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ff 653 July 65

ON NONLINEAR LONGITUDINAL DYNAMIC STABILITY

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Presented to the Flight Mechanics Panel
of Advisory Group for Aerospace
Research and Development

Cambridge, England
Sept. 20-23, 1966

[redacted]

N 68-27411
(ACCESSION NUMBER)
25
(PAGES)
TMX-59120
(NASA CR OR TMX OR AD NUMBER)
(THRU)
1
(CODE)
01
(CATEGORY)
FACILITY FORM 602

NATIONAL AERONAUTICS and SPACE ADMINISTRATION
WASHINGTON

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SUMMARY

Functional analysis is used to derive nonlinear counterparts of the linear stability derivative and superposition integral formulations for the aerodynamic pitching moment. The recurrence in the nonlinear formulation of integral terms accounting for the past is noted and explained on the basis of an energy balance. The results suggest that the equation of motion characteristic of free-oscillation wind-tunnel experiments may be a Liénard equation. A method of extracting the nonlinear elements of the equation from experimental records is shown to have a close connection with the energy relationships existing between free and forced oscillations.

FIGURE LEGENDS

Fig. 1.- Definition of nonlinear indicial response.

Fig. 2.- Summation of indicial responses.

Fig. 3.- Slowly varying motions.

Fig. 4.- Interpretation of $C_{N_{\dot{\alpha}}}$.

NOTATION

C_m	pitching-moment coefficient, $\frac{\text{pitching moment}}{q_o S l}$
$C_{m_\alpha}[\alpha(\xi),\ q(\xi);\ t,\tau]$	indicial pitching-moment response measured at t per unit step change in α occurring at τ, with q held fixed at $q(\tau)$
$C_{m_q}[\alpha(\xi),\ q(\xi);\ t,\tau]$	indicial pitching-moment response measured at t per unit step change in q occurring at τ, with α held fixed at $\alpha(\tau)$
C_N	normal-force coefficient, $\frac{N}{q_o S}$
$G[u(\xi),\ v(\xi)]$	functional notation: value at $\xi = t$ of a function $F(t)$ which is dependent on all the values taken by the two argument functions $u(\xi)$, $v(\xi)$ over the interval $0 \leq \xi \leq t$
l	characteristic length
N	normal force
q	dimensionless pitching-velocity parameter, $\frac{\dot{\theta} l}{V_o}$
q_o	dynamic pressure, $\frac{1}{2} \rho_o V_o^2$
S	characteristic area
t	time
t_a	time required for the indicial response to attain steady state following an instantaneous change in angle of attack or pitching velocity
V_o	flight speed
W	aircraft weight

α	angle of attack
α', α''	$\frac{d\alpha}{d\varphi}$, $\frac{d^2\alpha}{d\varphi^2}$
$\dot{\alpha}$	$\frac{d\alpha}{dt}$
θ	angle of pitch
ξ	running variable in time
ρ_o	atmospheric density
τ	value of time ξ at which a step change in α or q occurs
φ	number of characteristic lengths traveled in time t, $\frac{V_o t}{l}$
ω	circular frequency

When α, $\dot{\alpha}$, and q are used as subscripts with a normal-force or pitching-moment coefficient, a dimensionless derivative is indicated; thus

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}, \qquad C_{m_q} = \frac{\partial C_m}{\partial q}, \qquad C_{m_{\dot{\alpha}}} = \frac{\partial C_m}{\partial(\dot{\alpha} l/V_o)}$$

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1. INTRODUCTION

Many of the classical concepts of dynamic stability theory, for example, stability derivatives, indicial functions, and superposition, were initially derived under the assumption of linearity. The question can be asked: How, or to what extent, must one revise these concepts when the assumption of linearity becomes untenable? Results presented in (1) suggest the possibility of answering this question, at least in part, when it is studied within the framework of functional analysis.

It is of interest to situate functional analysis historically in the general line of development of dynamic stability theory. We shall try to show that it represents a logical continuation of this line from the linear into the nonlinear domain. Consider an aircraft undergoing a two-degree-of-freedom longitudinal motion consisting of arbitrary variations in time of angle of attack α and dimensionless pitching velocity q. The first formulation for the pitching-moment coefficient at time t, given by Bryan in 1911 (2), is shown as Eq. (1):

$$C_m(t) = \alpha(t)C_{m_\alpha}(\infty) + q(t)C_{m_q}(\infty) \tag{1}$$

This formulation is based on two assumptions; first, that the pitching moment depends only on the <u>instantaneous</u> values of α and q, and

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second, that it depends linearly on α and q. The coefficients in Eq. (1), the stability derivatives, are constants; the infinity symbol indicates that they are to be evaluated for steady flow, that is, as though the flow conditions existing at time t had existed for all previous time. An important development in the theory came with the recognition that the pitching moment depends not only on the instantaneous values of α and q, but also on their past values. Introducing the notion of indicial functions, that is, the responses in lift and moment to step changes in α and q, and that of superposition, led to a new formulation for the pitching moment (cf. (3) for a comprehensive bibliography), namely,

$$C_m(t) = C_m(0) + \int_0^t C_{m_\alpha}(t - \tau) \frac{d\alpha}{d\tau} d\tau + \int_0^t C_{m_q}(t - \tau) \frac{dq}{d\tau} d\tau \tag{2}$$

Here, the indicial pitching-moment responses are shown as $C_{m_\alpha}(t - \tau)$ and $C_{m_q}(t - \tau)$. One sees that $C_m(t)$ depends on the whole past of α and q, since all values taken by α and q in the interval zero to t appear within the integrals. Under the assumption of linearity, this formulation is exact; it eliminates entirely one of the two assumptions of Eq. (1). For the low reduced frequencies characteristic of aircraft motions, Eq. (2) may be reduced to an equation correct to the first order in frequency:

$$C_m(t) = \alpha(t)C_{m_\alpha}(\infty) + q(t)C_{m_q}(\infty) + \dot{\alpha}(t) \frac{l}{V_o} C_{m_{\dot{\alpha}}} \tag{3}$$

where

$$C_{m_{\dot{\alpha}}} = -\frac{V_o}{l} \int_0^\infty [C_{m_\alpha}(\infty) - C_{m_\alpha}(\tau)] d\tau$$

The reduction restores the form of Eq. (1) but includes the additional term $(\dot{\alpha}l/V_o)C_{m_{\dot{\alpha}}}$ which, in effect, accounts for the past within the order of

the approximation. Now, still under the assumption of linearity, it was the development of an integral form for $C_m(t)$, Eq. (2), that enabled one to see and remove the deficiencies of Eq. (1). Analogously, it is logical to seek a new integral form for $C_m(t)$, now independent of a linearity assumption, which might enable one to see and remove the deficiencies of Eq. (2) and (3). Since Eq. (2) is exact within the linearity assumption, the new form must include Eq. (2) as a special case, and so must contain those features of Eq. (2) which are independent of a linearity assumption. Accordingly, the new form for $C_m(t)$ must be an integral form which depends on α and q and on all the values taken by α and q within the time interval zero to t. This description of $C_m(t)$ corresponds precisely with the description of a functional, as originated by Volterra (4). The theory of functional analysis, in its most general form, does not depend on a linearity assumption. The theory, therefore, provides a logical framework for continuing the linear theory into the nonlinear domain.

2. ANALYSIS

It will be shown how the indicial responses and $C_m(t)$ may be defined as nonlinear functionals. Then, suitable approximations will be introduced which will reduce the equations to more tractable forms.

2.1 Definition of Nonlinear Indicial Responses

Two motions are considered, as shown on Fig. 1. First, beginning at $\xi = 0$, the aircraft undergoes the motion being studied, $\alpha(\xi)$, $q(\xi)$. At a certain time τ, the motion is constrained such that the

values of α and q at time τ remain constant thereafter. The pitching moment corresponding to this motion is measured at a fixed time t, subsequent to τ. Second, the aircraft undergoes precisely the same motion, beginning at $\xi = 0$ and constrained at $\xi = \tau$, except that at $\xi = \tau$ one of the two variables (α in Fig. 1) is given a step change over its previous value at $\xi = \tau$. Again, the pitching moment is measured at time t. The difference between the two measurements is divided by the incremental step $\Delta\alpha$; the limit of this ratio as the magnitude of the step approaches zero will be called the indicial pitching-moment response at time t per unit step change in α at time τ. The indicial pitching-moment response to a step change in q is defined analogously. Written in functional notation, the responses are defined as

$$\lim_{\Delta\alpha \to 0} \frac{\Delta C_m(t)}{\Delta\alpha} = C_{m_\alpha}[\alpha(\xi),\ q(\xi);\ t,\ \tau] \tag{4}$$

$$\lim_{\Delta q \to 0} \frac{\Delta C_m(t)}{\Delta q} = C_{m_q}[\alpha(\xi),\ q(\xi);\ t,\ \tau] \tag{5}$$

Eq. (4) and (5) generalize the notion of indicial functions. As the notation indicates, in general the indicial responses may depend not only on the levels of α and q at which the steps occur, but also on all the past values of α and q. In the most general case, then, the indicial responses are themselves functionals.

2.2 Summation of Responses

Just as in the linear case, one may break the time histories of α and q into a series of incremental step changes, as shown in Fig. 2.

Summing the incremental indicial responses to each of the step changes over the interval zero to t gives the desired integral form for $C_m(t)$:

$$C_m(t) = C_m(0) + \int_0^t C_{m_\alpha}[\alpha(\xi),\ q(\xi);\ t,\ \tau]\ \frac{d\alpha}{d\tau}\ d\tau$$

$$+ \int_0^t C_{m_q}[\alpha(\xi),\ q(\xi);\ t,\ \tau]\ \frac{dq}{d\tau}\ d\tau \qquad (6)$$

Eq. (6) is the nonlinear counterpart of Eq. (2). In the form given, it is exact, being in fact no more than an alternate, but suggestive, way of writing $C_m(t)$.

2.3 Approximate Formulation

At this point in the analysis, one may go in any of a number of directions. For some problems, in particular, those involving hysteresis, it may be necessary to retain, at least in part, the dependence on the past of the indicial responses. For a wide range of problems, however, it is possible to say that the indicial responses will not be strongly dependent on the past. This will be particularly true for flight at supersonic speeds and for dynamic stability analyses, where the motions generally are slowly varying. This is illustrated on Fig. 3. One sees that if the motion is slowly varying, and if the indicial responses depend at most only on the most immediate past, then so far as the indicial responses are concerned, the past is essentially time-invariant. In this case, the indicial responses now depend only on the <u>parameters</u> $\alpha(\tau)$, $q(\tau)$, rather than on the functions $\alpha(\xi)$, $q(\xi)$, and the responses become ordinary functions rather than functionals. The reduction is indicated specifically as

$$\left.\begin{aligned} C_{m_\alpha}[\alpha(\xi),\ q(\xi);\ t,\ \tau] &\approx C_{m_\alpha}(t - \tau;\ \alpha(\tau),\ q(\tau)) \\ C_{m_q}[\alpha(\xi),\ q(\xi);\ t,\ \tau] &\approx C_{m_q}(t - \tau;\ \alpha(\tau),\ q(\tau)) \end{aligned}\right\} \tag{7}$$

It will be noted in Eq. (7) that an additional simplification accrues with the assumption of an invariant past. For given levels $\alpha(\tau)$, $q(\tau)$, the responses will have the same form no matter when the steps occur. This means that the responses must be functions of the time <u>difference</u> $t - \tau$, rather than functions of t and τ separately.

When the use of Eq. (7) can be justified, the equation for $C_m(t)$ is simplified considerably. The equation for $C_m(t)$ takes the form of Eq. (8), where the deficiency functions F and G have been introduced, defined by Eq. (9).

$$C_m(t) = C_m(\infty;\ \alpha(t),\ q(t)) - \int_0^t F(t - \tau;\ \alpha(\tau),\ q(\tau))\ \frac{d\alpha}{d\tau}\ d\tau$$

$$- \int_0^t G(t - \tau;\ \alpha(\tau),\ q(\tau))\ \frac{dq}{d\tau}\ d\tau \tag{8}$$

where

$$\left.\begin{aligned} F(t - \tau;\ \alpha(\tau),\ q(\tau)) &= C_{m_\alpha}(\infty;\ \alpha(\tau),\ q(\tau)) - C_{m_\alpha}(t - \tau;\ \alpha(\tau),\ q(\tau)) \\ G(t - \tau;\ \alpha(\tau),\ q(\tau)) &= C_{m_q}(\infty;\ \alpha(\tau),\ q(\tau)) - C_{m_q}(t - \tau;\ \alpha(\tau),\ q(\tau)) \end{aligned}\right\} \tag{9}$$

The deficiency functions tend to vanish with increasing values of the argument $t - \tau$. Making use of the fact that aircraft motions generally will be of low reduced frequency, one can further reduce Eq. (8) to an equation correct to the first order in frequency. The result is

$$C_m(t) = C_m(\infty;\ \alpha(t),\ 0) + q(t)C_{m_q}(\infty;\ \alpha(t),\ 0)$$

$$+ \dot{\alpha}(t)\ \frac{l}{V_o}\ C_{m_{\dot{\alpha}}}(\alpha(t)) \tag{10}$$

where

$$C_{m_{\dot{\alpha}}}(\alpha(t)) = -\frac{V_o}{l}\int_o^\infty F(\tau;\ \alpha(t),\ 0)d\tau \tag{11}$$

Eq. (8) and (10) are considered to be the central results of this analysis. Eq. (8) should serve as a suitable generalization of the linear superposition equation, Eq. (2). Eq. (10) is the nonlinear counterpart of Eq. (3). One notes that Eq. (10) retains the familiar form of the linear stability derivative formulation, the principal difference being that now the coefficients are functions of α. This form might almost have been arrived at intuitively; it is reassuring, however, to have it emerge from within the framework of a rigorous theory.

3. APPLICATIONS

Some of the implications of the preceding results will be discussed. First, it will be shown why the term $C_{m_{\dot{\alpha}}}$ continues to appear in the form of Eq. (11). Then, an implication of Eq. (10) in regard to wind-tunnel experiments will be discussed, which will lead to a discussion of the problem of system identification.

3.1 Interpretation of $C_{N_{\dot{\alpha}}}$

In Eq. (10) and (11), just as in the linear theory, $C_{m_{\dot{\alpha}}}$, the term that accounts for the past, appears as an integral of the

deficiency function. It will be shown by a physical argument why this term, or more directly the analogous normal-force coefficient, $C_{N_{\dot{\alpha}}}$, continues to appear in this way. In Fig. 4, the aircraft on the left has been sinking, without pitching, for a long time at a constant rate. A force P must be applied to maintain the constant rate. The work done by the applied force over an arbitrarily large time interval zero to t_a is

$$Wk_1 = V_o \sin \alpha_o \int_o^{t_a} [W - N(\infty;\ \alpha_o)]d\tau \tag{12}$$

where W is the weight of the aircraft and $N(\infty;\ \alpha_o)$ is the steady-state normal force due to the constant angle of attack. Now, as shown on the right side of Fig. 4, let the same aircraft experience a step change in α at time zero, and then undergo the same motion as in the first case. The work done over the same time interval by the force applied to maintain a constant rate is

$$Wk_2 = V_o \sin \alpha_o \int_o^{t_a} [W - N(\tau;\ \alpha_o)]d\tau \tag{13}$$

The difference in work done is

$$Wk_1 - Wk_2 = -V_o \sin \alpha_o \int_o^{t_a} [N(\infty;\ \alpha_o) - N(\tau;\ \alpha_o)]d\tau \tag{14}$$

After identifying t_a with the time required for $N(\tau;\ \alpha_o)$ to reach steady state, one sees that the integral is the area enclosed by the indicial normal-force response curve and its steady-state value. That is to say, it is the area of the normal-force deficiency function, and it is therefore proportional to $C_{N_{\dot{\alpha}}}$. The energy of the aircraft is the same in both cases, since it undergoes the same

motion. The energy expended by the applied force is different in the two cases. The balance of energy, which is Eq. (14), therefore must have been given to or taken from the fluid. The term $C_{N_{\dot{\alpha}}}$ is a measure of the energy given to or taken from the fluid whenever the angle of attack changes from one level to another. This assertion holds, it will be noted, regardless of the magnitude of the angle of attack. This is why the terms $C_{N_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$ continue to appear as integrals of deficiency functions, even in the nonlinear analysis.

3.2 Wind-Tunnel Experiments

For free-oscillation wind-tunnel experiments, where the model oscillates about a fixed point, the angle of attack and the angle of pitch will be the same. In this case, Eq. (10) indicates that the equation of motion will have the form

$$\alpha''(\varphi) + \alpha'(\varphi)f(\alpha) + g(\alpha) = 0 \tag{15}$$

where

$$f(\alpha) \sim [C_{m_q}(\infty;\ \alpha(\varphi),\ 0) + C_{m_{\dot{\alpha}}}(\alpha(\varphi))]$$

$$g(\alpha) \sim C_m(\infty;\ \alpha(\varphi),\ 0)$$

$$\varphi = \frac{V_o t}{l}$$

If the theory is valid, Eq. (15), called a Liénard equation in nonlinear mechanics, is the one which should characterize oscillatory motions in the wind tunnel in the presence of nonlinear aerodynamic moments. The result suggests that there will be an increasing need for analytical methods capable of extracting the forms of f and g

from the results of experiments. The author has shown recently (5) how this might be done when the nonlinear elements in Eq. (15) are sufficiently small. It turns out that if one merely uses the well known Kryloff-Bogoliuboff method (6) in reverse, one obtains a pair of Abel integral equations; simple inversions yield f and g explicitly.

3.3 System Identification

The method mentioned above has been generalized recently to eliminate the necessity of the nonlinear elements being small (7). The results again appear as inversions of Abel integral equations. It is possible to explain physically why they continue to appear in this way. There follows an alternate development of the result for the damping coefficient f which will reveal its close connection with the energy relationships between free and forced oscillations.

Consider a system governed by Eq. (15), and let it be known that $f(\alpha)$ must be an even function and $g(\alpha)$ an odd function. Let the origin of the φ scale be placed at the start of a cycle of period Φ, and define the cycle so that $\alpha(0) = \alpha(\Phi) = 0$. Multiplying through by α' and integrating over the cycle gives an expression proportional to the energy loss over the cycle:

$$\Delta E = \frac{\alpha'^2(0) - \alpha'^2(\Phi)}{2} = \int_0^{\Phi} \alpha'^2(\varphi) f(\alpha) d\varphi \tag{16}$$

The solution of Eq. (15), presumed to be known from the results of experiments, can always be written in the form

$$\left.\begin{aligned} \alpha(\varphi) &= A(\varphi)\sin\psi(\varphi) \;; \qquad \psi = \omega\varphi + \Omega(\varphi) \\ \alpha'(\varphi) &= \omega A(\varphi)\cos\psi(\varphi) \\ \omega &= \frac{2\pi}{\Phi} \end{aligned}\right\} \tag{17}$$

Since $\sin \psi(0) = \sin \psi(\Phi) = 0$, Eq. (16) can be written more conveniently in terms of (A^2); that is

$$\frac{\alpha'^2(0) - \alpha'^2(\Phi)}{2} = \frac{\omega^2}{2} [A^2(0) - A^2(\Phi)] \tag{18}$$

or, equivalently,

$$\Delta E = - \frac{\Phi\omega^2}{2} \overline{(A^2)'} \tag{19}$$

where the bar signifies an average over the cycle (i.e., $\bar{y} \equiv (1/\Phi) \int_0^\Phi y \, d\varphi$).

Now consider forced periodic oscillations of the same system and let the period be the same as that of the particular cycle considered in the free oscillation. Over a cycle of oscillation, the energy expended by the forcing function will be proportional to

$$P = \int_0^\Phi \alpha'^2(\varphi) f(\alpha) d\varphi \tag{20}$$

For the class of forcing functions yielding periodic motions of period Φ, let the amplitude of each be such that each yields the same energy measure P; moreover, let this unique value of P equal the energy measure Eq. (19) of the cycle of the free oscillation. A particular member of this class of forcing functions will yield a <u>harmonic</u> motion

$$\alpha = \sqrt{\theta} \sin \omega\varphi \; ; \qquad \omega = \frac{2\pi}{\Phi} \tag{21}$$

with energy measure

$$P = \theta\omega \int_0^{2\pi} f(\sqrt{\theta} \sin u) \cos^2 u \, du \tag{22}$$

and amplitude $\sqrt{\theta}$ such that Eq. (22) equals Eq. (19). Then

$$\overline{(A^2)'} = - \frac{\theta}{\pi} \int_0^{2\pi} f(\sqrt{\theta} \sin u) \cos^2 u \, du \tag{23}$$

Since the left side of Eq. (23) can be presumed to be known from the results of experiments, Eq. (23) constitutes an integral equation for f. That it is a form of Abel's integral equation can be seen by letting

$$\overline{(A^2)'} = F(\theta) \tag{24}$$

and making the transformations

$$\left.\begin{aligned} \xi &= \theta \sin^2 u \\ \frac{f(\sqrt{\xi})}{\sqrt{\xi}} &= h(\xi) \end{aligned}\right\} \tag{25}$$

so that, after a differentiation

$$F'(\theta) = -\frac{1}{\pi} \int_0^\theta \frac{h(\xi)d\xi}{\sqrt{\theta - \xi}} \tag{26}$$

This is Abel's integral equation, which has a particularly simple inversion (8):

$$h(\theta) = -\frac{d}{d\theta} \int_0^\theta \frac{F'(\xi)d\xi}{\sqrt{\theta - \xi}} \tag{27}$$

The determination of h, and hence f, follows immediately, therefore, on expressing $\overline{(A^2)'}$ as a function of θ. Turning to this, and observing that

$$(A^2)' = fn(\varphi;\ A_o) \tag{28}$$

one sees the necessity of eliminating the dependence on the initial condition A_o; for otherwise $\overline{(A^2)'}$ and so ultimately f would show a dependence on A_o, which obviously would be incorrect. Now A^2 and ψ will be known as functions of φ and A_o; alternatively, A_o and φ can be written in terms of A^2 and ψ. Replacing A_o and φ by these dependencies in Eq. (28) yields $(A^2)'$ as a function of A^2 and ψ:

$$(A^2)' = K(A^2,\ \psi) \tag{29}$$

The <u>average</u> value of $(A^2)'$ is therefore

$$\overline{(A^2)'} = \frac{1}{\Phi} \int_0^{\Phi} K(A^2, \psi) d\varphi \tag{30}$$

This expression, a measure of the energy loss over the cycle, equally must be a measure of the energy per cycle for every member of the class of forced motions previously defined. In particular, it must be a measure of the energy for the <u>harmonic</u> motion, for which $A^2 = \theta$, $\psi = \omega\varphi$. On simply replacing A^2 by θ (a constant in the integration), ψ by $\omega\varphi$, one has

$$\overline{(A^2)'} = \frac{1}{\Phi} \int_0^{\Phi} K(\theta, \omega\varphi) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} K(\theta, u) du \tag{31}$$

The integration yields $\overline{(A^2)'}$ as a function of θ alone, and this is the function $F(\theta)$ required for use in Eq. (27).

Though the result is suggestive, it is necessary to remark that its practicability has not been demonstrated. Much more remains to be done in this field, which has come to be known under the general heading of "system identification" (see, e.g., (9)). Experimentalists especially could make very useful contributions. In particular, there is a need for a systematic series of experiments to determine in practice how much and what kind of data are required before the form of the nonlinearities can be defined with confidence.

REFERENCES

1. Tobak, Murray; and Pearson, Walter E.: A Study of Nonlinear Longitudinal Dynamic Stability. NASA TR R-209, 1964.

2. Bryan, George Hartley: Stability in Aviation. Macmillan, London, 1911.

3. Garrick, I. E.: Nonsteady Wing Characteristics. (Sec. F of Aerodynamic Components of Aircraft at High Speeds.) Vol. VII of High Speed Aerodynamics and Jet Propulsion, Allen F. Donovan and H. R. Lawrence, eds., Princeton University Press, Princeton, N. J., 1957.

4. Volterra, Vito: Theory of Functionals and of Integral and Integro-Differential Equations. Dover Pub., Inc., N. Y., 1959.

5. Tobak, Murray: On Deduction of Certain Nonlinear Differential Equations From Their Solutions. NASA TN D-2779, 1965.

6. Kryloff, Nikolai Mitrofanovich; and Bogoliuboff, Nikolai Nikolaevich: Introduction to Non-Linear Mechanics. Princeton University Press, Princeton, N. J., 1943.

7. Tobak, Murray: An Inverse Problem for a Class of Nonlinear Differential Equations. J. Math. Anal. Appl., vol. 15, no. 2, Aug. 1966.

8. Titchmarsh, Edward Charles: Introduction to the Theory of Fourier Integrals. 2nd ed., Clarendon Press, Oxford, 1948.

9. Arnold, Charles R.; and Narendra, Kumpati S.: The Characterization and Identification of Systems. Harvard Univ. Rep. TR-471, 1965.

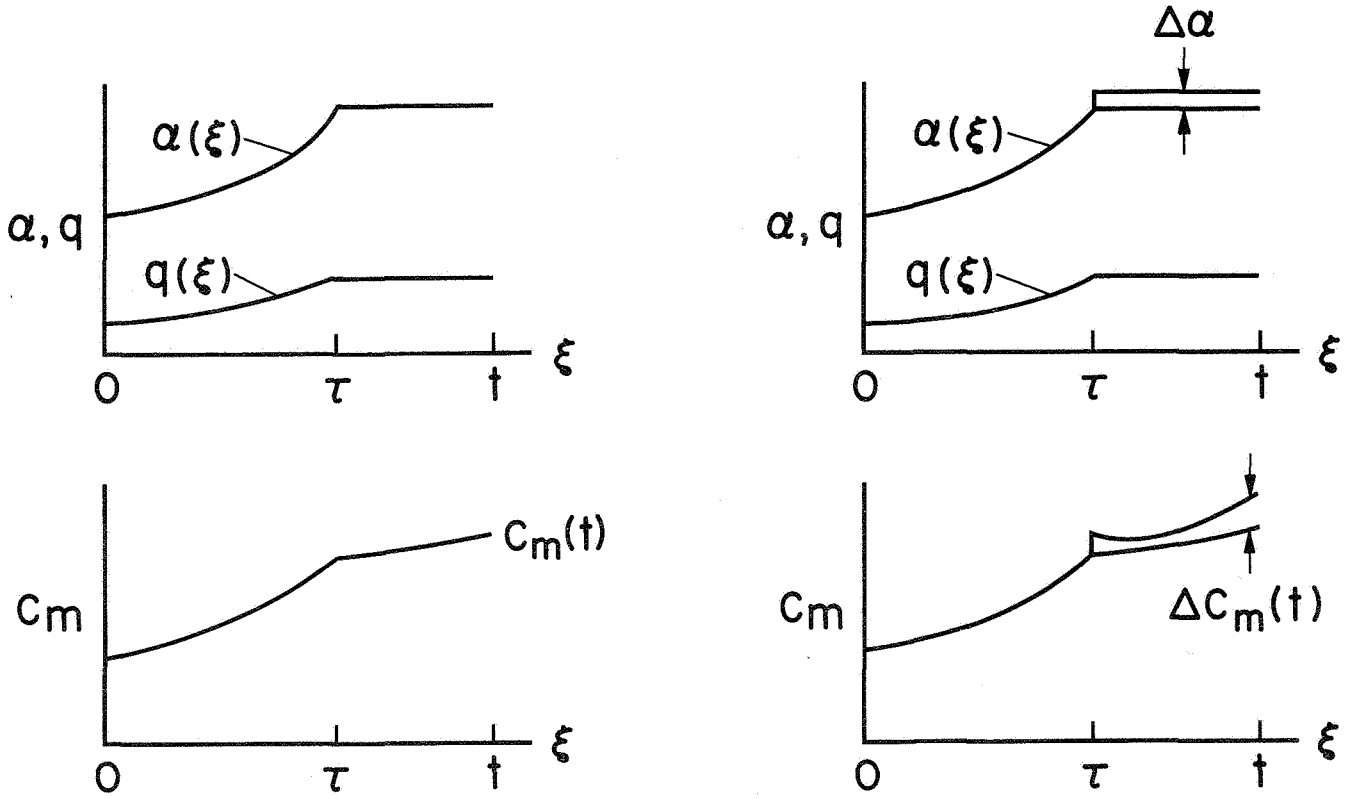

Fig. 1.- Definition of nonlinear indicial response.

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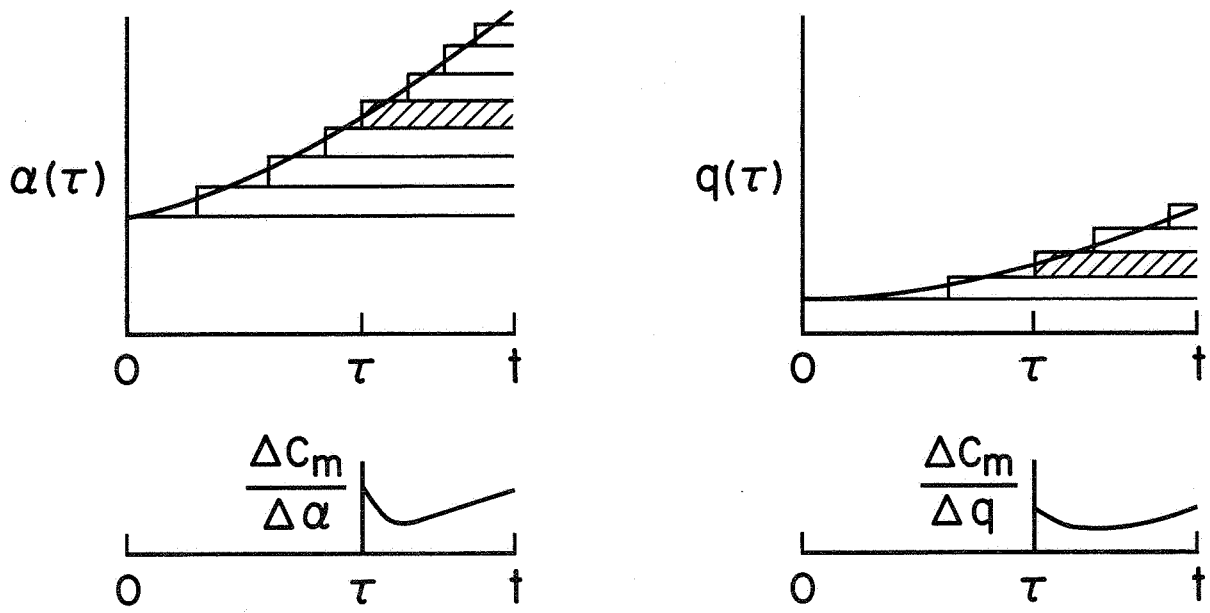

Fig. 2.- Summation of indicial responses.

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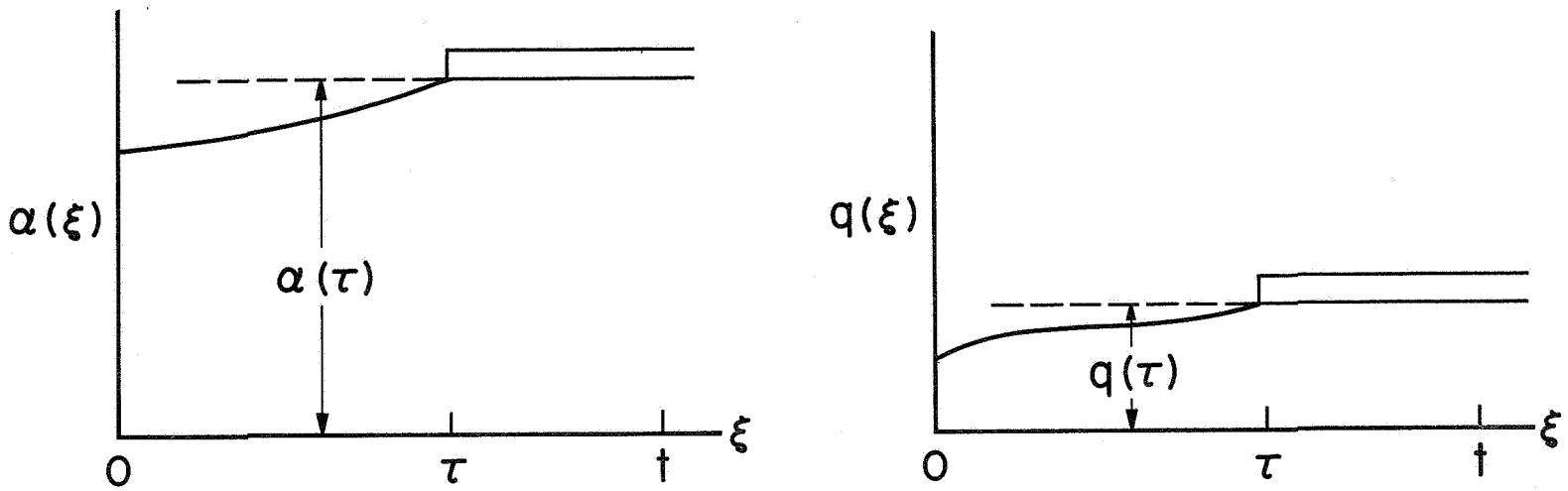

Fig. 3.- Slowly varying motions.

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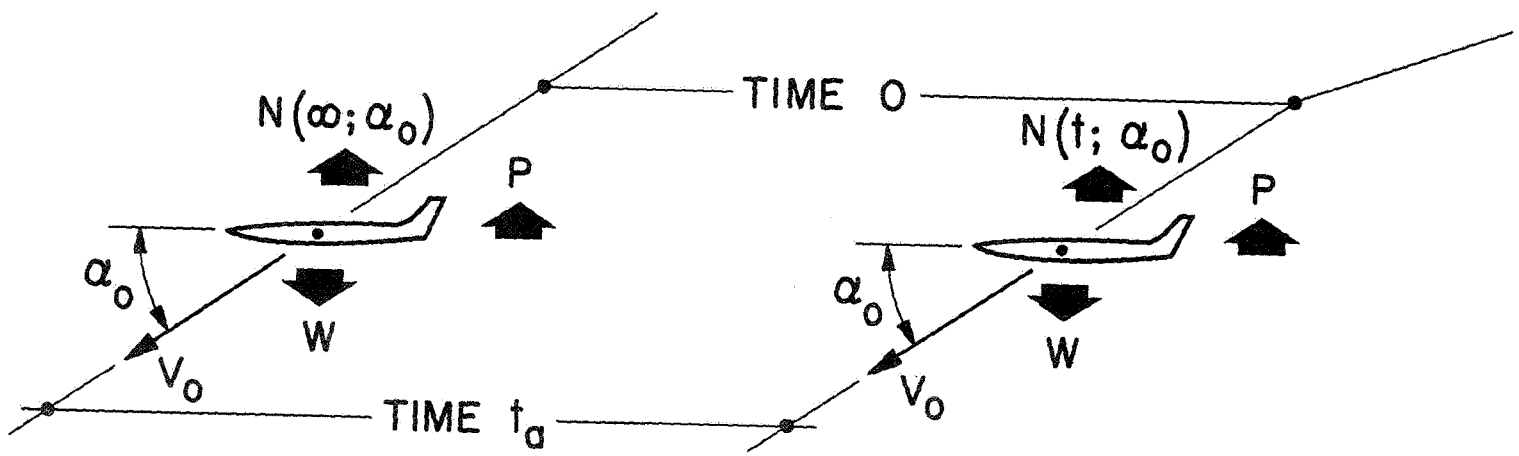

Fig. 4.- Interpretation of $C_{N_{\dot{\alpha}}}$.

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