NASA TECHNICAL NOTE

NASA TN D-4207

c.1

NASA TN D-4207

DESIGN AND ANALYSIS OF A RATE AUGMENTED DIGITAL-TO-ANALOG CONVERTER

by Sheldon Kopelson

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • NOVEMBER 1967

NASA TN D-4207

DESIGN AND ANALYSIS OF A RATE AUGMENTED DIGITAL-TO-ANALOG CONVERTER

By Sheldon Kopelson

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

DESIGN AND ANALYSIS OF A RATE AUGMENTED DIGITAL-TO-ANALOG CONVERTER

By Sheldon Kopelson
Langley Research Center

SUMMARY

A rate augmented digital-to-analog converter for computed time-dependent data is designed and evaluated. The converter produces a smoothed continuous function by digitally incrementing the function samples at a rate proportional to the predicted function change over each sample interval. The result is continuously converted to an analog voltage; this conversion produces an output which is, in effect, the sum of a linear ramp and the function sample. The converter input data are the computed sample and the change that is predicted by solving an nth-order extrapolation formula.

Results are presented of the performance of the rate augmented converter system for three different functions of time; both first- and second-order extrapolation formulas are used. In addition, the generalized transfer function of the formula-converter combination is derived and used to calculate transfer functions for several extrapolation formulas. For functions that are band limited to 0.05 times the sampling frequency, the experimental and theoretical results indicate that a significant improvement in smoothing can be obtained by using suitably chosen extrapolation formulas which are based upon three and four samples.

The digital extrapolation technique preserves the static accuracy of the converter system and permits a simple adjustment of the system for different sampling rates.

INTRODUCTION

The increasing complexity of aerospace simulation studies has produced, in certain critical computations, requirements for high precision which have been met by the use of digital computers. Precise samples of time-dependent solutions of the simulation equations are obtained at discrete intervals, the duration of which is in part determined by the time required to execute the entire solution of a set of simulation equations. The digital computers, however, are operated in conjunction with either analog computers or analog control equipment whenever a part of the simulation must remain in the continuous domain. One of the problems that arise in such simulations is the conversion of the quantities obtained from the digital computer at discrete intervals into accurate, continuous functions of real time.

In conventional, or zero-order-hold, digital-to-analog converter systems the value of one sample is held until the arrival of the next value. The result is a stair-step approximation of the continuous function. Smoothing this signal by conventional low-pass filters is generally undesirable because of the excessive time lag which is introduced. Smoothing may also be effected by a reduction of the step interval by programing the digital computer to calculate, in addition to the sample quantity, a set of extrapolated quantities for the succeeding sample interval. (A polynomial fit to a number of preceding samples is generally used.) These extrapolated quantities are transferred, in order, to the converter at submultiples of the sample interval. In a number of problems the level of smoothing provided by a few extrapolated points is acceptable. However, a relatively large number of extrapolation calculations and data transfers are required to obtain a high level of smoothing. In many problems the computational speed of the computer may not permit both the simulation and the extrapolation calculations within a sample interval that is consistent with the dynamics of the simulation.

Reference 1 describes a smoothing method in which an analog computing circuit is used, after the converter, to provide a continuous linear extrapolation of the last two data samples. Extension of this technique to higher-order extrapolation requires an increase in the number of analog computing elements. However, any inaccuracies which exist in the smoothing circuit degrade the net conversion accuracy for static as well as dynamic functions. In addition, the method has the operational disadvantage of requiring one or more individual adjustments per converted function when the sample interval is changed for different simulations.

An investigation of the smoothing that could be obtained by linear extrapolations whose slopes were determined by polynomial fitting to preceding samples led to the development of the subject converter system which uses the digital computer to determine the required extrapolation rate. At the start of each discrete time interval the computer furnishes two quantities to the converter system: the value of the function at that time and the predicted change in the function obtained by solving a selected nth-order extrapolation formula. During the sample interval the initial value is digitally incremented at a rate proportional to the predicted change concurrent with a continuous conversion of the result to an analog voltage.

The performance of a single-channel prototype converter system was studied for three different functions of time; both first-order and second-order extrapolation formulas were used. A review of the logical circuitry of the prototype led to the design of a multichannel converter system in which the number of components per channel is reduced by functionally relocating some of the elements used to generate the digital ramps.

The generalized transfer function of the formula-converter combination was derived in terms of the weighting coefficients used in an nth-order extrapolation formula. Calculated transfer functions for several specific extrapolation formulas are presented.

SYMBOLS

$c_0, c_1, c_2, \ldots, c_n$ weighting coefficients of Y_k, Y_{k-1}, Y_{k-2}, . . ., Y_{k-n} used in an extrapolation formula, where k, $k - 1$, $k - 2$, . . ., $k - n$ refer to the present and previous values of $y(t)$

E error

f frequency, Hz

f_c -3 dB frequency of a first-order filter, Hz

f_d damped natural frequency of a second-order filter, $f_n\sqrt{1 - \zeta^2}$, Hz

f_n natural frequency of a second-order filter, Hz

$G(j\omega)$ transfer function in the frequency domain

j unit imaginary vector, $j^2 = -1$

M numerical multiplier

N number of stages in a binary rate multiplier or bits in a digital-to-analog converter

t time, sec

t_k time of kth sample

T sample interval, $t_{k+1} - t_k$

T_{min} minimum sample interval at which converter system can be operated, sec

$y(t)$ continuous function of time, samples of which are obtained at finite intervals

$y_a(t)$ extrapolated approximation of $y(t)$

Y_k value of $y(t)$ at t_k

ΔY_k total predicted change in the interval $t_k \leqq t < t_{k+1}$

ζ ratio of actual damping to critical damping

Φ phase angle, radians

ω frequency, radians/sec

ω_s sampling frequency, $\frac{2\pi}{T}$, radians/sec

APPROXIMATION OF A CONTINUOUS FUNCTION BY LINEAR EXTRAPOLATION

A time-varying function $y(t)$, described only by a series of function values at fixed time intervals of spacing T, may be approximated by a set of linear extrapolations

$$y_a(t) = Y_k + \frac{\Delta Y_k}{T}\left(t - t_k\right) \qquad \left(t_k \leqq t < t_{k+1}\right) \tag{1}$$

With the exception of negligible quantization increments, the output of the rate augmented digital-to-analog converter is described by equation (1). The input data from the digital computer are Y_k, the value of the function at the start of the interval, and ΔY_k, the calculated total change over the interval.

The quantity ΔY_k is a weighted summation of Y_k and n preceding points; that is,

$$\Delta Y_k = c_0 Y_k + c_1 Y_{k-1} + c_2 Y_{k-2} + \ldots + c_n Y_{k-n} \tag{2}$$

The simplest method of determining the weighting coefficients for this extrapolation formula is to assume that Y_{k+1} will lie on the extension of the nth-order polynomial which fits the selected points. However, for second- and higher-order polynomials the coefficients can satisfy either of two criteria. One criterion is that the terminal value of the converter extrapolation be equal to the predicted next value of the function. The other criterion is that over the interval T the average difference between the linear extrapolation and the next segment of the fitted polynomial be zero. These two types of extrapolations are referred to as minimum terminal error and minimum average error, respectively. In either extrapolation the coefficients are derived by solving the equations that result from the choice of the polynomial and extrapolation criterion.

The design of suitable extrapolation formulas is not necessarily restricted to the polynomial fitting method. For example, the formula may be designed to provide good extrapolations near the peaks of a sine wave which is sampled at a specific rate (appendix C).

PROTOTYPE RATE AUGMENTED DIGITAL-TO-ANALOG CONVERTER

The fundamental principle of the digital rate augmentation technique is that the basic digital-to-analog converter (DAC) is driven by logic signals from a binary up/down counter rather than from a buffer register. At the start of each sample interval, the counter is preset to the value of the function. It is then incremented or decremented at a rate which is determined by the predicted change of the function during that sample interval. The predicted change is the result of a solution, by the digital computer, of some extrapolation formula. In order to convert both positive and negative function values to analog voltages without using an inverting amplifier and sign-controlled switch, the information in the counter must be in a numerical complement form rather than in sign-magnitude form. The prototype system was designed for the conversion of natural binary coded information with negative numbers expressed in two's complement form. As is shown subsequently the format of the predicted-change data must be converted to sign-magnitude form, within the system, in order to control the counting operation.

The block diagram of the prototype of the rate augmented conversion system, which was assembled from commercially available digital logic cards, is shown in figure 1. The control sequence of the prototype is shown in figure 2. The range of the system is $\pm\left(2^{11} - 1\right)$ (i.e., sign and 11 data bits). This range was chosen to provide resolution compatible with the accuracy of the available DAC modules.

The elements of the system below the horizontal line A-A in figure 1 have the capacity for serving additional converter channels. The computer tape processing equipment to the left of the vertical line B-B simulates an operating digital computer. Each block of data recorded on the magnetic tapes contains the two input quantities required by the system at the start of each sample interval: Y_k, the value of the function, and $\Delta Y_k/\left(2^{11} - 1\right)$, the predicted change of the function divided by one-half of the range of the system.

As a tape block is read, registers A and B are loaded in sequence with the quantities Y_k and $\Delta Y_k/\left(2^{11} - 1\right)$. When the end of the tape block occurs, the conversion operation for that sample is started. The transfer gates are enabled so that the counter can be set to Y_k which is stored in register A. Simultaneously, register C is set to $\Delta Y_k/\left(2^{11} - 1\right)$ which is stored in register B. The transfer gates are then disabled so that the counter can respond to its pulse input.

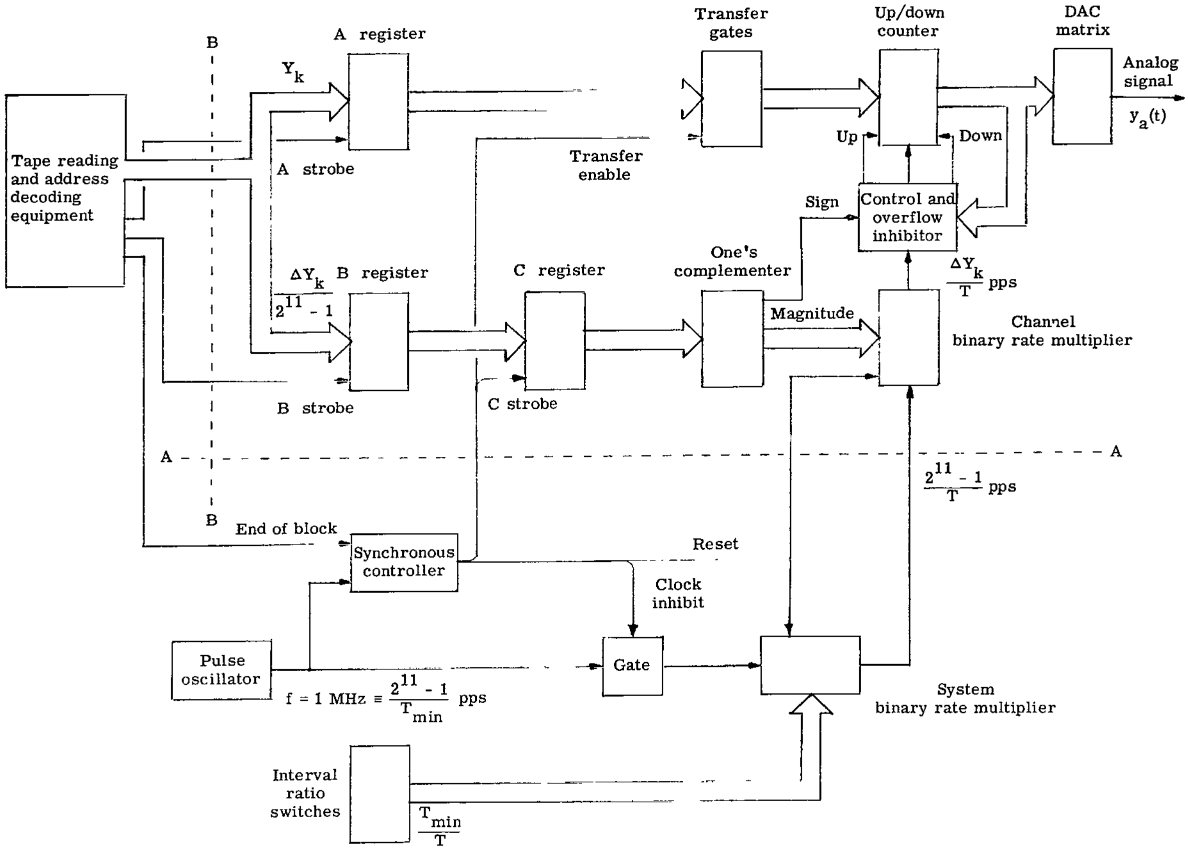


Figure 1.- Block diagram of prototype rate augmented digital-to-analog converter.

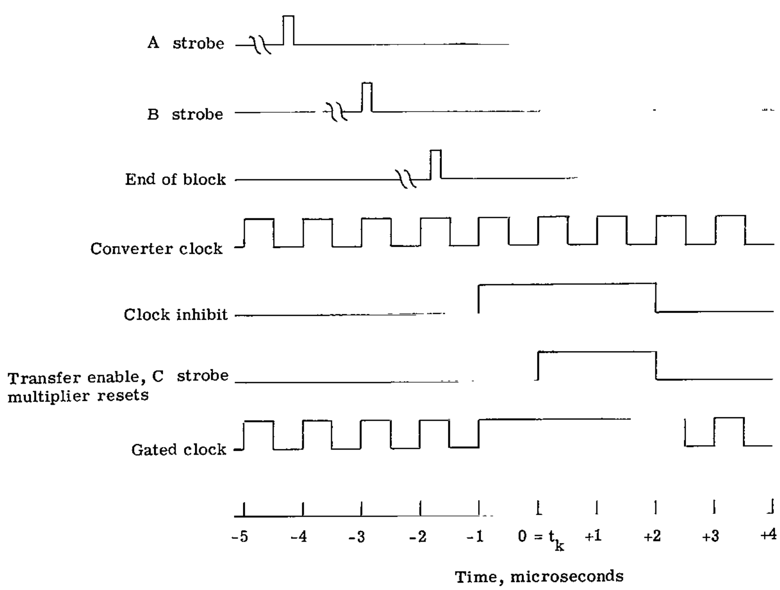


Figure 2.- Prototype converter control sequence.

Conversion of the ΔY_k information from two's complement to sign-magnitude representation is necessary to provide the counter with a count-up or count-down control signal and to develop a separate set of logical signals which determine the rate at which the counter is pulsed. The one's complementer performs a parallel one's complement to sign-magnitude conversion. Although this conversion causes an error of the least significant bit (when negative numbers are converted), the circuit was chosen because of its speed and simplicity.

The rate at which the up/down counter is pulsed is determined by the output of the channel binary rate multiplier (BRM). (See appendix A for a discussion of the BRM.) The BRM numerical multiplier input is the natural binary representation of the magnitude of $\Delta Y_k / \left(2^{11} - 1\right)$. Its multiplicand input is a pulse train of $\left(2^{11} - 1\right)/T$ pulses per second, where T is the sample interval. The output or product of this multiplier is a pulse train of $\Delta Y_k / T$ pulses per second. Consequently, in a sample interval $t_k \leqq t < t_{k+1}$, the counter and, therefore, the DAC follow (in one bit increments) equation (1).

In certain instances the numerical sum of the predicted change ΔY_k and of the actual function value Y_k exceeds the range of the counter. A counter overflow (which would result in a reversal of sign) is prevented by the overflow inhibitor shown in figure 1. This circuit tests the status of the counter and inhibits the count-up or the count-down control signal when the counter reaches plus or minus full scale, respectively.

The pulse source for the channel BRM is the output of the system BRM. The pulse input to the system BRM is a 1-MHz pulse train which is obtained, through a gate, from a crystal controlled pulse oscillator. (The maximum operating frequency of the logic elements used is 1 MHz.) For scaling purposes, this frequency is defined as the ratio of the converter half-range $\left(2^{11} - 1\right)$ to a minimum sampling interval T_{min}. The multiplier input is the ratio of T_{min} to the sampling interval T which is used for a given problem. The output of the system BRM is a pulse train at the rate of $\left(2^{11} - 1\right)/T$ pps which is the required channel BRM input. In the prototype, T is the measured interval between the reading of successive tape blocks and the binary code for T_{min}/T is determined by a set of 11 switches. In an operational converter system, T would be the iteration interval of a particular digital computer simulation program and the switches would be replaced by a register which would be set to T_{min}/T, by the computer, at the start of the simulation.

For a given maximum pulse oscillator frequency, the value of T_{min} is determined by the choice of the scaling constant $\left(2^{11} - 1\right)$ which is used to define the frequency as a BRM multiplicand. The value of this constant is limited by its additional use in the fractional representation of ΔY_k. In this use, the constant cannot be less than the maximum value of ΔY_k of a particular simulation problem. Considering all possible problems, foreknowledge of the maximum values of ΔY_k is not available. However, a difference between successive samples in excess of $\left(2^{11} - 1\right)$ would result in a predicted value for the next sample that would be off scale. It is assumed that the sampling rate in any problem will be such as to prevent this occurrence. In this case, the scale factor $\left(2^{11} - 1\right)$ may be used for all simulation problems.

TESTS OF THE PROTOTYPE

Qualitative tests of the prototype rate augmented digital-to-analog converter were made by comparing time histories of its output signal with those of a zero-order-hold converter. The input information for these tests was generated by computing the values of a series of points of analytically describable time-varying functions. These data were recorded on digital magnetic tape which was used as the actual input to the converters.

Three parameters were investigated: type of input function, sample interval, and choice of extrapolation formula. The selected test functions were a sine wave, the response of a first-order filter to a step function input, and the response of a second-order filter to a step function input. For each function, both first-order and second-order extrapolation formulas were used to compute the ΔY_k input for the rate augmented converter. The second-order extrapolation was of the minimum terminal error type. The zero-order-hold and the rate augmented converters were operated in parallel; the former received only the Y_k input sample and the latter received both the Y_k and the ΔY_k samples.

TEST RESULTS AND DISCUSSION

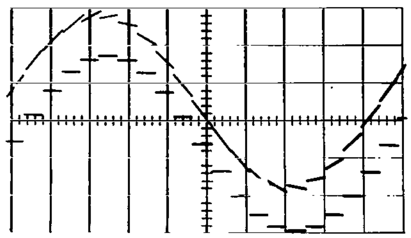
Figure 3 shows the oscilloscope traces of the output signals of both the rate augmented and the zero-order-hold converters for sine waves which were sampled at 20, 40, 60, and 120 samples per cycle. The first-order extrapolation formula

$$\Delta Y_k = Y_k - Y_{k-1} \tag{3}$$

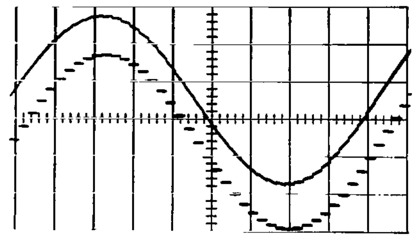
was used for this test. At all four sampling ratios the rate augmented converter provided a more accurate representation of the continuous function over most of each cycle. However, first-order extrapolation resulted in overextrapolations at the sine wave peaks. Calculated values range from 9 percent of peak amplitude at 20 samples per cycle to 0.2 percent at 120 samples per cycle. Converter outputs for sine wave samples, where the second-order extrapolation formula

$$\Delta Y_k = 2Y_k - 3Y_{k-1} + Y_{k-2} \tag{4}$$

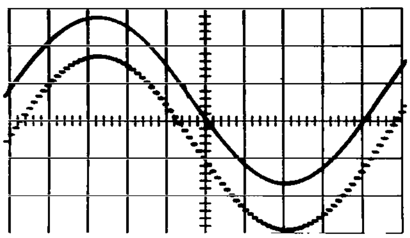
was used, are shown in figure 4. The overextrapolation errors for the four sampling ratios are reduced to a range from 1.4 percent to 0.002 percent.



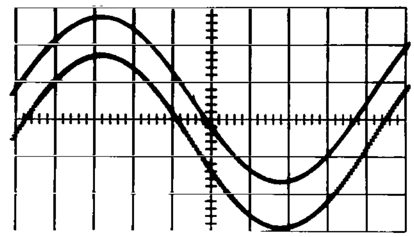
(a) 20 samples per cycle.



(b) 40 samples per cycle.

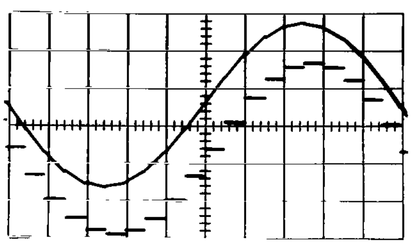


(c) 60 samples per cycle.

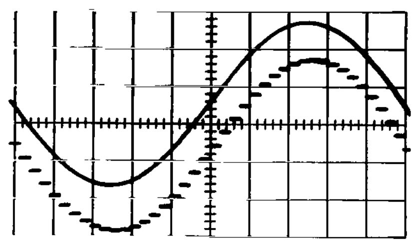


(d) 120 samples per cycle.

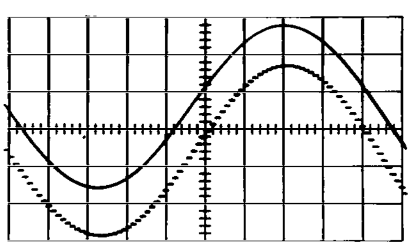
Figure 3.- Rate augmented DAC output using $\Delta Y_k = Y_k - Y_{k-1}$ compared with zero-order-hold output (lower trace) for sine wave.



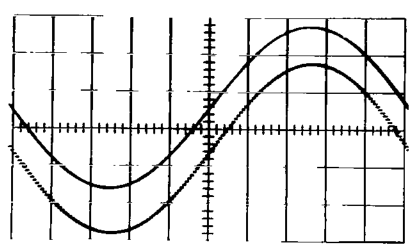
(a) 20 samples per cycle.



(b) 40 samples per cycle.



(c) 60 samples per cycle.



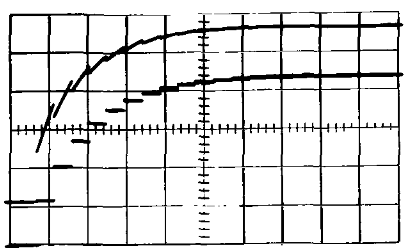
(d) 120 samples per cycle.

Figure 4.- Rate augmented DAC output using $\Delta Y_k = 2Y_k - 3Y_{k-1} + Y_{k-2}$ compared with zero-order-hold output (lower trace) for sine wave.

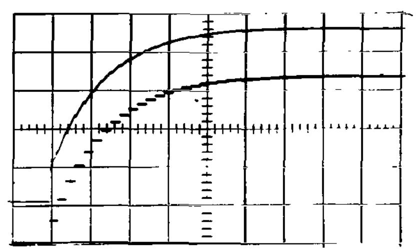
Converter outputs for samples of the response of a first-order filter to a step function disturbance as obtained by first-order and second-order extrapolation formulas are shown in figures 5 and 6, respectively. For these tests, a specific function frequency could not be defined. Thus, the sampling ratios chosen were $20f_c$, $40f_c$, $60f_c$, and $120f_c$. As can be seen in figure 6, the second-order extrapolation formula caused a significant error at the beginning of the step function response. By comparing figures 5 and 6, however, it can be seen that beyond this point the second-order extrapolation formula provided better smoothing than did the first-order formula.

Converter outputs for samples of the response of a second-order filter to a step function disturbance are shown in figures 7 and 8. The response was computed for a filter with 70.7 percent critical damping. The sampling ratios used were $20f_d$, $40f_d$, $60f_d$, and $120f_d$. Unlike that of the first-order filter, the response of the second-order filter to a step function disturbance does not have a sharp discontinuity. As can be seen by comparing figures 7 and 8, better smoothing was achieved for this time-varying function by using the second-order extrapolation formula with the rate augmented converter system.

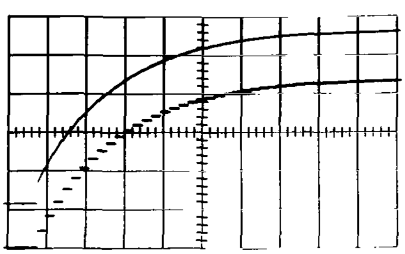
In addition to demonstrating the feasibility of the digital techniques which were utilized to provide rate augmentation, the tests afforded an empirical basis for comparing the effectiveness of the first- and second-order formulas which were used to determine the linear extrapolation rates. For the time-varying functions used, a more accurate



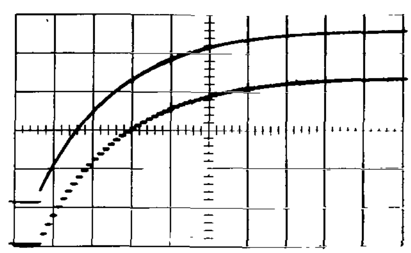
(a) 20 samples per cycle of f_c.



(b) 40 samples per cycle of f_c.

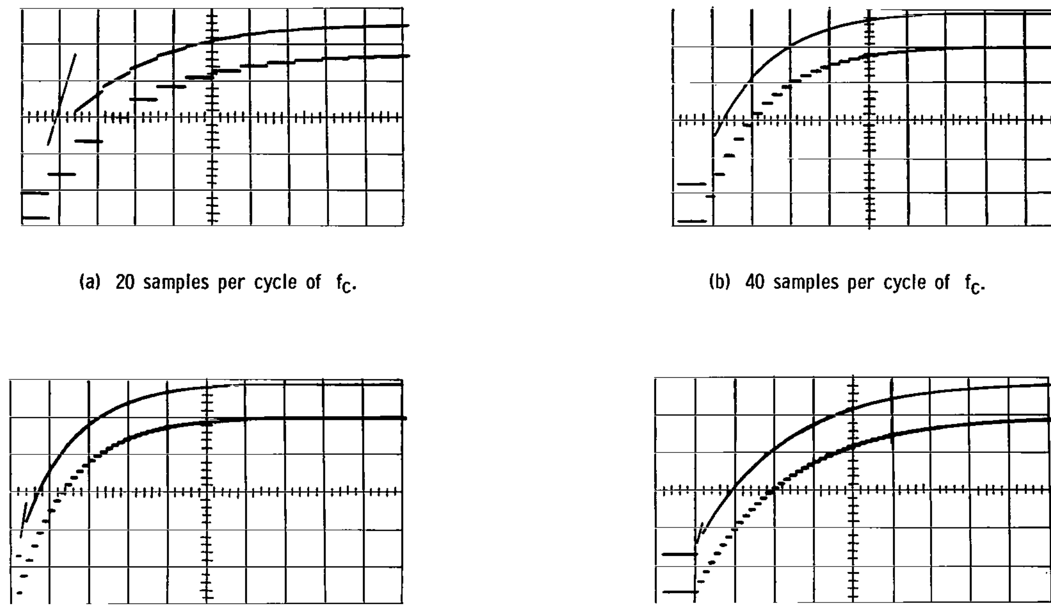


(c) 60 samples per cycle of f_c.



(d) 120 samples per cycle of f_c.

Figure 5.- Rate augmented DAC output using $\Delta Y_k = Y_k - Y_{k-1}$ compared with zero-order-hold output (lower trace) for first-order step response.



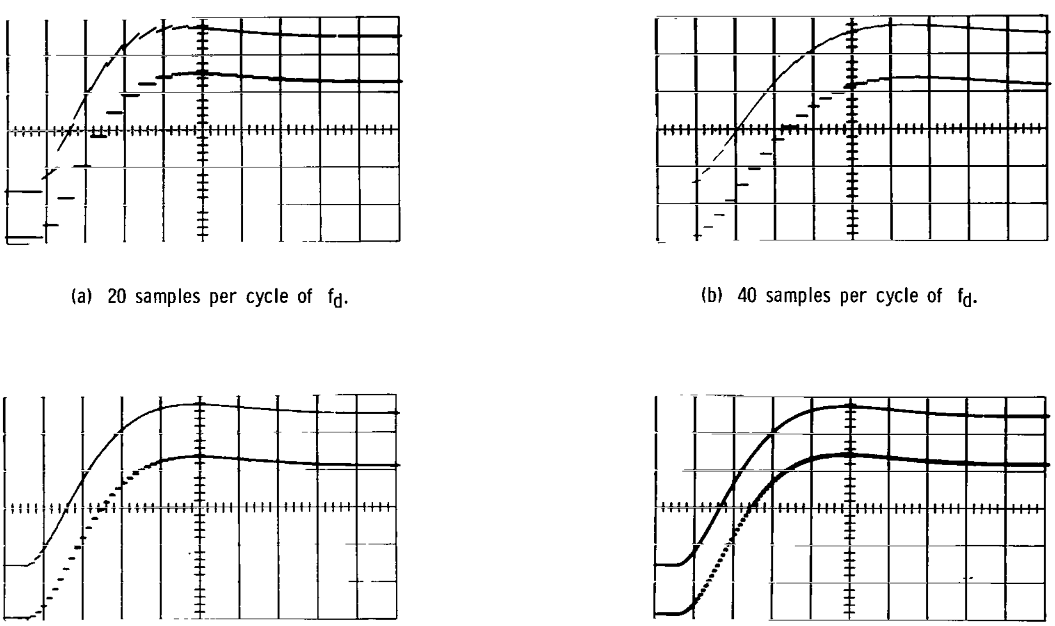
(a) 20 samples per cycle of f_c.

(b) 40 samples per cycle of f_c.

(c) 60 samples per cycle of f_c.

(d) 120 samples per cycle of f_c.

Figure 6.- Rate augmented DAC output using $\Delta Y_k = 2Y_k - 3Y_{k-1} + Y_{k-2}$ compared with zero-order-hold output (lower trace) for first-order step response.



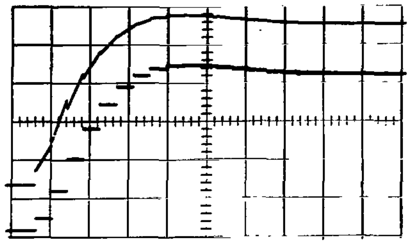
(a) 20 samples per cycle of f_d.

(b) 40 samples per cycle of f_d.

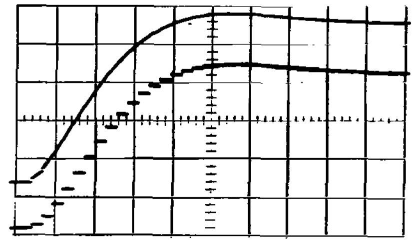
(c) 60 samples per cycle of f_d.

(d) 120 samples per cycle of f_d.

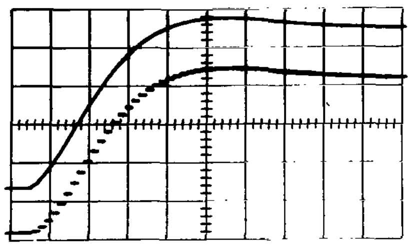
Figure 7.- Rate augmented DAC output using $\Delta Y_k = Y_k - Y_{k-1}$ compared with zero-order-hold output (lower trace) for second-order step response.



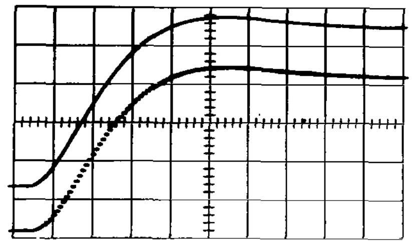
(a) 20 samples per cycle of f_d.



(b) 40 samples per cycle of f_d.



(c) 60 samples per cycle of f_d.



(d) 120 samples per cycle of f_d.

Figure 8.- Rate augmented DAC output using $\Delta Y_k = 2Y_k - 3Y_{k-1} + Y_{k-2}$ compared with zero-order-hold output (lower trace) for second-order step response.

representation of the continuous function was obtained, in general, with the second-order extrapolation formula. However, its effectiveness was reduced where the time-varying function had sharp discontinuities.

For $\Delta Y_k = 0$, the system counter acts as a static register which contains Y_k. The static accuracy of the converter system is therefore determined by that of the DAC modules which in the prototype is 0.06 percent of full range.

For nonzero ΔY_k, the converter system is subject to the following errors that are given in percent of the system range which is $\pm\left(2^{11} - 1\right)$ or $\left(2^{12} - 2\right)$:

(1) Clock inhibition: In figure 2, the system clock is inhibited one clock pulse before the counter is preset to a new starting value. The inhibition continues for the next two clock pulses. At the maximum extrapolation rate $\left(\Delta Y_k/T = 1 \text{ MHz}\right)$, this inhibition results in a constant error or 200/4094 percent to which is added an error of 100/4094 percent for the last clock interval.

(2) Incrementing error: The derivation of the maximum value of the incrementing error that results from the use of the binary rate multipliers is given in appendix A. This error is less than $\frac{4000}{9 \times 4094}$ percent.

(3) Propagation error: In the prototype system, the synchronous counting technique is used for all but the five least significant stages of the up/down counter. On even counts, an error occurs because of the propagation time (0.25 microsecond per stage) of these five stages. The error increases in logarithmic increments during the total propagation interval. In the worst case the total interval is 1.25 microseconds and the peak error (during the last 0.25 microsecond) is 3100/4094 percent.

AN OPERATIONAL MULTICHANNEL CONVERTER SYSTEM

After the prototype was tested, the economic value of certain design alterations became apparent. These changes are indicated in the block diagram of a proposed multichannel system (fig. 9). The system would be designed for a scale of N bits plus sign.

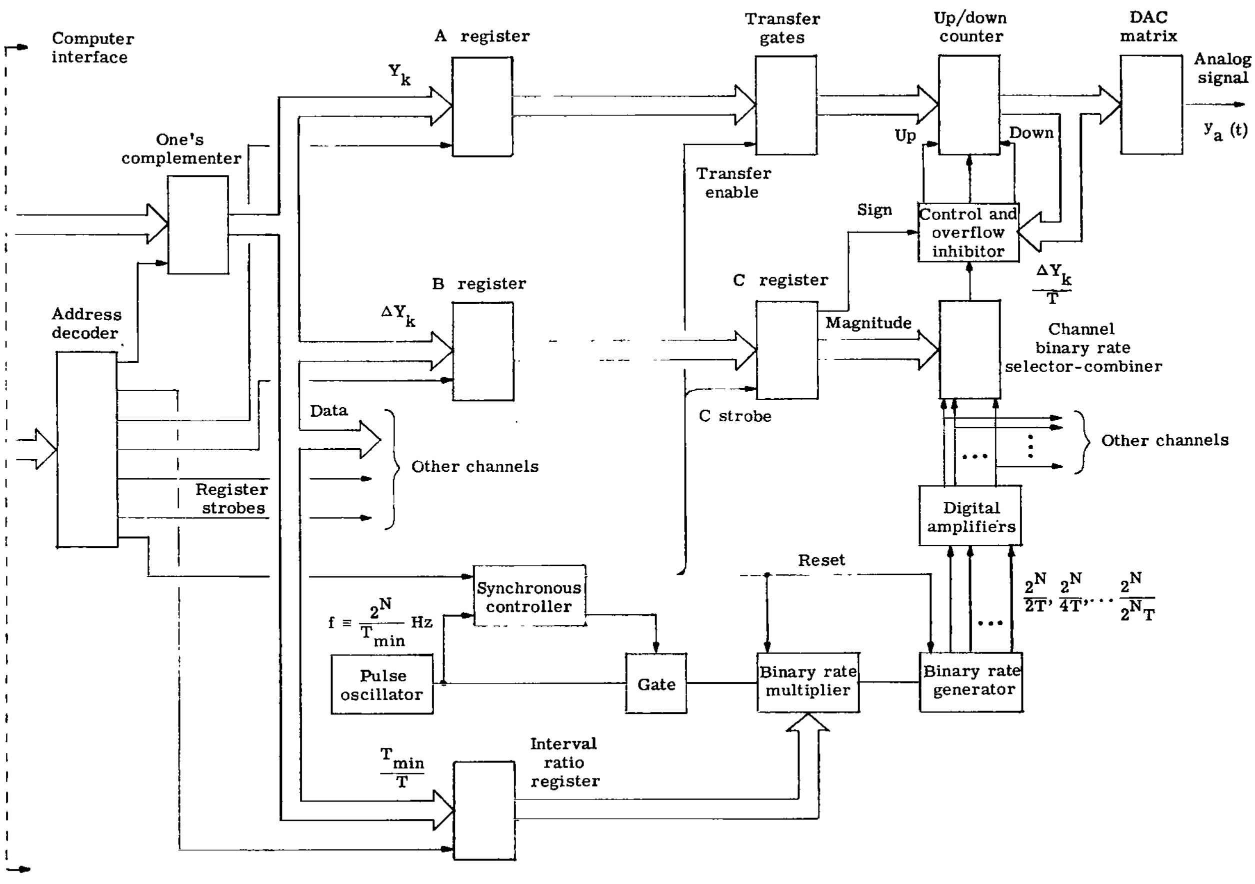


Figure 9.- Block diagram of operational multichannel rate augmented DAC system.

One change would eliminate one computer division operation required to prepare the input data for each channel by redefining the ΔY input fraction as $\Delta Y_k / 2^N$ and redefining T_{min} as 2^N divided by the oscillator frequency. Since the data transmitted from the computer to the converter system are in fixed-point binary format, an imaginary relocation of the binary point changes the computed ΔY_k to the binary fraction $\Delta Y_k / 2^N$.

Cost reduction of a multichannel system would be effected by relocation of certain logical functions into sections which serve all the channels. Accordingly, the one's complementer is located so that all data pass through it as they are addressed to the individual converter channels. In this equipment, the A and B (or ΔY_k) input registers of each channel may be considered to be independent, for addressing purposes, and even addresses may be assigned to all the B registers. A single one's complementer can then be made to serve all channels by adding to it the logic elements required to assure that only data addressed to even numbered registers are complemented.

Where sign-magnitude arithmetic format is used in the computer, it would be necessary to convert the format of the Y_k data while leaving unaltered that of the ΔY_k data. The format conversion would be inverted (i.e., sign-magnitude to one's complement), and the address logic would be changed to insure that the Y_k data were converted.

A second equipment cost reduction would be achieved by separating the functions of the channel binary rate multiplier, used in the prototype, into a binary rate generator and a binary rate selector-combiner. The binary rate generator would be located in the section which served all channels of a multichannel system, as shown in figure 9. Its output would be a parallel set of pulse trains at the rates of $2^N/2T$, $2^N/4T$, . . ., $2^N/2^N T$ pps. These pulse trains would be transmitted, through digital amplifiers, to all channels. Each channel would have a binary rate selector-combiner, whose output is a single pulse train. For a particular channel the pulse rate in pps would be ΔY_k, for that channel, divided by T.

Figure 9 also shows the interval ratio register which is used to change the time scaling whenever the simulation problem is changed. The fraction T_{min}/T, for the new problem, is transferred from the computer to the register as part of the initializing routine of the digital simulation program. The interval ratio operates on the system clock frequency; thus the time scaling for all channels is set simultaneously.

SYSTEM TRANSFER FUNCTIONS

The analysis of the combination of the computer-solved extrapolation formula and the rate augmented converter as the equivalent of a filter, the input of which is a series of impulse samples of a continuous function, is given in appendix B. In most instances, the available information regarding a sampled function, before performing the simulation, describes the frequency response characteristics of the simulated system from which it originates rather than the time domain characteristics of the function itself. In these cases, the response characteristics of the conversion filter are useful in estimating the smoothing effectiveness of a particular extrapolation formula.

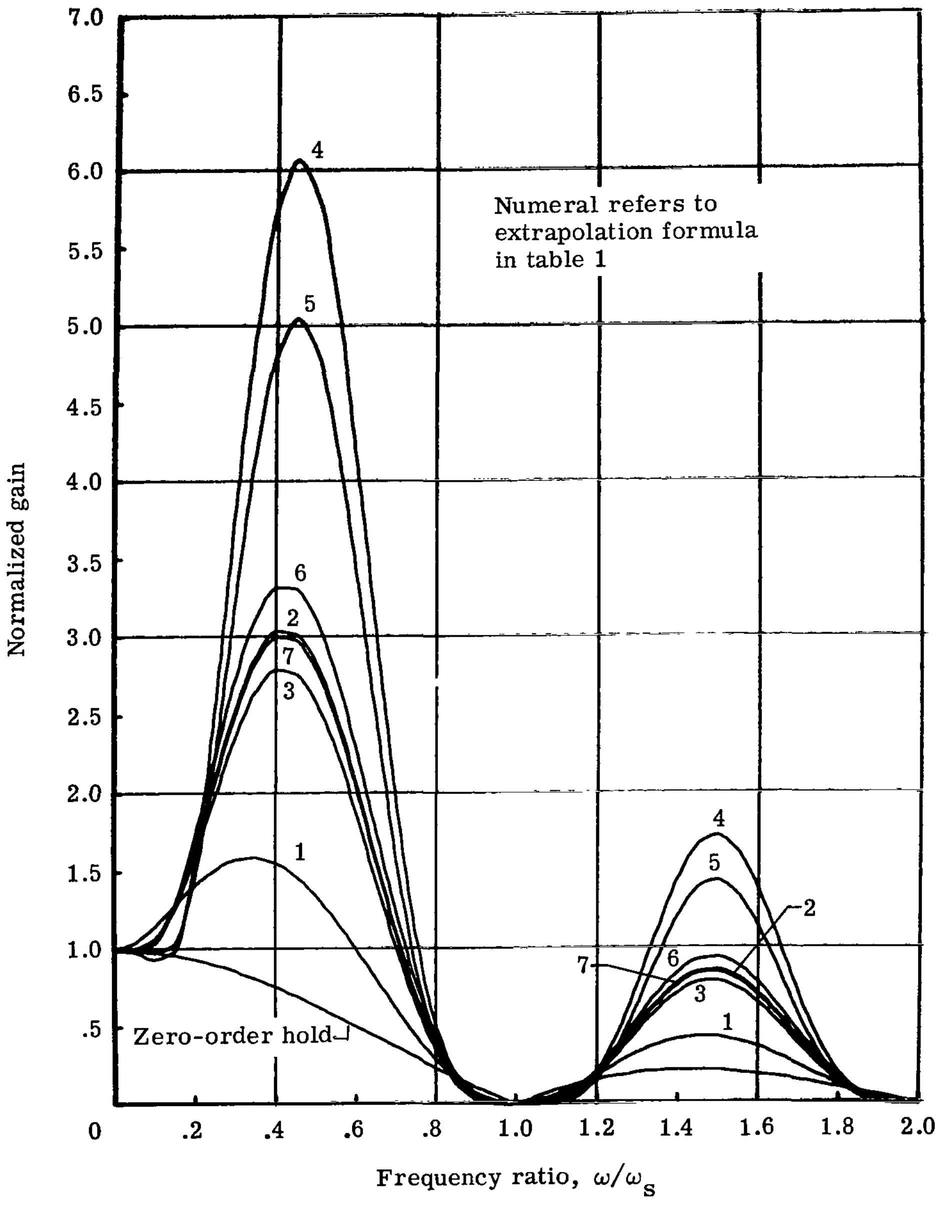
For the general extrapolation formula, the transfer function of the conversion filter is

$$
\begin{aligned}
G(j\omega) = \frac{T}{(\omega T)^2}\Bigg\{\Big[&-c_0 + \left(c_0 - c_1\right)\cos \omega T + \left(c_1 - c_2\right)\cos 2\omega T + \ldots + \left(c_{n-1} - c_n\right)\cos n\omega T \\
&+ c_n \cos(n + 1)\omega T + \left(1 + c_0\right)\omega T \sin \omega T + c_1\omega T \sin 2\omega T + c_2\omega T \sin 3\omega T \\
&+ \ldots + c_n\omega T \sin(n + 1)\omega T\Big] + j\Big[-\omega T + \left(1 + c_0\right)\omega T \cos \omega T + c_1\omega T \cos 2\omega T \\
&+ c_2\omega T \cos 3\omega T + \ldots + c_n\omega T \cos(n + 1)\omega T - \left(c_0 - c_1\right)\sin \omega T \\
&- \left(c_1 - c_2\right)\sin 2\omega T - \ldots - \left(c_{n-1} - c_n\right)\sin n\omega T - c_n \sin(n + 1)\omega T\Big]\Bigg\}
\end{aligned}
\tag{5}
$$

where T is the period between samples. For a specific extrapolation formula, both the normalized gain $\left|G(j\omega)/T\right|$ and the phase response of the conversion filter can be readily computed as a function of the ratio of signal frequency to sampling frequency ω/ω_s. This normalized form is convenient since the transfer function of the sampling process by which the input data are obtained has an overall gain factor $1/T$, which can be canceled by the factor T of equation (5).

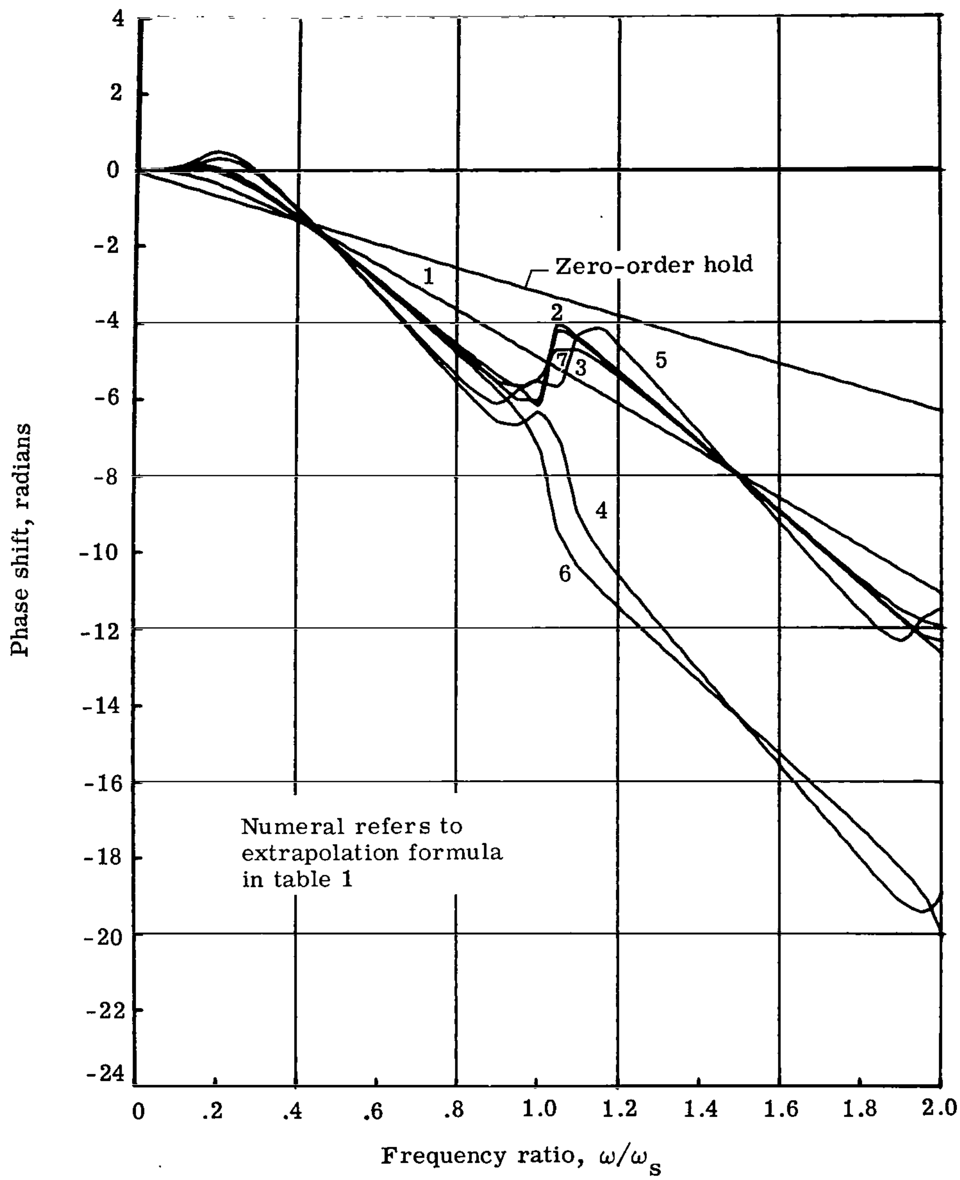
Gain and phase responses of the conversion filter were computed for the extrapolation formulas in table 1. The first five formulas are based on conventional polynomial curve fitting and include the two formulas for which experimental results were obtained. The last two formulas are designed to provide good extrapolations near the peaks of sine waves sampled approximately 20 times per cycle and, like polynomial-based formulas, provide exact extrapolations for linear functions. These last two formulas are derived in appendix C.

The computed gain and phase responses are shown in figure 10, where the range beyond $\omega/\omega_s = 0.5$ describes the response to the first two of the infinite number of complementary signals (ref. 2) which are created by the sampling process. The response of zero-order-hold, or conventional, converter is included for comparison. In figure 11, these responses are plotted to a larger scale for $\omega/\omega_s = 0$ to 0.07.



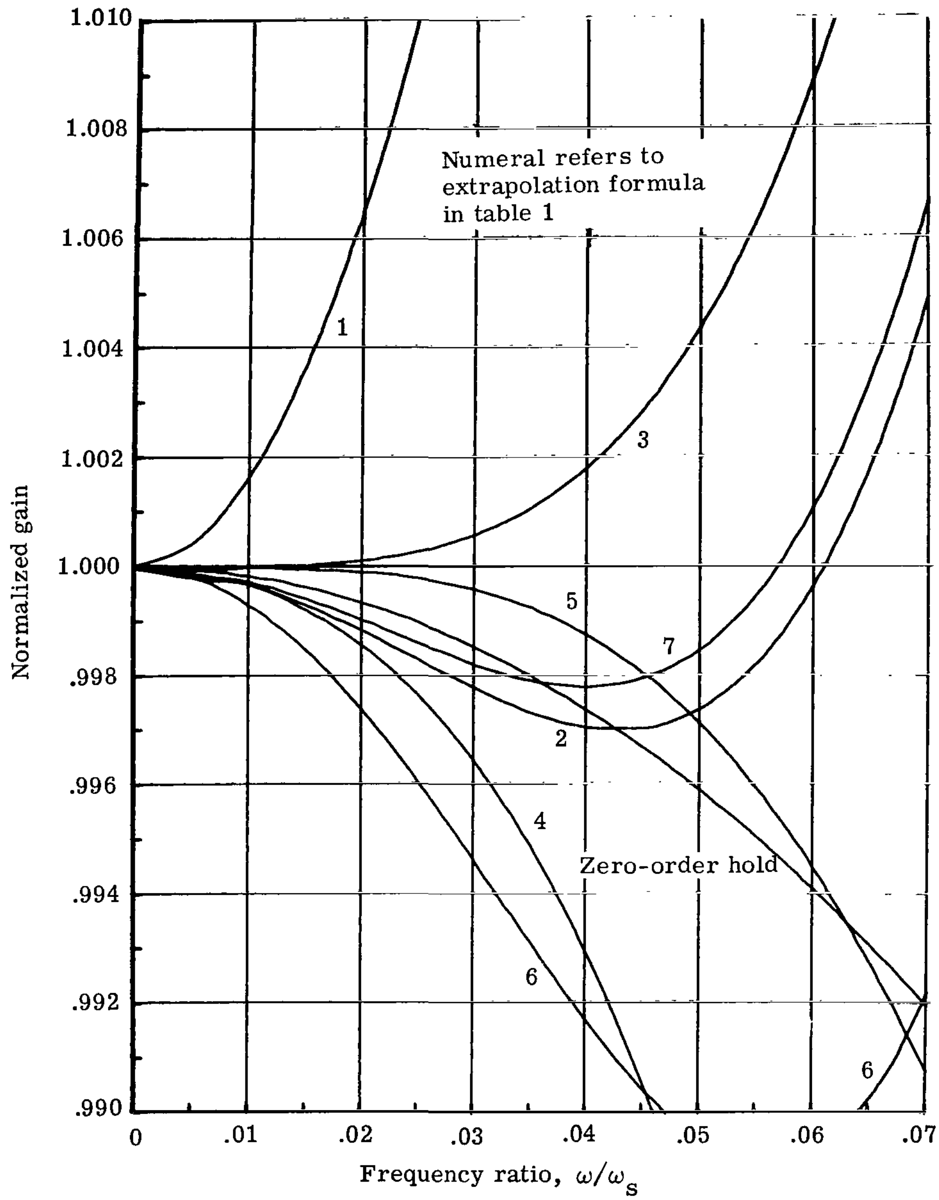
(a) Gain responses.

Figure 10.- Conversion filter responses.



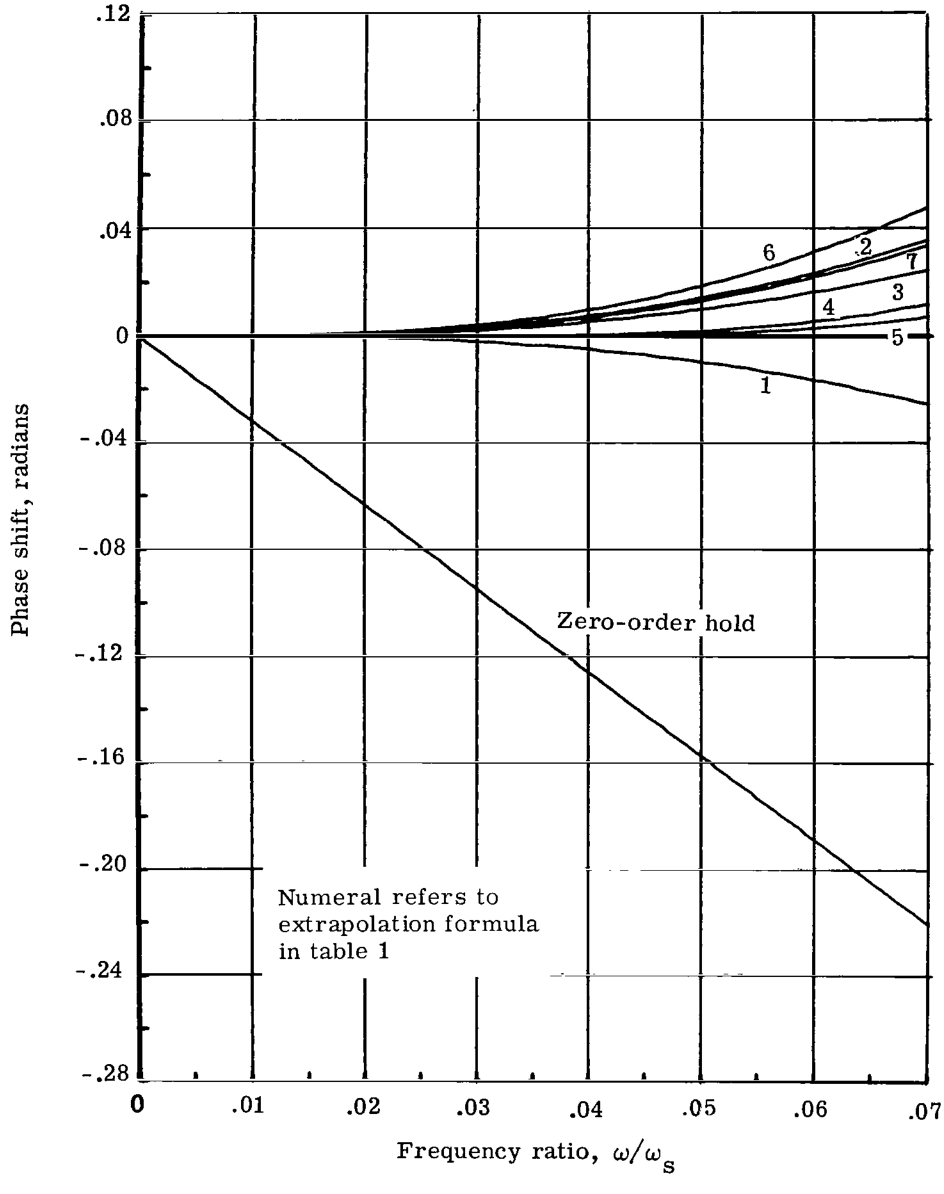
(b) Phase responses.

Figure 10.- Concluded.



(a) Gain responses.

Figure 11.- Conversion filter responses at low frequencies.



(b) Phase responses.

Figure 11.- Concluded.

Table 2 lists the gain and phase responses at $\omega/\omega_S = 0.05$ as well as the gain at the corresponding first complementary frequency $(\omega/\omega_S = 0.95)$. A relative evaluation of the extrapolation formulas can be made by comparing the response characteristics and by correlating the characteristics with available experimental results. For example, a comparison of the conventional converter with the first-order-formula—rate-augmented-converter combination at $\omega/\omega_S = 0.05$ indicates that the large errors which the conventional converter exhibits in the time domain (fig. 3(a)) are caused by its relatively large phase lag, at this frequency, and poor attenuation of complementary signals. The first-order-formula—rate-augmented-converter combination has both lower phase lag and better complementary signal attenuation. However, its gain at the data frequency is 4 percent greater than unity. The result is seen in the noticeable overshoots near the peak of the test sine wave.

For samples of arbitrary functions which are band restricted to a radian frequency of approximately $0.05\omega_S$, four of the formulas in table 1 provide good information frequency response (both amplitude and phase) coupled with good attenuation of complementary signals. In the order of which the responses approach the ideal conversion filter, they are the second-order minimum terminal error, three-point sine wave fitted minimum average error, third-order minimum terminal error, and the third-order minimum average error formulas. For this frequency range, the response characteristics (figs. 10 and 11, and table 2) for formula 5 (and the converter) are extremely close to those of an ideal conversion filter. However, the gain responses (fig. 10(a)) begin to rise rapidly above $\omega/\omega_S = 0.05$. The choice of an extrapolation formula should, therefore, be conditioned by the gain roll-off characteristics of the simulated systems from which the function samples are obtained. The magnitudes of the gain peaks (fig. 10(a)) are indicative of the relative sharpness required in the roll-off. This point is corroborated by the experimental results obtained for the second-order minimum terminal error formula for step functions passed through both first-order and second-order low-pass filters (figs. 6 and 7).

CONCLUDING REMARKS

A rate augmented digital-to-analog converter (DAC) has been developed and utilized in a conversion technique for calculated time-dependent data which uses a computer-solved extrapolation (or interpolation) formula to determine the change of the continuous function between successive sampled data points. Extrapolation formulas would be used for closed-loop or real-time processes such as simulation. Experimental and analytical studies of the technique indicate that a significant increase in the accuracy with which a varying function of time is reproduced as a continuous signal can be obtained by extrapolation formulas which are based upon three or more preceding samples. These formulas may be either a minimum terminal error or a minimum average error type. In addition, a choice of formula design may be made between polynomial fitting and sine wave fitting. Calculation of the formula-converter gain and phase characteristics from the generalized transfer function provides a simple and effective means of validating the formula choice.

The use of digital circuitry to provide the rate augmentation in the circuit before the basic DAC module resulted in two additional features:

1. The static accuracy of the system is entirely determined by the DAC module.

2. In a multichannel converter system, only one adjustment is required to match the basic extrapolation rate, for all channels, to the rate at which solutions are obtained from the computer. This adjustment may be simply automated and included in the initialization part of the digital computer program.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., March 30, 1967,
125-19-06-01-23.

APPENDIX A

THE BINARY RATE MULTIPLIER

The logic circuit of the binary rate multiplier (BRM) which was used in the rate augmented digital-to-analog converter is shown in figure 12. The BRM multiplicand input is the pulse train input f_0 to the counter and its multiplier input is the binary code representing a given fraction M. The BRM logic circuit is similar to the circuit described in reference 3 except that the high repetition rate of the BRM input and the length of the counter dictated the use of the synchronous counting technique. This technique, however, facilitated the generation of pulses which marked the transition of each counter flip-flop to the set state. These pulses, when gated in accordance with the code representing M and then combined, make up the output pulse train.

A typical timing diagram for a BRM for which the pulse input occurs at regular intervals is shown in figure 13 for $M = \frac{19}{32}$.

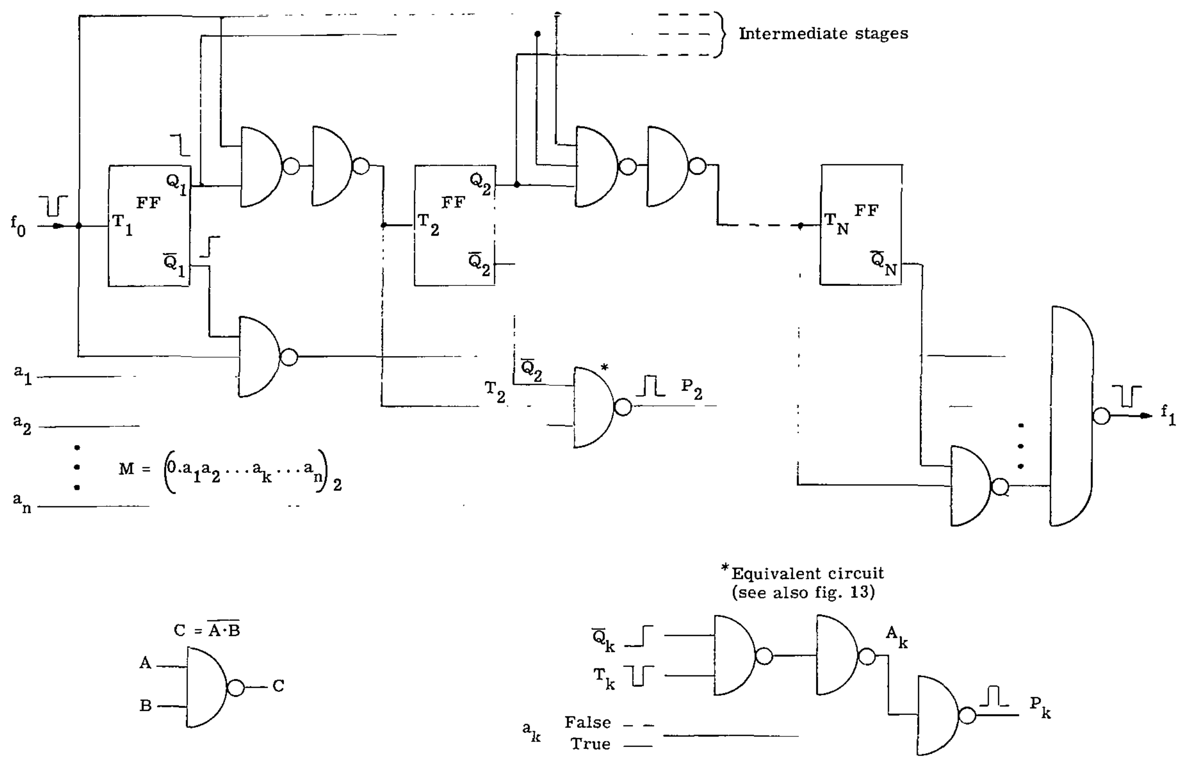


Figure 12.- Logic circuit of binary rate multiplier.

If the pulse rate of the multiplicand is f_0, the multiplier is defined as

$$M = \left(0.a_1 a_2 \ . \ . \ . \ a_k \ . \ . \ . \ a_n\right)_2 \tag{A1}$$

and a_k is the least significant bit that is a "1", then for 2^k input pulses the BRM will emit $M2^k$ pulses. Thus, for input and output rates averaged over integral multiples of the period $2^k/f_0$, the circuit performs the exact multiplication

$$f_1 = Mf_0 \tag{A2}$$

Within these intervals the output may be viewed as a frequency-modulated pulse train of carrier frequency f_1.

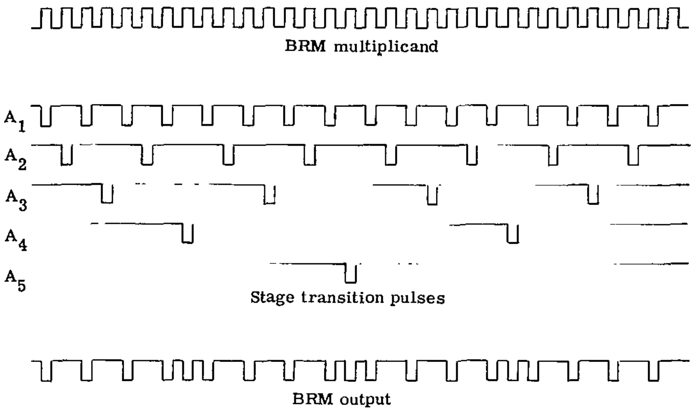


Figure 13.- Binary rate multiplier signals. $M = \frac{19}{32}$ (that is, 0.10011_2).

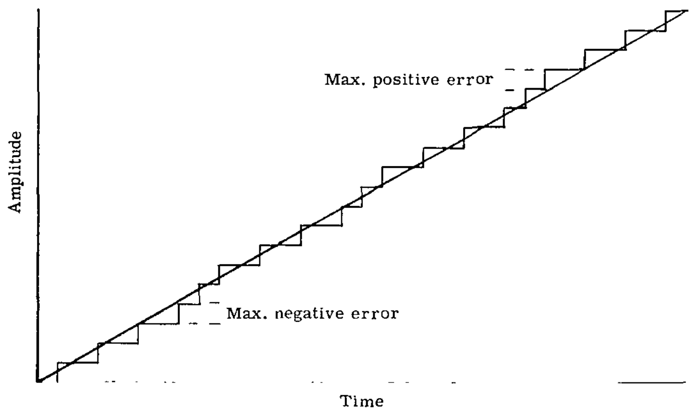


Figure 14.- Sum of BRM pulses compared with continuous ramp.

In the rate augmented digital-to-analog converter, the output of a BRM increments a binary counter to approximate a continuous ramp. If the counter were incremented at constant pulse spacing, unidirectional errors of one count would occur. However, the "frequency-modulated" output of a BRM produces both positive and negative errors. At a given point the magnitude of the error is a function of both M and the number of pulses n into the BRM; this is illustrated in figure 14 for $M = \frac{19}{32}$.

Reference 3 presents a method of determining the error for a single BRM preceding the counter. For a BRM containing N stages, the value of the maximum possible error E_{max} is

$$E_{max} = \frac{7}{18} + \frac{N}{6} + \frac{\left(-\frac{1}{2}\right)^N}{9} \tag{A3}$$

Table 3 lists both the time of occurrence of the maximum positive errors and the causative multiplier as a function of the number of stages.

APPENDIX A

For the rate augmented digital-to-analog converter, the error of interest is the maximum error which may occur when the counter is preceded by two BRM's in series. A general analytic solution for the exact value of this error is not available in the literature. However, the limiting value of the error can be derived by expressing the error E(n,M) (input count of n pulses, multiplier of M) calculated for each BRM (treated as an independent single unit) as an error in the timing of the output pulse relative to the period of the ideal output frequency. This timing error is illustrated in figure 15 for an "unmodulated" input to the first BRM at frequency f_0 and for multipliers $M_1 = \frac{5}{8}$ and $M_2 = \frac{7}{8}$. The ideal output of the first BRM is represented in figure 15(c). Each output pulse lags its corresponding pulse in the actual output (fig. 15(b)) by $\frac{E(n_1,M_1)}{f_1}$ seconds, where subscript 1 denotes the position of the BRM in the series.

If the pulse train in figure 15(c) were the output of the first BRM and, hence, the input to the second BRM, the output of the latter BRM would be that in figure 15(d). The ideal output of this multiplier, at frequency $M_1M_2f_0$, is shown in figure 15(e). Each pulse in the ideal output lags the corresponding pulse in figure 15(d) by

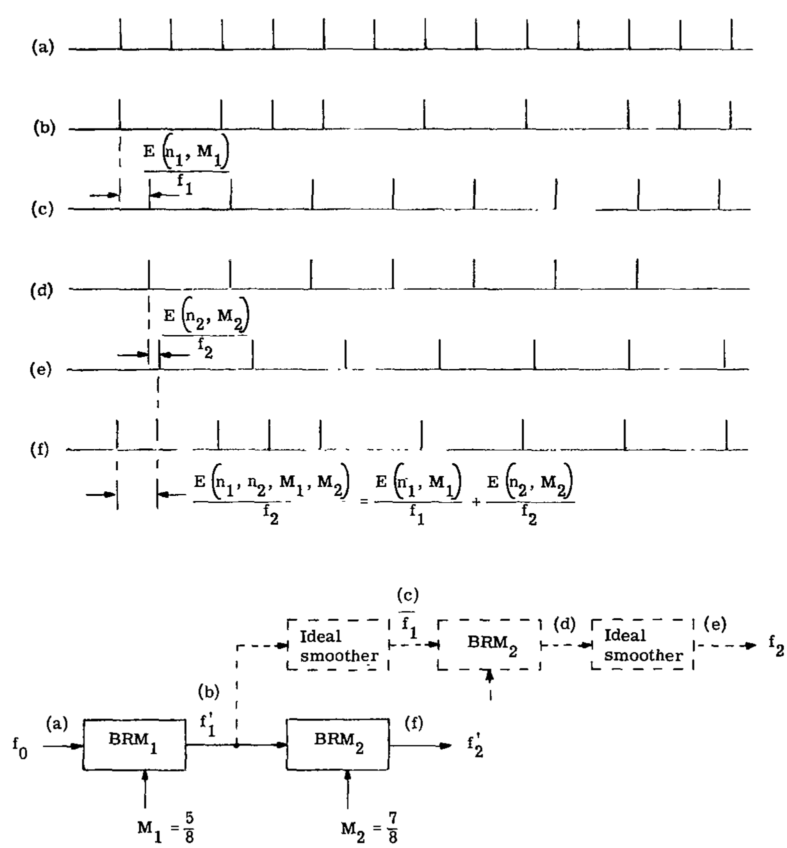


Figure 15.- Pulse trains for two binary rate multipliers in series. (Primes indicate "unmodulated" pulse trains.)

$\frac{E(n_2,M_2)}{f_2}$ seconds. However, the actual output of the second BRM is the pulse train in figure 15(f). Further, each pulse of figure 15(f) occurs at the same time as a pulse of figure 15(b) and each pulse of figure 15(d) occurs at the same time as a pulse of figure 15(c). Thus, the lag between corresponding pulses of the ideal and actual outputs of the second BRM is

$$\frac{E(n_1,n_2,M_1,M_2)}{f_2} = \frac{E(n_1,M_1)}{f_1} + \frac{E(n_2,M_2)}{f_2} \tag{A4}$$

By substituting f_2/M_2 for f_1, the error at a given pulse of the output of the series of BRM's is

$$E(n_1,n_2,M_1,M_2) = M_2E(n_1,M_1) + E(n_2,M_2) \tag{A5}$$

For given multipliers M_1 and M_2, the maximum output error occurs when the individual maxima coincide. The amplitude of this error is maximized if M_1 is either of the two values that permits a maximum possible error for the first BRM. It is not necessarily maximized by setting M_2 at the upper value that permits a maximum possible error for the second BRM since a higher value may result in a greater increase in the first term than the decrease in the second term. However, since $M_2 < 1$ and the error of either multiplier cannot exceed that calculated from equation (A3), the limit of the error for the two BRM's in series can be stated as

$$\left|E_{1,2}\right| < \left|E_{N_1}\right| + \left|E_{N_2}\right| \tag{A6}$$

where N_1 and N_2 denote the number of stages in the first and second BRM's, respectively.

APPENDIX B

GENERALIZED TRANSFER FUNCTION OF THE CONVERSION SYSTEM

The rate augmented digital-to-analog converter requires two inputs to produce the output $y_a(t)$ and, therefore, it does not have a transfer function in the conventional sense. However, the total system, which includes implementing an extrapolation formula within the digital computer and the converter, does have a single input, which is the series of computed values (or samples) Y_k. The system therefore has a definable transfer function which can be determined by the conventional method of dividing the Laplace transform of its response to a disturbance by the transform of the disturbance and then replacing the Laplacian operator s with $j\omega$.

As stated in the literature, such as reference 2, the series of samples is a train of unit impulses each of which is multiplied by the value of the function at the time of sampling. The transform of one unit impulse is unity. The system transfer function, in Laplace notation, is thus equal to the transform of its response to one unit impulse.

The response of the system to an impulse may be determined from the equation which defines its output in response to a set of samples – that is, from

$$y_a(t) = Y_k + \frac{\left(c_0 Y_k + c_1 Y_{k-1} + \ldots + c_n Y_{k-n}\right)\left(t - t_k\right)}{T} \qquad \left(t_k \leqq t < t_{k+1}\right) \tag{B1}$$

where the weighting coefficients are determined by the choice of extrapolation formula. It is evident that a specific sample assumes different weights in each succeeding interval with the response to that sample extending over a number of intervals equal to the number of weighting coefficients.

The normalized response of the system to a single sample is shown in figure 16. The transform of this response is

$$\begin{aligned} F(s) = \frac{T}{(sT)^2}\Bigg\{&\left(c_0 + sT\right) - sT\left[\left(1 + c_0\right)e^{-sT} + c_1 e^{-2sT} + c_2 e^{-3sT} + \ldots + c_n e^{-(n+1)sT}\right] \\ &- \left[\left(c_0 - c_1\right)e^{-sT} + \left(c_1 - c_2\right)e^{-2sT} + \ldots + \left(c_{n-1} - c_n\right)e^{-nsT} + c_n e^{-(n+1)sT}\right]\Bigg\} \end{aligned} \tag{B2}$$

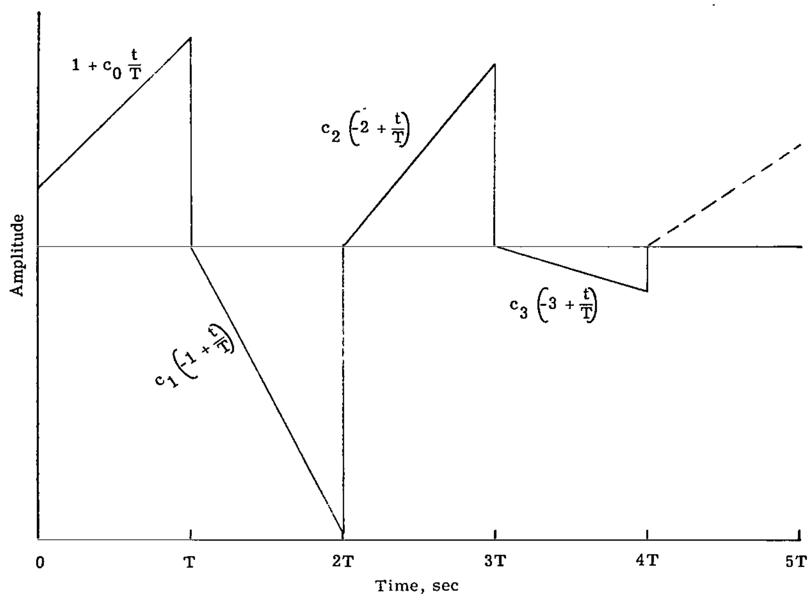


Figure 16.- Unit impulse response of a linear extrapolation filter.

The generalized transfer function of the conversion system is therefore

$$
\begin{aligned}
G(j\omega) = \frac{T}{(\omega T)^2}\Big\{\Big[&-c_0 + \left(c_0 - c_1\right)\cos \omega T + \left(c_1 - c_2\right)\cos 2\omega T + \ldots + \left(c_{n-1} - c_n\right)\cos n\omega T \\
&+ cn \cos(n + 1)\omega T + \left(1 + c_0\right)\omega T \sin \omega T + c_1\omega T \sin 2\omega T + c_2\omega T \sin 3\omega T \\
&+ \ldots + c_n\omega T \sin(n + 1)\omega T\Big] + j\Big[-\omega T + \left(1 + c_0\right)\omega T \cos \omega T + c_1\omega T \cos 2\omega T \\
&+ c_2\omega T \cos 3\omega T + \ldots + c_n\omega T \cos(n + 1)\omega T - \left(c_0 - c_1\right)\sin \omega T \\
&- \left(c_1 - c_2\right)\sin 2\omega T - \ldots - \left(c_{n-1} - c_n\right)\sin \omega T - cn \sin(n + 1)\omega T\Big]\Big\}
\end{aligned}
\qquad \text{(B3)}
$$

APPENDIX C

NONPOLYNOMIAL EXTRAPOLATION FORMULAS

Extending a parabola fitted to three equally spaced points on a function in order to predict the succeeding point results in overshoot at the peak values of sine waves. However, an exact peak prediction can be made for a specific ratio of sample interval to sine wave period if the coefficients of the extrapolation formula

$$\Delta Y_k = c_0 Y_k + c_1 Y_{k-1} + c_2 Y_{k-2} \tag{C1}$$

are chosen such that

$$1 - \sin\left[\frac{\pi}{2}\left(1 - \frac{4\omega}{\omega_s}\right)\right] = c_0 \sin\left[\frac{\pi}{2}\left(1 - \frac{4\omega}{\omega_s}\right)\right] + c_1 \sin\left[\frac{\pi}{2}\left(1 - \frac{8\omega}{\omega_s}\right)\right] + c_2 \sin\left[\frac{\pi}{2}\left(1 - \frac{12\omega}{\omega_s}\right)\right] \tag{C2}$$

where the sample interval, expressed in radians, is $2\pi\omega/\omega_s$. However, two constraints are required in addition to equation (C2) to define the coefficients. A useful pair of constraints is that the extrapolation formula yield exact extrapolations when $y(t)$ is linear and that this occur regardless of the value of Y_k. For a linear function $y(t)$ the first constraint results in

$$\Delta Y_k = \Delta Y_{k-1} = \Delta Y_{k-2} \; . \; . \; . \; Y_{k-n} \equiv \Delta Y$$

and, for the general form of the extrapolation formula, in

$$\Delta Y = c_0 Y_k + c_1\left(Y_k - \Delta Y\right) + c_2\left(Y_k - 2\Delta Y\right) + \; . \; . \; . + c_n\left(Y_k - n\Delta Y\right) \tag{C3}$$

which reduces to

$$\Delta Y = Y_k\left(c_0 + c_1 + \; . \; . \; . + c_n\right) - \Delta Y\left(c_1 + 2c_2 + \; . \; . \; . + nc_n\right) \tag{C4}$$

Since no restraint is placed on Y_k,

$$\sum_{0}^{j=n} c_j = 0 \tag{C5}$$

and

$$\sum_{1}^{j=n} jc_j = -1 \tag{C6}$$

APPENDIX C

The solution of equations (C2), (C5), and (C6) for $n = 2$ and a specific ratio of ω/ω_s provides a minimum terminal error extrapolation formula. The formula which yields exact peak values for sine waves sampled every 18.48° (approximately 20 samples per cycle) and which also has rational coefficients is

$$\Delta Y_k = \frac{35}{16} Y_k - \frac{54}{16} Y_{k-1} + \frac{19}{16} Y_{k-2} \tag{C7}$$

By comparison, a second-order minimum terminal error extrapolation formula gives peak overshoots of 1.5 percent at this sampling ratio.

A minimum average error extrapolation is defined as one for which, over the interval $t_k \leqq t = t_{k+1}$, the average error between the linear approximation and $y(t)$ is equal to zero and which is expressed by

$$\int_{t_k}^{t_{k+1}} \left[Y_k + \frac{\Delta Y_k (t - t_k)}{T} \right] dt = \int_{t_k}^{t_{k+1}} y(t) dt \tag{C8}$$

However, $y(t)$ is only known to be some function which passes through its sampled values and which is expected to pass near a predicted next value. The most convenient function to assume in order to solve equation (C8) is a polynomial whose order is one less than the number of points to which it is fitted. Although equation (C7) is based upon three function samples, it is not derived from a polynomial. Therefore, the predicted next value

$$Y_{k+1} = Y_k + \left(\frac{35}{16} Y_k - \frac{54}{16} Y_{k-1} + \frac{19}{16} Y_{k-2} \right) \tag{C9}$$

may be used with the sampled values Y_{k-2} through Y_k to define $y(t)$ as a third-order polynomial. If this third-order polynomial is used to define $y(t)$, the solution of equation (C8) results in the minimum average error extrapolation formula

$$\Delta Y_k = \frac{379}{192} Y_k - \frac{566}{192} Y_{k-1} + \frac{187}{192} Y_{k-2} \tag{C10}$$

REFERENCES

1. Training and Education Group: Data Sampling and Filtering for Hybrid Computation. Lecture 7 of Introduction to Hybrid Computation. Electron. Associates, Inc., c.1963.

2. Truxal, John G.: Automatic Feedback Control System Synthesis. McGraw-Hill Book Co., Inc., 1955, pp. 500-506.

3. [Mergler, H. W.]: Notes on Digital Control Systems Engineering. Vol. 2, Eng. Div. of Case Inst. Technol., pp. 10-1 – 10-57.

TABLE 1.- WEIGHTING COEFFICIENTS OF EXTRAPOLATION FORMULAS

$$\left[\Delta Y_k = c_0 Y_k + c_1 Y_{k-1} + c_2 Y_{k-2} + \cdots c_n Y_{k-n}\right]$$

Formula	Extrapolation type	c_0	c_1	c_2	c_3
1	1st order	1	-1	0	0
2	2d-order, minimum terminal error	2	-3	1	0
3	2d-order, minimum average error	11/6	-16/6	5/6	0
4	3d-order, minimum terminal error	3	-6	4	-1
5	3d-order, minimum average error	31/12	-59/12	37/12	-9/12
6	3-point sine wave fitted, minimum terminal error	35/16	-54/16	19/16	0
7	3-point sine wave fitted, minimum average error	379/192	-566/192	187/192	0

TABLE 2.- SYSTEM RESPONSE AT $\omega/\omega_s = 0.05$ AND 0.95

Formula	Extrapolation type	$\omega/\omega_s = 0.05$		$\omega/\omega_s = 0.95$
		Normalized gain, $\left\lvert\frac{G(j\omega)}{T}\right\rvert$	Phase angle, Φ, rad	Normalized gain, $\left\lvert\frac{G(j\omega)}{T}\right\rvert$
	Conventional converter (zero-order hold)	0.996	-0.1571	0.0524
1	1st order	1.040	-.0097	.0166
2	2d-order, minimum terminal error	.997	+.0143	.0080
3	2d-order, minimum average error	1.004	+.0101	.0076
4	3d-order, minimum terminal error	.988	+.0024	.0040
5	3d-order, minimum average error	.997	+.0013	.0036
6	3-point sine wave fitted, minimum terminal error	.989	+.0190	.0095
7	3-point sine wave fitted, minimum average error	.998	+.0136	.0078

TABLE 3.- OCCURRENCE OF POSITIVE MAXIMUM MULTIPLIER ERROR FOR AN N STAGE BINARY RATE MULTIPLIER

	Multiplier	Error occurrence, input counts
N odd	$\frac{2^{N+1} - 1}{3 \times 2^N}$	$\frac{2^{N+1} - 1}{3}$
	$\frac{5 \times 2^{N-1} + 1}{3 \times 2^N}$	$\frac{5 \times 2^{N-1} + 1}{3}$
N even	$\frac{5 \times 2^{N-1} - 1}{3 \times 2^N}$	$\frac{2^{N+1} + 1}{3}$
	$\frac{2^{N+1} + 1}{3 \times 2^N}$	$\frac{5 \times 2^{N-1} - 1}{3}$

NASA-Langley, 1967 —— 8 L-4738

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546