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ON SOME POSSIBLE SIMPLIFICATIONS AND CHANGES IN HANSEN'S LUNAR THEORY

by Peter Musen

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Greenbelt, Md.

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ABSTRACT

The work presented here is a further development of the ideas expressed in this author's earlier work on the modification of Hansen's lunar theory. We introduce here several devices which permit us to reduce the number of quantities which must be re-computed with each iteration cycle. We refer the motion of the satellite to an "almost ideal" system of coordinates with two axes lying in the mean orbital plane. These axes are selected in such a way that the differential equations of the relative motion differ from the equation of motion in Hansen's ideal system only by the terms of higher orders. The position of the satellite relative to the almost ideal system is determined by Hansen's coordinates $1 + \nu$ and $n_0 \, \delta z$ of the projection of the satellite on its mean orbital plane and by the "elevation" ζ of the satellite relative to this plane. The position of the almost ideal system itself with respect to an inertial system is determined by two uniform rotations, one around the normal to the orbital plane of the sun, and one around the normal to the mean orbital plane of the satellite. The problem of integration is reduced to solving a linear partial differential equation by means of successive approximation, or to expanding the integrating operator into a series of products of two linear operators. One of these operators is a linear partial differential operator and the other is the inverse of a linear partial differential operator with the constant coefficients. We propose here, as in the previous work, to compute Hansen elements Ξ, Υ and Ψ separately and to fuse them into $\bar{W}$ bypassing Hansen's function W.

The application of Hansen's theory to the Jovian satellites is a topic of considerable astronomical interest and importance. M. Charnow has programmed the modified Hansen's theory in order to produce the yearly ephemerides of the X^{th} satellite of Jupiter.

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ON SOME POSSIBLE SIMPLIFICATIONS AND CHANGES IN HANSEN'S LUNAR THEORY

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INTRODUCTION

This work represents a further development of possible modifications of Hansen's (1838, 1862) lunar theory which were presented by the author in a previous work (Musen, 1963). We direct our attention to Hansen's lunar theory because it is applicable with some changes to satellites moving in orbits with high inclinations relative to the orbital plane of the Sun. In particular, its possible application to the motions of Jovian satellites (Mullholland, 1965; Charnow, 1967) is a topic of considerable astronomical interest and importance.

The main kinematical characteristics of Hansen's theory are: 1) the separation of the periodic perturbations *in* the orbital plane from those *of* the orbital plane itself; and 2) the use of a fixed rotating ellipse as an intermediate orbit. Normally, the periodic perturbations *in* the orbital plane are larger than the periodic perturbations *of* the orbital plane itself. These two types of effects are not completely independent, but their mutual influence is of higher order and it can be calculated.

The differential equations of motions of the satellite in its osculating orbital plane are referred to a rotating reference frame rigidly connected to this plane. This frame of reference Hansen terms as "ideal," because the relative and the absolute velocities coincide and because the Coriolis and the centrifugal forces cancel each other. Thus, the vectorial differential equation of the relative motion has the same form as in an inertial system. An intermediate orbit is an ellipse of constant shape located in the osculating orbital plane and rotating with constant angular velocity relative to the ideal frame around the axis normal to this plane. A fictitious "auxiliary" satellite moves in this ellipse in accordance with Kepler's laws with respect to pseudo-time. The position vector of the real satellite at any given time is determined as its deviation from the position vector of the auxiliary satellite in time and space.

In Hansen's theory all the angular perturbations in the orbital plane are combined into one single angle $n_0 \delta z$, the perturbations of the mean anomaly. The radial perturbations are given by $1 + \nu$, the ratio of the radius vector of the real satellite to the radius vector of the fictitious satellite. The position of the osculating orbital plane and, consequently, of the ideal system of coordinates is determined by the mean position of the orbital plane and by three additional parameters which absorb all the small periodic oscillations of the osculating orbital plane around its mean position. The final output in the classical form of Hansen's theory are the trigonometric expansions of the perturbations in the sine of the latitude, of $1 + \nu$ and of $n_0 \delta z$.

The author has suggested (Musen, 1959) that the expansion of the sine of the latitude be discarded if the inclination is large. Instead, he proposes to use a set of Euler's four "redundant" parameters to determine the periodic deviations of the osculating orbital plane from its mean position. Euler's parameters, like Hansen's, carry all the periodic effects in the node, the inclination, and the position of the departure point. The position of the mean orbital plane itself relative to the inertial frame is influenced only by the secular effects in the node, the perigee and the departure point. The use of Euler's parameters makes all the angular arguments of the theory linear relative to time from the outset and the expansion of the disturbing function becomes more symmetrical, as well as more algebraic in form, as compared to the expansion in terms of Hansen's parameters.

Developing the theory of satellites in terms of Brendel's coordinates (Brendel, 1925), the author has noticed (Musen, 1967) the existence of an "almost ideal" reference frame in the *mean* orbital plane, such that the differential equations of motion of the satellite relative to this frame differ from the equations in the ideal system only by the terms of higher orders. The use of the mean orbital plane instead of the osculating one, as well as the introduction of the "almost ideal" system, permits one to discard the use of Hansen's or Euler's parameters. In addition no "redundant" parameters appear in the reformulation of the theory. The position of the satellite is determined by Hansen's coordinates of its projection on the mean orbital plane and by its "elevation" ζ relative to this plane. This elevation is very small and its use permits one to contract the iteration process considerably. This system leads to an expansion of the disturbing function such that the coefficients of ζ and $1 + \nu$ remain unaltered from the iteration to iteration. This circumstance, consequently, represents a substantial simplification as compared to the system based on the use of Hansen or Euler parameters.

Hansen's original lunar theory contains the derivatives of the disturbing function relative to the eccentricity and relative to the parameters associated with the position of the osculating orbital plane. Evidently, the literal expansions of the disturbing function and of its derivatives are required before the process can be made completely numerical. We retain here our previous system (Musen, 1963) which makes use of expansions of the components of disturbing force and which somewhat resembles Hansen's planetary theory. This system permits the numerical substitution of the mean eccentricity and of the mean inclination from the very start. We expand the disturbing force into an infinite series in terms of the mean anomalies of both the satellite and of the sun, and in powers of $1 + \nu$ and ζ. Expansion in terms of the disturbed eccentric anomaly E of the satellite and of the true anomaly f' of the sun is also possible. The advantage of this last expansion is that the disturbing function can be represented as a series of trigonometric polynomials in E and f'. However, if the higher order effects are being sought, then the problem of integration is much simpler if the mean anomalies are employed. The determination of the mean motion of the perigee in Hansen's theory involves division by the eccentricity. Thus, an inconvenience arises if the mean eccentricity is very small. The situation can be remedied by keeping the first power of the eccentricity in literal form—in other words, by representing each term of the trigonometric series as a "vector"

$A\epsilon + B$, with the following laws of multiplication for the "unit vectors":

$$\epsilon \cdot 1 = 1 \cdot \epsilon = \epsilon , \qquad \epsilon^2 = \epsilon e_0 ,$$

where e_0 is the *numerical* value of the mean eccentricity. The motion of the perigee is obtained from the condition that no constant for the form ϵA is present in the derivative of Hansen's element Ψ. After the work is completed ϵ should be replaced by e_0.

The main charm of Hansen's lunar theory is that all the perturbations in the orbital plane are determined by one single function $\overline{W}$. As an auxiliary step Hansen introduces a second function W. In this second function the "elliptic" mean anomaly, which describes the motion of the fictitious satellite, and the mean anomaly taken as the argument in the expansion of perturbations are separated. Then the differential equation for W depends only upon the derivatives of the osculating elements, but not upon the motion of the fictitious satellite directly. After the integration of the expansion of dW/dt is completed, the distinction between the two types of mean anomalies is removed; and by applying Hansen's "bar-operator" W becomes $\overline{W}$. With respect to these two functions we take the same approach as before (Musen, 1963): *only $\overline{W}$ has a direct kinematical meaning;* W is merely an artificial device to blend three independent series into one.

However, neither the number of terms nor the computing time can be diminished by using W instead of the independent three series. We propose here, as in the previous work, to compute the three series for Hansen's elements Ξ, Υ, Ψ separately, without forming the W-function. After they are computed, we fuse them together to form the $\overline{W}$-function. We suggest the method of iteration to solve the problem, partly because the starting approximation might include some perturbation effects already, and thus the input information might go beyond Hansen's moving ellipse, and partly because the programming is more uniform if the iteration process is employed. An additional reason for employing the process of iteration is its close association with the fixed point theorems in the modern theory of differential equations. These theorems are now being used widely to establish the existence and the properties of solutions.

A considerable portion of Hansen's lunar theory constitutes the expansion in powers of the perturbations in the mean anomaly $n_0 \,\delta z$. Almost every step requires this expansion and the determination of $\overline{W}$ even requires the expansion in powers of $n_0 \,\delta z$ twice. We can abbreviate the process of expansion considerably by developing the theory in terms of the *disturbed* mean anomaly directly without resorting to the expansion in powers of $n_0 \,\delta z$ at almost every step. Then the problem of integration relative to time can be reduced either to solving of a linear partial differential equation by means of successive approximations or by expanding the integrating operator $(d/dt)^{-1}$ into a series of products of two linear operators. One of these operators is a linear partial differential operator and the other is the inverse of a linear partial differential operator with constant coefficients at the operators of the partial derivatives with respect to the four basic arguments.

Several numerical inaccuracies were found in Hansen's expansions of the coordinates of the Moon. However, this circumstance does not affect the correctness of his basic theoretical thought.

The theory is applicable to the natural planetary satellites when solar effects are dominant. Several small, but observable and physically important effects of non-solar origin, which are so important in the motion of the Moon, have no significance in the computation of yearly ephemerides of the satellites of the outer planets.

The Hill-Brown lunar theory makes use of the variational solution as an intermediary orbit. This solution is free from the influence of the eccentricity and of the inclination and represents a natural start if one wants to expand the coordinates and the frequencies in powers of the constants of integration. This does not mean, however, that in general the variational solution contains the effects which are the most important numerically. The choice of the rotating ellipse as an intermediary is a natural one in a purely numerical approach, because this ellipse introduces the mean motions of the node and of the perigee from the start, and because the large terms in the perturbations have the eccentricity of the satellite as a factor.

The unorthodox manner of treating the perturbations, the absence of the direct appeal to the method of the variation of the astronomical constants and, especially, the use of the "disturbed" time, and the rather difficult style of Hansen's exposition, make the reading of Hansen's work rather demanding to modern readers. But the reader willing to overcome these obstacles will in the final analysis find Hansen's basic idea to be admirably clear and well adaptable to the use of modern electronic machines. M. Charnow (1966) has programmed an accurate expansion of Hansen's perturbations using the modified Hansen's theory as developed by the author. On the basis of his expansion the yearly ephemerides of the Xth satellite of Jupiter are being computed. Since 1958 this satellite was not observed because of the absence of the accurate theory and of the ephemerides. On the basis of Charnow's ephemeris the satellite was re-found in 1967 by E. Roemer in Arizona. The application of this expansion to the XIth satellite of Jupiter is being planned.

DIFFERENTIAL EQUATIONS OF MOTION OF THE SATELLITE

Let $\mathbf{k}$ be the unit-vector normal to the orbital plane of the planet and let us consider this plane as fixed. Let $\mathbf{P}$ be the unit-vector along the ascending node of the mean orbital plane, $\mathbf{R}$ be the normal to this plane and $\mathbf{Q} = \mathbf{R} \times \mathbf{P}$. The mean elements are selected in such a way that no purely secular or mixed terms will appear in the expansions of the rectangular coordinates. The mean orbital plane rotates around $\mathbf{k}$ with constant angular velocity

$$n_0 \boldsymbol{\gamma} = n_0 \gamma \mathbf{k}$$

where n_0, the mean anomalistic motion of the projection of the satellite on the mean orbital plane, must be given in advance as a constant of the theory. Let I be the inclination of the mean orbital plane toward the orbital plane of the planet; it is also one of the basic constants of the theory. We have:

$$n_0 \boldsymbol{\gamma} = n_0 \beta \mathbf{Q} + n_0 \alpha \mathbf{R} \tag{1}$$

where

$$\alpha = \gamma \cos \mathrm{I} , \qquad \beta = \gamma \sin \mathrm{I} . \tag{2}$$

Let E be the mass of the planet, m' the mass of the sun, $\mathbf{r}$ the position vector of the satellite, and $\mathbf{r}'$ the position vector of the sun. The gravitational constant is put equal to one. We consider the mass of the satellite as negligible. The mutual distance $|\mathbf{r}' - \mathbf{r}|$ between the sun and the satellite we designate by Δ. The equation of motion of the satellite relative to the rotating frame $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ has the form:

$$\frac{d^2 \mathbf{r}}{dt^2} + 2 n_0 \boldsymbol{\gamma} \times \frac{d\mathbf{r}}{dt} = -\frac{E}{r^3} \mathbf{r} + E \,\mathrm{grad}_{\mathbf{r}} R \tag{3}$$

where

$$R = \frac{m'}{E}\left(\frac{1}{\Delta} - \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3}\right) + \frac{1}{2}\frac{n_0^2}{E}(\boldsymbol{\gamma} \times \mathbf{r})^2 . \tag{4}$$

Let $\boldsymbol{\rho}$ be the projection of $\mathbf{r}$ on the mean orbital plane and ζ be the "elevation" of the satellite with respect to this plane. We have

$$\mathbf{r} = \boldsymbol{\rho} + \zeta \mathbf{R} . \tag{5}$$

Substituting Equation 5 into Equation 3 and taking Equation 1 into consideration, we have

$$\frac{d^2 \boldsymbol{\rho}}{dt^2} + 2 n_0 \alpha \mathbf{R} \times \frac{d\boldsymbol{\rho}}{dt} + 2 n_0 \beta \mathbf{P} \frac{d\zeta}{dt} = -\frac{E}{r^3} \boldsymbol{\rho} + E \,\mathrm{grad}_{\boldsymbol{\rho}} R \tag{6}$$

and

$$\frac{d^2 \zeta}{dt^2} - 2 n_0 \beta \mathbf{P} \cdot \frac{d\boldsymbol{\rho}}{dt} = -\frac{E}{r^3} \zeta + E \frac{\partial R}{\partial \zeta} . \tag{7}$$

In the mean orbital plane we choose a second reference frame rotating with the constant velocity $-n_0 \alpha$ relative to the frame $(\mathbf{P}, \mathbf{Q})$ around the $\mathbf{R}$-vector. In accordance with the basic rules of kinematics we have to replace in the Equations 6 and 7

$$\frac{d\boldsymbol{\rho}}{dt} \text{ by } \frac{d\boldsymbol{\rho}}{dt} - n_0 \alpha \mathbf{R} \times \boldsymbol{\rho} ,$$

$$\frac{d^2 \boldsymbol{\rho}}{dt^2} \quad \text{by} \quad \frac{d^2 \boldsymbol{\rho}}{dt^2} - 2 n_0 \alpha \mathbf{R} \times \frac{d\boldsymbol{\rho}}{dt} - n_0^2 \alpha^2 \boldsymbol{\rho} ,$$

where $d\boldsymbol{\rho}/dt$ and $d^2 \boldsymbol{\rho}/dt^2$ now designate the relative velocity and the relative acceleration of the satellite in the new rotating system.

Equation 6 takes the form:

$$\frac{d^2 \boldsymbol{\rho}}{dt^2} = - \frac{E}{\rho^3} \boldsymbol{\rho} + E \boldsymbol{\Phi} , \tag{8}$$

where

$$\boldsymbol{\Phi} = \text{grad}_{\boldsymbol{\rho}} \Omega - \frac{2 n_0 \alpha}{E} \mathbf{P} \frac{d\zeta}{dt} \tan I \tag{9}$$

and

$$\Omega = \frac{m'}{E} \left(\frac{1}{\Delta} - \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3} \right) + \left(\frac{1}{r} - \frac{1}{\rho} \right) + \frac{n_0^2}{2E} \left[(\boldsymbol{\gamma} \times \mathbf{r})^2 - \alpha^2 \boldsymbol{\rho}^2 \right] . \tag{10}$$

Let ξ, η, ζ be the coordinates of the satellite relative to the system $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$: then

$$\mathbf{r} = \xi \mathbf{P} + \eta \mathbf{Q} + \zeta \mathbf{R} , \tag{11}$$

$$\boldsymbol{\rho} = \xi \mathbf{P} + \eta \mathbf{Q} . \tag{12}$$

Let us define the mean semi-major axis of the orbit of the satellite by means of the equation

$$n_0^2 a_0^3 = E . \tag{13}$$

We have also for the corresponding equation for the sun

$$n_0'^2 a_0'^3 = m' + E \tag{14}$$

or, in a more convenient form,

$$m' = \left(1 - \frac{E}{m' + E} \right) n_0'^2 a_0'^3 .$$

Substituting Equations 11 through 14 into Equations 9 and 10 we deduce:

$$a_0 \mathbf{\Phi} = \mathrm{grad}_{\boldsymbol{\rho}} a_0 \Omega - \frac{2\alpha}{n_0 a_0^2} \mathbf{P} \frac{d\zeta}{dt} \tan I , \tag{15}$$

and

$$a_0 \Omega = \left(1 - \frac{E}{m' + E}\right) m^2 \frac{a_0^3}{\alpha_0^3} \left(\frac{1}{\Delta} - \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3}\right)$$

$$+ a_0 \left(\frac{1}{r} - \frac{1}{\rho}\right) + \frac{\alpha^2}{a_0^2} \left[\frac{1}{2} \left(\xi^2 + \zeta^2\right) \tan^2 I - \eta \zeta \tan I\right] \tag{16}$$

where

$$m = \frac{n_0'}{n_0}$$

and α_0 is the parallactic factor

$$\alpha_0 = \frac{a_0}{a_0'} .$$

All the terms in Equations 15 and 16, excepting the first term in Ω, are of higher orders. Thus the equation of motion in the form given by Equation 8 differs from the corresponding equation of motion in a nearly ideal system by the terms of higher orders. This is why we designate the second rotating system of coordinates as an "almost ideal."

Equation 7 can be written in the form:

$$\frac{d^2 \zeta}{dt^2} = E \frac{\partial \Omega}{\partial \zeta} + 2 n_0 \alpha \frac{d\xi}{dt} \tan I . \tag{17}$$

Making use of Equation 13 we can write this equation in the form:

$$\frac{d^2 (\zeta/a_0)}{dt^2} = n_0^2 \frac{\partial (a_0 \Omega)}{\partial (\zeta/a_0)} + 2 n_0 \alpha \frac{d\xi/a_0}{dt} \tan I .$$

To facilitate the integration we put this last equation in the form:

$$\frac{d^2(\zeta/a_0)}{dt^2} = -\frac{E}{\bar{\rho}^3}\cdot\frac{\zeta}{a_0} + n_0^2\,\frac{\partial(a_0 V)}{\partial(\zeta/a_0)} + 2\,n_0\,\alpha\,\frac{d\xi/a_0}{dt}\,\tan I\ , \tag{18}$$

where

$$a_0 V = a_0 \Omega + \frac{1}{2}\left(\frac{a_0}{\bar{\rho}}\right)^3\left(\frac{\zeta}{a_0}\right)^2 .$$

The Equation 8 of motion relative to the almost ideal system of coordinates of the projection of the satellite on its mean orbital plane has the same external form as the equation of the disturbed planar motion in an inertial or in an ideal system. It is clear, that because of this identity of forms the process of integration of Equation 8 will go formally along the same line as, say, in the ideal system of coordinates. Similarly, as in the ideal system, all theory of osculation, and the definitions of the elements as well as of Hansen's intermediary ellipse and of Hansen's coordinates, can be transferred automatically to the motion in an almost ideal system. Of course, the elements which we introduce are rather the constants of integration and they are not identical to the osculating elements of motion in an inertial system. In the following exposition when referring to the "elements" we shall understand consistently such "relative" elements in the almost ideal frame.

The intermediary orbit in our theory is the ellipse of constant shape rotating uniformly with constant angular velocity $n_0 y$ relative to the almost ideal system around the normal to the mean orbital plane. The auxiliary satellite P_1 is moving in this ellipse in accordance with Kepler's laws and its position in the ellipse is determined by the standard set of equations:

$$\frac{\bar{\rho}}{a_0}\cos\bar{f} = \cos E - e_0\ , \tag{19}$$

$$\frac{\bar{\rho}}{a_0}\sin\bar{f} = \sqrt{1-e_0^2}\,\sin E\ , \tag{20}$$

$$\frac{\bar{\rho}}{a_0} = 1 - e_0\cos E\ , \tag{21}$$

$$E - e_0\sin E = n_0\,z(t) + c_0\ , \tag{22}$$

where a_0 is the semi-major axis of the auxiliary ellipse, e_0 its eccentricity, and $\bar{r}$, $\bar{f}$, E, $n_0 z + c_0$ respectively the radius-vector, the true, the eccentric and the mean anomalies of P_1. At a given time t the projection of the real satellite P on the mean orbital plane will have the same direction as the position of P_1 at the time z. Designating by $\mathbf{r}(t)$ the position vector of P, by $\boldsymbol{\rho}(t)$ the position vector of the projection of P on the mean orbital plane, and by $\bar{r}(z)$ the position vector of P_1, we can write the basic relations of the theory in the form:

$$\mathbf{r}(t) = \boldsymbol{\rho}(t) + \zeta \mathbf{R} , \tag{23}$$

$$\boldsymbol{\rho}(t) = (1+\nu)\,\bar{\boldsymbol{\rho}}(z) , \tag{24}$$

$$z = z(t) . \tag{25}$$

The position of the perigee of the auxiliary ellipse relative to the x-axis of the almost ideal system is $\pi_0 + n_0\, yt$, where π_0 is a constant. Designating by v the polar angle of the projection of P with respect to the x-axis of the almost ideal system, we can split Equation 24 into two standard relations of Hansen's theory:

$$\rho = (1+\nu)\,\bar{\rho} \tag{26}$$

$$v = \bar{f} + \pi_0 + n_0\, yt . \tag{27}$$

We introduce now the elements osculating relative to the almost ideal system in a sense we discussed before. Let a be the relative osculating semi-major axis, e the relative osculating eccentricity, χ the longitude of the relative osculating pericenter reckoned from the x-axis of the almost ideal system. The osculating mean motion is defined as usually by the equation:

$$n = \frac{E}{a^{3/2}} . \tag{28}$$

We also make use of Hansen's elements

$$\frac{h_0}{h} \quad \text{and} \quad \frac{h}{h_0} \tag{29}$$

and

$$\begin{aligned} \Xi &= -1 - \frac{h_0}{h} + 2\frac{h}{h_0} - 3 e_0 \frac{h}{h_0} \cdot \frac{e \cos\left(\chi - \pi_0 - n_0\, yt\right) - e_0}{1 - e_0^2} , \\ &= -1 - \frac{h_0}{h} + 2\frac{h}{h_0} - \frac{3}{2} e_0 \Upsilon , \end{aligned} \tag{30}$$

$$\Upsilon = 2\frac{h}{h_0} \cdot \frac{e \cos\left(\chi - \pi_0 - n_0\, yt\right) - e_0}{1 - e_0^2} , \tag{31}$$

$$\Psi = 2\frac{h}{h_0} \cdot \frac{e \sin\left(\chi - \pi_0 - n_0 yt\right)}{1 - e_0^2} . \tag{32}$$

where

$$h = \frac{\sqrt{E}}{\sqrt{a\left(1 - e^2\right)}} = \frac{na}{\sqrt{1 - e^2}} \tag{33}$$

and, similarly,

$$h_0 = \frac{\sqrt{E}}{\sqrt{a_0\left(1 - e_0^2\right)}} = \frac{n_0 a_0}{\sqrt{1 - e_0^2}} . \tag{34}$$

Designating by $E\,S$ the radial component of the disturbing force and by $E\,T$ its orthogonal component we have (Hansen, 1838) with some small changes in notation:

$$\frac{d}{dt}\frac{h_0}{h} = \frac{n_0}{\sqrt{1 - e_0^2}}\,\rho\left(a_0 T\right) , \tag{35}$$

$$\frac{d\Upsilon}{dt} = n_0 y \Psi + \frac{2n_0}{\sqrt{1 - e_0^2}}\left\{\left[\left(\frac{a_0}{\rho} + \frac{h^2}{h_0^2} \cdot \frac{1}{1 - e_0^2}\right)\cos\bar{f} + \frac{e_0}{1 - e_0^2} \cdot \frac{h^2}{h_0^2}\right]\rho\left(a_0 T\right) + \frac{a_0}{\rho}\sin\bar{f} \cdot \rho\left(a_0 S\right)\right\} , \tag{36}$$

$$\frac{d\Psi}{dt} = -n_0 y\left(\Upsilon + 2\frac{h}{h_0} \cdot \frac{e_0}{1 - e_0^2}\right) + \frac{2n_0}{\sqrt{1 - e_0^2}}\left\{\left(\frac{a_0}{\rho} + \frac{h^2}{h_0^2} \cdot \frac{1}{1 - e_0^2}\right)\sin\bar{f} \cdot \rho\left(a_0 T\right) - \frac{a_0}{\rho}\cos\bar{f} \cdot \rho\left(a_0 S\right)\right\} . \tag{37}$$

From the expression (15) for $a_0 \boldsymbol{\Phi}$ we obtain:

$$\rho\left(a_0 S\right) = \boldsymbol{\rho} \cdot a_0 \boldsymbol{\Phi} = \rho\frac{\partial\left(a_0 \Omega\right)}{\partial\rho} - \frac{2\alpha}{n_0 a_0^2}\frac{d\zeta}{dt}\xi \tan I , \tag{38}$$

$$\rho\left(a_0 T\right) = \mathbf{R} \cdot \boldsymbol{\rho} \times a_0 \boldsymbol{\Phi} = \frac{\partial\left(a_0 \Omega\right)}{\partial v} + \frac{2\alpha}{n_0 a_0^2}\frac{d\zeta}{dt}\eta \tan I . \tag{39}$$

Designating by σ the angular distance of the ascending node of the mean orbital plane from the x-axis of the almost ideal system of coordinates we have:

$$\sigma = n_0 \alpha t + \sigma_0 \tag{40}$$

where σ_0 is a constant, and taking Equation 27 into consideration

$$\xi = \rho\cos(v-\sigma) = (1+\nu)\,\bar{\rho}\cos(\bar{f}+\omega)\ , \tag{41}$$

$$\eta = \rho\sin(v-\sigma) = (1+\nu)\,\bar{\rho}\sin(\bar{f}+\omega)\ , \tag{42}$$

where we set

$$\omega = (\pi_0 - \sigma_0) + n_0\,(y-\alpha)\,t\ . \tag{43}$$

Substituting Equations 41 and 42 into Equations 38 and 39 we obtain:

$$\rho\cdot(a_0\,S) = \rho\,\frac{\partial\,a_0\,\Omega}{\partial\rho} - \frac{2\alpha(1+\nu)}{n_0}\,\frac{d(\zeta/a_0)}{dt}\,p\tan I\ , \tag{44}$$

$$\rho\cdot(a_0\,T) = \frac{\partial\,a_0\,\Omega}{\partial\bar{f}} + \frac{2\alpha(1+\nu)}{n_0}\,\frac{d(\zeta/a_0)}{dt}\,q\tan I\ , \tag{45}$$

where we set

$$p = \frac{\bar{\rho}}{a_0}\cos(\bar{f}+\omega)\ , \tag{46}$$

$$q = \frac{\bar{\rho}}{a_0}\sin(\bar{f}+\omega)\ . \tag{47}$$

Making use of Equations 44 and 45, the Equations 35 through 37 take the form:

$$\frac{d}{dt}\,\frac{h_0}{h} = \frac{n_0}{\sqrt{1-e_0^2}}\cdot\frac{\partial(a_0\,\Omega)}{\partial\bar{f}} + \frac{2(1+\nu)\alpha q}{\sqrt{1-e_0^2}}\cdot\frac{d(\zeta/a_0)}{dt}\tan I\ , \tag{48}$$

$$\begin{aligned}\frac{d\Upsilon}{dt} = n_0\,y\Psi + \frac{2\,n_0}{\sqrt{1-e_0^2}}\Bigg\{&\left[\left(\frac{a_0}{\rho} + \frac{h^2}{h_0^2}\cdot\frac{1}{1-e_0^2}\right)\cos\bar{f} + \frac{e_0}{1-e_0^2}\cdot\frac{h^2}{h_0^2}\right]\cdot\frac{\partial(a_0\,\Omega)}{\partial\bar{f}} + \frac{a_0}{\rho}\sin\bar{f}\cdot\rho\,\frac{\partial(a_0\,\Omega)}{\partial\rho}\\ &+ \frac{4\alpha}{\sqrt{1-e_0^2}}\left[\sin\omega + \frac{1+\nu}{1-e_0^2}\cdot\frac{h^2}{h_0^2}\cdot q(\cos\bar{f}+e_0)\right]\frac{d(\zeta/a_0)}{dt}\tan I\Bigg\},\end{aligned} \tag{49}$$

$$\frac{d\Psi}{dt} = -n_0 y\left(\Upsilon + 2\frac{h}{h_0}\cdot\frac{e_0}{1-e_0^2}\right) + \frac{2n_0}{\sqrt{1-e_0^2}}\left\{\left(\frac{a_0}{\rho} + \frac{h^2}{h_0^2}\cdot\frac{1}{1-e_0^2}\right)\sin\bar{f}\cdot\frac{\partial(a_0\Omega)}{\partial\bar{f}} - \frac{a_0}{\rho}\cos\bar{f}\cdot\rho\frac{\partial(a_0\Omega)}{\partial\rho}\right.$$

$$\left. + \frac{4\alpha}{\sqrt{1-e_0^2}}\left(\cos\omega + \frac{1+\nu}{1-e_0^2}\frac{h^2}{h_0^2}\cdot q\sin\bar{f}\right)\frac{d(\zeta/a_0)}{dt}\tan I\right\}. \tag{50}$$

Making use of

$$\frac{\partial\Omega}{\partial\bar{f}} = \frac{1}{\sqrt{1-e_0^2}}\left(\frac{\bar{\rho}^2}{a_0^2}\frac{\partial\Omega}{\partial\ell} - \frac{\bar{\rho}}{a_0}\cdot\frac{e_0\sin\bar{f}}{\sqrt{1-e_0^2}}\cdot\rho\frac{\partial\Omega}{\partial\rho}\right),$$

and of the basic relation

$$\frac{\bar{\rho}}{a_0}\sin\bar{f} + \frac{\bar{\rho}^2}{a_0^2}\frac{\sin\bar{f}}{1-e_0^2} = \int\left(2\frac{\bar{\rho}}{a_0}\cos\bar{f} + 3e_0\right)d\ell\bigg/\sqrt{1-e_0^2}\ ,$$

where ℓ is the disturbed mean anomaly

$$\ell = n_0 z + c_0\ ,$$

we can transform Equations 48 through 50 to a form resembling that contained in the author's previous work (Musen, 1963), but with some additional terms. They appear because we choose the almost ideal system to be the basic reference frame, instead of the ideal one. We obtain:

$$\frac{d\Upsilon}{dt} = +n_0 y\Psi + M_1\frac{\partial(a_0\Omega)}{\partial\ell} + N_1\rho\frac{\partial(a_0\Omega)}{\partial\rho} + K_1\frac{d\zeta/a_0}{dt}\ , \tag{51}$$

$$\frac{d\Psi}{dt} = -n_0 y\left(\Upsilon + 2\frac{h}{h_0}\cdot\frac{e_0}{1-e_0^2}\right) + M_2\frac{\partial(a_0\Omega)}{\partial\ell} + N_2\rho\frac{\partial(a_0\Omega)}{\partial\rho} + K_2\frac{d\zeta/a_0}{dt}\ , \tag{52}$$

$$\frac{d}{dt}\frac{h_0}{h} = M_3\frac{\partial(a_0\Omega)}{\partial\ell} + N_3\rho\frac{\partial(a_0\Omega)}{\partial\rho} + K_3\frac{d\zeta/a_0}{dt}\ , \tag{53}$$

where

$$M_1 = \frac{2n_0}{1-e_0^2}\cdot\frac{1}{e_0}\cdot\left[\left(1-e_0^2-\frac{\bar{\rho}^2}{a_0^2}\right) - \frac{\nu}{1+\nu}\left(1-e_0^2-\frac{\bar{\rho}}{a_0}\right) + \left(\frac{h^2}{h_0^2}-1\right)\frac{\bar{\rho}}{a_0}\left(1-\frac{\bar{\rho}}{a_0}\right)\right], \tag{54}$$

$$N_1 = \frac{2 n_0}{1 - e_0^2} \cdot \frac{\bar{\rho}}{a_0} \cdot \frac{\sin \bar{f}}{\sqrt{1 - e_0^2}} \left[1 - \frac{a_0}{\rho} \cdot \frac{\nu}{1 + \nu} - \left(\frac{h^2}{h_0^2} - 1 \right) \left(\frac{a_0}{\bar{\rho}} - 1 \right) \right] , \tag{55}$$

$$K_1 = \frac{4\alpha}{\sqrt{1 - e_0^2}} \left[\sin \omega + \frac{h^2}{h_0^2} \cdot \frac{1 + \nu}{1 - e_0^2} q \left(\cos \bar{f} + e_0 \right) \right] \tan I , \tag{56}$$

$$M_2 = \frac{2 n_0}{1 - e_0^2} \left[\frac{1}{\sqrt{1 - e_0^2}} \int \left(2 \frac{\bar{\rho}}{a_0} \cos \bar{f} + 3 e_0 \right) d\ell - \frac{\nu}{1 + \nu} \cdot \frac{\bar{\rho}}{a_0} \sin \bar{f} + \left(\frac{h^2}{h_0^2} - 1 \right) \cdot \frac{\bar{\rho}^2}{a_0^2} \cdot \frac{\sin \bar{f}}{1 - e_0^2} \right] , \tag{57}$$

$$N_2 = \frac{2 n_0}{(1 - e_0^2)^{3/2}} \left[- \left(\frac{\bar{\rho}}{a_0} \cos \bar{f} + 2 e_0 \right) + \frac{\nu}{1 + \nu} \sqrt{1 - e_0^2} \frac{d}{d\ell} \left(\frac{\bar{\rho}}{a_0} \sin \bar{f} \right) + \left(\frac{h^2}{h_0^2} - 1 \right) e_0 \frac{\bar{\rho}}{a_0} \frac{\sin \bar{f}}{\sqrt{1 - e_0^2}} \frac{d}{d\ell} \left(\frac{\bar{\rho}}{a_0} \cos \bar{f} \right) \right] , \tag{58}$$

$$K_2 = \frac{4\alpha}{\sqrt{1 - e_0^2}} \left(\cos \omega + \frac{h^2}{h_0^2} \frac{1 + \nu}{1 - e_0^2} q \sin \bar{f} \right) \tan I , \tag{59}$$

$$M_3 = \frac{n_0}{1 - e_0^2} \frac{\bar{\rho}^2}{a_0^2} , \tag{60}$$

$$N_3 = - \frac{n_0}{1 - e_0^2} \cdot \frac{\bar{\rho}}{a_0} \cdot \frac{e_0 \sin \bar{f}}{\sqrt{1 - e_0^2}} , \tag{61}$$

$$K_3 = \frac{2\alpha (1 + \nu) q}{\sqrt{1 - e_0^2}} \tan I . \tag{62}$$

EXPANSION OF THE DISTURBING FUNCTION AND THE FORCE-COMPONENTS

Let the x-axis of the inertial system be directed along the line of apsides, toward the pericenter of the sun, the y-axis be normal to the line of apsides, in the plane of the solar orbit. The z-axis will be normal to this plane. Let ω' be the angular distance of the pericenter of the sun from the ascending node of the mean orbital plane of the satellite. We have

$$\omega' = \omega_0' - n_0 \gamma t . \tag{63}$$

The position vector $\boldsymbol{\rho}'$ of the sun can be represented as

$$\mathbf{r}' = \boldsymbol{\rho}' = \mathbf{P} \rho' \cos (f' + \omega') + \mathbf{k} \times \mathbf{P} \rho' \sin (f' + \omega') , \tag{64}$$

where f' is the true anomaly of the sun. Substituting

$$\mathbf{k} = \mathbf{Q} \sin I + \mathbf{R} \cos I$$

into the last equation we deduce

$$\boldsymbol{\rho}' = \mathbf{P} \rho' \cos(f' + \omega') + \mathbf{Q} \rho' \cos I \sin(f' + \omega') - \mathbf{R} \rho' \sin I \sin(f' + \omega') . \tag{65}$$

The position vector of the satellite can be represented in accordance with Equations 41 and 42 as:

$$\mathbf{r} = \mathbf{P} \rho \cos(\overline{f} + \omega) + \mathbf{Q} \rho \sin(\overline{f} + \omega) + \mathbf{R} \zeta . \tag{66}$$

Making use of the two last equations we deduce:

$$\mathbf{r} \cdot \boldsymbol{\rho}' = \rho \rho' \cos H - \rho' \zeta \sin(f' + \omega') \sin I$$

and

$$\Delta^2 = |\mathbf{r} - \boldsymbol{\rho}'|^2 = \left(\rho^2 + \rho'^2 - 2 \rho \rho' \cos H\right) + 2 \rho' \zeta \sin(f' + \omega') \sin I + \zeta^2 , \tag{67}$$

where

$$\cos H = \cos(\overline{f} + \omega) \cos(f' + \omega') + \sin(f + \omega) \sin(f' + \omega') \cos I . \tag{68}$$

Setting

$$1 + \frac{\rho^2}{\rho'^2} - 2 \frac{\rho}{\rho'} \cos H = \Delta_0^2 , \tag{69}$$

$$\frac{a_0'}{\rho'} = s' , \qquad \frac{a_0'}{\rho'} \cos(f' + \omega') = p' , \qquad \frac{a_0'}{\rho'} \sin(f' + \omega') = q' , \tag{70}$$

we can rewrite Equation 67 as

$$\Delta^2 = \rho'^2 \Delta_0^2 \left[1 + 2 \left(\frac{\zeta/a_0}{\Delta_0} \alpha_0 s'\right) \frac{q' \sin I}{s' \Delta_0} + \left(\frac{\zeta/a_0}{\Delta_0} \alpha_0 s'\right)^2\right] .$$

From the last equation we obtain:

$$\frac{a_0}{\Delta} = \frac{\alpha_0 s'}{\Delta_0} \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \frac{\alpha_0^j s'^j}{\Delta_0^j} P_j \left(\frac{q' \sin I}{s' \Delta_0}\right) . \tag{71}$$

Taking into account

$$P_j(x) = \sum_{k=0}^{[j/2]} (-1)^k A_{j,k} x^{j-2k} ,$$

where

$$A_{j,k} = \frac{1}{2^j} \cdot \frac{(2j-2k)!}{k!\,(j-k)!\,(j-2k)!} ,$$

we have

$$\frac{a_0}{\Delta} = \alpha_0 \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \alpha_0^j \sum_{k=0}^{[j/2]} (-1)^k A_{j,k} (q' \sin I)^{j-2k} \frac{s'^{2k+1}}{\Delta_0^{2j-2k+1}} . \tag{72}$$

Making use of the expansion

$$\frac{1}{\Delta_0^{2\lambda}} = \sum_{n=0}^{+\infty} \frac{\rho^n}{\rho'^n} C_n^{\lambda} (\cos H)$$

in terms of Gegenbauer polynomials, we deduce

$$\frac{a_0}{\Delta} = \alpha_0 s' \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \alpha_0^j \sum_{n=0}^{+\infty} \sum_{k=0}^{[j/2]} (-1)^k A_{j,k} (q' \sin I)^{j-2k} s'^{2k} \cdot \left(\frac{\rho}{\rho'}\right)^n C_n^{j-k+1/2} (\cos H) . \tag{73}$$

We set

$$S_1 = \frac{\bar{\rho}}{a_0} \cdot \frac{a_0'}{\rho'} \cos(\bar{f} - f' + \omega - \omega') = p p' + q q' , \tag{74}$$

$$S_2 = \frac{\bar{\rho}}{a_0} \cdot \frac{a_0'}{\rho'} \cos(\bar{f} + f' + \omega + \omega') = p p' - q q' , \tag{75}$$

$$S_0 = S_1 \cos^2 \frac{I}{2} + S_2 \sin^2 \frac{I}{2} = \frac{\bar{\rho}}{a_0} \cdot \frac{a_0'}{\rho'} \cos H , \tag{76}$$

where the first term in Equation 76 is the main part,

$$K_0 = s s' = \frac{\bar{\rho}}{a_0} \cdot \frac{a_0'}{\rho'} \tag{77}$$

and

$$\left(\frac{\bar{\rho}}{a_0}\right)^n \cdot \left(\frac{a_0'}{\rho'}\right)^n C_n^{\lambda}(\cos H) = T_n^{\lambda} . \tag{78}$$

We deduce from Equation 73

$$\frac{a_0}{\Delta} = \alpha_0 s' \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \sum_{n=0}^{+\infty} \alpha_0^{n+j} (1+\nu)^n \Omega_{j,n} , \tag{79}$$

where

$$\Omega_{j,n} = \sum_{k=0}^{[j/2]} (-1)^k A_{j,k} (q' \sin I)^{j-2k} s'^{2k} T_n^{j-k+1/2} , \tag{80}$$

the useless term $\Omega_{0,0} = +1$ can be omitted. From

$$C_n^{\lambda}(\cos H) = \sum_{m=0}^{[n/2]} (-1)^m B_{n,m}^{\lambda} \cos^{n-2m} H ,$$

where

$$B_{n,m}^{\lambda} = \frac{2^{n-2m} (\lambda, n-m)}{m! \, (n-2m)!}$$

and taking Equations 76 and 77 into account, we deduce the formula which is convenient for obtaining the expansion of T_n^λ on the electronic computer:

$$T_n^\lambda = \sum_{m=0}^{[n/2]} (-1)^m B_{n,m}^\lambda S_0^{n-2m} K_0^{2m} . \tag{81}$$

We have the following relations between the series T_n^λ which are a paraphrase of the recurrent relations between Gegenbauer polynomials:

$$T_{n+2}^\lambda = \frac{2\lambda + 2n + 2}{n + 2} S_0 T_{n+1}^\lambda - \frac{2\lambda + n}{n + 2} K_0^2 T_n^\lambda , \tag{82}$$

$$T_n^\lambda = \frac{2\lambda}{n} \left(S_0 T_{n-1}^{\lambda+1} - K_0^2 T_{n-2}^{\lambda+1}\right) , \tag{83}$$

$$T_n^\lambda = \frac{2\lambda}{n + 2\lambda} \left(T_n^{\lambda+1} - S_0 T_{n-1}^{\lambda+1}\right) , \tag{84}$$

$$T_n^\lambda = \frac{n - 1 + 2\lambda}{n} S_0 T_{n-1}^\lambda - \frac{2\lambda}{n} \left(K_0^2 - S_0^2\right) T_{n-2}^{\lambda-1} . \tag{85}$$

The application of the relations in Equations 82 through 85 can simplify the actual expanding of T_n^λ into trigonometric series with four basic arguments: ℓ, ℓ', ω, ω'. The expansion on the computer can begin with the expansion of the basic expressions

$$\xi_0 = \frac{\bar\rho}{a_0} \cos \bar f , \qquad \eta_0 = \frac{\bar\rho}{a_0} \sin \bar f , \qquad s = \frac{\bar\rho}{a_0}$$

and

$$\xi_0' = \frac{a_0'}{\rho'} \cos f' , \qquad \eta_0' = \frac{a_0'}{\rho'} \sin f' , \qquad s' = \frac{a_0'}{\rho'}$$

into trigonometric series in terms of ℓ and respectively of ℓ'. Then

$$p = \xi_0 \cos\omega - \eta_0 \sin\omega , \qquad q = \xi_0 \sin\omega + \eta_0 \cos\omega ,$$

$$p' = \xi_0' \cos\omega' - \eta_0' \sin\omega' , \qquad q' = \xi_0' \sin\omega' + \eta_0' \cos\omega' ,$$

are formed. After that in succession are formed

$$S_1 , \qquad S_2 , \qquad S_0 , \qquad K_0 , \qquad T_n^\lambda$$

and powers of $q' \sin I$ and s' and, finally, $\Omega_{j,n}$ and a_0/Δ. The expansions are trigonometric series in four basic arguments and with purely numerical coefficients. The numerical values of the mean eccentricity of the satellite, of the eccentricity of the solar orbit, of the mutual inclination I and of the parallactic factor a_0 can be substituted from the outset.

For the indirect term of the disturbing function we have

$$a_0 \frac{\mathbf{r} \cdot \boldsymbol{\rho}'}{\rho'^3} = a_0 \left[\frac{\rho \cos H}{\rho'^2} - \frac{\zeta}{\rho'^2} \sin(f' + \omega') \sin I\right]$$

or, in a slightly different form,

$$a_0 \frac{\mathbf{r} \cdot \boldsymbol{\rho}'}{\rho'^3} = \alpha_0^2 s' \quad (1+\nu)\Omega_{0,1} - \frac{\zeta}{a_0} \Omega_{1,0} \quad . \tag{86}$$

Substituting Equations 79 and 86 into Equation 16 we obtain

$$a_0 \Omega = \left(1 - \frac{E}{m' + E}\right) m^2 s' \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \sum_{n=0}^{+\infty} \alpha_0^{n+j-2} (1+\nu)^n \Omega_{j,n}$$

$$+ \sum_{j=1}^{+\infty} \left(\frac{\zeta}{a_0}\right)^{2j} (1+\nu)^{-2j-1} \omega_j$$

$$+ \alpha^2 \left\{\frac{1}{2}\left[(1+\nu)^2 p^2 + \left(\frac{\zeta}{a_0}\right)^2\right] \tan^2 I - (1+\nu) q \frac{\zeta}{a_0} \tan I\right\}, \tag{87}$$

where

$$\omega_j = \frac{(-1)^j}{2^j} \cdot \frac{(2j-1)!!}{j!} s^{-2j-1}$$

and for $a_0 V$ we obtain, after an easy transformation:

$$a_0 V = \left(1 - \frac{E}{m' + E}\right) m^2 s' \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \sum_{n=0}^{+\infty} \alpha_0^{n+j-2} (1+\nu)^n \Omega_{j,n}$$

$$+ \frac{1}{2}\left(\frac{\zeta}{a_0}\right)^2 s^{-3} \left[\frac{3\nu}{(1+\nu)^2} + \frac{\nu^3}{(1+\nu)^3}\right] + \sum_{j=2}^{+\infty} \left(\frac{\zeta}{a_0}\right)^{2j} (1+\nu)^{-2j-1} \omega_j$$

$$+ \alpha^2 \left[\frac{1}{2}\left(\frac{\zeta}{a_0}\right)^2 \tan^2 I - (1+\nu) q \frac{\zeta}{a_0} \tan I\right] . \tag{88}$$

In the expansions in Equations 87 and 88 we set:

$$\Omega_{0,0} = \Omega_{1,0} = \Omega_{0,1} = 0 .$$

The expansions to be substituted into Equations 51 through 53 and Equation 18 are

$$\frac{\partial (a_0 \Omega)}{\partial \ell} = \left(1 - \frac{E}{m' + E}\right) m^2 s' \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \sum_{n=0}^{+\infty} \alpha_0^{n+j-2} (1+\nu)^n \frac{\partial \Omega_{j,n}}{\partial \ell}$$

$$+ \sum_{n=1}^{+\infty} \left(\frac{\zeta}{a_0}\right)^{2n} (1+\nu)^{-2n-1} \frac{\partial \omega_n}{\partial \ell}$$

$$+ \alpha^2 \left[\frac{1}{2} (1+\nu)^2 \frac{\partial p^2}{\partial \ell} \tan^2 I - (1+\nu) \frac{\partial q}{\partial \ell} \cdot \frac{\zeta}{a_0} \cdot \tan I\right] \quad (89)$$

and

$$\rho \frac{\partial (a_0 \Omega)}{\partial \rho} = \left(1 - \frac{E}{m' + E}\right) m^2 s' \sum_{j=0}^{+\infty} (-1)^j \left(\frac{\zeta}{a_0}\right)^j \sum_{n=0}^{+\infty} n\, \alpha_0^{n+j-2} (1+\nu)^n \Omega_{j,n}$$

$$- \sum_{n=1}^{+\infty} \left(\frac{\zeta}{a_0}\right)^{2n} (2n+1)(1+\nu)^{-2n-1} \omega_n$$

$$+ \alpha^2 \left[(1+\nu)^2 q^2 \tan^2 I - (1+\nu) q \frac{\zeta}{a_0} \tan I\right], \quad (90)$$

$$\frac{\partial (a_0 V)}{\partial \zeta / a_0} = \left(1 - \frac{E}{m' + E}\right) m^2 s' \sum_{j=1}^{+\infty} j (-1)^j \left(\frac{\zeta}{a_0}\right)^{j-1} \sum_{n=0}^{+\infty} \alpha_0^{n+j-2} (1+\nu)^n \Omega_{j,n}$$

$$+ \frac{\zeta}{a_0} s^{-3} \left[\frac{3\nu}{(1+\nu)^2} + \frac{\nu^3}{(1+\nu)^3}\right] + \sum_{j=2}^{+\infty} 2j \left(\frac{\zeta}{a_0}\right)^{2j-1} (1+\nu)^{-2j-1} \omega_j$$

$$+ \alpha^2 \left[\frac{\zeta}{a_0} \tan^2 I - (1+\nu) q \tan I\right]. \quad (91)$$

In fact, only few powers of ζ/a_0 are to be kept in the expansions in Equations 89 through 91. The explicit form of $\Omega_{j,n}$ and of T_n^{λ} which are sufficient for the purpose of the actual expansion up to the fourth power in ζ/a_0 are:

$$\Omega_{0,n} = T_n^{1/2} ,$$

$$\Omega_{1,n} = (q' \sin I) T_n^{3/2} ,$$

$$\Omega_{2,n} = \frac{3}{2} (q' \sin I)^2 T_n^{5/2} - \frac{1}{2} s'^2 T_n^{3/2} ,$$

$$\Omega_{3,n} = \frac{5}{2} (q' \sin I)^3 T_n^{7/2} - \frac{3}{2} (q' \sin I) s'^2 T_n^{5/2} ,$$

$$\Omega_{4,n} = \frac{35}{8} (q' \sin I)^4 T_n^{9/2} - \frac{15}{4} (q' \sin I)^2 s'^2 T_n^{7/2} + \frac{3}{8} s'^4 T_n^{5/2} ,$$

. . . .

and

$$T_1^{1/2} = S_0 ,$$

$$T_2^{1/2} = \frac{3}{2} S_0^2 - \frac{1}{2} K_0^2 ,$$

$$T_3^{1/2} = \frac{5}{2} S_0^3 - \frac{3}{2} S_0 K_0^2 ,$$

$$T_4^{1/2} = \frac{35}{8} S_0^4 - \frac{15}{4} S_0^2 K_0^2 + \frac{3}{8} K_0^4 ,$$

$$T_1^{3/2} = 3 S_0 ,$$

$$T_2^{3/2} = \frac{15}{2} S_0^2 - \frac{3}{2} K_0^2 ,$$

$$T_3^{3/2} = \frac{35}{2} S_0^3 - \frac{15}{2} S_0 K_0^2 ,$$

$$T_1^{5/2} = 5 S_0 ,$$

$$T_2^{5/2} = \frac{35}{2} S_0^2 - \frac{5}{2} K_0^2 ,$$

$$T_1^{7/2} = 7 S_0 ,$$

. . . .

THE PROBLEM OF INTEGRATION

The integration of the differential Equations 51 through 53 and Equation 18 can be accomplished by their reduction to integral equations and then by applying the method of iteration. The operator

$$D = \frac{d}{dt} = \frac{d\, n_0 z}{dt} \frac{\partial}{\partial \ell} + n_0' \frac{\partial}{\partial \ell'} + \dot{\omega} \frac{\partial}{\partial \omega} + \dot{\omega}' \frac{\partial}{\partial \omega'} , \tag{92}$$

where

$$\dot{\omega} = n_0 (y - \alpha) , \qquad \dot{\omega}' = n_0 \alpha \sec I ,$$

can be represented as a sum of the operators

$$D = D_0 + D_1 ,$$

where

$$D_0 = n_0 \frac{\partial}{\partial \ell} + n_0' \frac{\partial}{\partial \ell'} + \dot{\omega} \frac{\partial}{\partial \omega} + \dot{\omega}' \frac{\partial}{\partial \omega'} \tag{93}$$

and

$$D_1 = \frac{d\, n_0 \delta z}{dt} \frac{\partial}{\partial \ell} \tag{94}$$

and the value of $d\, n_0 \delta z/dt$ is given by Equation 97. Each of the differential Equations 51 through 53 and the equation for $d\, n_0 \delta z/dt$ have the form

$$\frac{d\varphi}{dt} = \Phi\left(\ell, \ell', \omega, \omega'; 1 + \nu, \zeta/a_0; h/h_0, \Upsilon, \Psi\right) .$$

They can be re-written as integral equations of the form

$$\varphi = D_0^{-1} \left(\Phi - D_1 \varphi\right)$$

to which the method of iteration can be applied.

The process of formal integration of Equations 51 through 53 and of the equation for the perturbations of the mean anomaly is thus reduced to the simple application of the inverse of a linear partial differential operator with constant coefficients to trigonometrical series. In the lunar problem each series is either a purely sine or a purely cosine series in four basic arguments. If the result of the integration is a cosine series, then an additive constant of integration must be introduced. Such constants appear in h_0/h and in Υ. The series for Ψ and $n_0\,\delta z$ are pure sine series and additive constants are equal to zero. The value of $n_0\,y$ at each integration step is obtained in such a way that no constant term appears in the right side of Equation 52. This is done to avoid the presence of a secular term in Ψ. The iterative process is continued until the computation of series of $1+\nu$, $n_0\,\delta z$, ζ/a_0, h_0/h, Υ, Ψ, y and α leads, with accepted numerical accuracy, to the same results.

At this step we shall introduce Hansen's $\overline{W}$ function in a manner similar to the classical theory. We set

$$\overline{W} = \Xi + \Upsilon\left(\frac{\overline{\rho}}{a_0}\cos\overline{f} + \frac{3}{2}e_0\right) + \Psi\frac{\overline{\rho}}{a_0}\sin\overline{f}\,. \tag{95}$$

The analogy is however, only external. The $\overline{W}$ function as defined above is *not* identical with the corresponding function of the classical theory. The classical $\overline{W}$ is associated with the perturbations of the auxiliary satellite moving in the *osculating* orbital plane and the influence of the periodic perturbations appears in $\overline{W}$ only indirectly. Our $\overline{W}$ is associated with the perturbations of the auxiliary satellite moving in the *mean* orbital plane and the influence of the periodic perturbations of the osculating orbital plane is reflected in our $\overline{W}$ directly by the presence of terms of the form $K_i\left(d\,\zeta/a_0/dt\right)$ in Equations 51 through 53.

Thus, in the present theory the $\overline{W}$ function represents a more or less formal device whose introduction is suggested by the form in Equation 8 of the equation of motion of the satellite in an almost ideal system of coordinates. Equations of type 8 permit one to transfer to our problem all the formalism of the classical theory, or at least that which concerns the perturbations of the radius vector and of the mean anomaly of the projection of the satellite on its mean orbital plane. We have, for example, the Hill formula (1881) in the form

$$\frac{d\,n_0\,\delta z}{dt} = n_0\,\frac{\overline{W}+\nu^2}{1-\nu^2} - \frac{n_0\,y}{\sqrt{1-e_0^2}}\,\frac{\overline{\rho}^2}{a_0^2} \tag{96}$$

or

$$\frac{d\,n_0\,\delta z}{dt} = n_0\,\overline{W} + \frac{n_0\,\nu^2\,(1+\overline{W})}{1-\nu^2} - \frac{n_0\,y}{\sqrt{1-e_0^2}}\,\frac{\overline{\rho}^2}{a_0^2}\,. \tag{97}$$

Let

$$\left[\frac{h_0}{h}\right], \qquad [\Upsilon]\,, \qquad [\Psi]$$

be the values obtained by the formal integration toward the end of an iteration step. Of course, these values will be different for each specific cycle of the iterative process, but in order not to complicate the writing we shall use these symbols indiscriminantly for all cycles. Similarly as in the previous modification of Hansen's theory and in accordance with the remarks made above, we have

$$\frac{h_0}{h} = 1 + c_1 + \left[\frac{h_0}{h}\right] = 1 + \Delta \,, \tag{98}$$

$$\Upsilon = c_2 + [\Upsilon] \quad \text{and} \quad \Psi = [\Psi] \tag{99}$$

where c_1 and c_2 are the constants of integration. From Equation 30 we obtain

$$\Xi = \left(-3c_1 - \frac{3}{2} e_0 c_2\right) + [\Xi] \,, \tag{100}$$

where we set

$$[\Xi] = -3\left[\frac{h_0}{h}\right] - \frac{3}{2} e_0 [\Upsilon] + 2\left(\Delta^2 - \Delta^3 + \cdots\right) \,, \tag{101}$$

and Δ^2, Δ^3, · · · may be taken from the previous iteration. Taking into account Equations 99 and 100 and setting

$$[\bar{W}] = [\Xi] + \left(\frac{\bar{\rho}}{a_0} \cos \bar{f} + \frac{3}{2} e_0\right) [\Upsilon] + \frac{\bar{\rho}}{a_0} \sin \bar{f} \, [\Psi] \tag{102}$$

we have

$$\bar{W} = -n_0\left(3c_1 + \frac{3}{2} e_0 c_2\right) + n_0 c_2\left(\frac{\bar{\rho}}{a_0} \cos \bar{\varphi} + \frac{3}{2} e_0\right) + [\bar{W}] \,, \tag{103}$$

and Equation 97 takes the form

$$\frac{d\, n_0 \delta z}{dt} = -n_0\left(3c_1 + \frac{3}{2} e_0 c_2\right) + n_0 c_2\left(\frac{\bar{\rho}}{a_0} \cos \bar{\varphi} + \frac{3}{2} e_0\right) + Q \,, \tag{104}$$

where

$$Q = [\bar{W}] + \frac{n_0 \nu^2 (1 + \bar{W})}{1 - \nu^2} - \frac{n_0 y}{\sqrt{1 - e_0^2}} \cdot \frac{\bar{\rho}^2}{a_0^2} \,. \tag{105}$$

The constants c_1 and c_2 must be determined in such a way that the constant term and the term of the form $A \cos \ell$ do not appear in the right side of Equation 104. The determination of these constants is fully discussed in the author's previous paper and therefore is omitted here. The integral equation to be solved by the method of iteration has the form:

$$n_0\, \delta z = D_0^{-1}\left\{-n_0\left(3c_1 + \frac{3}{2} e_0 c_2\right) + n_0 c_2\left(\frac{\bar{\rho}}{a_0} \cos \bar{\varphi} + \frac{3}{2} e_0\right) + Q - D_1 n_0\, \delta z\right\}, \tag{106}$$

or

$$n_0\, \delta z = D_0^{-1}\left\{-n_0\left(3c_1 + \frac{3}{2} e_0 c_2\right) + n_0 c_2\left(\frac{\bar{\rho}}{a_0} \cos \bar{\varphi} + \frac{3}{2} e_0\right) + Q - \frac{d n_0\, \delta z}{dt} \frac{\partial n_0\, \delta z}{\partial \ell}\right\}. \tag{106'}$$

The values of ν^2, W and of $n_0\, \delta z$ in the right sides of Equations 105 and 106 again can be taken from the previous cycle of iteration.

The perturbations in the radius vector are obtained without integration, as in the author's previous work, using the formula

$$\nu = \frac{1}{2}(\Delta - \bar{W}) - \frac{1}{2}(\Delta + \bar{W})\nu .$$

The integration of the differential Equation 18 requires some special considerations. We rewrite this equation in the form which favors the application of the variation of constants:

$$\left(D_0^2 + \frac{E}{\bar{\rho}^3}\right) \frac{\zeta}{a_0} = Z , \tag{107}$$

where

$$Z = n_0^2 \frac{\partial a_0 V}{\partial \zeta / a_0} + 2 n_0 \alpha \frac{d(1+\nu)p}{dt} \tan I - D_2 \frac{\zeta}{a_0} , \tag{108}$$

and

$$D_2 = D_0 D_1 + D_1 D_0 + D_1^2 .$$

The solutions of the homogeneous equation corresponding to Equation 107 are

$$\frac{\zeta_1}{a_0} = p , \qquad \frac{\zeta_2}{a_0} = q .$$

By applying the method of variation of constants we obtain the integral equation

$$\frac{\zeta}{a_0} = \frac{1}{\sqrt{1-e_0^2}} \left[q D_0^{-1} (pZ) - p D_0^{-1} (qZ) \right] , \tag{109}$$

which we shall solve, by the method of iteration. At each iteration step the approximation to $n_0 \alpha$ must be determined in such a way that no term of the form $A \sin(\ell + \omega)$ appears in ζ/a_0. The convergence of the iteration process for ζ/a_0, because of its smallness, is fast as compared to the speed of convergence for $n_0 \delta z$. In fact, only a few powers of ζ/a_0 are to be kept in the right side of Equation 109.

COMPUTATION OF THE POSITION VECTOR

The computation of the position vector at the given time t is based on the numerical evaluation of $1 + \nu$, $n_0 \delta z$ and ζ using the corresponding series. It starts with the determination of ℓ and $n_0 \delta z$ by means of successive approximation using the expression

$$\ell = n_0 t + c_0 + n_0 \delta z \tag{110}$$

and the series for $n_0 \delta z$, and beginning with

$$\ell = n_0 t + c_0$$

as the first approximation. The numerical values of three other basic arguments, ℓ, ω and ω' are obtained from Equation 92. After ℓ and $n_0 \delta z$ are computed we evaluate the numerical values of $1 + \nu$ and ζ from their series. Then the position vector of the satellite is obtained by using the formulas:

$$\mathbf{r} = A_3(-\omega') \cdot A_1(\cos I) \cdot A_3(\omega) \begin{bmatrix} (1+\nu) a_0 \left(\cos E - e_0\right) \\ (1+\nu) a_0 \sqrt{1-e_0^2} \sin E \\ \zeta \end{bmatrix} ,$$

$$E - e_0 \sin E = \ell ,$$

where A_1 and A_3 are matrices of the form:

$$A_1(\alpha) = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos\alpha & -\sin\alpha \\ 0 & +\sin\alpha & -\cos\alpha \end{bmatrix} , \qquad A_3(\alpha) = \begin{bmatrix} +\cos\alpha & -\sin\alpha & 0 \\ +\sin\alpha & +\cos\alpha & 0 \\ 0 & 0 & +1 \end{bmatrix} .$$

There is no real necessity to invert the series for $1 + \nu$, $n_0 \delta z$ and ζ in order to represent the perturbations in terms of four arguments linear in time, because the solution of Equation 110 for ℓ can be accomplished very rapidly and without any inversion using an electronic computer.

CONCLUSION

By referring the motion of the satellite to an almost ideal system of coordinates it is possible to put Hansen lunar theory into a form which gives the coordinates in a more direct manner. The number of auxiliary constants is less than in the classical version and it is also less than in the author's previous version. The necessity of a literal expansion of the disturbing function is removed and the development is made a purely numerical one.

We decided to use the general perturbations method to represent the motion of satellites because such a method, when it can be used, gives a better physical description of the motion as an oscillatory process, as compared to the method of numerical integration. The proximity of Jupiter X in 1967 to its predicted position serves as a check of the theoretical thought and shows that the machines can be used to perform analytical expansions in celestial mechanics.

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