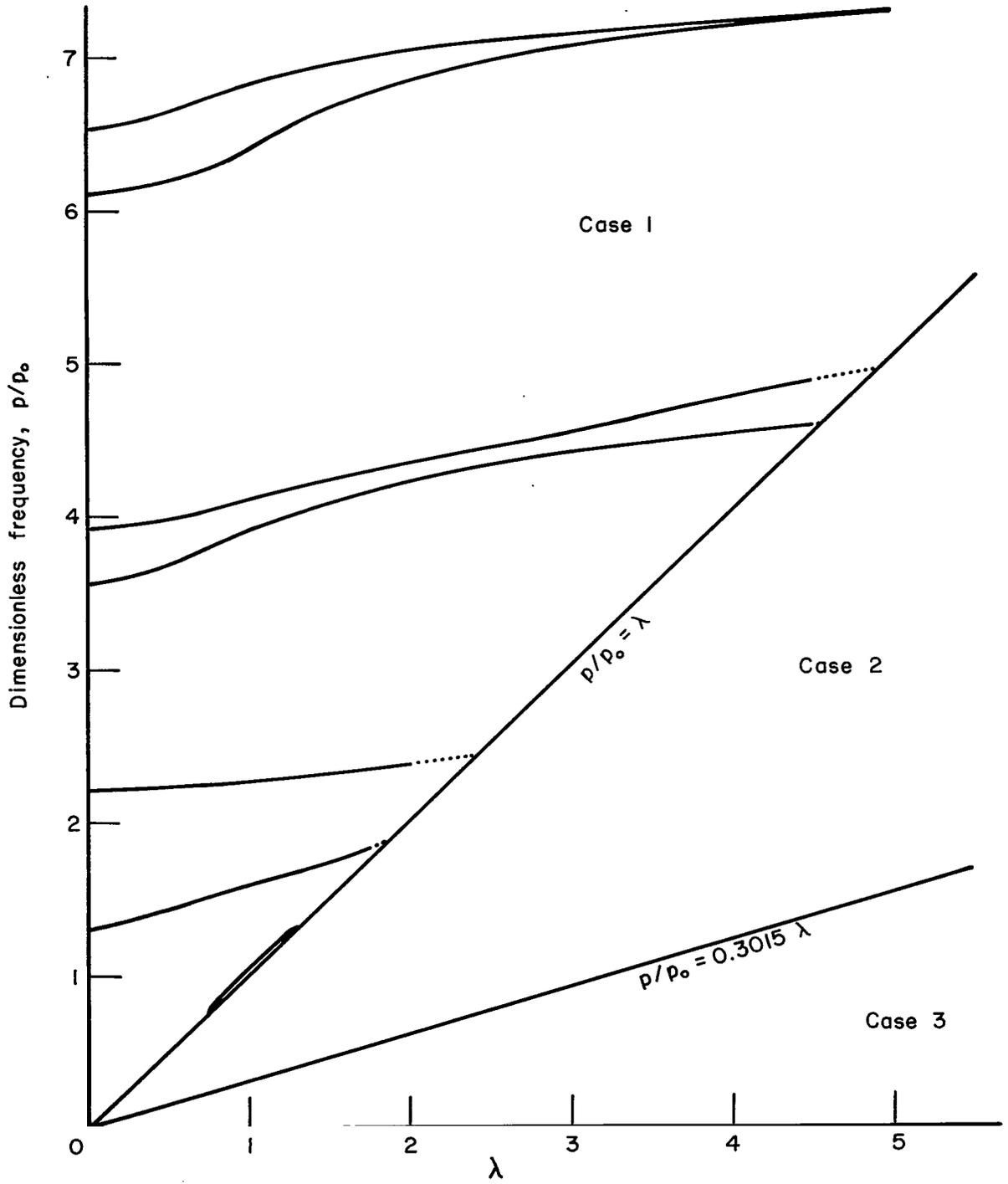


Figure 5.- Frequency versus λ ; $m = 1$, $a/b = 1.5$.

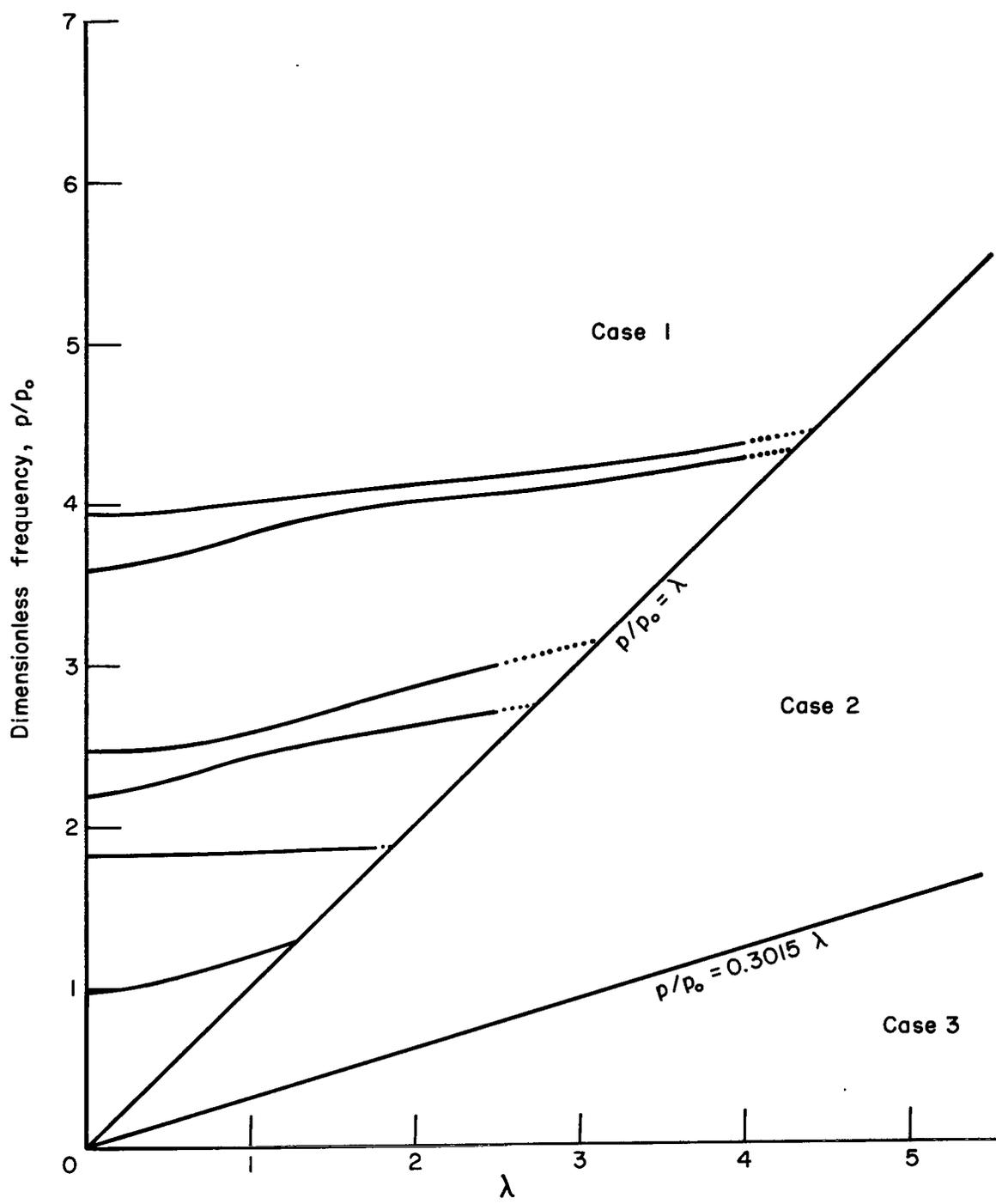


Figure 6.- Frequency versus λ ; $m = 1$, $a/b = 2.5$.



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