A RE-EVALUATION OF THE THEORY FOR
THE HYDROSTATIC FIGURE OF THE EARTH

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ABSTRACT

The equations of the classical theory for the hydrostatic figure of the earth are modified in order to make their development independent of the external potential theory. These modifications are essential in order to study any discrepancy which may exist between the actual flattening $f$ and the hydrostatic flattening $f_h$. An expression is obtained giving the hydrostatic flattening explicitly in terms of other parameters, which seems to have the advantage of easy numerical manipulation. The results obtained from the modified equations disagree with previous findings, giving a hydrostatic flattening of $1/296.70 \pm 0.05$ in contrast to the value of $1/299.86 \pm 0.05$ as found from the classical hydrostatic equations of de Sitter or the recently proposed method of Jeffreys. These results, if true, lead to the conclusion that the hydrostatic flattening of the earth is greater than its actual flattening, i.e., $f_h^{-1} - f^{-1} < 0$, in contradiction to the currently prevalent opinion that $f_h^{-1} - f^{-1} > 0$. Some difficulty is likely to arise with certain of the geophysical hypotheses that are constructed on the basis of the currently accepted belief that the actual flattening of the earth is greater than the corresponding hydrostatic flattening for it.
INTRODUCTION

The hydrostatic theory of the earth, in its present form, was first
developed by Clairaut (1743). Legendre (1795) simplified Clairaut's
differential equation by making an important substitution, and the theory
has since been used primarily in estimating the flattening of the earth
spheroid, on the assumption that the value of flattening obtained from it
would give the best available approximation to the real flattening of the
earth spheroid. Doubt was cast on this assumption, however, when Tisserand
(1891) and Poincaré (1910), on the basis of the hydrostatic theory,
derived $f^{-1} = 297.3$, in contrast to the then-accepted value of $f^{-1} = 293.5$
which A. R. Clarke had obtained from arc measurements in 1880 (Jeffreys,
1962, p. 152). Later, Hayford (1909, 1910) using more extensive data,
obtained $f^{-1} = 297.0$, which was quite consistent with the value obtained
from hydrostatic theory within the limits of accuracy. This reaffirmed
the then-prevalent opinion that the hydrostatic theory method for computing
the flattening of the earth was more accurate than any other available at
that time (de Sitter, 1924, 1933; Jeffreys, 1948, 1962; Bullard, 1948;
Jones, 1954) and was an interesting finding—in view of the apparent
departure of the shape of the physical surface of the earth from that
which hydrostatic conditions would require (Jeffreys, 1952, p. 145).
However, it was based on the fact that since hydrostatic equilibrium
probably existed throughout the earth's interior except for the upper
crust (which forms only a negligible part of the whole earth), the assumption
of hydrostatic equilibrium for the whole earth was not unjustified. This
belief persisted until artificial satellites made possible the direct
determination of the geopotential coefficient $J$, from which the flattening
f of the best-fitting spheroid could be accurately computed using the external potential theory, without recourse either to geodetic or gravity data or to the hydrostatic equilibrium theory. The currently accepted value of f computed in this way is 1/298.25.

At about the same time Hendriksen (1960), using de Sitter’s (1924) equations, found \( f_n^{-1} = 500.0 \) corresponding to \( J = 1622.4 \times 10^{-6} \). Later, Jeffreys (1964), using \( J = (1624.17 \pm 0.05) \times 10^{-6} \), as obtained by King-Hele, Cook, and Rees (1963), found \( f_n^{-1} = 299.67 \pm 0.05 \), confirming the discrepancy between f and \( f_n^{-1} \). In a previous paper (Khan, 1967a) I used the improved values of the constant of precession (Rabe, 1950) and of moon-earth mass ratio (1/81.303) and obtained \( f_n^{-1} = 299.80 \pm 0.05 \) by employing Jeffreys’ (1964) and approach, which has both the advantages of speed and simplicity. Although \( f_n^{-1} - f^{-1} \) is greater than the uncertainty in the

\(^2\)Actually it is erroneous to use the satellite-determined J in the hydrostatic equations and the problem of hydrostatic equilibrium of the earth can be and should be solved with any knowledge of J. However, this possibility does not seem to have been explored and the trend of using a pre-determined J (either from satellite motion as in post-satellite times or from internal density distribution as in pre-satellite times) in the solution of hydrostatic equations persists up till now. The results reported in this paper are obtained accepting this trend as correct (which it is not). The flaws of this method will be discussed in a subsequent paper (Khan, 1967b) in which a correct solution of the problem will be given.
determination of either \( f \) or \( f_n \), yet in magnitude it is so small that it is worthwhile to ensure that the discrepancy is genuine and that it does not arise merely from an error in some of the parameters involved in the calculations or even from some undetected flaw in the hydrostatic theory itself, before attempting to construct the possible geophysical hypotheses to explain it. For this reason I have re-evaluated the hydrostatic theory and give first a brief summary of de Sitter's method followed by a very brief account of Henriksen's (1960) post-satellite approach. When de Sitter (1924) worked out his hydrostatic equations, the geopotential coefficient \( J \) was not known accurately and he eliminated it from the right hand side of his equations by means of a well-known relation derived from external potential theory, hopefully presuming that there was no significant difference between the value of \( J \) for the real earth and the hydrostatic \( J \). Henriksen (1960) seems to have used these equations without either detecting this fact or pointing it out. The satellite determination of \( J \) has proved the above assumption to be incorrect, and its elimination is no longer desirable (as explained later) because it tends to introduce an implicit \( f_n = f \) (in case of the presently adopted approach to compute \( f_n \)). For this reason I have rewritten de Sitter's equation so that \( J \) appears explicitly in them, rather than being eliminated as was done earlier (Darwin, 1900; de Sitter, 1924). I call these equations 'the modified equations of hydrostatic theory' in this paper. As will be shown later, the use of these modified equations has led to a value of hydrostatic flattening which differs considerably from the previous results. These modified equations are solved simultaneously in order to obtain an explicit expression for \( f_n \). It is important to note that usually the
hydrostatic flattening is defined as the flattening which the earth, with its actual values of geopotential coefficient $J$ and the dynamical flattening $H$, would have if it were in hydrostatic equilibrium. Since $H$ and $J$ for the real earth are not compatible on the basis of hydrostatic equilibrium, this situation is physically impossible and, consequently, hydrostatic flattening is a hypothetical number. It is in this sense that the term 'hydrostatic flattening' has been used throughout this paper. This is obviously not a realistic definition of the hydrostatic flattening in terms of the actual situation and the mathematical realities but this point will not be discussed any further in this paper lest it lead to confusion. I will discuss this aspect of the problem in detail in another paper (Khan, 1967b).

**DYNAMICAL FLATTENING $H$**

As in my previous paper (Khan, 1967a), I use Rabe's (1950) value for the constant of precession which he obtained from the 1950 opposition of Mars and the recently improved value of the moon-earth mass ratio obtained from Ranger-shot data. Taking

$$\text{moon-earth mass ratio} = 1/81.303$$

and

$$\text{constant of precession} = 5492.791'' \text{ (1900 epoch),}$$

the value of $H$ can be calculated using well-known formulas (de Sitter, 1920). In Table 1 my value for $H$ is compared with those of other investigators. It may be pointed out at this stage that the slight difference in my value of $H$ and that of others, does not critically affect the value of the hydrostatic flattening and hence does not change in any
way the nature of the discussion to follow.

CLASSICAL EQUATIONS OF HYDROSTATIC THEORY

Pre-satellite method. The classical hydrostatic theory is given in detail in a number of papers (Darwin, 1900; de Sitter, 1924, 1933; Jeffreys, 1953, 1962, 1964; Bullard, 1948; Jones, 1954). We, therefore, mention here only those points which are necessary for convenience of reference without giving a full account of the theory.

We can readily obtain the following relation from the external potential theory.

\[ J = f - \frac{1}{2} \frac{1}{m} - \frac{1}{2} f^2 + \frac{1}{7} m f + \frac{4}{7} \kappa \]  

(1)

where

\[ m = \frac{u^2 r_m^3}{GM} \]

\( f \) = actual flattening of the best-fitting spheroid (Such a spheroid is sometimes called normal surface.)

\( \kappa \) = a constant indicating the departure of the spheroidal surface from the corresponding ellipsoidal surface

\( J = \frac{3}{2} J_2 \) where \( J_2 \) is the coefficient of the second harmonic term in the spherical harmonic expansion of the geopotential

\( \omega \) = the angular velocity and

\( r_m \) = the mean radius of the earth.
Equation (1) gives $f$ if $J$ and $k$ are known. However, since direct determinations of the precise value of $J$ were not available in pre-satellite times, $J$ had to be found by some other means. The earth was assumed to be in hydrostatic equilibrium so that the resulting outer surface, often referred to as the "ideal surface," had the same flattening as the "normal surface," i.e., $f_n = f$. One could then use the well-known relation

$$J = q H \text{ where } q = \frac{3}{2} \frac{C}{M a_e^2},$$

(2)

In the above formula $C$ denotes the polar moment of inertia, $M$ the mass and $a_e$ the equatorial radius of the earth.

$H$ could be calculated from the constant of precession using formulas given by de Sitter (1938).

By considering the expansion of potential at a point inside the earth and then assuming that the surfaces of equal density coincide with the equipotential surfaces (which is the condition of hydrostatic equilibrium), the equation for $q$ could be written as (de Sitter, 1924):

$$q = 3 \left(1 - \frac{2}{3} f_n\right) \int_0^1 \delta \beta^4 \, d\beta + \frac{2}{3} J,$$

(3)

which, with the help of equation (1) could be rewritten$^3$ as

$$q = 1 - \frac{1}{3} m - 2 \left(1 - \frac{2}{3} f_n\right) \int_0^1 D \beta^4 \, d\beta,$$

(4)

$^3$Note the introduction of an implicit $f_n = f$ if the observed value of $J$ is used to compute $f_n$. 

and finally transformed into

\[ q = 1 - \frac{1}{3} m - 2 \left( \frac{1 - \frac{2}{3} f_n}{1 + \frac{4}{3} f_n} \right) \frac{\sqrt{1 + \eta_s}}{1 + \lambda_s} \]  \hspace{1cm} (5)\]

where use was made of the relation

\[ \int_0^1 D \beta^4 d\beta = \frac{1}{5} \frac{\sqrt{1 + \eta_s}}{1 + \lambda_s} \]  \hspace{1cm} (6)\]

In the above expressions, \( \delta \) is the density expressed in terms of the mean density as a unit, \( D \) the mean density within the surface \( \delta \) expressed in the same units, and \( \eta_s \) and \( \lambda_s \) are the surface values of the parameters \( \eta \) and \( \lambda \) which depend upon the internal density distribution of the earth.

Equation (5) will give \( q \) if \( \eta_s \) and \( \lambda_s \) are known. The considerations which led to equation (5), also gave a boundary condition for the determination of \( \eta_s \). This can be written as

\[ \eta_s f' \left( \frac{1 + \frac{4}{3} f_n}{7 - \frac{4}{21} m} \right) = 3 f' \left( \frac{1 + \frac{2}{3} f_n}{7 - \frac{4}{3} f_n} \right) - 5 \left( \frac{1 + \frac{2}{3} f_n}{3 - \frac{4}{3} f_n} \right) \]  \hspace{1cm} (7)\]

which with the help of equation (1) can be transformed into

\[ \eta_s f' = -\frac{5}{2} m - 2 f' + \frac{10}{21} m^2 + \frac{4}{7} f_n^2 - \frac{6}{7} m f_n \]  \hspace{1cm} (8)\]

\[ \text{Note again the introduction of an implicit } f_n = f \text{ if observed value of } f \text{ is used to compute } f_n. \]
where
\[ f' = f_n - \frac{5}{42} f_n + \frac{4}{7} \]

Note that the transformation of equation (3) into (5) and of (7)
into (8) is only possible through the use of equation (1).

Equations (5) and (8) are the same as de Sitter's equations
numbered (21) and (22) in his 1924 paper. In the process of deriving
equation (8) de Sitter made an algebraic error (Jeffreys, 1953) but his
final result\(^5\) [stated in equation (8)] does not appear to have been
affected by it.

In the pre-satellite procedure, equation (8) was used to find the
best value of \( \eta \) by an iteration procedure (in conjunction with some
other equations not given here, see de Sitter, 1924; Bullard, 1948; or
Jeffreys, 1962, 1964 for a complete account). The corresponding value
of \( \lambda \) was obtained from equation (6). These values were used to compute \( \varphi \)
from equation (5). \( J \) could then be computed from equation (2) and \( f_n \) from
equation (1). Thus derived, \( f_n \) was then regarded as the best available
approximation to the real flattening \( f \). This method is discussed in further
detail from a slightly different point of view in another paper (Khan,
1967a).

Classical post-satellite method. Using the satellite determination of
\( J \), Honriksen (1960) and O'Keefe (1960) were the first to compute hydrostatic

\(^5\) He has also omitted the term \( \frac{4}{9} f^2 \) in deriving equation (5) but this term
ultimately becomes \( O(f^2) \) by a subsequent multiplication and drops out
anyway. However, it is desirable to include it to show where it really
drops out. Note that none of the errors seems to be important to our
discussion.
It appears they found $q$ from equation (2), put $\lambda_g = 0$ in equation (5), substituted the value of $\eta_s$ found therefrom in equation (8) and solved the resulting quadratic in $f_n$. Jeffreys (1964) also developed an excellent numerical procedure for computing $f_n$ using a simplified density model. Details of these methods are available in the above-quoted references.

**MODIFIED EQUATIONS OF THE HYDROSTATIC THEORY**

Since de Sitter (1924) had derived equations (5) and (8) from equations (3) and (7) respectively [by eliminating $J$ with the help of equation (1)], Henriksen (1960) appears to have used equation (1) indirectly in the computation of $f_n$. Note that there is nothing wrong with using equation (1) in hydrostatic theory if the true hydrostatic value of $J$ (Hill, 1967b) is used in the computations, but since the observed value of $J$ is usually used in equations (5) and (8) to find the corresponding $f_n$, and since the observed value of $J$ in equation (1) relates the real flattening to the observed $J$ (in case of an earth not in hydrostatic equilibrium), it is apparent that equations (5) and (8) contain an implicit $f_n = f$.

Jeffreys' (1964) numerical approach is valid if hydrostatic value of $J$ is used but is not valid when non-hydrostatic $J$ is used.

Since in our present method, in order to study any discrepancy between hydrostatic flattening $f_n$ and actual flattening $f$, it is important to distinguish between $f$ and $f_n$, we must correct for any intermixing of these two in de Sitter's equations (such as the one pointed out above) before using them to obtain numerical results. For this purpose I have modified de Sitter's equations, expressing them explicitly in terms of $J$, 


thus completely avoiding the use of any relation derived from the external potential theory.

To get the modified equations, we write equation (3), correct to the second order, in the form

$$q = 3 \left( 1 - \frac{2}{3} \frac{1}{h} + \frac{4}{9} \frac{f^2}{h^2} \right) \int_0^1 \delta \beta \, d\beta + \frac{2}{3} J$$ \hspace{1cm} (9)

As stated before, if $D$ is the mean density within the surface $\beta$ expressed in terms of the mean density as a unit,

$$D = \frac{3}{2^3} \int_0^\beta \delta \beta \, d\beta$$

which on differentiation with respect to $\beta$ gives

$$-\frac{\beta}{D} \frac{dD}{d\beta} = 3 \left( 1 - \frac{\delta}{D} \right)$$

or

$$\delta = D \left( 1 + \frac{1}{3} \frac{\beta \, d\beta}{D \, d\beta} \right)$$

Substituting this value of $\delta$ in equation (9), we obtain

$$q = 3 \left( 1 - \frac{2}{3} \frac{1}{h} + \frac{4}{9} \frac{f^2}{h^2} \right) \int_0^1 D \left( 1 + \frac{1}{3} \frac{\beta \, d\beta}{D \, d\beta} \right) \beta \, d\beta + \frac{2}{3} J$$
which, with the help of equation (6), can be finally transformed into

$$\zeta = 1 - \frac{2}{3} \bar{r}_h + \frac{2}{9} \bar{r}_h^2 - \frac{4}{5} \left( 1 - \frac{2}{3} \bar{r}_h + \frac{4}{9} \bar{r}_h^2 \right) \frac{\sqrt{1 + \eta_s}}{1 + \lambda_s}$$  \hspace{1cm} (10)

Equation (7) can be simplified to the form

$$\eta_s \bar{r}' = \frac{6}{7} \bar{r}_h^2 + \frac{4}{7} \bar{r}_h - J \left( \frac{10}{21} \bar{r}_h + \frac{20}{21} m \right)$$  \hspace{1cm} (11)

$\lambda$ is defined as the departure from unity of the average value of a certain function $F(\eta)$ over the range of integration. This function $F(\eta)$ which occurs in the classical hydrostatic theory is given by

$$F(\eta) = \frac{1 + \frac{1}{2} \eta - \frac{1}{10} \eta^2 + \frac{2}{105} \zeta \xi}{\sqrt{1 + \eta}}$$  \hspace{1cm} (12)

where

$$\zeta = 3 \left( 1 - \frac{\delta}{D} \right)$$

and

$$\xi = 7 (m/D) (1 + \eta) - 3 \bar{r}_h (1 + \eta)^2 - 4 \bar{r}_h$$

The function $F(\eta)$ has the remarkable property that its value always lies very near unity, the maximum deviation being of the order of $10^{-4}$. $\lambda_s$, which is the value of $\lambda$ for the outer surface, is given by Bullard (1943) as
\[ \lambda_s = (1.6 \times 1.8)10^{-4} \]

This estimate, however, is based on the density distribution suggested by Sutton (1940, 1942). Jeffreys (1964) using a simplified density model, finds \( \lambda_s = 1.3 \times 10^{-4} \) and points out that if \( \lambda_s = 0 \) instead, the resulting \( f_h \) is greater by \( 6 \times 10^{-7} \) only. It can be shown that \( P(\eta) < 1 \) for \( \eta > 0.53 \). As will be shown later, my value of \( \eta_s \) (the value of \( \eta \) for the outer surface) is considerably greater than that of Bullard (1943) or Jeffreys (1964). Since \( P(\eta) < 1 \) for \( \eta > 0.53 \), it is logical to expect that the mean value of the function \( P(\eta) \) for this new range of integration that is extending further into the domain \( \eta - 0.53 > 0 \), will be still closer to unity. This will revise the estimate of \( \lambda_s \) even further downwards and its effect on the value of \( f_h \) will be even less important.\(^6\) It seems legitimate, therefore, to take \( \lambda_s = 0 \) for initial calculations. Note that Henniker (1960) also took \( \lambda_s = 0 \).

From equation (10) one obtains

\[ \eta_s = (1 + \lambda_s)^2 \left[ \left| \frac{1 - q - \frac{2}{3} f_h + \frac{2}{3} J + \frac{4}{9} f_h^2}{\frac{2}{3} \left( 1 - \frac{2}{3} f_h + \frac{4}{9} f_h^2 \right)} \right| \right]^{-1} \]  

(13)

With \( \lambda_s = 0 \), one can substitute \( \eta_s \) from equation (13) into equation (11) and solve the resulting equation for \( f_h \). This value of \( f_h \) can then be used to find \( \eta_s \) from equation (13) corresponding to \( \lambda_s = 0 \). Following this

\(^6\)Slight variations in the value of \( \lambda_s \) do not affect the value of \( f_h \) significantly. This is apparent from Table 2 and Figure 1 given later.
procedure, one can compute a set of values for $f_h$ corresponding to arbitrary values of $\lambda_s$.

Although the procedure described above is adequate, it is instructive to derive an expression which gives $f_h$ explicitly in terms of other parameters. This can be done by writing equation (13) as

$$
\eta_s = \frac{25}{4} \epsilon^2 \sigma^2 \left( \frac{1 - \Delta_1}{1 - \Delta_2} \right)^2 - 1
$$

(14)

where

$$
\sigma' = 1 - \sigma
$$

$$
\Delta_1 = \frac{2}{3} \left[ \frac{f_h - J - \frac{2}{3} f_h^2}{f_h} \right]
$$

$$
\Delta_2 = \frac{2}{3} \left[ \frac{f_h - \frac{2}{3} f_h^2}{f_h} \right]
$$

$$
F = 1 + \lambda_s
$$

It is instructive to note that $\Delta_1$ and $\Delta_2$ are both of the order of flattening.

Simplifying equation (14), one obtains

$$
\eta_s = \frac{25}{4} \epsilon^2 \sigma^2 \left[ \frac{1}{1 - \frac{2(f_h - \Delta_1)}{f_h} + \left( \frac{\Delta_1^2}{f_h^2} - \frac{3\Delta_2^2}{f_h} - 4\Delta_1\Delta_2 \right) - 1
$$

$$
= \eta_0 + \eta_1 + \eta_2
$$

(16)
where

\[ \eta_0 = \frac{25}{4} r^2 q'^2 - 1 \]

\[ \eta_1 = \frac{25}{2} r^2 q'^2 (\Delta_2 - \Delta_1) = \frac{25}{3} r^2 q' \left[ J - q \tilde{z}_n + \frac{2}{3} q \tilde{z}_n^2 \right] \]  \hspace{1cm} (17)

and

\[ \eta_2 = \frac{25}{4} r^2 q'^2 \left( \Delta_1^2 + 3 \Delta_2^2 - 4 \Delta_1 \Delta_2 \right) \]

Note that the quantity \( \eta_1 \) is of the order of \( \tilde{z}_n \) while \( \eta_2 \) is of the order of \( \tilde{z}_n^2 \).

Using this value of \( \eta_2 \), equation (11) can be written as

\[ A \tilde{z}_n^2 + (\eta_0 - 3 + \delta_1) \tilde{z}_n + 5J + \delta_2 = 0 \]  \hspace{1cm} (18)

where we have put

\[ \Delta = \frac{17}{14} - \frac{5}{42} \eta_0 - \frac{25}{3} r^2 q' \]

\[ \delta_1 = \frac{25}{3} r^2 q' J - \frac{4}{7} \tilde{m} + \frac{10}{21} \]

and

\[ \delta_2 = \frac{4}{7} \eta_0 \tilde{k} - \frac{12}{7} \tilde{k} + \frac{20}{21} \tilde{m} J \]  \hspace{1cm} (19)
Note that $\delta_1$ is approximately of the order of $f_h$ while $\delta_2$ is of the order of $f_h^2$.

Equation (18) gives the required expression for $f_h$ which, correct to the second order of small quantities, is

$$f_h = \frac{1}{(n_0 - 3)} \left[ - (5J + \delta_2) + \frac{(5J + \delta_2) \delta_1}{n_0 - 3} - \frac{25 A J^2}{(n_0 - 3)^2} \right]$$  \hspace{1cm} (20)

It is interesting to see that in this development, the expression for $f_h$ corresponding to the first order theory is

$$f_h = \frac{5J}{3 - n_0}$$  \hspace{1cm} (21)

Equation (20) is relatively easier to manipulate numerically. It also seems more straightforward to compute $f_h$ corresponding to arbitrary values of $\lambda_0$ than to compute it from the usual equations. This can be done by writing the equation (20) as

$$f_h = \frac{Q}{n_0 - 3}$$

where

$$Q = \left[ - (5J + \delta_2) + \frac{(5J + \delta_2) \delta_1}{n_0 - 3} - \frac{25 A J^2}{(n_0 - 3)^2} \right].$$

$Q$ appears to have the remarkable property that, with other parameters fixed, it remains practically constant, (changing only in the ninth place after decimal) for a reasonable range of variation of $\lambda_0$. In fact, with $\lambda_0$ ranging from $-8 \times 10^{-4}$ to $-8 \times 10^{-5}$, the corresponding range of
variation of $Q$ (with all other parameters fixed) is only from $-6 \times 10^{-9}$ to $+6 \times 10^{-9}$. Thus $\tilde{F}_h$ practically depends only on $\eta_0$ which can be easily computed for arbitrary values of $\lambda_s$. We have tabulated below $\tilde{F}_h$ corresponding to a few values of $\lambda_s$.

The data of Table 2 are plotted in Figure 1. It can be seen that the value of $\tilde{F}_h$ is not very sensitive to reasonable changes in the value of $\lambda_s$. Also note that for a reasonable range of variation of $\lambda_s$, $\tilde{F}_h$ varies linearly with $\lambda_s$.

The value of $\kappa$ has been obtained from the following equation:

$$K = \frac{24}{7} \kappa + 3\tilde{F}_h^2 - \frac{15}{7} m \tilde{f}$$

where $K$ is satellite-determined. The value of $K$ used in these calculations was obtained from Konz’s (1964) value of $J_4 = -1.649 \times 10^{-5}$. Consequently, $\kappa$ becomes

$$\kappa = 78 \times 10^{-3}$$

This gives a maximum depression of 4.9 meters (at latitude 45°) of the earth spheroid from the corresponding ellipsoidal surface. This value of $\kappa$ should be more reliable than that derived by de Sitter (1924) or Ballard (1946) from the solution of a second-order differential equation in $\kappa$ (originally given by Darwin, 1899) which involved assumptions about the internal density distribution. However, any reasonable variations in $\kappa$ do not appear to change $\tilde{F}_h$ noticeably. Note that in equation (22), $\tilde{f}$ is the actual flattening.
Table 3 summarizes some of the more important values of $\lambda_3$, $\kappa$, and $\eta_3$ as obtained by different investigators.

In Table 4 I have compared different values of the hydrostatic flattening obtained both in the pre-satellite and the post-satellite times. The correct value of the hydrostatic flattening obtained from the modified hydrostatic equations is $1/296.70$ corresponding to $\eta_3 = 1032.645 \times 10^{-6}$ (Kozai, 1964), as opposed to $1/299.36$ obtained (with $m$ and $\eta$ fixed in both cases) from de Sitter's (1924) classical equations (used by Honriken, 1960). The error in the previous value of hydrostatic flattening arose from the fact (pointed out earlier) that de Sitter (1924) had eliminated the geopotential coefficient $J$ from the right-hand side of his hydrostatic equations because accurate determinations of the value of $J$ were not available when he worked out his equations. This elimination, most probably, was not detected in the post-satellite investigations (Honriken, 1960; O'Keefe, 1960; Han, 1967a) in which de Sitter's hydrostatic equations appear to have been used as such without correcting for the error introduced by these eliminations. The modified equations of course, contain the geopotential coefficient $J$ explicitly and hence are not susceptible to this source of error. It may be emphasized that the difference between my value of $f_n$ and the previous ones cannot be attributed to the slightly different values of $\eta$ and $m$ used by me. To show this I have listed two values of $f_n$ in Table 4 using two different sets of data which are given in the footnote of Table 4.

The change of sign in the value of $f_n^{-1} - f^{-1}$ will create some difficulty with some of the geophysical hypotheses (Tink and Macdonald, 1960; Wang, 1963; Tsuchi, 1963; Tsuchi and Kato, 1964) constructed on the previous belief that the hydrostatic flattening is smaller than the real
flattening. As is obvious from Table 4, the hydrostatic flattening in actuality turns out to be greater than the real flattening. I must emphasize, however, that this definition of the hydrostatic flattening (used throughout this paper and hitherto used by all the post-satellite investigators on hydrostatic theory) is not a realistic one. In the case of the earth, the true hydrostatic equilibrium will exist only when the hydrostatic and external potential solutions (independently obtained) converge to a single solution corresponding to a given rate of rotation of the earth, because equation (1) neither assumes nor discounts the existence of hydrostatic equilibrium in the earth's interior and hence should be valid for both hydrostatic or non-hydrostatic conditions. However, even with this definition the hydrostatic flattening turns out to be greater than the real flattening of the earth. This point along with some other related aspects of the hydrostatic equilibrium theory is discussed in detail in another paper (Khatt, 1967b).

Table 5 gives the values of the geopotential coefficients $J_2$ and $J_4$, corresponding to my value of the hydrostatic flattening, the flattening of the international reference ellipsoid, and the currently accepted value of the hydrostatic flattening. Note that the satellite-determined value of $J_2$ is $J_2 = 1062.645 \times 10^{-6}$ (Kaula, 1964). It does not seem to be very realistic to use the $J_2$ (corresponding to the hydrostatic flattening) given in this table, to estimate the hydrostatic stresses existing in the earth. This point will be discussed in detail in another paper (Khan, 1967b).
SUMMARY AND CONCLUSIONS

The equations of hydrostatic theory are modified in order to make their development independent of the external potential theory. The value of hydrostatic flattening found from these modified equations is $f_{\text{H}}^{-1} = 296.70 \pm 0.05$ as opposed to the currently accepted value of $f_{\text{H}}^{-1} = 299.85 \pm 0.05$ found from de Sitter's equations in which the geopotential coefficient $J$ was eliminated by de Sitter by making use of a relation derived from the external potential theory. The value of $\eta_s$ is found to be $\eta_s = 0.5569$ from these modified equations and is greater (refer to Table 3) than that of Bullard (1948) or Jeffreys (1964). This, if correct, appears to revise even further downwards the value of the parameter $\lambda_s$ which Bullard (1948) has given as $\lambda_s = (1.6 \pm 1.8) \times 10^{-4}$ and Jeffreys (1964) as $\lambda_s = 1.3 \times 10^{-4}$. Small changes in the value of $\lambda_s$ do not affect critically (refer to Table 2 and Fig. 1) the value of $f_{\text{H}}^{-1}$.

In order to make the hydrostatic flattening coincide with the actual flattening by merely varying the value of $\lambda_s$, one has to assume very high negative values of $\lambda_s$ which seem improbable. Hence these results appear to confirm that there does exist a discrepancy between the actual flattening of the earth and the flattening it would have if it were in hydrostatic equilibrium. However, contrary to the currently held belief, it appears that the hydrostatic flattening is in fact greater than the actual flattening. The error in the previous values of the hydrostatic flattening (given by Henriksen, 1960; O'Keefe, 1960; Jeffreys, 1964; Khan, 1967a) appears to have arisen from the elimination of $J$ by means of a relation derived from the external potential theory. This can now be avoided by
modifying the classical equations to make their development independent of the external potential theory.

These results would appear to create some difficulty with some of the geophysical hypotheses constructed on the previous belief, which attempted to explain the then-assumed excess of the actual flattening of the earth over the corresponding hydrostatic flattening as being due to a time-lag in the adjustment of the rocky earth to its decreased rate of rotation (Shilk and MacDonald, 1960), or to the additional flattening of the earth because of the heavy load of polar ice caps (Huang, 1966), or to a rigid lower mantle and a viscous layer in the upper mantle, implying a long recovery-time because of the confinement of the flow to the upper mantle (Takeuchi, 1963; Takeuchi and Masogawa, 1964). More work on some other aspects of the problem of the hydrostatic equilibrium of the earth is in progress and the results will be reported shortly in another paper (Shilk, 1967b).

ACKNOWLEDGMENT

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REFERENCES


Claixs, L. C., Théorie de la Fig.ure de la Terre, Paris, 1743.


<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Flattening</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeffreys</td>
<td>1964</td>
<td>0.0032730</td>
</tr>
<tr>
<td>Henriksen</td>
<td>1960</td>
<td>0.0032793</td>
</tr>
<tr>
<td>Jeffreys</td>
<td>1950</td>
<td>0.0032729</td>
</tr>
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<td>Bullard</td>
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<td>0.0032737</td>
</tr>
<tr>
<td>de Sitter</td>
<td>1924</td>
<td>0.00327942</td>
</tr>
<tr>
<td>Khan</td>
<td>1967a</td>
<td>0.00327384</td>
</tr>
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</table>

*The uncertainty in these values is in the seventh place after the decimal.*
<table>
<thead>
<tr>
<th>$\lambda_0$</th>
<th>$f_n$</th>
<th>$f_n^{-1}$</th>
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<tbody>
<tr>
<td>$+3 \times 10^{-4}$</td>
<td>0.00337397</td>
<td>293.387</td>
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<td>$+6 \times 10^{-4}$</td>
<td>0.00337568</td>
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<td>0.00337219</td>
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<td>$-6 \times 10^{-4}$</td>
<td>0.00336776</td>
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<tr>
<td></td>
<td>$\lambda_s$</td>
<td>$\kappa$</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------</td>
<td>----------------</td>
</tr>
<tr>
<td>de Sitter (1924)</td>
<td>(4.4 ± 1.5) $10^{-4}$</td>
<td>$50 \times 10^{-3}$</td>
</tr>
<tr>
<td>Ballard (1948)</td>
<td>(1.8 ± 1.3) $10^{-6}$</td>
<td>$63 \times 10^{-3}$</td>
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<tr>
<td>Jeffrey (1954)</td>
<td>1.3 $10^{-4}$</td>
<td>$64 \times 10^{-3}$</td>
</tr>
<tr>
<td>My values</td>
<td>---</td>
<td>$75 \times 10^{-3}$</td>
</tr>
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</table>

*The use of modified equations does not affect the value of $r_s$ to any appreciable degree. If we use de Sitter's classical equations as Henrøeøn (1960) did, $r_s = 0.5809(5)$. Thus the value of $r_s$ does not seem to be the cause of the discrepancy in Henrøeøn's and my value of $\lambda_s$ (mentioned later).
### TABLE 4. Comparison of Hydrostatic Flattening Values

<table>
<thead>
<tr>
<th>Results obtained prior to satellite-determination of $\eta$:</th>
<th>$f - \eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>de Sitter (1924)</td>
<td>296.92 ± 0.136</td>
</tr>
<tr>
<td>de Sitter (1933)</td>
<td>296.783 ± 0.036</td>
</tr>
<tr>
<td>Chandrasekhar (1943)</td>
<td>287.332 ± 0.050</td>
</tr>
<tr>
<td>Jeffreys (1952)</td>
<td>297.295 ± 0.072</td>
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<tr>
<td>Jeffreys (1964)$^a$</td>
<td>255.75 ± 0.65</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Results obtained after the satellite-determination of $\eta$:</th>
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</thead>
<tbody>
<tr>
<td>$f^{-1} = 258.25$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f^{-1}_n$</th>
<th>$f^{-1} - f^{-1}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Henríquez (1960)</td>
<td>339.0 ± 1.75</td>
</tr>
<tr>
<td>O'Keefe (1960)$^a$</td>
<td>299.8 ± 1.55</td>
</tr>
<tr>
<td>Jeffreys (1964)</td>
<td>299.37 ± 0.05 ± 1.42</td>
</tr>
<tr>
<td>Khan (1967)</td>
<td>299.86 ± 0.05 ± 1.61</td>
</tr>
<tr>
<td>My present values</td>
<td>296.70 ± 0.05 ± 1.55</td>
</tr>
<tr>
<td></td>
<td>297.04 ± 0.05 ± 1.21</td>
</tr>
</tbody>
</table>

Using the pre-satellite approach.

**Henríquez's calculations.**

Based on $m = 0.00044500$ (Khan, 1967a; Jeffreys, 1964), $\eta = 0.00027384$ (Khan, 1967a) and $J_2 = 0.001052643$ (Kossin, 1964).

**Based on $m = 0.00044992$ (Henríquez, 1960), $\eta = 0.00027373$ and $J_2 = 0.00105270$.**
TABLE 5. Geopotential Coefficients Corresponding to Different Values of Flattening

<table>
<thead>
<tr>
<th>Flattening 1</th>
<th>Flattening</th>
<th>$J_2$</th>
<th>$J_4$</th>
<th>(corresponding to $y$ value of hydrostatic flattening)</th>
</tr>
</thead>
<tbody>
<tr>
<td>296.70</td>
<td>0.003337040</td>
<td>1094.321 x 10^-6</td>
<td>-2.444 x 10^-6</td>
<td></td>
</tr>
<tr>
<td>297.0</td>
<td>0.00336700</td>
<td>1092.06 x 10^-6</td>
<td>-2.412 x 10^-6</td>
<td>(corresponding to flattening of the international reference ellipsoid)</td>
</tr>
<tr>
<td>298.80</td>
<td>0.00333356</td>
<td>1079.158 x 10^-6</td>
<td>-2.325 x 10^-6</td>
<td>(corresponding to the currently accepted value of the hydrostatic flattening)</td>
</tr>
<tr>
<td>298.25</td>
<td>0.00335299</td>
<td>1082.645 x 10^-6</td>
<td>-2.380 x 10^-6</td>
<td>(corresponding to the actual flattening)</td>
</tr>
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</table>

*Compare this value with observed $J_4 = -1.449 x 10^{-6}$ (Egan, 1984).