

NASA TM X- 55966

NUMERICAL INTEGRATION OF ORBITS IN MULTIREVOLUTION STEPS

C. E. VELEZ

JANUARY 1967



GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

FACILITY FORM 602	N67-39995	
	(ACCESSION NUMBER)	(THRU)
	22	1
	(PAGES)	(CODE)
TMX-55966	30	
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)	

X-542-67-341

NUMERICAL INTEGRATION OF ORBITS IN
MULTIREVOLUTION STEPS

C. E. Velez

January 1967

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

NUMERICAL INTEGRATION OF ORBITS IN
MULTIREVOLUTION STEPS

C. E. Velez

ABSTRACT

A predictor-corrector algorithm applicable to the numerical integration of an artificial earth satellite in multirevolution steps is reviewed. The multirevolution predictor and corrector formulas are developed to yield general recursive relations for the coefficients, which are similar to those for the Adams' integration formulas. In addition, the applicability of the algorithm to orbit calculations is demonstrated numerically.

PRECEDING PAGE BLANK NOT FILMED.

CONTENTS

	<u>Page</u>
ABSTRACT	iii
INTRODUCTION	1
FORMULATION	1
THE MULTIREVOLUTION ALGORITHM	9
NUMERICAL RESULTS	11
SUMMARY AND CONCLUSIONS	15
REFERENCES	15
APPENDIX—THE EQUATIONS OF MOTION	17

PRECEDING PAGE BLANK NOT FILMED.

NUMERICAL INTEGRATION OF ORBITS IN MULTIREVOLUTION STEPS

INTRODUCTION

Many artificial satellite orbit determination problems, such as lifetime studies or computing long-range predictions, require the numerical integration of the equations of motion over many periods of revolution. An algorithm designed to save computing time in solving such problems has been proposed in a paper by Cohen and Hubbard (1960). Essentially, the method involves combining the numerical integration with a procedure which "steps" the calculations ahead in multirevolution increments. This "stepping" procedure, both in formulation and usage, is similar to the Adams' predictor-corrector process in numerical integration. Briefly, a cycle in the algorithm consists of first extrapolating or "predicting" the orbital elements n -revolutions ahead. Then, starting with these extrapolated values, the equations of motion are integrated over one revolution. Finally, the extrapolated values are improved by using a corrector formula along with the information obtained from the single revolution integration.

The multirevolution "predictor" or extrapolation formula was introduced in a paper by Mace and Thomas (1960). In this paper it was demonstrated that if successive revolutions are similar, so that the successive descending nodes are close together, one may extrapolate without loss of accuracy as many revolutions as will make the change in the orbital elements from node to extrapolated node about the same as the change in step by step integration over one revolution. Supplementing this work, Cohen and Hubbard developed a corrector formula for this n -revolution extrapolation. In addition, it was also shown that this corrector formula reduces to the Adams' formula in the limiting case, i.e., the coefficients of the corrector were shown to be polynomials in $1/n$ with the constant terms being the Adams' coefficients.

In this paper, the similarity in derivation between the multirevolution formulas and the Adams' predictor-corrector formulas will be explored and used to develop recursive formulas for the coefficients. In addition, the multirevolution algorithm will be detailed and examples demonstrating its usage will be given.

FORMULATION

Let f be a function defined on an interval $[x_0, x_m]$ with equal spacing h (i.e., $x_i - x_{i-1} = h$, $i = 1, 2, \dots, m$) and assume f is continuously differentiable.

Consider the difference operators ∇ , Δ , E^a , D , 1 defined by

$$\nabla f(x) = f(x) - f(x-h) \quad (\text{backward difference})$$

$$\Delta f(x) = f(x+h) - f(x) \quad (\text{forward difference})$$

$$E^a f(x) = f(x+ah) \quad (\text{shifting operator})$$

$$Df(x) = f'(x) \quad (\text{differentiating operator})$$

$$1 f(x) = f(x) \quad (\text{identity})$$

where $x \in [x_0, x_m]$. Also, if L is any one of these operators, define L^n by

$$L^n f(x) = L(L^{n-1} f(x)).$$

Some basic relations between these operators are*

$$\Delta = \frac{\nabla}{1 - \nabla} \quad (1)$$

$$E = 1 + \Delta \quad (2)$$

$$E = e^{hD} \quad (3)$$

We begin by deriving the Adams' integration formulas. The predictor can be derived by expressing the forward difference operator in terms of backward

*Equality interpreted in the sense of operators. For verification of these formulas and justification for the algebraic manipulation of these operators see *Hildebrand*, pp. 128-134.

differences of the derivative. We have from (2) and (3) that

$$hD = -\log(1-\nabla) \quad (4)$$

and hence, using (1),

$$\begin{aligned} \Delta &= \frac{\nabla}{1-\nabla} = \frac{\nabla}{1-\nabla - \log(1-\nabla)} hD \\ &= h \left\{ \sum_{i=0}^{\infty} \alpha_i \nabla^i \right\} D \end{aligned} \quad (5)$$

where the coefficients α_i can be determined as follows:

Using the expressions

$$\frac{x}{1-x} = x \sum_{i=0}^{\infty} x^i \quad (6)$$

and

$$-\log(1-x) = x \sum_{i=0}^{\infty} \frac{x^i}{i+1} \quad (7)$$

we have

$$\frac{\nabla}{1-\nabla - \log(1-\nabla)} = \frac{\sum_{i=0}^{\infty} \nabla^i}{\sum_{i=0}^{\infty} \frac{\nabla^i}{i+1}} = \sum_{i=0}^{\infty} \alpha_i \nabla^i$$

where the α_i are given recursively by

$$\left. \begin{aligned} \alpha_0 &= 1 \\ \alpha_i &= 1 - \sum_{j=1}^i \beta_j \alpha_{i-j} \quad i = 1, 2, 3, \dots \end{aligned} \right\} \quad (8)$$

where

$$\beta_j = \frac{1}{j+1} \quad (9)$$

Hence, using (5) and (8), our formula emerges in the form

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ &= h \left\{ 1 + \frac{1}{2} \nabla + \frac{5}{12} \nabla^2 + \dots \right\} f'(x), \end{aligned} \quad (10)$$

which is the familiar Adams-Bashforth formula used to obtain a "predicted" value of $f(x+h)$ when solving 1st order initial value problems. The corresponding corrector formula can be derived by expressing the backward difference operator in terms of backward differences of the derivative:

As above, using (4), we have

$$\nabla = \frac{\nabla}{-\log(1-\nabla)} hD = h \left\{ \sum_{i=0}^{\infty} \alpha_i^* \nabla^i \right\} D \quad (11)$$

where the coefficients α_i^* are given recursively by

$$\left. \begin{aligned} \alpha_0^* &= 1 \\ \alpha_i^* &= - \sum_{j=1}^i \beta_j \alpha_{i-j}^* \end{aligned} \right\} \quad (12)$$

where the β_j are given by (9).

We have then, using (11) and (12),

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h) \\ &= h \left\{ 1 - \frac{1}{2} \nabla - \frac{1}{12} \nabla^2 - \frac{1}{24} \nabla^3 - \dots \right\} f'(x),\end{aligned}\quad (13)$$

which is the Adams-Moulton formula used to obtain a corrected value of $f(x)$ in the numerical integration process.

We note that from (5) and (11) we have

$$[1 + \nabla + \nabla^2 + \dots][\alpha_0^* + \alpha_1^* \nabla + \dots] = \alpha_0 + \alpha_1 \nabla + \dots$$

so equating coefficients of like powers, we have the relationship

$$\sum_{i=0}^k \alpha_i^* = \alpha_k \quad (14)$$

We now wish to derive the multirevolution predictor and corrector formulas. This is done precisely as above, except that differences of the derivative are replaced by differences at a large interval of differences at a small interval. To this end, we define the differences Δ_n and ∇_n by

$$\Delta_n f(x) = f(x+nh) - f(x)$$

$$\nabla_n f(x) = f(x) - f(x-nh),$$

where n is same fixed integer, i.e., Δ_n and ∇_n are the forward and backward differences taken at n times the step of Δ and ∇ . A relation between ∇_n and Δ is then given by

$$(1 - \nabla_n) = (1 + \Delta)^{-n}$$

since

$$\begin{aligned}
 (1 - \nabla_n) f(x) &= f(x - nh) . \\
 &= \mathbf{E}^{-n} f(x) \\
 &= (1 + \Delta)^{-n} f(x) .
 \end{aligned}$$

So we have, in analogy with (4),

$$\Delta = (1 - \nabla_n)^{-1/n} - 1 , \tag{15}$$

and derive the multirevolution predictor or extrapolation formula by expressing the forward difference Δ_n in terms of backward differences ∇_n^i of the forward difference Δ . We have from (1) and (15), in analogy with (5),

$$\begin{aligned}
 \Delta_n &= \frac{\nabla_n}{1 - \nabla_n} = \frac{\nabla_n}{1 - \nabla_n} \frac{1}{(1 - \nabla_n)^{-1/n} - 1} \Delta \\
 &= n \left\{ \sum_{i=0}^{\infty} \gamma_i \nabla_n^i \right\} \Delta
 \end{aligned} \tag{16}$$

where the coefficients γ_i , which are polynomials in $1/n$, can be determined using series (6) and the expansion

$$(1 - x)^\alpha = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} x^i . \tag{17}$$

We have

$$\begin{aligned}
 (1 - \nabla_n)^{-1/n} - 1 &= \sum_{i=1}^{\infty} (-1)^i \binom{-1/n}{i} \nabla_n^i \\
 &= \frac{\nabla_n}{n} \sum_{i=0}^{\infty} b_i \nabla_n^i
 \end{aligned}$$

where $b_0 = 1$ and

$$b_k = (-1)^k \frac{\prod_{j=1}^k \left(-\frac{1}{n} - j\right)}{(k+1)!} \quad k = 1, 2, 3, \dots \quad (18)$$

and hence

$$\begin{aligned} \frac{\nabla_n}{1 - \nabla_n} \frac{1}{(1 - \nabla_n)^{-1/n} - 1} &= \frac{n \sum_{i=0}^{\infty} \nabla_n^i}{\sum_{i=0}^{\infty} b_i \nabla_n^i} \\ &= n \sum_{i=0}^{\infty} \gamma_i \nabla_n^i \end{aligned}$$

where the γ_i are given recursively by

$$\left. \begin{aligned} \gamma_0 &= 1 \\ \gamma_i &= 1 - \sum_{j=1}^i b_j \gamma_{i-j} \quad i = 1, 2, 3, \dots \end{aligned} \right\} \quad (19)$$

Using (16) and (19), we then obtain

$$\begin{aligned} \Delta_n f(x) &= f(x + nh) - f(x) \\ &= \left\{ 1 + \frac{1}{2} \left(1 - \frac{1}{n}\right) \nabla_n + \frac{1}{12} \left(5 - \frac{6}{n} + \frac{1}{n^2}\right) \nabla_n^2 \right. \\ &\quad \left. + \dots \right\} n \Delta f(x), \quad (20) \end{aligned}$$

which is the formula used to compute the "predicted" or extrapolated value of $f(x + nh)$ by using the backward differences ∇_n^i of the forward difference of $f(x)$ at $1/n^{\text{th}}$ the step.

Again, in analogy with the integration formulas, the corresponding multi-revolution corrector can be derived by expressing the backward difference ∇_n in terms of backward differences ∇_n^i of Δ .

We have (see (11)),

$$\nabla_n = \frac{\nabla_n}{(1 - \nabla_n)^{1/n} - 1} \Delta = n \left\{ \sum_{i=0}^{\infty} \gamma_i^* \nabla_n^i \right\} \Delta \quad (21)$$

where, as above, we find that the coefficients γ_i^* are given recursively by

$$\left. \begin{aligned} \gamma_0^* &= 1 \\ \gamma_i^* &= - \sum_{j=1}^i b_j \gamma_{i-j}^* \quad i = 1, 2, 3, \dots \end{aligned} \right\} \quad (22)$$

where the b_i are given by (18).

Using (21) and (22) we obtain

$$\begin{aligned} \nabla_n f(x) &= f(x) - f(x - nh) \\ &= \left\{ 1 - \frac{1}{2} \left(1 + \frac{1}{n} \right) \nabla_n - \frac{1}{12} \left(1 - \frac{1}{n^2} \right) \nabla_n^2 - \dots \right. \\ &\quad \left. - \dots \right\} n \Delta f(x), \quad (23) \end{aligned}$$

which is the formula used to compute a corrected value of $f(x)$ by using a forward difference of an extrapolated value for $f(x)$.

We note the similarity between the coefficients given by (8), (19) and (12), (22). We also note, in analogy with (13), the relation

$$\sum_{i=0}^k \gamma_i^* = \gamma_k .$$

Finally, we remark that in practice, an algebraic equivalent of Equations (10) and (13) known as the "summed" form of the difference formulae is often used. It has been established (Henrici, 1962) that this modification considerably reduces the propagated round-off error associated with these formulae. Formally, one can obtain this modification by applying the inverse operator ∇^{-1} to both sides of Equations (10) and (13). In like manner, one can obtain the "summed" form of the extrapolation formulae by applying the operator ∇_n^{-1} to both sides of Equations (20) and (23).

THE MULTIREVOLUTION ALGORITHM

Let f_j denote the value of an orbital element at the descending node of the j^{th} revolution and let n be the number of revolutions to be "stepped." Also let k be the order of the highest backward difference ∇_n^i to be retained in formulas (16) and (21). We then have the multirevolution formulas:

$$\Delta_n f_j = n \sum_{i=0}^k \gamma_i \nabla_n^i (\Delta f_j) \quad (24)$$

$$\nabla_n f_j = n \sum_{i=0}^k \gamma_i^* \nabla_n^i (\Delta f_j) \quad (25)$$

We see that, in analogy with the Adams' integration process, a set of starting values must be computed before these formulas can be used. Assume then that the values $\Delta f_i = f_{i+1} - f_i$, $i = 0, n, 2n, 3n, \dots, kn$, have been computed, i.e., we assume knowledge of the orbital elements at descending nodes $0, 1, n, n+1, 2n, 2n+1, \dots, kn, kn+1$. (See table on next page). Using these values, we can compute the differences

$$\nabla_n^i (\Delta f_{kn}) , \quad i = 1, 2, \dots, k ,$$

i.e.,

$$\nabla_n (\Delta f_{kn}) = \Delta f_{kn} - \Delta f_{(k-1)n}$$

$$\nabla_n^2 (\Delta f_{kn}) = \Delta f_{kn} - 2\Delta f_{(k-1)n} + \Delta f_{(k-2)n} ,$$

etc. We can then form a "starting" table (entries above the diagonal):

f_0	Δf_0			
f_1				
\vdots				
\vdots				
f_n	Δf_n			
f_{n+1}				
\vdots				
\vdots				
f_{2n}	Δf_{2n}			
f_{2n+1}				
\vdots				
\vdots		$\nabla_n (\Delta f_{(k-1)n})$		$\nabla_n^k (\Delta f_{kn})$
$f_{(k-1)n}$	$\Delta f_{(k-1)n}$		$\nabla_n^2 (\Delta f_{kn})$	\vdots
$f_{(k-1)n+1}$		$\nabla_n (\Delta f_{kn})$		$\nabla_n^k (\Delta f_{(k+1)n})$
\vdots				\vdots
\vdots				\vdots
f_{kn}	Δf_{kn}		$\nabla_n^2 (\Delta f_{(k+1)n})$	\vdots
f_{kn+1}		$\nabla_n (\Delta f_{(k+1)n})$		
\vdots				
\vdots				
$f_{(k+1)n}$	$\Delta f_{(k+1)n}$			
$f_{(k+1)n+1}$				

Figure 1

The multirevolution cycle is then:

(a) Extrapolate the orbital elements n revolutions ahead using (24) with $j = nk$, i.e., we compute a "predicted" value for $f_{(k+1)n}$, the element at the descending node of the $(k+1)n^{\text{th}}$ revolution.

(b) Starting with these extrapolated values, we integrate one revolution to obtain the elements at descending node $(k+1)n+1$ and form the forward difference

$$\Delta f_{(k+1)n} = f_{(k+1)n+1} - f_{(k+1)n} .$$

(c) We form the backward differences $\nabla_n^i (\Delta f_{(k+1)n})$, $i = 1, 2, \dots, k$ and use formula (25) with $j = (k+1)n$ to obtain a corrected value for $f_{(k+1)n}$, and the cycle is complete. The process is then repeated with $j = (k+1)n$ in (a).

We make the following remarks concerning the computation procedure:

(i) Values of the orbital elements at the descending node of any revolution can be obtained by inverse interpolation, i.e., first, a direct interpolation is made to find the time of nodal crossing, followed by an inverse interpolation to obtain the values of the orbital elements.

(ii) Other points in the orbit such as perigee, apogee or ascending node could be used instead of the descending node.

(iii) The time associated with the extrapolated orbital elements can also be obtained by the extrapolation process, allowing one to obtain values for the orbital elements at any point in a computed revolution.

(iv) For any particular orbit, the number of revolutions which can be "stepped" by this process and yet maintain some predictable accuracy can be determined by comparing the change in the orbital elements from node to node with the change required by the short step integration.

(v) For any particular orbit, the number of differences $\nabla_n^{(i)}$ to be retained in the extrapolation formulas can be determined precisely as in the case of the integration formula, i.e., retaining differences of sufficiently high order so that the "truncated" terms are within the accuracy of the computations.

NUMERICAL RESULTS

The equations of motion considered in the numerical computations are of the form

$$\ddot{\bar{x}} = -\frac{\mu\bar{x}}{R^3} + \bar{F}_1 + \bar{F}_2 , \quad (26)$$

where $\bar{x} = (x, y, z)$, $R = (x^2 + y^2 + z^2)^{1/2}$, \bar{F}_1 and \bar{F}_2 perturbation functions, and μ is a constant. The formulation of these perturbations and the values of the related constants which were used are given in the appendix.

Before beginning, we make the following remarks concerning the numerical results:

(i) All short-step numerical integrations were performed with an Adams-Cowell 13th order predictor-corrector process (Henrici, pp. 192-194, 291-293), using a Runge-Kutta type starter, so that this process was used both for the comparison integrations (Table A), and for the short-step integrations required by the multirevolution algorithm.

(ii) A predictor-corrector tolerance of 1×10^{-12} was used for all the Cowell integrations. The number of derivative evaluations tabulated in the tables below was obtained by multiplying the number of steps taken at the Cowell time step h by the average number of predictor-corrector iterations taken over the range of integration.

(iii) For the examples, three orbits were used where in each case the integrations were performed for approximately 100 revolutions:

<u>Initial Elements*</u>	<u>Example I</u>	<u>Example II</u>	<u>Example III</u>
Semi-major axis (earth radii)	1.15	1.26	6.71
Eccentricity	.075	.072	.003
Inclination	1.52	1.03	.0004
Mean anomaly	6.03	3.71	3.80
Arg. of perigee	1.15	3.14	0.31
Long. of ascending node	4.76	6.16	2.29
Period (mins.)	104.25	120.37	1469.46
Range of integration (mins.)	11,000	12,500	148,000

(iv) All computations were performed on the UNIVAC 1108 computer using double-precision arithmetic.

In Table A, the results of the comparison integrations are tabulated. Error estimates were obtained by integrating the Equations (26) with $F_1 = F_2 = 0$, and comparing with the Keplerian solution. For the cases considered, the approximate ratio of the change in the position which occurred at the Cowell time step h , and the corresponding change which occurred from descending node to descending node, was found to be 10 or greater.

*Angles in radians.

Table A

Orbit	h (mins.)	No. of Rev. Computed	Total No. of Der. Eval.	Estimated Error	Computation Time (mins.)
I	1.0	105	17,337	2×10^{-9}	1.14
II	.8	104	15,617	4×10^{-12}	1.09
III	22.0	100	7,257	9×10^{-9}	.71

We now consider the results obtained by applying the multirevolution algorithm to our test cases. The elements used in the extrapolation process were the position and velocity vectors themselves. (The time associated with the extrapolated values was also obtained in this manner.) We begin by tabulating the results obtained by using the predictor formula (24) only (Table B). The error estimate given is the difference between the position at the node of the 100th revolution obtained by the comparison integration (Table A) and by the multirevolution process. In each case, for each n , the order of the process (k) was increased until no change in accuracy occurred.

Table B

Orbit	n	k	Total No. of Der. Eval.	Estimated Error	Computation Time (mins.)
I	3	2	6049	1×10^{-8}	.55
		4	6737	3×10^{-9}	.59
	5	2	4586	3×10^{-8}	.39
		4	5934	2×10^{-9}	.48
	7	2	4337	8×10^{-8}	.35
		4	6343	4×10^{-9}	.48
II	9	2	4542	1×10^{-7}	.36
		4	7209	1×10^{-9}	.52
	3	2	5650	9×10^{-8}	.51
		4	6268	8×10^{-11}	.55
	5	2	4263	5×10^{-7}	.36
		4	5483	6×10^{-11}	.44
7	2	4013	1×10^{-6}	.33	
		4	5834	4×10^{-10}	.45
	6	7656	8×10^{-11}	.57	
		4188	3×10^{-6}	.33	
9	2	6611	2×10^{-9}	.49	
		6	9034	4×10^{-10}	.65
	3	2	2329	6×10^{-7}	.62
		2	1803	5×10^{-7}	.45
7	2	1815	5×10^{-7}	.40	
		2	1923	5×10^{-7}	.37

Examining these results, we find that significant gains in efficiency were obtained by application of the algorithm to the numerical integrations, with no significant loss in accuracy.

We remark that a good portion of the computation time tabulated was used in forming the starting table (Figure 1), and in restarting the short-step integrations required by the process, since these were done with the Runge-Kutta, Adams-Cowell process.

In Table C, the results obtained by applying the predictor (24), followed by an application of the corrector (25), are tabulated. As in Table B, for each n , the order (k) was increased until no change in accuracy occurred.

Table C

Orbit	n	k	Total No. of Der. Eval.	Estimated Error	Computation Time (mins.)
I	3	2	10,903	2×10^{-9}	.99
	5	2	7,317	5×10^{-9}	.65
		4	8,361	3×10^{-9}	.70
	7	2	6,157	1×10^{-8}	.52
		4	7,860	1×10^{-9}	.62
	9	2	5,907	2×10^{-8}	.48
		4	8,271	2×10^{-9}	.62
	II	3	2	10,194	2×10^{-8}
4			10,528	8×10^{-11}	.94
5		2	6,819	7×10^{-8}	.59
		4	7,755	7×10^{-11}	.65
7		2	5,717	2×10^{-7}	.48
		4	7,254	6×10^{-11}	.58
9		2	5,466	4×10^{-7}	.44
		4	7,605	1×10^{-10}	.58
III	3	6	9,744	2×10^{-11}	.72
		2	4,086	6×10^{-7}	.99
	5	2	2,781	5×10^{-7}	.68
		2	2,442	5×10^{-7}	.56
9	2	2,387	5×10^{-7}	.48	

Examining these results, we find that an application of the corrector yielded slight gains in accuracy, but at some expense in efficiency. Again, as in Table B, this cost in efficiency was due to the process used for all the short-step integration required by the algorithm.

At this point, several considerations should be made:

(1) As suggested in References 1 and 4, an iterative process using the multi-revolution formulas could be used to form the required starting table (Figure 1), instead of the short-step integration process. Such a scheme would increase the efficiency of both the predictor and predictor-corrector processes used to obtain the results.

(2) Since much of the computation time, especially in Table C, was used in restarting the short-step integrations by the Runge-Kutta technique an increase in efficiency could be obtained by extrapolating the required values for these integrations, as well as the nodal points.

(3) With more efficient starting processes, it could be feasible to iterate on the corrector (25) to improve the accuracy when using large n . This would be analogous to the Adams' integration process.

SUMMARY AND CONCLUSIONS

The multirevolution algorithm has been reviewed and found to be an effective tool applicable to the orbit determination problem.

The recursive formulas obtained for the coefficients required by the process have been found to be an effective and simple means of obtaining these coefficients for any given n and k . The numerical results have indicated that the "predictor only" scheme can be applied to the common multi-step integration process with a Runge-Kutta starter to yield considerable gains in efficiency and, if more efficient starting processes were used, gains in accuracy due to an application of the corrector could be obtained at little, or no expense in efficiency.

REFERENCES

1. Cohen, C. and Hubbard, E., "An Algorithm Applicable to Numerical Integration of Orbits in Multirevolution Steps," *Astronomical J.*, Vol. 65, October 1960, pp. 454-456.
2. Henrici, P., "Discrete Variable Methods in Ordinary Differential Equations," John Wiley and Sons, Inc., New York, 1962.
3. Hildebrand, F., "Intro. to Numerical Analysis," McGraw-Hill, 1956.
4. Mace, D. and Thomas, L., "An Extrapolation Formula for Stepping the Calculation of an Artificial Satellite Several Revolutions Ahead at a Time," *Astronomical J.*, Vol. 65, June 1960, pp. 300-303.

APPENDIX

THE EQUATIONS OF MOTION

The equations of motion considered in this paper are of the form

$$\ddot{\bar{\mathbf{x}}} = -\frac{\mu\bar{\mathbf{x}}}{R^3} + \bar{\mathbf{F}}_1 + \bar{\mathbf{F}}_2$$

$$\bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0$$

$$\dot{\bar{\mathbf{x}}}(t_0) = \dot{\bar{\mathbf{x}}}_0$$

where $\mathbf{x} = (x, y, z)$, $R = |\bar{\mathbf{x}}| = (x^2 + y^2 + z^2)^{1/2}$, μ is a constant, $\bar{\mathbf{x}}_0, \dot{\bar{\mathbf{x}}}_0$ the given initial position and velocity vectors, $\bar{\mathbf{F}}_1$ the perturbation due to the non-sphericity of the earth, and $\bar{\mathbf{F}}_2$ the perturbation due to drag.

$$\bar{\mathbf{F}}_1 = (F_x, F_y, F_z)$$

is given by

$$F_x = \frac{\mu x}{R^6} \left[J R^2 (5s - 1) + H z (7s - 3) + \frac{K}{6} (-63s^2 + 42s - 3) \right]$$

$$F_y = \frac{\mu y}{R^6} \left[J R^2 (5s - 1) + H z (7s - 3) + \frac{K}{6} (-63s^2 + 42s - 3) \right]$$

$$F_z = \frac{\mu z}{R^6} \left[J R^2 (5s - 1) + \frac{K}{6} (-63s^2 + 70s - 15) \right] + \frac{\mu H}{5R^5} (35s^2 - 30s + 3)$$

PRECEDING PAGE BLANK NOT FILMED.

where $s = (z/R)^2$ and J, H, K are the 2nd, 3rd and 4th harmonics of the earth's potential field. \bar{F}_2 is of the form

$$\frac{C_D A}{2M} \rho |\bar{V}_r| \bar{V}_r$$

where V_r is the relative velocity of the satellite, ρ the atmospheric density* at the satellite position, and A, M, C_D are the cross-sectional area, mass, and drag coefficient of the satellite respectively.

Numerical constants:

Unit of Time = 806.832 secs.

Unit of Length = 6378.388 kms.

$\mu = 1$

$J = +0.162 \times 10^{-2}$

$H = -0.640 \times 10^{-5}$

$K = +0.690 \times 10^{-5}$

$A = +0.116 \times 10^5 \text{ cm}^2$

$M = +0.158 \times 10^6 \text{ gm}$

$C_D = +2.3$

*The density was computed (from a standard model) for satellite heights of less than 1000 kms; otherwise ρ was set to zero. Hence, in our examples, only orbit I was influenced by \bar{F}_2 .