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ABSTRACT

A question that often arises when one is attempting to understand the principles underlying the Brown-Twiss stellar interferometer is "How can there be any kind of interference phenomena with light produced by incoherent sources?" Starting from the familiar interference pattern produced by two coherent sources we may proceed in simple steps to a picture of two incoherent sources producing an interference pattern that jumps about at random. This randomly jumping pattern leaves behind a "footprint," however, in the form of the intensity autocorrelation function. We will see how the autocorrelation function for an extended, incoherent source may be constructed and that it is this function that is measured by the Brown-Twiss stellar interferometer.

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I. Introduction

The physical principles underlying interference phenomena such as the pattern produced by a diffraction grating or a double slit interferometer are understood by practically everyone who has studied physical optics. On the other hand ten or more years after the basic papers of Hanbury, Brown and Twiss[1,2] describing the principles and operation of their post-detection stellar interferometers a considerable area of confusion still exists in the minds of many physicists, both students and professionals, concerning the fundamental concepts employed in their operation. This condition persists in spite of the considerable number of papers which have since been published[3] that investigate this phenomenon in great detail.

There appear to be two main areas of confusion in this matter. The first concerns the correlation of arrival times of two separate photons at two separate detectors. This appears to be in contradiction with a basic principle of quantum electrodynamics which is, as stated by Dirac,[4] "Each photon then interferes only with itself. Interference between two different photons never occurs."

This question has been answered quite clearly by Purcell[5] and by Mandel[6] to the effect that while two photons can not interfere with each other they may be correlated by virtue of their being bosons and their tendancy to clump together in phase space.

It would be difficult to improve on the clarity of the treatment of the quantum problem found in references 5 and 6. A brief discussion of this matter is given in Appendix I, therefore, only for background. In fact, this treatment leans heavily on that of Mandel.[6] The primary result of this discussion is that the operation of the stellar interferometer can be understood entirely from the point of view of classical electromagnetic theory, a point that has been emphasized by Hanbury Brown and Twiss.[2] In fact the validity of the semi-classical approach to a wide variety of electromagnetic phenomena has recently been demonstrated by Mandel and Wolf.[7]

The second question is purely classical in nature and is essentially: "How is it possible to have any kind of interference phenomena with light from incoherent sources?" Although this question has been extensively discussed,[8] to those not familiar with the rather technical literature on the coherence properties of radiation from incoherent sources it remains somewhat of a puzzle. It is to this question that the present paper is addressed.

In Section II we shall develop a treatment of the spatial dependence of intensity correlations starting with the simple, first-order interference pattern produced by two coherent sources. From this simple concept we will proceed in simple steps to (it is hoped) an understanding of the physical principles of the post detection stellar interferometer of Hanbury Brown and Twiss.

II. The Spatial Dependence of Intensity Fluctuation Correlation

The essential element in the post-detection interferometer is a detector of electromagnetic radiation whose response is proportional to the intensity, I, of the radiation. This can be a radio receiver with a "square-law" detector circuit or, in the optical region of the spectrum, a photo-electric cell. In Appendix I it is shown that, although the photo-electric cell responds to individual photons, for the purpose of studying the statistical correlation between the output of two separate detectors we may ignore this fact and assume that its output is directly proportional to the classical intensity.

Actually no detector has a zero resolution time and rather than having an output, $O(t)$, directly proportional to the instantaneous intensity, $I(t)$, the appropriate quantity will be $u(t)$ where

$$u(t) \propto \int_{t-T}^{t} I(t)\,dt. \tag{1}$$

In this expression T represents the resolution time of the detector. If we define the coherence time, Δt, of the radiation to be the characteristic time during which the intensity varies only slightly and if we choose $T \ll \Delta t$ we have

$$u(t) \propto T\,I(t). \tag{2}$$

This assumption is actually quite reasonable for the radio-interferometer[1] but not in the optical case.[2] The complications involved in the case where $T \geq \Delta t$ in no way change the basic principles that we shall consider but they do make the treatment more difficult. For this reason we shall assume in what follows that the output of any detector under discussion is directly proportional to the instantaneous intensity of radiation falling upon it.

In the operation of the post-detection interferometer two detectors, A and B, are separated by a distance, δ. Their outputs are passed through filters which remove the steady component and allow only the fluctuations to pass. In other words the output of a filter is proportional to $I(t) - \langle I \rangle$ where the brackets $\langle \; \rangle$ represent a time average. The outputs from the two filters are multiplied together and time averaged in a circuit called a correlator: the output of the correlator, C_{AB}, is therefore,

$$C_{AB} \propto \Big\langle \big(I_A(t) - \langle I_A(t) \rangle \big) \big(I_B(t) - \langle I_B(t) \rangle \big) \Big\rangle$$
$$= \langle I_A \, I_B \rangle - \langle I_A \rangle \langle I_B \rangle . \tag{3}$$

Figure 1 is a schematic representation of this arrangement.

It is clear that we now wish to examine the relationship between the correlation of intensity fluctuations in two separated detectors, on the one hand, and the structure of the source of radiation, on the other.

We start with the simplest possible picture of two sources, 1 and 2, located at $\mathbf{r}_1$ and $\mathbf{r}_2$ each radiating a monochromatic wave of frequency ω and amplitudes A_1 and A_2. At first we shall consider their amplitudes to be constant. However, each source will have a phase, $\phi_1(t)$ and $\phi_2(t)$, that will change with time in a random way but with $d\phi/dt \approx 1/\Delta t \ll \omega$. The important point to note is that although the two sources have no fixed relative phase, at any given time a certain instantaneous phase difference, $\phi_1(t) - \phi_2(t)$, does exist. Neglect of this point causes much of the confusion concerning the possibility of interference phenomena existing for incoherent sources.

At the origin of our coordinate system the amplitude from both sources is given by

$$A = A_1 \exp i\ [r_1 + \omega t + \phi_1 (t - r_1/c)]$$
$$+ A_2 \exp i\ [r_2 + \omega t + \phi_2 (t - r_2/c)]. \tag{4}$$

In the above expression the amplitudes, A_1 and A_2, are the amplitudes at the origin from source 1 and 2 respectively, and r_1 and r_2 are measured in wavelengths, c/ω, for simplicity. The corresponding intensity will be given by

$$I(t) = A_1^2 + A_2^2 + 2A_1A_2 \cos(r_1 - r_2 + \phi_1(t - r_1/c) - \phi_2(t - r_2/c)). \tag{5}$$

If we move to a point a distance, $\mathbf{x}$, away from the origin keeping t fixed, and assume that $x \ll r_1, r_2$ (and for observations made on the earth of astronomical objects this is extremely valid), the distances from the sources to the new position are given by

$$r_1' \approx r_1 - \hat{\mathbf{r}}_1 \cdot \mathbf{x};\ r_2' \approx r_2 - \hat{\mathbf{r}}_2 \cdot \mathbf{x}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$, a unit vector.

We have for the intensity at the point, $\mathbf{x}$, (since we may safely take $A_1(\mathbf{x}) = A_1$ etc.)

$$I(\mathbf{x}, t) = A_1^2 + A_2^2 + 2A_1A_2 \cos(r_1 - r_2 - \mathbf{x} \cdot \boldsymbol{\theta}_{12} + \Phi_{12}(\mathbf{x}, t)), \tag{6}$$

where $\boldsymbol{\theta}_{12} = \hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2$, and for small values is just the angular separation between sources 1 and 2, $\Phi_{12}(\mathbf{x}, t)$ is the new relative phase angle given by

$$\Phi_{12}(\mathbf{x}, t) = \phi_1(t - r_1/c + \hat{\mathbf{r}}_1 \cdot \mathbf{x}/c) - \phi_2(t - r_2/c + \hat{\mathbf{r}}_2 \cdot \mathbf{x}/c).$$

We may now ask to what extent equation (6) represents a well defined interference pattern. We note first of all that we do not know the overall phase of the cosine function, since we do not know $r_1 - r_2$ (certainly not to within a wavelength) and we do not know the phase factor Φ_{12}. However, the behaviour of (6) as a function of $\mathbf{x}$ is well defined provided the phase angles do not change appreciably (for if they do, they change in a random way and we are lost). To assure this we must have $\hat{\mathbf{r}}_1 \cdot \mathbf{x} \approx \hat{\mathbf{r}}_2 \cdot \mathbf{x} \ll c\Delta t$, since $\phi(t)$ is essentially constant for times small compared to Δt. We can achieve this by requiring $\hat{\mathbf{r}}_1 \cdot \mathbf{x} = \hat{\mathbf{r}}_2 \cdot \mathbf{x} = 0$, but this would make $\mathbf{x} \cdot \boldsymbol{\theta}_{12} = 0$ and we would have no dependence on $\mathbf{x}$ at all. The best compromise, therefore, is to set $\frac{1}{2}\,\mathbf{x} \cdot (\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2) = 0$, or in other words to stay on a plane perpendicular to the average direction of the sources (see Figure 2). In practice[1,2] this is achieved by inserting a time delay in the circuitry of the detector at $\mathbf{x}$ equal to $-\delta t = 1/2\,(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2) \cdot \mathbf{x}/c$ which has the same effect on the arguments of ϕ_1 and ϕ_2.

Using either approach we may write the arguments of ϕ_1 and ϕ_2 as, $t - r_1/c + \hat{\mathbf{r}}_1 \cdot \mathbf{x}/c - 1/2\,(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2) \cdot \mathbf{x}/c = t - r_1/c + \boldsymbol{\theta}_{12} \cdot \mathbf{x}/2c$, and $t - r_2/c - \boldsymbol{\theta}_{12} \cdot \mathbf{x}/2c$ respectively. We now require that $\boldsymbol{\theta}_{12} \cdot \mathbf{x} \ll 2c\Delta t$. But we have required $d\varphi/dt \approx 1/\Delta t \ll \omega$ and since $\omega = c$ in our units we have $2c\Delta t \gg 1$. Therefore $\boldsymbol{\theta}_{12} \cdot \mathbf{x}$ may vary through many times 2π without appreciably changing ϕ_1 or ϕ_2.

With these considerations we may write (6) as

$$I(\mathbf{x}, t) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\boldsymbol{\theta}_{12} \cdot \mathbf{x} + \Phi_{12}(t)), \tag{7}$$

where we have dropped the constant (and unknown) factor, $r_1 - r_2$, and have written I_1 and I_2 for A_1^2 and A_2^2 respectively. We see that what we have is just a standard coherent interference pattern, but one that will not stand still. It jumps from place to place staying in one spot only for times less than Δt.

It would be extremely difficult to determine its structure, and hence $\boldsymbol{\theta}_{12}$ which would be of astronomical interest, by taking its photograph[9] or by scanning it with a single photo tube.

On the other hand, it is important to realize that the spatial pattern represented by (7) has a certain internal structure which is independent of its overall position. The internal structure of a function can be described in terms of its autocorrelation function.[10] The autocorrelation function is a measure of how the values of a function at two points, separated by a fixed amount, are related.

For a given function, $F(x)$, the autocorrelation function, $\psi(\delta)$, is defined

$$\psi(\delta) = \lim_{x\to\infty} \frac{1}{x} \int_{-x/2}^{x/2} F(x') F(x' + \delta) \, dx' . \tag{8}$$

It is easy to see that if $F(x)$ is a periodic function with a period X we have

$$\psi(\delta) = \frac{1}{X} \int_{x}^{x+X} F(x') F(x' + \delta) \, dx' \tag{9}$$

which is independent of x. If we now form this quantity for our intensity pattern (7) we will have

$$\psi(\boldsymbol{\delta}) = \frac{1}{|\mathbf{X}|} \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{X}} I(\mathbf{x}', t) \, I(\mathbf{x}' + \boldsymbol{\delta}, t) \, d\mathbf{x}' = (I_1 + I_2)^2 + 2 I_1 I_2 \cos(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta})$$

where

$$\mathbf{X} = 2\pi \, \boldsymbol{\theta}_{12} / |\theta_{12}|^2 \tag{10}$$

which we see is independent of $\mathbf{x}$ and the phase, $\Phi_{12}(t)$. This quantity, viewed as a function of δ, allows one to extract the physically interesting parameter, θ_{12}, even though we have no information at all about the position of the interference pattern.

From our point of view the significance of the quantity $\psi(\delta)$ lies in the fact that changing the phase angle, Φ_{12}, is completely equivalent to changing $(\theta_{12}\cdot\mathbf{x})$. Therefore we may replace the integral over $\mathbf{x}$ in (10) by an integral over Φ_{12};

$$\psi(\delta) = \frac{1}{2\pi}\int_0^{2\pi} I(\mathbf{x}',\ t)\ I(\mathbf{x}' + \delta,\ t)\ d\Phi_{12}. \tag{11}$$

If we take a time average of the integrand, $I(\mathbf{x}, t)\, I(\mathbf{x}+\delta, t)$, the phase angle, Φ_{12}, will vary randomly and uniformly over the range 0 to 2π and we will have performed a Monte-Carlo integration of (11). Therefore we may write

$$\langle I(\mathbf{x},\ t)\ I(\mathbf{x}+\delta,\ t)\rangle = \psi(\delta) = (I_1 + I_2)^2 + 2I_1 I_2 \cos(\theta_{12}\cdot\delta). \tag{12}$$

Equation (12) demonstrates the underlying principle of the stellar interferometer of Hanbury Brown and Twiss. It shows that two sources of radiation that have no fixed phase relationship can, nevertheless, cause the intensity at two different points to be correlated. Furthermore the correlation depends on the separation of the detectors, δ, and the separation of the sources, θ_{12}, in a way that is very reminiscent of the interference pattern that would have been obtained had the sources been coherent. We can now see that this is not surprising since the correlation pattern is closely related to the underlying interference pattern that is jumping about too rapidly to be seen. The picture of the sources that we have considered so far is very simple and not very realistic. We shall refine it as we proceed until we have a picture that resembles a real astronomical object. Nevertheless the basic principle will remain that expressed in equation (12).

III. Refinements of the Model

In the last section we considered sources that radiated a scalar amplitude A. Our description therefore corresponded to a completely polarized source or to measurements made with aligned polarizing filters placed over the detectors. The treatment of partially polarized sources is rather complicated and will not be attempted here. However the extension to completely unpolarized sources is almost trivial if one assumes that waves of opposite polarization are completely uncorrelated and also remembers that they do not interfere with each other. With this in mind we have for completely unpolarized sources, $I_{||} = I_{\perp} = \frac{1}{2} I$.

$$I(\mathbf{x}, t) = I_1 + I_2 + \sqrt{I_1 I_2}\,[\cos(\boldsymbol{\theta}_{12} \cdot \mathbf{x} + \Phi_{12(||)}(t)) + \cos(\boldsymbol{\theta}_{12} \cdot \mathbf{x} + \Phi_{12(\perp)})] \tag{7'}$$

From this we obtain

$$\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \boldsymbol{\delta}, t)\rangle = (I_1 + I_2)^2 + I_1 I_2 \cos(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}). \tag{12'}$$

We may combine (12) and (12') into one equation

$$\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \boldsymbol{\delta}, t)\rangle = (I_1 + I_2)^2 + \gamma I_1 I_2 \cos(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}) \tag{12''}$$

where $\gamma = 1$ for completely unpolarized light and $\gamma = 2$ for completely polarized light.

It is a straightforward matter to generalize expression (12") to the case of N sources with direction vectors, $\hat{\mathbf{r}}_i$

$$\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \boldsymbol{\delta}, t)\rangle = \left(\sum_{i=1}^{N} I_i\right)^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} (I_i I_j \cos(\boldsymbol{\theta}_{ij} \cdot \boldsymbol{\delta})) \tag{13}$$

where we keep

$$\frac{1}{N} \left[\sum_{i}^{N} \hat{\mathbf{r}}_i\right] \cdot \boldsymbol{\delta} = 0 .$$

At this point we should notice a rather peculiar fact. Until now we have been picturing our point oscillators as radiating a constant amplitude wave with only the phase varying at random. Therefore for a single source we have I_i = const. and $\langle I_i^2\rangle = \langle I_i\rangle^2$. However, if we allow N of our point sources in expression (13) to come together at a point to form a single source we have, setting $\boldsymbol{\theta}_{ij} = 0$

$$\begin{aligned}
\langle I(\mathbf{x})\, I(\mathbf{x} + \boldsymbol{\delta})\rangle &= \langle I^2\rangle \\
&= \left(\sum_{i}^{N} I_i\right)^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} I_i I_j \\
&= \langle I\rangle^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} I_i I_j
\end{aligned} \tag{14}$$

This indicates that we no longer have a simple point source with constant amplitude but instead we have one whose intensity fluctuates in time so that $\langle I^2\rangle > \langle I\rangle^2$.

If we realize that any macroscopic source of radiation (where macroscopic means of dimension very large compared to the wavelength) is in fact made up of many microscopic oscillators we see that our "point" sources should be considered as being made up of many small identical oscillators. To answer the question of how many oscillators should contribute to one source we consider equation (14) for N equal oscillators where $I_i = \langle I \rangle / N$. We have then

$$\langle I^2 \rangle = \langle I \rangle^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} I_i I_j$$

$$= \langle I \rangle^2 \left(1 + \frac{N(N-1)}{N^2} \frac{\gamma}{2}\right)$$

$$= \langle I \rangle^2 \left(1 + \frac{\gamma}{2} - \frac{\gamma}{2N}\right) \tag{15}$$

If we choose $N = \infty$ we have for any point oscillator the relation

$$\langle I_i^2 \rangle = \left(1 + \frac{\gamma}{2}\right) \langle I_i \rangle^2 \tag{16}$$

It may be objected that this picture of a point source as made up of a very large number of oscillators, all oscillating with a steady amplitude but constantly changing phase is not very realistic. This is true. However, in Appendix II it is shown that a far more realistic model of a radiation source also yields equation (16). This is not too surprising; it is merely one more example of the statistical law of large numbers. This law asserts that the probability distributions of a quantity which is a sum of N random quantities approaches a

Gaussian as N becomes infinite. In our case this means that the statistical properties of the output from N oscillators become independent of the properties of the individual oscillators as N becomes very large.

We now see that in deriving our expression for the autocorrelation of the intensity, expression (12''), we should have taken time averages of the individual terms taking into account the fluctuations of I_1 and I_2. Equation (12'') becomes

$$\langle I(\mathbf{x},\ t)\, I(\mathbf{x}+\boldsymbol{\delta},\ t)\rangle = \langle (I_1 + I_2)^2\rangle + \gamma \langle I_1 I_2\rangle \cos(\boldsymbol{\theta}_{12}\cdot\boldsymbol{\delta}) \tag{17}$$

and instead of (13) we have

$$\langle I(\mathbf{x},\ t)\, I(\mathbf{x}+\boldsymbol{\delta},\ t) = \left\langle \left(\sum_{i=1}^{N} I_i\right)^2 \right\rangle + \frac{\gamma}{2} \sum_{i\neq j=1}^{N} \langle I_i I_j\rangle \cos(\boldsymbol{\theta}_{ij}\cdot\boldsymbol{\delta})$$

$$= \left\langle \sum_{i=1}^{N} I_i^2 \right\rangle + \sum_{i\neq j=1}^{N} \langle I_i I_j\rangle \left(1 + \frac{\gamma}{2}\cos(\boldsymbol{\theta}_{ij}\cdot\boldsymbol{\delta})\right) \tag{18}$$

Applying equation (16) and noting that $\langle I_i I_j\rangle = \langle I_i\rangle\langle I_j\rangle$ because the intensity fluctuations of two different sources are independent we have

$$\langle I(\mathbf{x},\ t)\, I(\mathbf{x}+\boldsymbol{\delta},\ t)\rangle = \left(1 + \frac{\gamma}{2}\right) \sum_{i=1}^{N} \langle I_i\rangle^2$$

$$+ \sum_{i\neq j=1}^{N} \langle I_i\rangle \langle I_j\rangle \left(1 + \frac{\gamma}{2}\cos(\boldsymbol{\theta}_{ij}\cdot\boldsymbol{\delta})\right) \tag{19}$$

It is interesting to see that if we allow these sources to come together to form one source we have

$$\langle I^2\rangle = \left(1+\frac{\gamma}{2}\right)\sum_{i=1}^{N}\langle I_i\rangle^2 + \left(1+\frac{\gamma}{2}\right)\sum_{i\neq j=1}^{N}\langle I_i\rangle\langle I_j\rangle$$

$$= \left(1+\frac{\gamma}{2}\right)\sum_{i,j=1}^{N}\langle I_i\rangle\langle I_j\rangle = \left(1+\frac{\gamma}{2}\right)\left(\sum_{i=1}^{N}\langle I_i\rangle\right)^2$$

$$= \left(1+\frac{\gamma}{2}\right)(\langle I\rangle)^2 \tag{20}$$

which is the same as (16) for a point source. This just shows that an infinity of oscillators plus several other infinities of oscillators behaves like an infinity of oscillators.

Equation (19) can be cast into a form that is quite suggestive; recalling that $\boldsymbol{\theta}_{ij} = \hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j$ we have

$$\langle I(\mathbf{x})\, I(\mathbf{x}+\boldsymbol{\delta})\rangle = \sum_{i,j=1}^{N}\langle I_i\rangle\langle I_j\rangle\left(1+\frac{\gamma}{2}\cos\left(\hat{\mathbf{r}}_i\cdot\boldsymbol{\delta} - \hat{\mathbf{r}}_j\cdot\boldsymbol{\delta}\right)\right)$$

$$= \left(\sum_{i}^{N}\langle I_i\rangle\right)^2 + \frac{\gamma}{2}\sum_{i,j=1}^{N}\langle I_i\rangle\langle I_j\rangle \exp\left(i\hat{\mathbf{r}}_i\cdot\boldsymbol{\delta} - i\hat{\mathbf{r}}_j\cdot\boldsymbol{\delta}\right)$$

$$= \langle I\rangle^2 + \frac{\gamma}{2}\left|\sum_{i=1}^{N}\langle I_i\rangle \exp\left(i\hat{\mathbf{r}}_i\cdot\boldsymbol{\delta}\right)\right|^2 \tag{21}$$

The generalization from N discrete sources to a continuous distribution of sources is immediate. With the substitutions $\langle I_i \rangle \rightarrow I(\hat{\mathbf{r}})\, d\Omega$, $\sum_i \rightarrow \int$ equation (21) becomes

$$\langle I(\mathbf{x})\, I(\mathbf{x}+\boldsymbol{\delta})\rangle = \langle I\rangle^2 + \frac{\gamma}{2}\left|\int I(\hat{\mathbf{r}}) \exp(i\hat{\mathbf{r}}\cdot\boldsymbol{\delta})\, d\Omega\right|. \tag{22}$$

where $I(\hat{\mathbf{r}})$ is the source brightness per solid angle in the direction $\hat{\mathbf{r}}$.

Returning to equation (3) we may identify I_A with $I(\mathbf{x})$ and I_B with $I(\mathbf{x}+\boldsymbol{\delta})$. It is easy to verify that $\langle I(\mathbf{x})\rangle = \langle I(\mathbf{x}+\boldsymbol{\delta})\rangle = \langle I\rangle$ and we may therefore substitute equation (22) into (3) to obtain for the output of the correlator

$$C_{AB} \propto \left\{\langle I(\mathbf{x})\, I(\mathbf{x}+\boldsymbol{\delta})\rangle - \langle I(\mathbf{x})\rangle\langle I(\mathbf{x}+\boldsymbol{\delta})\rangle\right\}$$

$$= \left\{\frac{\gamma}{2}\left|\int F(\hat{\mathbf{r}}) \exp(i\hat{\mathbf{r}}\cdot\boldsymbol{\delta})\, d\Omega\right|^2\right\}. \tag{23}$$

It is of interest to note that the quantity $\tilde{F}(\boldsymbol{\delta}) = \int F(\hat{\mathbf{r}}) \exp(i\hat{\mathbf{r}}\cdot\boldsymbol{\delta})\, d\Omega$ is the quantity determined with the Michelson interferometer. By determining this quantity as a function of $\boldsymbol{\delta}$ the complete brightness distribution over the face of a star $F(\hat{\mathbf{r}})$ may be determined using the inverse Fourier transform

$$F(\hat{\mathbf{r}}) = (2\pi)^{-2}\int \tilde{F}(\boldsymbol{\delta}) \exp(-i\hat{\mathbf{r}}\cdot\boldsymbol{\delta}). \tag{24}$$

With the post-detection interferometer we may determine only $|F(\boldsymbol{\delta})|^2$ and we have no information about the phase of $F(\boldsymbol{\delta})$. We are, therefore, not able to uniquely reconstruct $F(\hat{\mathbf{r}})$ without the use of additional assumptions.

The above result has been obtained under the condition that $d\phi/dt$ be small compared to the frequency ω. This is equivalent to assuming that the source is quasi-monochromatic, since any change in ϕ introduces additional frequencies to the wave and the condition $d\phi/dt \ll \omega$ is the same as $\Delta\omega \ll \omega$ or in other words we are dealing with a narrow band of frequencies. We will now examine what happens if we relax this condition.

We see immediately that equation (7) is no longer valid since we may not consider the phase difference Φ_{12} to be independent of $\mathbf{x}$. Because of this, equation (12) should be replaced by

$$\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \boldsymbol{\delta}, t)\rangle = (I_1 + I_2)^2 + 2I_1 I_2 C(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}) \tag{25}$$

where

$$C(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}) = 2\left\langle \cos\left(\boldsymbol{\theta}_{12} \cdot \mathbf{x} + \Phi_{12}(\mathbf{x}, t)\right) \cos\left(\boldsymbol{\theta}_{12} \cdot \mathbf{x} + \boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta} + \Phi_{12}(\mathbf{x} + \boldsymbol{\delta}, t)\right)\right\rangle \tag{26}$$

If we write $\Phi_{12}(\mathbf{x} + \boldsymbol{\delta}, t) = \Phi_{12}(\mathbf{x}, t) + \Delta\Phi_{12}(\mathbf{x} + \boldsymbol{\delta}, t)$ we may expand the cosine functions to obtain

$$C(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}) = \Big\langle \cos(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}) \Big\{ \cos^2(\boldsymbol{\theta}_{12} \cdot \mathbf{x}) \left[\cos^2 \Phi_{12} \cos \Delta\Phi_{12}\right.$$

$$\left. - \cos\Phi_{12} \sin\Phi_{12} \sin\Delta\Phi_{12}\right] + \sin^2(\boldsymbol{\theta}_{12} \cdot \mathbf{x}) \left[\sin^2 \Phi_{12} \cos \Delta\Phi_{12}\right.$$

$$\left. + \sin\Phi_{12} \cos\Phi_{12} \sin\Delta\Phi_{12}\right] - \cos(\boldsymbol{\theta}_{12} \cdot \mathbf{x}) \sin(\boldsymbol{\theta}_{12} \cdot \mathbf{x}) \left[\cos^2 \Phi_{12} \sin\Delta\Phi_{12}\right.$$

$$- \sin^2 \Phi_{12} \sin \Delta\Phi_{12} + 2 \sin \Phi_{12} \cos \Phi_{12} \cos \Delta\Phi_{12} \Big] \Big\}$$

$$- \sin\left(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}\right) \Big\{ \cos^2\left(\boldsymbol{\theta}_{12} \cdot \mathbf{x}\right) \Big[\cos^2 \Phi_{12} \sin \Delta\Phi_{12}$$

$$+ \cos \Phi_{12} \sin \Phi_{12} \cos \Delta\Phi_{12} \Big] + \sin^2\left(\boldsymbol{\theta}_{12} \cdot \mathbf{x}\right) \Big[\sin^2 \Phi_{12} \sin \Delta\Phi_{12}$$

$$- \sin \Phi_{12} \cos \Phi_{12} \cos \Delta\Phi_{12} \Big] + \cos\left(\boldsymbol{\theta}_{12} \cdot \mathbf{x}\right) \sin\left(\boldsymbol{\theta}_{12} \cdot \mathbf{x}\right) \Big[\cos^2 \Phi_{12} \cos \Delta\Phi_{12}$$

$$- \sin^2 \Phi_{12} \cos \Delta\Phi_{12} - 2 \sin \Phi_{12} \cos \Phi_{12} \sin \Delta\Phi_{12} \Big] \Big\} \Big\rangle \tag{27}$$

where the arguments of Φ_{12} and $\Delta\Phi_{12}$ are understood. We shall now assume that Φ_{12} performs a random walk such that the distribution of $\Delta\Phi_{12}$ does not depend on Φ_{12}. The primary justification for such an assumption is that any natural model of a light source would have this property.

If we recall that uniform distribution of Φ_{12} implies

$$\left\langle \cos^2 \Phi_{12} \right\rangle = \left\langle \sin^2 \Phi_{12} \right\rangle = \frac{1}{2}$$

and

$$\left\langle \sin \Phi_{12} \cos \Phi_{12} \right\rangle = 0$$

equation (27) becomes

$$C(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}) = \cos(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta})\langle\cos\Delta\Phi_{12}\rangle - \sin(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta})\langle\sin\Delta\Phi_{12}\rangle$$

$$= R_e\left\{\exp(i\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta})\left\langle\exp(i\Delta\Phi_{12})\right\rangle\right\} \tag{28}$$

where R_e means the real part and $\left\langle\exp(i\Delta\Phi_{12})\right\rangle$ can depend only on $\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}$ (not on $\mathbf{x}$) since we have assumed throughout the discussion that the statistical properties of the source do not depend on time and the arguments of Φ_{12} are just retarded time.

The development from this point on proceeds exactly as before with the simple substitution of our function $C(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta})$ for $\cos(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta})$. Recalling that $\Delta\Phi_{12} = \Delta\phi_2 - \Delta\phi_1$, we see that equation (21) becomes

$$\langle I(\mathbf{x})\, I(\mathbf{x}+\boldsymbol{\delta})\rangle = R_e\left\{\sum_{i,j=1}^{N}\langle I_i\rangle\langle I_j\rangle\left(1+\frac{\gamma}{2}\exp\left(i\hat{\mathbf{r}}_i \cdot \boldsymbol{\delta} - i\hat{\mathbf{r}}_j \cdot \boldsymbol{\delta}\right)\left\langle\exp\left(i\Delta\phi_i - i\Delta\phi_j\right)\right\rangle\right)\right\}$$

$$= \langle I\rangle^2 + \frac{\gamma}{2}\sum_{i,j=1}^{N}\langle I_i\rangle\exp(i\hat{\mathbf{r}}_i \cdot \boldsymbol{\delta})\left\langle\exp(i\Delta\phi_i)\right\rangle \times \langle I_j\rangle\exp(-i\hat{\mathbf{r}}_j \cdot \boldsymbol{\delta})\left\langle\exp(-i\Delta\phi_j)\right\rangle$$

$$= \langle I\rangle^2 + \frac{\gamma}{2}\left|\sum_{i=1}^{N}\langle I_i\rangle\left\langle\exp(i\Delta\phi_i)\right\rangle\exp(i\hat{\mathbf{r}}_i \cdot \boldsymbol{\delta})\right|^2 \tag{29}$$

Before proceeding, we should pause and consider the function $\left\langle \exp\left(i\Delta\phi_i\right)\right\rangle$. If we form the autocorrelation function of the amplitude $A_i(t)$ we have

$$\psi(\tau) \equiv \left\langle A_i^*(t)\, A(t+\tau)\right\rangle \Big/ |A|^2$$

$$= \exp(i\omega\tau)\left\langle \exp\left(i\Delta\phi(\tau)\right)\right\rangle = \exp(i\omega\tau)\,\psi'(\tau) \tag{30}$$

The Fourier transform of $\psi(\tau)$ is

$$\tilde{\psi}(\omega') = \int_{-\infty}^{\infty} \psi(\tau)\exp(i\omega'\tau)\, d\tau = \tilde{\psi}'(\omega'+\omega) \tag{31}$$

where $\tilde{\psi}'(\omega')$ is the transform of $\psi'(\tau)$. The function $\tilde{\psi}(\omega')$ is called the normalized power spectrum[10] of the amplitude $A_i(t)$. It has this name because it represents the relative amount of power radiated in the frequency interval ω' and $\omega' + d\omega'$.[10] If we therefore write

$$\left\langle \exp\left(i\Delta\phi_i(\tau)\right)\right\rangle = (2\pi)^{-1}\int_{-\infty}^{\infty} \tilde{\psi}'(\omega')\exp(-i\omega'\tau)\, d\omega'$$

and substitute this expression in (29) we have

$$\left\langle I(\mathbf{x})\, I(\mathbf{x}+\delta)\right\rangle - \left\langle I\right\rangle^2$$

$$= \frac{\gamma}{4\pi}\left|\sum_{i=1}^{N}\left\langle I_i\right\rangle \int_{-\infty}^{\infty} \exp\left(i\hat{\mathbf{r}}_i\cdot\boldsymbol{\delta} - i\omega'\tau_i\right)\tilde{\psi}_i'(\omega')\, d\omega'\right|^2 \tag{32}$$

If we now abandon our practice of writing lengths in units of the wave length we have $\delta \rightarrow \delta/\lambda = \delta\omega/c$. We also have $\tau_i = \hat{\mathbf{r}}_i \cdot \delta/c$ and therefore we may write

$$\langle I(\mathbf{x})\, I(\mathbf{x}+\delta)\rangle - \langle I\rangle^2$$

$$= \frac{\gamma}{4\pi}\left|\sum_{i=1}^{N}\langle I_i\rangle \int_{-\infty}^{\infty} \exp\left[i\hat{\mathbf{r}}_i \cdot \delta(\omega-\omega')/c\right]\tilde{\psi}'(\omega')\, d\omega'\right|^2 \tag{33}$$

If we now assume that all of our sources have the same spectral distribution, i.e. $\tilde{\psi}_i'(\omega') = \psi'(\omega')$ for all i we may go to the limit of a continuously distributed source and write as the equivalent of equation (23)

$$C_{AB} \propto \left\{\frac{\gamma}{4\pi}\left|\iiint_{-\infty}^{\infty} F(\hat{\mathbf{r}}) \exp\left[i\hat{\mathbf{r}} \cdot \delta(\omega-\omega')/c\right]\tilde{\psi}'(\omega')\, d\omega'\, d\Omega\right|^2\right\}$$

$$= \left\{\frac{\gamma}{4\pi}\left|\iiint_{-\infty}^{\infty} F(\hat{\mathbf{r}}) \exp\left[i\hat{\mathbf{r}} \cdot \delta\omega'/c\right]\tilde{\psi}(\omega')\, d\omega' d\Omega\right|^2\right\}$$

$$= \left\{\frac{\gamma}{4\pi}\left|\int_{-\infty}^{\infty} \tilde{F}(\delta\omega'/c)\,\psi(\omega')\, d\omega'\right|^2\right\} \tag{34}$$

This result is valid for any spectral distribution assuming that the detector itself is not frequency dependent in its characteristics. In any real situation $\tilde{\psi}(\omega')$ should be multiplied by the frequency response function of the detector since any real detector is equivalent to one of our ideal detectors with a frequency filter placed in front of it.

We can see from (34) that if $\tilde{\psi}(\omega')$ is sharply peaked about a particular frequency (quasi-monochromatic) we obtain our previous result equation (23). It is fairly easy to see that the finite bandwidth of the input radiation causes a loss of information in the sense that the function $\tilde{F}(\delta\omega/c)$ gets "smeared out" over the frequency distribution. For example, if the characteristic size of the source is Θ, $\tilde{F}$ will have structure over values of δ of order $c/\omega\Theta$ or larger. If the spread in frequency $\Delta\omega$ is such that $\Delta(c/\omega\Theta) = (c/\omega\Theta)\,\Delta\omega/\omega \approx c/\omega\Theta$ or $\Delta\omega/\omega \approx 1$ the structure will be lost and even the overall size of the source cannot be determined with any great accuracy.

We see therefore that even though our initial requirement $d\phi/dt \ll \omega$ is not strictly required it certainly indicates the best operating conditions and if it is violated such that $d\phi/dt \approx \omega$ it is very difficult to obtain any information about the source.

IV. Conclusions

We have seen that starting with the simple notion of an interference pattern that jumps randomly around we may proceed through a series of simple steps to the spatial correlation pattern of light from an extended source. This correlation pattern exists because what we call incoherent sources are not really totally incoherent. There exist short periods of time during which some phase relations exist between the various parts of the source. During this same short period of time therefore an interference pattern exists at the point of observation. Since these phase relations between the various parts of the source do not persist this interference pattern is constantly shifting and changing. However, during this change certain internal structures of the interference pattern are constant due to the overall structure of the source. The correlation pattern that we observe with the Hanbury Brown and Twiss type of stellar interferometer is just a measure of this constant, internal structure.

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APPENDIX I

THE CONNECTION BETWEEN QUANTUM AND CLASSICAL DESCRIPTION

The primary method of detection of a photon in the optical range is through the use of a detector that employs the photo-electric effect in its operation (e.g., a photo-electric tube). In fact we may define the detection of a photon in a light beam as the ejection of an electron from the photo-cathode of some suitable detector. In this context the concepts of photon and classical electromagnetic field are related through a single consideration, namely, that for a photo-cathode illuminated by a light beam the instantaneous probability per unit time of emission of a photo electron (detection of a photon) is proportional to the instantaneous intensity of the electro-magnetic wave falling on the photo-cathode;

$$d\mathcal{P}(1) = \alpha I(t)\, dt \tag{AI-1}$$

where $d\mathcal{P}(1)$ is the probability for one and only one photon to be detected in a time dt and α is the proportionality coefficient. The probability for two photons to be detected is of second order in dt and is considered negligible. From probability theory we know that for a finite interval of time between t_1 and t_2 the probability for the detection of n photons is given by the Poisson distribution

$$\mathcal{P}_u(n) = \frac{[u(t_1, t_2)]^n \exp[-u(t_1, t_2)]}{n!} \tag{AI-2}$$

where

$$u(t_1, t_2) \equiv \int_{t_1}^{t_2} \alpha I(t)\, dt .$$

Unfortunately, unless the intensity $I(t)$ is a constant, the distribution $\mathbb{P}_u(n)$ is not observable, for if one took statistics on successive intervals (t_1, t_2), (t_2, t_3) - - - (t_n, t_{n+1}) one would find that in general the value of $u(t_n, t_{n+1})$ would vary from interval to interval. Expression (SI-2) would describe the distribution only for the collection of intervals that had a common value, say α_1, for the variable u (see figure 3).

If the variable u varies from interval to interval such that the probability of any interval having a particular value of u is given by $\mathbb{P}(u)$, the probability of detecting n photons in any interval picked at random will be given by the Poisson distribution averaged over all possible values of u ie.

$$\mathbb{P}(n) = \int_0^{\infty} \mathbb{P}(u) \frac{u^n \exp(-u)\, du}{n!} \tag{AI-3}$$

It would be well at this point to ask what possible meaning can we give to the instantaneous intensity $I(t)$ or its integral over an interval $u(t_n, t_{n+1})$ since there seems to be no operational way to determine their values. The number of photons detected in any given interval has no unique relationship to the value of u for that interval. The answer must be that I and u are calculational devices used to determine the statistical properties of photo-electrons from a photo-cathode. If one calculates the probability per unit time for ejection of a photo-electron due to a source of electromagnetic radiation some distance away and does so strictly from the theory of quantum electrodynamics one finds that this probability is proportional to the absolute value squared of a vector quantity $\mathbf{A}$. Furthermore, it turns out that if the source is made up of a large number of elementary quantum systems and is some distance away the quantity $\mathbf{A}$ may be calculated to a very good degree of approximation by using Maxwell's equations and considering $\mathbf{A}$ to be the classical vector potential and hence $|\mathbf{A}|^2$ to be the classical intensity $I(t)$.

We see, therefore, that although the quantities $I(u)$ and $u(t_n, t_{n+1})$ are, strictly speaking, not observables (although we shall see that their average values may be operationally defined) nevertheless they are conceptually and calculationally useful quantities.

If we now consider a large number of non-overlapping intervals of length T we may inquire about the mean value and rms fluctuation of the number n of detected photons in each interval. Employing the definition of a mean value

$$\langle f(n) \rangle = \sum_{n=0}^{\infty} f(n)\, \mathbb{P}(n) \tag{AI-4}$$

and making use of the relation

$$\sum_{n=0}^{\infty} \frac{n!}{(n-p)!} \frac{u^n}{n!} \exp(-u) = u^p \tag{AI-5}$$

we obtain

$$\langle n \rangle = \sum_{n=0}^{\infty} \int_0^{\infty} \mathcal{P}(u) \frac{n\, u^n}{n!} \exp(-u)\, du$$

$$= \int_0^{\infty} \mathcal{P}(u)\, u\, du = \langle u \rangle . \qquad \text{(AI-6)}$$

This may be taken to be the operational definition of $\langle u \rangle$, the average value of u.

The mean square fluctuation of n about its mean value will be given by

$$\begin{aligned} \langle \Delta n^2 \rangle = \langle (n - \langle n \rangle)^2 \rangle &= \langle n^2 \rangle - \langle n \rangle^2 \\ &= \langle u^2 \rangle + \langle u \rangle - \langle u \rangle^2 \\ &= \langle n \rangle + \langle (u - \langle u \rangle)^2 \rangle \\ &= \langle n \rangle + \langle \Delta u^2 \rangle . \qquad \text{(AI-7)} \end{aligned}$$

The first term on the right hand side of (AI-7) is just what would be given if light were a stream of classical particles, for then the statistics of the photons would be just Poissonian. The second term $\langle \Delta u^2 \rangle$ which is a measure of the fluctuations of the wave field due to interference phenomena of one sort or another shows that the deviation of photon statistics from Poissonian due to their boson nature can be determined solely from an investigation of the associated classical electromagnetic field. In fact it can be shown[11] that the very existence of the classical electromagnetic field is intimately connected with the boson character of the photon.

We now turn to the situation where two different detectors A and B are situated at different points of space. For fixed values of u_A and u_B the emission of electrons from the two cathodes is completely independent so we have

$$\mathcal{P}_{u_A u_B}(n_A, n_B) = \frac{(u_A)^{n_A}(u_B)^{n_B}}{n_A!\, n_B!} \exp[-u_A - u_B]$$

$$= \mathcal{P}_{u_A}(n_A)\, \mathcal{P}_{u_B}(n_B) \qquad \text{(AI-8)}$$

and

$$\mathcal{P}(n_A, n_B) = \iint \mathcal{P}(u_A, u_B)\, \mathcal{P}_{u_A, u_B}(n_A, n_B)\, du_A\, du_B . \qquad \text{(AI-9)}$$

Expression (AI-8) indicates the complete independence of the distributions of n_A and n_B for given values of u_A and u_B. However, if u_A and u_B are not distributed independently ie. $\mathcal{P}(u_A, u_B) \neq \mathcal{P}(u_A)\mathcal{P}(u_B)$ then $\mathcal{P}(n_A, n_B) \neq \mathcal{P}(n_A)\mathcal{P}(n_B)$ in general and n_A and n_B are not distributed independently. In other words a correlation between u_A and u_B imposes a correlation between n_A and n_B.

In the operation of the stellar interferometer of Hanbury Brown and Twiss the fluctuations in the "output" of two different detectors are multiplied together in a circuit called a correlator and this product is averaged over time. The "output" of a photoelectric detector will be a current or voltage that is proportional to the number of photo-electrons ejected from its cathode during some interval of time T which is essentially the resolution time of the detector.

The averaged output of the correlator will, therefore, be proportional to

$$\langle (n_A - \langle n_A \rangle)(n_B - \langle n_B \rangle) \rangle = \langle n_A n_B \rangle - \langle n_A \rangle \langle n_B \rangle$$

$$= \langle u_A u_B \rangle - \langle u_A \rangle \langle u_B \rangle \qquad \text{(AI-10)}$$

and we see that this output is directly expressible in terms of the fluctuations of the classical electromagnetic intensity. If the fluctuations of u_A and u_B are independent then $\langle u_A u_B \rangle = \langle u_A \rangle \langle u_B \rangle$ and there is no averaged output from the correlator.

APPENDIX II

A DETAILED SOURCE MODEL

We now consider a model of a radiation source that is considerably more realistic than the one employed in Section III. Instead of constructing the source out of a large number of oscillators running with a constant amplitude and phases that change randomly over a period of time Δt we shall construct our source from a collection of oscillators that turn on at random times with random starting phases and then after a time Δt turn off again.

This model is suggested, of course, by the picture of a hot gas whose atoms are constantly being excited through collisions and then radiating their extra energy away during a short time Δt.

The method we shall use was employed by Rice[12] in his calculations of the "shot effect," however, for a different treatment of the same problem the reader is referred to a paper by Janossy.[13]

Consider a collection of oscillators situated at a point S. If the i^{th} oscillator at S turns on at $t = 0$ it will produce a field at R given by

$$\mathbf{A}_i(t) = \hat{\mathbf{e}}_i \exp(i\,\phi_i)\, F(t - |SR|/c) \qquad \text{(AII-1)}$$

where ϕ_i is an overall phase angle and $\hat{e}_i$ is a unit polarization vector perpendicular to the line joining S and R. $F(t)$ describes the characteristic output of an oscillator and has the properties $F(t) = 0$ for $t < 0$ and $F(t) \approx 0$ for $T \gg \Delta t$

where ≈ 0 is to mean "equal to zero for all practical purposes." Since the retardation factor $|SR|/c$ will apply equally to all times at R we shall drop it in subsequent calculations merely keeping in mind that any event at time t at S corresponds to an effect at R at time $t + |SR|/c$.

The total field at R due to many oscillators turning on at times t_i with phase angle ϕ_i and polarization $\hat{e}_i$ is given by

$$A(t) = \sum_i \hat{e}_i \exp(i\phi_i) F(t - t_i). \qquad \text{(AII-2)}$$

The field produced by N oscillators will be considered to be a random function of the 3N random variables ϕ_i, $\hat{e}_i$, and t_i. If the random variables have probability distribution $\wp(\phi_i)$, $\wp(\hat{e}_i)$, and $\wp(t_i)$ we may define the average of a random function as the weighted integral of the function over all values of the random variables,

$$\langle F(\phi_i, \hat{e}_i, t_i) \rangle = \prod_i \int \wp(\phi_i) d\phi_i \int \wp(\hat{e}_i) d\hat{e}_i \int \wp(t_i) dt_i F(\phi_i, \hat{e}_i, t_i), \qquad \text{(AII-3)}$$

Before proceeding further we will define the normalized autocorrelation function of the function $F(t)$ as

$$\Phi(\tau) = \frac{\int_{-\infty}^{\infty} F^*(t)\, F(t+\tau)\, dt}{\int_{-\infty}^{\infty} |F(t)|^2\, dt} \qquad \text{(AII-4)}$$

where it is assumed that the integrals exist. It is easy to verify that $\Phi(0) = 1$, and $\Phi(-\tau) = \Phi^*(\tau)$.

Since $F(t)$ is essentially zero everyone except for $0 \leq t \leq \Delta t$ we see that $\Phi(\tau)$ is essentially zero everywhere except for $-\Delta t \leq \tau \leq \Delta t$.

We now return to our amplitude function, expression (AII-2). For a process that is stationary in time the sum should be over a infinite number of oscillators over all times. Since it would be difficult to calculate averages with this quantity directly we shall calculate with the subsidiary quantity $A_T(t)$ which is the amplitude produced by all of those oscillators with turn on times t_i such that $-T/2 \leq t_i \leq T/2$. At the end of the calculation we shall always take the limit $T \rightarrow \infty$.

We shall assume that the t_i are independently and uniformly distributed with a probability per unit time η where ηT is the average number turning on in a time T. The phase angles ϕ_i are independently and uniformly distributed between 0 and 2π. We shall leave consideration of the distribution of $\hat{e}_i$ until later.

We shall now proceed to calculate our averages in three steps; first calculating $\langle F \rangle_{TN}$ for the case where exactly N oscillators turn on in time $-T/2 \leq t \leq T/2$. We will then calculate

$$\langle F \rangle_T = \sum_{N=0}^{\infty} \wp_T(N) \langle F \rangle_{TN}$$

where $\wp_T(N)$ is the Poisson's distribution

$$\frac{(\eta T)^N}{N!} \exp(-\eta T).$$

Finally we have

$$\langle F \rangle = \lim_{T \to \infty} \langle F \rangle_T .$$

With these remarks in mind we consider first of all the intensity

$$I(t) = A^*(t)\; A(t)$$

$$= \sum_{i\,j} (\hat{e}_i \cdot \hat{e}_j) \exp(i\phi_j - i\phi_i) F^*(t - t_i)\, F(t - t_j) \qquad \text{(AII-5)}$$

$$\langle I(t) \rangle_{NT} = \left\{ \prod_{i=1}^{N} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int \wp(\hat{e}_i)\, d\hat{e}_i \int_{-T/2}^{T/2} \frac{dt_i}{T} \right\}$$

$$\times \left(\sum_{i,j=1}^{N} (\hat{e}_i \cdot \hat{e}_j) \exp(i\phi_j - i\phi_i)\, F^*(t - t_i)\, F(t - t_j) \right) \qquad \text{(AII-6)}$$

We can see immediately that only those terms in the sum where $i = j$ give a non-zero contribution after the phase angle averaging so we have (recalling $\hat{e}_i \cdot \hat{e}_i = 1$)

$$\langle I(t) \rangle_{NT} = \frac{N}{T} \int_{-T/2}^{T/2} dt_i\, |F(t - t_i)|^2 \qquad \text{(AII-7)}$$

Since the time interval T can be as large as we wish, as long as t is not too close to $-T/2$ or $T/2$ we may replace the integral over $|F|^2$ by the infinite integral to obtain

$$\langle I(t) \rangle_{NT} = \frac{N}{T} \int_{-\infty}^{\infty} |F(t)|^2\, dt \qquad \text{(AII-8)}$$

$$\langle I(t)\rangle_T = \sum_{N=0}^{\infty} \langle I(t)\rangle_{NT} \; \wp_T(N)$$

$$= \frac{\eta T}{T} \int_{-\infty}^{\infty} |F(t)|^2 \, dt = \eta \int_{-\infty}^{\infty} |F(t)|^2 \, dt \qquad \text{(AII-9)}$$

The passage to the limit $T \to \infty$ is now trivial and we have

$$\langle I(t)\rangle = \eta \int_{-\infty}^{\infty} |F(t)|^2 \, dt \qquad \text{(AII-10)}$$

The average intensity is thus seen to be just the intensity produced by a single oscillator multiplied by the average number of oscillators turned on per unit time; a rather intuitive result. We note that the result is independent of the state of polarization, i.e., independent of $\wp(\hat{e}_i)$.

We turn now to the autocorrelation function of the intensity $\langle\psi(\tau)\rangle$ defined as $\psi(\tau) = \langle I(t)\, I\,(t+\tau)\rangle$.

We have

$$(\psi(\tau))_{NT} = \left\{ \prod_{i=1}^{N} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int \wp(\hat{e}_i)\, d\hat{e}_i \int_{-T/2}^{T/2} \frac{dt_i}{T} \right\}$$

$$\times \left\{ \sum_{i,j,k,l=1}^{N} (\hat{e}_i \cdot \hat{e}_j)(e_k \cdot e_l) \; \exp i(\phi_j - \phi_i + \phi_\ell - \phi_k) \right.$$

$$\left. \times \, F^*(t-t_i)\, F(t-t_j)\, F^*(t+\tau-t_k)\, F(t+\tau-t_l) \right\}. \qquad \text{(AII-11)}$$

Once again we see that phase angle averaging gives a zero result unless $i = j$ and $k = 1$, or $i = 1$ and $j = k$, or $i = j = k = 1$. This gives

$$(\psi(\tau))_{NT} = \sum_{i \neq k=1}^{N} \frac{1}{T}\left(\int_{-T/2}^{T/2} |F(t - t_i|^2\, dt_i\right)\left(\frac{1}{T}\int_{-T/2}^{T/2} |F(t+\tau-t_k)|^2 dt_k\right)$$

$$+ \sum_{i \neq k=1}^{N} \iint \mathcal{P}(\hat{e}_i)\,\mathcal{P}(\hat{e}_k)\,(\hat{e}_i \cdot \hat{e}_k)^2\, d\hat{e}_i\, d\hat{e}_k$$

$$\times \left(\frac{1}{T}\int_{-T/2}^{T/2} F^*(t - t_i)F(t+\tau-t_i)\, dt_i\right)\left(\frac{1}{T}\int_{-T/2}^{T/2} F^*(t+\tau-t_k)\right.$$

$$\left. \times\; F(t - t_k)\, dt_k\right)$$

$$+ \sum_{i}^{N} \frac{1}{T}\int_{-T/2}^{T/2} |F(t - t_i)|^2\; |F(t+\tau-t_i|^2\, dt_i \qquad \text{(AII-12)}$$

Once again converting the integrals to infinite integrals under the assumption that both t and $t + \tau$ are far removed from $-T/2$ and $T/2$ we obtain

$$(\psi(\tau))_{NT} = \frac{N(N-1)}{T^2}\left(\int_{-\infty}^{\infty} |F(t)|^2\, dt\right)^2$$

$$+ \frac{N(N-1)}{T^2}\left\langle(\mathbf{e}_i \cdot \mathbf{e}_k)^2\right\rangle \left|\int_{-\infty}^{\infty} F^*(t)\, F(t+\tau)dt\right|^2$$

$$+ \frac{N}{T}\int_{-\infty}^{\infty} |F(t)|^2\; |F(t+\tau)|^2\, dt \qquad \text{(AII-13)}$$

where

$$\langle(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k)^2\rangle$$

$$= \iint \mathbb{P}(\hat{\mathbf{e}}_i)\, \mathbb{P}(\hat{\mathbf{e}}_k)(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k)^2 \, d\hat{\mathbf{e}}_i \, d\hat{\mathbf{e}}_n$$

Averaging over N and taking the limit $T \to \infty$ is once again straightforward giving

$$\psi(\tau) = \eta^2 \left(\int_{-\infty}^{\infty} |F(t)|^2 \, dt \right)^2$$

$$+ \eta^2 \left| \int_{-\infty}^{\infty} F^*(t) \, F(t + \tau) \, dt \right|^2 \langle(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k)^2\rangle$$

$$+ \eta \int_{-\infty}^{\infty} |F(t)|^2 \, |F(t + \tau)|^2 \, dt \qquad \text{(AII-14)}$$

Turning now to the quantity $\langle(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k)^2\rangle$ we define two states of polarization, completely polarized by $\mathbb{P}(\hat{\mathbf{e}}_i) = \delta(\hat{\mathbf{e}}_i - \hat{\mathbf{e}}')$ and completely unpolarized by $\mathbb{P}(\hat{\mathbf{e}}_i) = (2\pi)^{-1}$. It is completely straightforward to verify that $\langle(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k)^2\rangle = 1$ for completely polarized $\langle(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k)^2\rangle = 1/2$ for completely unpolarized. We may thus simplify (AII-14) to be

$$\psi(\tau) = \langle I\rangle^2 \left[1 + \frac{\gamma}{2} |\Phi(\tau)|^2 \right]$$

$$+ \eta \int_{-\infty}^{\infty} |F(t)|^2 \, |F(t + \tau)|^2 \, dt \qquad \text{(AII-15)}$$

where $\gamma = 1$ for unpolarized and $= 2$ for polarized.

In the limit of a very large η (this is equivalent to a large number of oscillators being on at any given time) we may neglect the term linear in η compared to $\langle I \rangle^2$ which is quadratic in η giving

$$\psi(\tau) = \langle I \rangle^2 \left[1 + \frac{\gamma}{2} |\Phi(\tau)|^2 \right] \qquad \text{(AII-16)}$$

a familiar result.[14]

For $\tau = 0$ $\Phi(\tau) = 1$ and we have

$$\psi(0) = \langle I^2(t) \rangle = \left(1 + \frac{\gamma}{2} \right) \langle I(t) \rangle^2 \qquad \text{(AII-17)}$$

our desired result.

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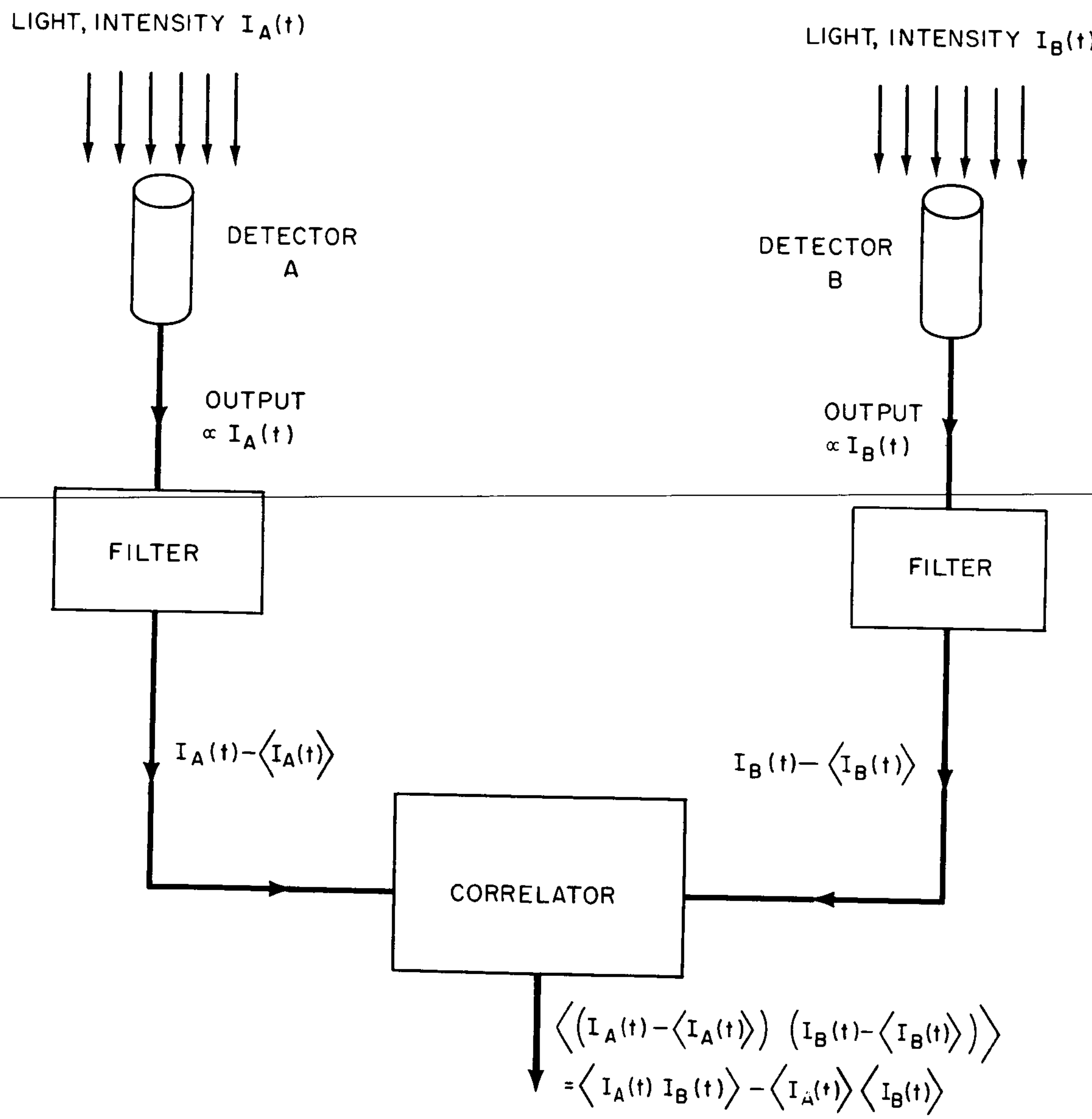
FIGURE CAPTIONS

Fig. 1 Schematic of Post Detection Interferometer.

Fig. 2 Instantaneous Interference Pattern Produced by Two Sources.

Fig. 3 Sequence of Time Intervals Having Different Values of $u = \int_{t_i}^{t_{i+1}} I(t)\,dt$.



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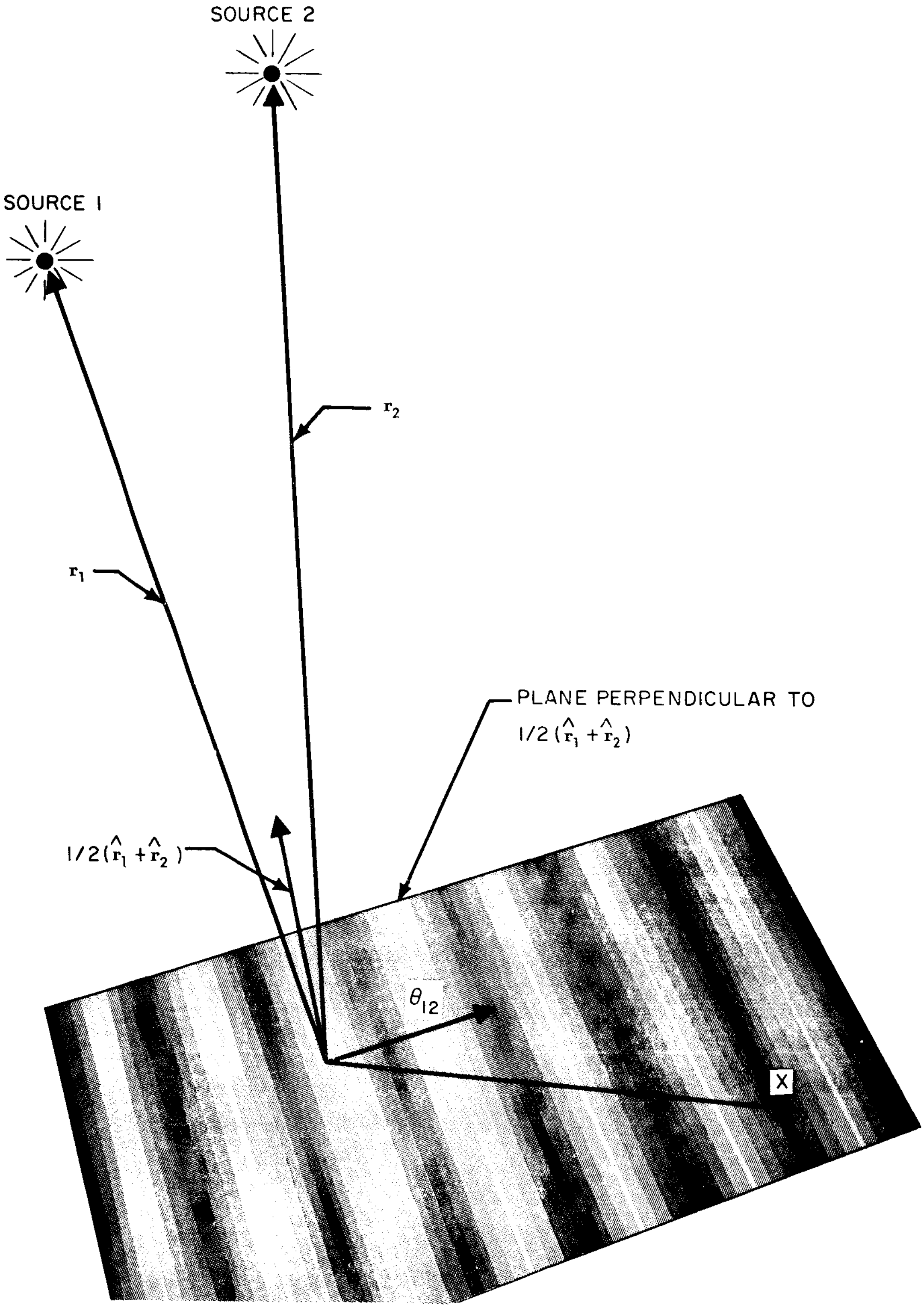

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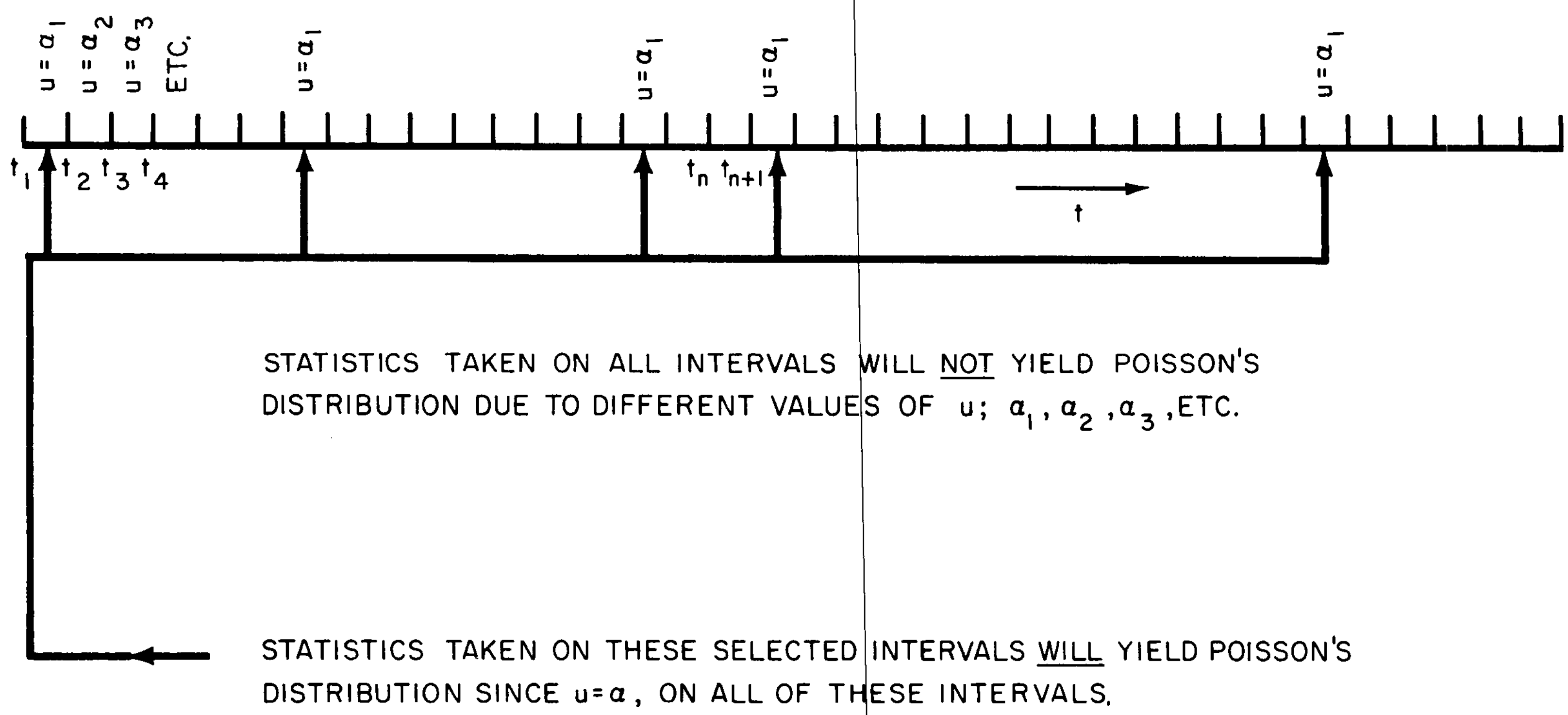

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