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PROBABILITY DISTRIBUTIONS FOR THE ERRORS IN THE PARAMETERS OF NEAR-EARTH CIRCULAR ORBITS WITH APOLLO APPLICATIONS

C. W. MURRAY, JR.

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GODDARD SPACE FLIGHT CENTER
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April 22, 1968

Mission and Systems Analysis Branch
Mission and Trajectory Analysis Division

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ABSTRACT

A statistical technique is described for analyzing the effects of insertion errors (actual minus nominal) on the parameters of near-earth circular orbits. Probability distributions are obtained numerically using Keplerian two-body equations of motion, and it is shown that the errors in perigee, apogee, and eccentricity are non-Gaussian with non-zero means. The technique is applied to a nominal Apollo earth parking orbit. The analysis indicates that the actual near-earth parking orbit will be close to the nominal 100 n.mi. circular orbit. For example, there is a 99.5% probability that the actual perigee height will exceed 97.5 n.mi., and a 99.5% probability that the actual apogee height will be less than 102.4 n.mi. Insertion errors (actual minus nominal) and insertion ship tracking errors (calculated minus actual) are combined as a further example.

PROBABILITY DISTRIBUTIONS FOR THE ERRORS IN THE PARAMETERS OF NEAR-EARTH CIRCULAR ORBITS WITH APOLLO APPLICATIONS

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SUMMARY

A statistical technique is described for analyzing the effects of insertion errors (actual minus nominal) on the parameters of near-earth circular orbits. The error in the state vector (position and velocity) at insertion is assumed to have a multivariate normal distribution with a zero mean and given covariance matrix. Probability distributions for some of the errors in the parameters are obtained numerically using Keplerian two-body equations of motion, and it is shown that the error in perigee, apogee, eccentricity, and the angle between the actual position vector and the nominal position vector at insertion are non-Gaussian with non-zero means.

The technique is applied to a nominal Apollo earth parking orbit using an expected covariance matrix of insertion errors typical of the performance of the Saturn V Launch Vehicle at insertion into a near-earth nominally circular parking orbit of 100 n.mi. The analysis indicates that the actual parking orbit will be close to the nominal 100 n.mi. orbit. For example, there is a 99.5% probability that the actual perigee height will exceed 97.5 n.mi., and also a 99.5% probability that the actual apogee height will be less than 102.4 n.mi.

Insertion errors (actual minus nominal) and insertion ship tracking errors (measured or calculated minus actual) are combined as a further example. Results indicate that under worst case conditions for the coefficients of correlation between the tracking errors in insertion height, speed, and flight path angle (coefficients of +0.9), there is a 90% probability that the calculated perigee height will exceed 91 n.mi. The three sigma values for the insertion ship tracking errors in insertion height, speed, and flight path angle used in the analysis are (Reference 1):

$$3\sigma_{\Delta r_1} = 2.4 \text{ n.mi. } (4.44 \text{ km})$$

$$3\sigma_{\Delta v_1} = 16 \text{ ft/sec } (4.87 \text{ m/sec})$$

$$3\sigma_{\Delta \gamma_1} = 0.16^\circ \ (2.79 \text{ mrad}).$$

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PROBABILITY DISTRIBUTIONS FOR THE ERRORS IN THE PARAMETERS OF NEAR-EARTH CIRCULAR ORBITS WITH APOLLO APPLICATIONS

INTRODUCTION

The near-earth orbit of a space vehicle can be determined from the position and velocity vectors of the vehicle at insertion using Keplerian two-body equations of motion. Due to insertion errors (errors in the xyz components of these vectors within an inertial Cartesian coordinate system at the time of insertion), the actual orbit will deviate from the desired nominal.

The purpose of this report is two-fold: (1) to study the probability distributions of some of the errors in the parameters of near-earth circular orbits assuming the insertion errors are correlated Gaussian errors with zero means; (2) to demonstrate a statistical technique using numerical integration for determining the probability distributions of non-Gaussian errors.

By stating that the errors in the Cartesian coordinates of the vehicle's position and velocity vectors at insertion are Gaussian, we are making a more basic assumption—that these particular errors can be represented as linear combinations of a number of error sources within the guidance and control system of the launch vehicle (gyro drift, accelerometer errors, etc.) which are Gaussian (but not necessarily independent).* The assumption of linearity is reasonable if second order effects (and higher) of these error sources are quite

*A numerical procedure for determining the coefficients of the error sources in the linear expression for each of the errors in the state (Δx, Δy, Δz, $\Delta\dot{x}$, $\Delta\dot{y}$, $\Delta\dot{z}$) is to take each error source one at a time, and using its one sigma value, determine the error in the six-dimensional state vector through a digital computer powered flight program which simulates the guidance system in the launch vehicle. If there are N sources of error, N + 1 computer simulations would have to be made (one run for the nominal state). In matrix notation, if $\Delta\mathbf{X}$ is a (6×1) vector representing the error in the state (position and velocity) of the vehicle at insertion, $\mathbf{Q}$ is a $(6 \times N)$ matrix, and $\Delta\mathbf{W}$ is an $(N \times 1)$ vector of error sources,

$$\Delta\mathbf{X} = \mathbf{Q}\Delta\mathbf{W}$$

and the above procedure is equivalent to determining one column of the $\mathbf{Q}$ matrix at each step. Then, the covariance matrix of the error in the state will be given by:

$$\mathbf{M} = \mathbf{QPQ}^T$$

where $\mathbf{P}$ is the correlation matrix (1's down the main diagonal and the coefficients of correlation as off-diagonal elements) of the $(N \times 1)$ error vector. If $\Delta\mathbf{W}$ has mean vector $\boldsymbol{\xi}$ (error sources with biases), then $\Delta\mathbf{X}$ will have mean vector $\mathbf{Q}\boldsymbol{\xi}$. Usually these error sources are taken to be independent and to have zero mean.

small compared to first order effects (certainly this would be true in a well-designed guidance system). The Gaussian assumption is reasonable if one considers these error sources to be unpredictable—constant for one flight but varying in a random fashion over many flights in accordance with a Gaussian distribution. Even if this is not exactly true, the assumption that the error in the state (position and velocity) vector at insertion has a multivariate normal distribution with an associated covariance matrix and zero mean provides us with a working tool to comparatively analyze the performance of one space vehicle with another and to obtain bounds on the errors in the parameters of the orbit for a particular mission.

It will be seen that some of the errors in the parameters of the orbit can be expressed to a good approximation as linear combinations of the insertion errors, and therefore, will be normally distributed (Gaussian). Others cannot, and the probability distributions of these will be non-Gaussian, for which a numerical technique is necessary to obtain the distribution.

An example will be given which applies the technique to a nominal Apollo earth parking orbit.

Determination of the probability distribution of a random variable is important for two reasons: (1) It gives more information than just the mean and standard deviation, in particular, for non-Gaussian distributions; (2) It may be important for orbital parameters having critical upper or lower bounds such as perigee height in the case of the Apollo parking orbit. (Reference 2).

1. GENERAL

Let $\mathbf{X}$ represent the nominal state vector (position and velocity) of a space vehicle (within an inertial coordinate system) at insertion into a near-earth orbit, and let η be one of the parameters of the orbit. Then, using Keplerian two-body equations of motion, we can express η as a function of $\mathbf{X}$

$$\eta = \eta(\mathbf{X}) \tag{1}$$

Due to insertion errors, the actual state $\mathbf{X}_p$ will differ from the nominal state $\mathbf{X}$ by an amount $\Delta\mathbf{x}$

$$\mathbf{X}_p = \mathbf{X} + \Delta\mathbf{x} \tag{2}$$

We may therefore write an error $\Delta\eta$ in the parameter η as

$$\Delta\eta = \eta(\mathbf{X}_p) - \eta(\mathbf{X}) = \eta(\mathbf{X} + \Delta\mathbf{x}) - \eta(\mathbf{X}) \tag{3}$$

Let $\Delta\mathbf{X}$ be a random vector representing the error in the state vector at insertion.

Then

$$\Delta H = \eta(\mathbf{X} + \Delta\mathbf{X}) - \eta(\mathbf{X}) \tag{4}$$

is a random variable* representing the error in η.

In this analysis we will assume that the nominal orbit is circular and that $\Delta\mathbf{X}$ has a multivariate normal distribution with zero mean and a given covariance matrix.

It will be seen that some of the errors in the parameters of the orbit can be expressed to a very good approximation as linear combinations of the components of $\Delta\mathbf{X}$, and, therefore, will be normally distributed. Other errors (e.g., perigee error and apogee error) cannot be expressed as linear combinations, and for these a numerical technique must be used to determine the distribution.

2. THE $(\mathbf{U}^0\ \mathbf{V}^0\ \mathbf{W}^0)$ COORDINATE SYSTEM

The $(\mathbf{U}^0\ \mathbf{V}^0\ \mathbf{W}^0)$ system is a right-handed inertial Cartesian Coordinate System frequently used to define the position and velocity of a space vehicle at insertion. It may be seen in Figure 1.

The $\mathbf{U}^0$ vector is a unit vector in the $\mathbf{R}$ (position vector) direction. The $\mathbf{W}^0$ vector is a unit vector in the $(\mathbf{R} \times \mathbf{V})$ direction ($\mathbf{V}$ is the velocity vector), and the $\mathbf{V}^0$ vector is a unit vector in the $(\mathbf{W}^0 \times \mathbf{U}^0)$ direction.

*It should be noted that $\Delta\mathbf{x}$ is a value which the random vector $\Delta\mathbf{X}$ can take. Also, $\Delta\eta$ is a value which the random variable ΔH can take.

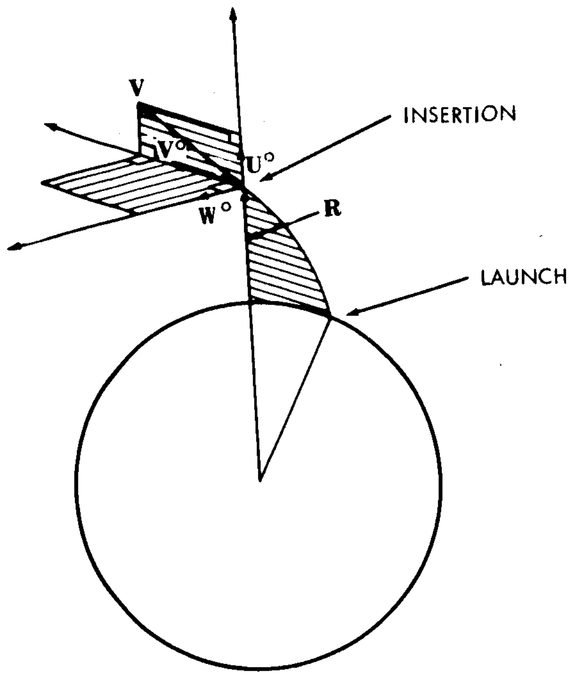

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Figure 1. The $(U^0\ V^0\ W^0)$ Coordinate System

In the $(U^0\ V^0\ W^0)$ System the **R** and **V** vectors for a nominally circular orbit are

$$\mathbf{R} = \begin{bmatrix} r_0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

$$\mathbf{V} = \begin{bmatrix} 0 \\ \sqrt{\mu / r_0} \\ 0 \end{bmatrix} \tag{6}$$

where r_0 is the radius at insertion (radius of the orbit) and μ is the gravitational constant. The perturbed (or actual) position and velocity vectors are then

$$\mathbf{R}_p = \begin{bmatrix} r_0 + \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix}$$

$$\mathbf{V}_p = \begin{bmatrix} \Delta x_4 \\ v_0 + \Delta x_5 \\ \Delta x_6 \end{bmatrix} \qquad (7)$$

where $v_0 = \sqrt{\mu/r_0}$.

3. EXPRESSIONS FOR SOME OF THE ERRORS IN THE PARAMETERS*

Insertion Radius Error

The error in insertion radius is

$$\Delta r_0 = r_0(\mathbf{X} + \Delta\mathbf{x}) - r_0 = \sqrt{\left(r_0 + \Delta x_1\right)^2 + \left(\Delta x_2\right)^2 + \left(\Delta x_3\right)^2} - r_0$$

$$= \left(r_0 + \Delta x_1\right)\left\{1 + \left(\frac{1}{2}\right)\left(\frac{\left(\Delta x_2\right)^2 + \left(\Delta x_3\right)^2}{\left(r_0 + \Delta x_1\right)^2}\right) - \cdots\right\} - r_0$$

$$\approx \Delta x_1$$

neglecting 2^{nd} order terms and higher. Or, in random variable notation

$$\Delta R_0 \approx \Delta X_1 \qquad (8)$$

*Throughout this Section capital letters will denote random variables. Bold letters represent vectors and matrices.

Insertion Speed Error

The error in insertion speed is

$$\Delta v_0 = v_0\,(\mathbf{X} + \Delta\mathbf{x}) - v_0 = \sqrt{(\Delta x_4)^2 + (v_0 + \Delta x_5)^2 + (\Delta x_6)^2} - v_0$$

$$= (v_0 + \Delta x_5)\left\{1 + \left(\frac{1}{2}\right)\left(\frac{(\Delta x_4)^2 + (\Delta x_6)^2}{(v_0 + \Delta x_5)^2}\right) - \cdots\right\} - v_0$$

$$\approx \Delta x_5$$

neglecting 2^{nd} order terms and higher. Or,

$$\Delta V_0 \approx \Delta X_5 \tag{9}$$

The Error in Flight Path Angle at Insertion

The error in flight path angle at insertion is

$$\Delta\gamma_0 = \gamma_0\,(\mathbf{X} + \Delta\mathbf{x}) = \sin^{-1}\left(\frac{\mathbf{R}_p \cdot \mathbf{V}_p}{|\mathbf{R}_p|\ |\mathbf{V}_p|}\right)$$

$$\approx \left(r_0^{-1}\right)\Delta x_2 + \left(v_0^{-1}\right)\Delta x_4$$

or

$$\Delta\Gamma_0 \approx \left(r_0^{-1}\right)\Delta X_2 + \left(v_0^{-1}\right)\Delta X_4 \tag{10}$$

neglecting 2^{nd} order terms and higher and noting that the flight path angle is zero for a nominally circular orbit.

The Error in Energy Per Unit Mass

The error in energy per unit mass is

$$\Delta c_3 = c_3(\mathbf{X} + \Delta \mathbf{x}) - c_3(\mathbf{X})$$

$$= \left(v_0 + \Delta v_0\right)^2 - \left(\frac{2\mu}{r_0 + \Delta r_0}\right) - v_0^2 + \left(\frac{2\mu}{r_0}\right)$$

$$= v_0^2 + 2 v_0 \Delta v_0 + \left(\Delta v_0\right)^2 - \left(\frac{2\mu}{r_0}\right)\left\{1 - \left(\frac{\Delta r_0}{r_0}\right) + \left(\frac{\Delta r_0}{r_0}\right)^2 - \cdots\right\}$$

$$- v_0^2 + \left(\frac{2\mu}{r_0}\right)$$

$$\approx 2 v_0 \Delta x_5 + \left(\frac{2\mu}{r_0^2}\right) \Delta x_1$$

or

$$\Delta C_3 \approx 2 v_0 \Delta X_5 + \left(\frac{2\mu}{r_0^2}\right) \Delta X_1 \tag{11}$$

neglecting 2nd order terms and higher.

Semi-major Axis Error

For ease in analysis let $\lambda = \left(r_0 v_0^2/\mu\right)$. Then, the error in semi-major axis may be written

$$\Delta a = a(\mathbf{X} + \Delta \mathbf{x}) - a(\mathbf{X}) = \left(\frac{r_0 + \Delta r_0}{2 - (\lambda + \Delta\lambda)}\right) - \left(\frac{r_0}{2 - \lambda}\right)$$

$$\Delta a = \left(\frac{r_0 + \Delta r_0}{2}\right)\left\{1 + \left(\frac{\lambda}{2} + \frac{\Delta\lambda}{2}\right) + \left(\frac{\lambda}{2} + \frac{\Delta\lambda}{2}\right)^2 + \cdots\right\}$$

$$- \left(\frac{r_0}{2}\right)\left\{1 + \frac{\lambda}{2} + \left(\frac{\lambda}{2}\right)^2 + \cdots\right\}$$

$$= \left(\frac{r_0}{2-\lambda}\right) + \left(\frac{\Delta r_0}{2-\lambda}\right) + \left(\frac{r_0}{(2-\lambda)^2}\right)\Delta\lambda - \left(\frac{r_0}{2-\lambda}\right) + \text{higher order terms}$$

But, for a circular orbit $\lambda \doteq 1$. Hence

$$\Delta a \doteq \Delta r_0 + (r_0)\,\Delta\lambda + \text{higher order terms}$$

But,

$$\Delta\lambda = \left(2v_0^{-1}\right)\Delta v_0 + \left(r_0^{-1}\right)\Delta r_0 + \text{higher order terms}$$

Thus we can write

$$\Delta a \approx 2\Delta x_1 + \left(2r_0\, v_0^{-1}\right)\Delta x_5$$

or

$$\Delta A \approx 2\Delta X_1 + \left(2r_0\, v_0^{-1}\right)\Delta X_5 \qquad (12)$$

The Angle Between the Nominal and the Perturbed Position Vector at Insertion

The Angle between $\mathbf{R}$ and $\mathbf{R}_p$ is

$$\beta = \beta\left(\mathbf{X}, \mathbf{X}_p\right) = \cos^{-1}\left(\frac{\mathbf{R}_p \cdot \mathbf{R}}{|\mathbf{R}_p|\;|\mathbf{R}|}\right)$$

$$= \cos^{-1}\left(\frac{r_0 + \Delta x_1}{\sqrt{(r_0 + \Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{(\Delta x_2)^2 + (\Delta x_3)^2}}{(r_0 + \Delta x_1)}\right)$$

$$\approx r_0^{-1} \sqrt{(\Delta x_2)^2 + (\Delta x_3)^2}$$

or

$$B \approx r_0^{-1} \sqrt{(\Delta X_2)^2 + (\Delta X_3)^2} \tag{13}$$

The Error in Eccentricity

The eccentricity e is a function of r_0, v_0, and γ_0

$$e = e(r_0, v_0, \gamma_0) = \sqrt{\sin^2 \gamma_0 + \left(\frac{r_0 v_0^2}{\mu} - 1\right)^2 \cos^2 \gamma_0}$$

But for a circular orbit $e = 0$. Therefore, the error in eccentricity Δe is

$$\Delta e = e(r_0 + \Delta r_0, v_0 + \Delta v_0, \gamma_0 + \Delta\gamma_0)$$

or

$$\Delta E = e\left(r_0 + \Delta R_0,\ v_0 + \Delta V_0,\ \gamma_0 + \Delta \Gamma_0\right) \tag{14}$$

The Error in Perigee

The perigee radius r_p is a function of r_0, v_0, and γ_0

$$r_p = r_p\left(r_0, v_0, \gamma_0\right) = \frac{r_0}{2 - \left(\frac{r_0 v_0^2}{\mu}\right)}\left\{1 - \sqrt{\sin^2 \gamma_0 + \left(\frac{r_0 v_0^2}{\mu} - 1\right)^2 \cos^2 \gamma_0}\right\}$$

For a circular orbit $r_p = r_0$. Therefore, the error in perigee is

$$\Delta r_p = r_p\left(r_0 + \Delta r_0,\ v_0 + \Delta v_0,\ \gamma_0 + \Delta \gamma_0\right) - r_0$$

or

$$\Delta R_p = r_p\left(r_0 + \Delta R_0,\ v_0 + \Delta V_0,\ \gamma_0 + \Delta \Gamma_0\right) - r_0 \tag{15}$$

The Error in Apogee

The apogee radius r_a is a function of r_0, v_0, and γ_0

$$r_a = r_a\left(r_0, v_0, \gamma_0\right) = \frac{r_0}{2 - \left(\frac{r_0 v_0^2}{\mu}\right)}\left\{1 + \sqrt{\sin^2 \gamma_0 + \left(\frac{r_0 v_0^2}{\mu} - 1\right)^2 \cos^2 \gamma_0}\right\}$$

For a circular orbit $r_a = r_0$. Therefore, the error in apogee is

$$\Delta r_a = r_a\left(r_0 + \Delta r_0,\ v_0 + \Delta v_0,\ \gamma_0 + \Delta\gamma_0\right) - r_0$$

or

$$\Delta R_a = r_a\left(r_0 + \Delta R_0,\ v_0 + \Delta V_0,\ \gamma_0 + \Delta\Gamma_0\right) - r_0 \tag{16}$$

4. THE VARIANCES AND COVARIANCES BETWEEN THE ERRORS IN INSERTION RADIUS, SPEED AND FLIGHT PATH ANGLE*

Using (8), (9), and (10) we may easily obtain expressions for the variances of ΔR_0, ΔV_0, and $\Delta\Gamma_0$

$$\sigma^2_{\Delta r_0} \approx \sigma^2_{\Delta x_1} \tag{17}$$

$$\sigma^2_{\Delta v_0} \approx \sigma^2_{\Delta x_5} \tag{18}$$

$$\sigma^2_{\Delta\gamma_0} \approx \left(r_0^{-2}\right)\sigma^2_{\Delta x_2} + \left(v_0^{-2}\right)\sigma^2_{\Delta x_4} + 2\left(r_0\, v_0\right)^{-1}\ \sigma_{\Delta x_2 \Delta x_4} \tag{19}$$

The covariances are

$$\sigma_{\Delta r_0 \Delta v_0} \approx \sigma_{\Delta x_1 \Delta x_5} \tag{20}$$

*If ΔX_i and ΔX_j are random variables, $\sigma_{\Delta x_i \Delta x_j} = \rho_{\Delta x_i \Delta x_j}\sigma_{\Delta x_i}\sigma_{\Delta x_j}$ is the covariance between ΔX_i and ΔX_j, $\rho_{\Delta x_i \Delta x_j}$ is the coefficient of correlation between them, and $\sigma_{\Delta x_i}$ and $\sigma_{\Delta x_j}$ are the standard deviations of ΔX_i and ΔX_j respectively.

$$\sigma_{\Delta v_0 \Delta \gamma_0} \approx \left(r_0^{-1}\right) \sigma_{\Delta x_2 \Delta x_5} + \left(v_0^{-1}\right) \sigma_{\Delta x_4 \Delta x_5} \tag{21}$$

$$\sigma_{\Delta r_0 \Delta \gamma_0} \approx \left(r_0^{-1}\right) \sigma_{\Delta x_1 \Delta x_2} + \left(v_0^{-1}\right) \sigma_{\Delta x_1 \Delta x_4} \tag{22}$$

5. THE CUMULATIVE DISTRIBUTION FUNCTIONS OF THE ERRORS IN THE PARAMETERS*

The Cumulative Distribution Functions of the Errors in Insertion Radius, Speed, and Flight Path Angle, the Error in Energy Per Unit Mass, and the Error in Semi-Major Axis

From Equations (8), (9), (10), (11) and (12) we see that ΔR_0, ΔV_0, $\Delta \Gamma_0$, ΔC_3, and ΔA (the errors in insertion radius, speed, and flight path angle, the error in energy per unit mass, and the error in semi-major axis) are linear combinations of the components of ΔX, and are therefore normally distributed.

The Cumulative Distribution Functions of the Errors in Eccentricity, Perigee Radius, and Apogee Radius

From Equations (14), (15), and (16)

$$\Delta E = e\left(r_0 + \Delta R_0,\ v_0 + \Delta V_0,\ \gamma_0 + \Delta \Gamma_0\right) \tag{14}$$

$$\Delta R_p = r_p\left(r_0 + \Delta R_0,\ v_0 + \Delta V_0,\ \gamma_0 + \Delta \Gamma_0\right) - r_0 \tag{15}$$

$$\Delta R_a = r_a\left(r_0 + \Delta R_0,\ v_0 + \Delta V_0,\ \gamma_0 + \Delta \Gamma_0\right) - r_0 \tag{16}$$

---

*The cumulative distribution function $F = F(\Delta\eta)$ of the random variable ΔH is a function of $\Delta\eta$ and gives the probability that ΔH is less than or equal to $\Delta\eta$. Mathematically, we write

$$F = F(\Delta\eta) = \Pr(\Delta H \leq \Delta\eta) = \int_{-\infty}^{\Delta\eta} f(x)\,dx$$

where $f(x)$ is the probability density function of ΔH.

we see that ΔE, ΔR_p, and ΔR_a are nonlinear functions of the components of $\Delta \mathbf{X}$, and, therefore, are not normally distributed. However, the probability distributions of these variables can be calculated.

Since ΔR_0, ΔV_0, and $\Delta\Gamma_0$ have a trivariate normal distribution, they can be expressed as linear functions of three uncorrelated normal random variables Y_1, Y_2, Y_3:

$$
\begin{aligned}
\Delta R_0 &= a_{11} Y_1 + a_{12} Y_2 + a_{13} Y_3 \\
\Delta V_0 &= a_{21} Y_1 + a_{22} Y_2 + a_{23} Y_3 \\
\Delta\Gamma_0 &= a_{31} Y_1 + a_{32} Y_2 + a_{33} Y_3
\end{aligned}
\tag{23}
$$

And, by the inverse linear transformation, we can express the Y_i's as functions of ΔR_0, ΔV_0, $\Delta\Gamma_0$

$$
\begin{aligned}
Y_1 &= b_{11} \Delta R_0 + b_{12} \Delta V_0 + b_{13} \Delta\Gamma_0 \\
Y_2 &= b_{21} \Delta R_0 + b_{22} \Delta V_0 + b_{23} \Delta\Gamma_0 \\
Y_3 &= b_{31} \Delta R_0 + b_{32} \Delta V_0 + b_{33} \Delta\Gamma_0
\end{aligned}
\tag{24}
$$

Thus, the variances of the Y_i's (σ_{y_i}'s) can be calculated as functions of the variances and covariances between the correlated variables using (24). Since uncorrelated normal random variables are independent, the joint density function of the Y_i's is given by the product of their marginal frequency functions. This is the essence of the technique.

Each probability density function for Y_i is approximated by a discrete distribution having probability mass points p_{i_k} over a range of values y'_{i_k} extending from $-L\sigma_{y_i}$ to $L\sigma_{y_i}$ and spaced $(L/N)\,\sigma_{y_i}$ apart.* Thus,

$$p_{i_k} \approx P\left[y_{i_k}' - \left(\frac{L}{2N}\right)\sigma_{y_i} < Y_i \le y_{i_k}' + \left(\frac{L}{2N}\right)\sigma_{y_i}\right] \tag{25}$$

$$(i = 1,\ 2,\ 3;\ k = 1,\ 2, \cdots, 2N+1)$$

where

$$p_{i_k} = \left(S_i^{-1}\right) q_{i_k}$$

$$q_{i_k} = \left(2\pi\sigma_{y_i}^2\right)^{-\frac{1}{2}} e^{-\left(y_{i_k}'\right)^2 / 2\sigma_{y_i}^2}$$

$$y_{i_k}' = \left(\frac{L}{N}\right)\left[k - (N+1)\right]\sigma_{y_i}$$

for $(i = 1,\ 2,\ 3;\ k = 1,\ 2,\ 3, \cdots,\ 2N+1)$

and

$$S_i = \sum_{k=1}^{2N+1} q_{i_k}$$

The larger the integer N for a fixed L, the better the approximation in (25).

---

*σ_{y_i} is the standard deviation of Y_i.

Since there are $(2N+1)$ possible values for each density function approximation, there will be a total of $(2N+1)^3$ possible combinations of values for the joint density approximation to the Y_i's. Further, since the Y_i's are independent, the probability of occurrence of p_{jkl} of each combination of values for y'_{1_j}, y'_{2_k}, y'_{3_l}, $(j, k, l = 1, 2, 3, \cdots, 2N+1)$ is given by the product

$$p_{jkl} = p_{1_j} p_{2_k} p_{3_l} \tag{26}$$

For each set of values for Y_1, Y_2, and Y_3, there is a corresponding set of values for ΔR_0, ΔV_0, and $\Delta\Gamma_0$ by the equations in (23), and a value for one of the variables ΔE, ΔR_p, and ΔR_a (by Equations (14), (15), and (16)) having the probability p_{jkl} $(j, k, l = 1, 2, 3, \cdots, 2N+1)$. Thus, the probability density function approximation for ΔE, ΔR_p, and ΔR_a will each have $(2N+1)^3$ possible values.

For purposes of machine computation, the range of each random variable ΔE, ΔR_p, and ΔR_a can be divided into a number of mutually exclusive intervals. For all values of the variable falling within one of these intervals, the associated probability as given by Equation (26) can be summed since all of the $(2N+1)^3$ possible combinations of values for ΔR_0, ΔV_0, and $\Delta\Gamma_0$ are mutually exclusive. In this way we obtain an approximation for the probability that the random variable $(\Delta E, \Delta R_p, \Delta R_a)$ assumes values within the particular interval, and thus an approximation for the probability density function.

The cumulative distribution function is obtained by summing the probability density function.

The Cumulative Distribution Function of B

By Equation (13)

$$B \approx r_0^{-1} \sqrt{(\Delta X_2)^2 + (\Delta X_3)^2}$$

we see that B is not a linear function of the components of $\Delta \mathbf{X}$ and therefore, not normally distributed. The cumulative distribution function of B can be obtained in similar fashion as for ΔE, ΔR_p, and ΔR_a. However, in this case B is a function of only two components of $\Delta \mathbf{X}$, ΔX_2 and ΔX_3.

6. AN EXAMPLE WITH APPLICATION TO A NOMINAL APOLLO EARTH PARKING ORBIT

In order to illustrate the statistical technique described in Section 5, we will consider a nominal Apollo earth parking orbit—a nominally circular orbit of 100 n.mi.

Reference 3 describes the dispersions in position and velocity of the Saturn V Launch Vehicle at insertion due to 30 navigation parameters. Reference 4 describes how the covariance matrix of insertion errors shown in Table 1 and obtained from Reference 5 was constructed from the uncertainties (standard deviations) of these navigation parameters. The matrix, therefore, expresses the deviation of the onboard estimate from the nominal insertion condition due to errors in the onboard navigation during the launch phase. It can be considered typical of the performance of the Saturn V Launch Vehicle for a nominal Apollo Mission at insertion into a near-earth nominally circular parking orbit of 100 n.mi. as of the date of this analysis (Reference 6). The matrix is given in the $\mathbf{U}^0\ \mathbf{V}^0\ \mathbf{W}^0$ Coordinate System (Figure 1). The elements along the main diagonal are the variances of ΔX_i (the components of $\Delta\mathbf{X}$) expressed in units of $(\mathrm{ft})^2$ and $(\mathrm{ft/sec})^2$. The off-diagonal elements are the covariances between ΔX_i and ΔX_j $(i \neq j)$ and are expressed in units of $(\mathrm{ft})^2$, $(\mathrm{ft/sec})^2$, and $(\mathrm{ft})^2/\mathrm{sec}$. The normalized covariance matrix or correlation matrix corresponding to the matrix in Table 1 is shown in Table 2. It is also given in the $\mathbf{U}^0\ \mathbf{V}^0\ \mathbf{W}^0$ Coordinate System. However, the elements along the main diagonal are unit variances while the off-diagonal elements are the coefficients of correlation between ΔX_i and ΔX_j $(i \neq j)$.

It is necessary to mention at this point that we are assuming throughout this analysis that $\Delta\mathbf{X}$ has a multivariate normal distribution with zero mean and the covariance matrix shown in Table 1.

Table 1

Expected Saturn V Insertion Covariance Matrix* ($\mathbf{U}^0\ \mathbf{V}^0\ \mathbf{W}^0$ Coordinate System)

$$\begin{bmatrix} 977736.00 & -745996.00 & -3162.1120 & 3935.6520 & -2060.5760 & -15.106120 \\ & 743820.00 & 1493.9920 & -3123.1360 & 1961.3040 & 8.9479200 \\ & & 1180016.0 & -18.365520 & 5.3787600 & 3499.8720 \\ & & & 16.197720 & -8.6568800 & -0.0769048 \\ \text{Symmetric} & & & & 5.2908400 & 0.02838016 \\ & & & & & 11.023800 \end{bmatrix}$$

*The elements along the main diagonal are the variances of ΔX_i expressed in units of $(\mathrm{ft})^2$ and $(\mathrm{ft/sec})^2$, the off-diagonal elements are the covariances between the ΔX_i, expressed in units of $(\mathrm{ft})^2$, $(\mathrm{ft/sec})^2$, and $(\mathrm{ft})^2/\mathrm{sec}$.

Table 2

Correlation Matrix* for Matrix in Table 1

$$\begin{bmatrix} 1.00000000 & -0.87476575 & -0.0029438962 & 0.98896049 & -0.90597355 & -0.0046012567 \\ & 1.00000000 & 0.0015946684 & -0.89976693 & 0.98866389 & 0.0031248020 \\ & & 1.00000000 & -0.0042008032 & 0.0021526623 & 0.97038159 \\ & & & 1.00000000 & -0.93513032 & -0.0057552056 \\ \text{Symmetric} & & & & 1.00000000 & 0.0037160955 \\ & & & & & 1.00000000 \end{bmatrix}$$

*The elements along the main diagonal are the unit variances of the ΔX_i. The off-diagonal elements are the coefficients of correlation between the ΔX_i.

One may ask, "What actually does a covariance matrix of insertion errors represent?" And, "What is the usefulness of such a matrix?"

The covariance matrix is actually a measure of the uncertainty in the state vector (position and velocity) at insertion due to uncertainties in the actual values of navigation parameters. If the guidance and control system is working properly, we could still expect this much variation or dispersion in the position and velocity of the vehicle at insertion.

The usefulness of such a matrix can best be seen in its providing us with a tool for determining expected variations and bounds (along with their associated probabilities under the Gaussian assumption) of some of the parameters of the orbit.

Using the expressions for the variances and covariances between ΔR_0, ΔV_0, and Γ_0 in Equations (17) through (22) as well as the covariance matrix in Table 1, the covariance matrix of ΔR_0, ΔV_0, and $\Delta\Gamma_0$ can be obtained, and is shown in Table 3. Table 4 shows a normalized matrix where the elements along the main diagonal are the standard deviations of ΔR_0, ΔV_0, and $\Delta\Gamma_0$, and the off-diagonal elements are the coefficients of correlation between these random variables.

Table 3

Covariance Matrix Σ_0 of ΔR_0, ΔV_0, and $\Delta\Gamma_0$*

$$\begin{bmatrix} 0.02644932 & -0.33891053 & -0.00100201 \\ & 5.29084000 & 0.01588871 \\ \text{Symmetric} & & (0.49300523)10^{-4} \end{bmatrix}$$

*The elements along the main diagonal are the variances of ΔR_0, ΔV_0, and $\Delta\Gamma_0$, expressed respectively in units of $(\text{n.mi.})^2$, $(\text{ft/sec})^2$, and $(\text{deg})^2$. The off-diagonal elements are the covariances between ΔR_0, ΔV_0, and $\Delta\Gamma_0$, expressed in units of (n.mi.)(ft/sec), (n.mi.)(deg), and (ft)(deg)(sec).

The cumulative distribution functions for some of the errors in the parking orbit are shown in Figures 2 through 10, and were obtained as described in Section 5.

From these figures we can see that the angular error in the position vector at insertion (Figure 7), the error in eccentricity (Figure 8), the perigee error (Figure 9), and the apogee error (Figure 10), are non-Gaussian and do not have zero means.

Table 4

Normalized Covariance Matrix
of ΔR_0, ΔV_0, $\Delta \Gamma_0$*

$$\begin{bmatrix} 0.16263246 & -0.90597355 & -0.87748401 \\ & 2.30018258 & 0.98378570 \\ \text{Symmetric} & & (0.70214331)10^{-2} \end{bmatrix}$$

*The elements along the main diagonal are the standard deviations of ΔR_0, ΔV_0, and $\Delta \Gamma_0$, expressed in (n.mi.), (ft/sec), and (deg). The off-diagonal elements are the coefficients of correlation between ΔR_0, ΔV_0, and $\Delta \Gamma_0$.

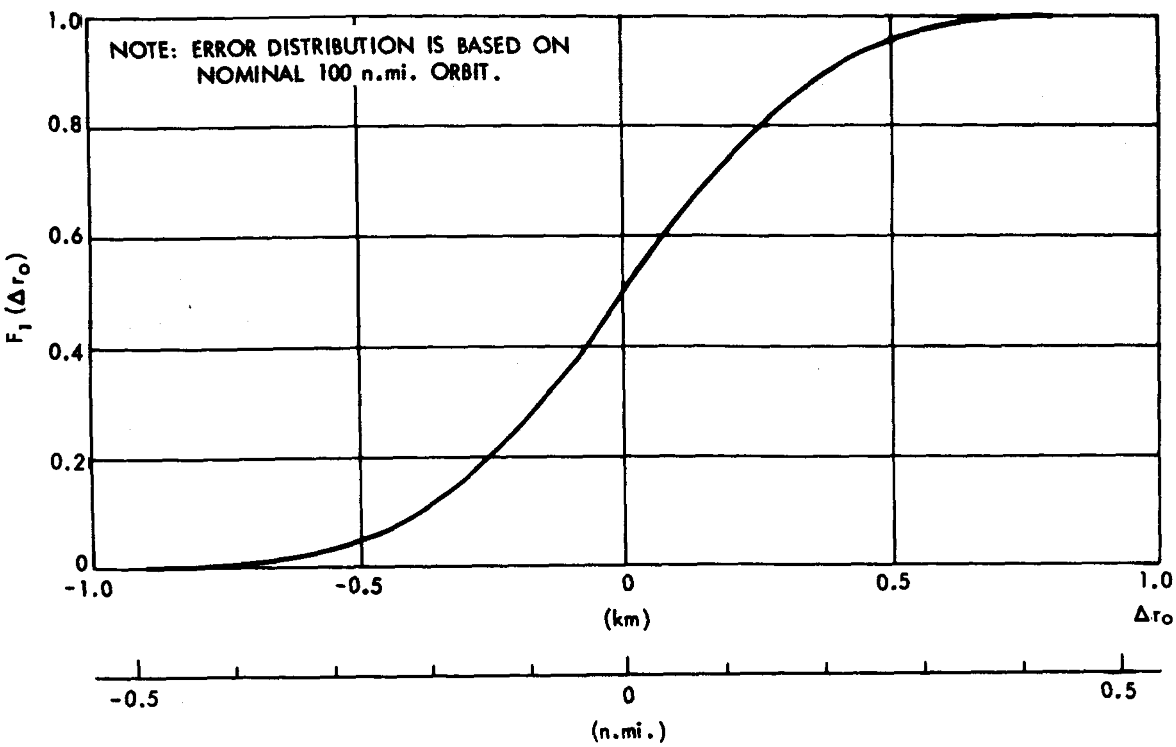

Figure 2. The Cumulative Distribution Function of the Error in Insertion Radius ΔR_0

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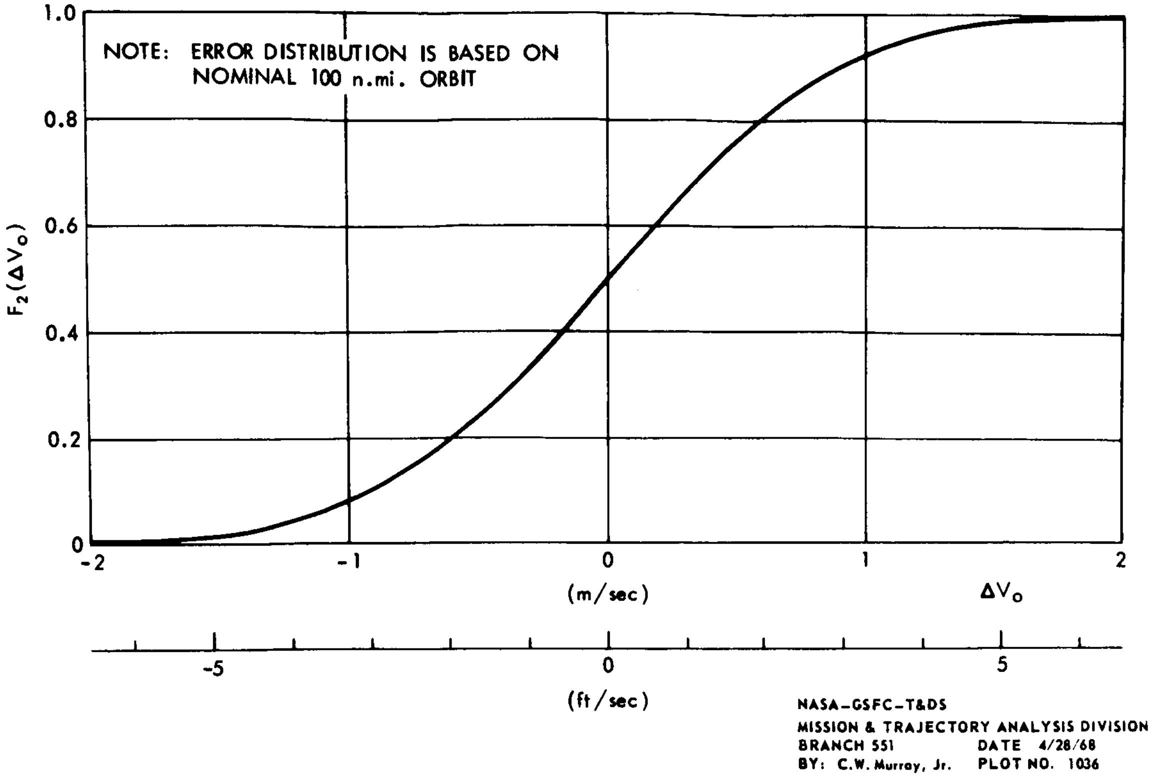

Figure 3. The Cumulative Distribution Function of the Error in Insertion Speed ΔV_0

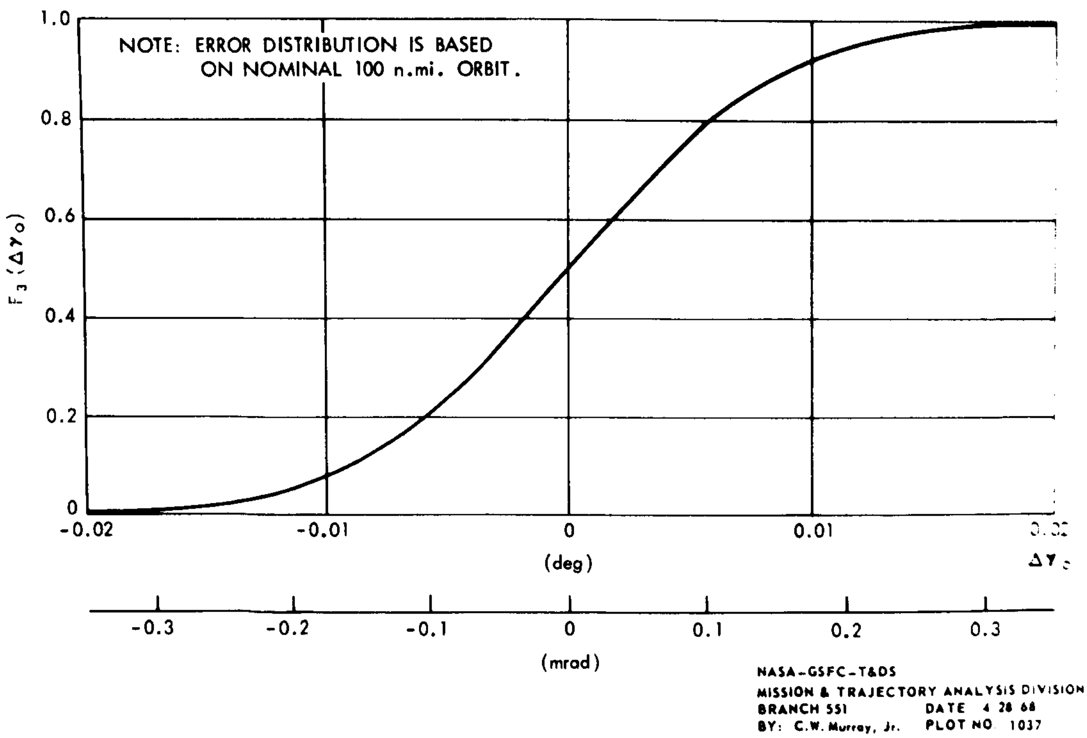

Figure 4. The Cumulative Distribution Function of the Error in Insertion Flight Path Angle $\Delta\Gamma_0$

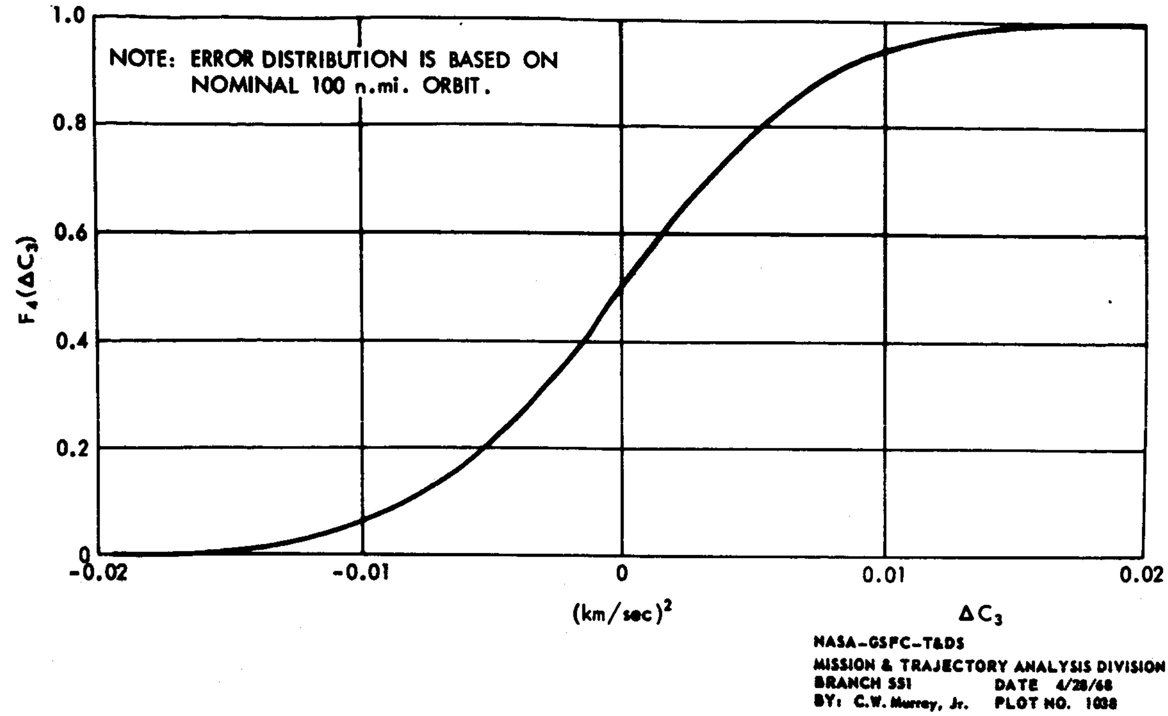

Figure 5. The Cumulative Distribution Function of the Error in Energy Per Unit Mass ΔC_3

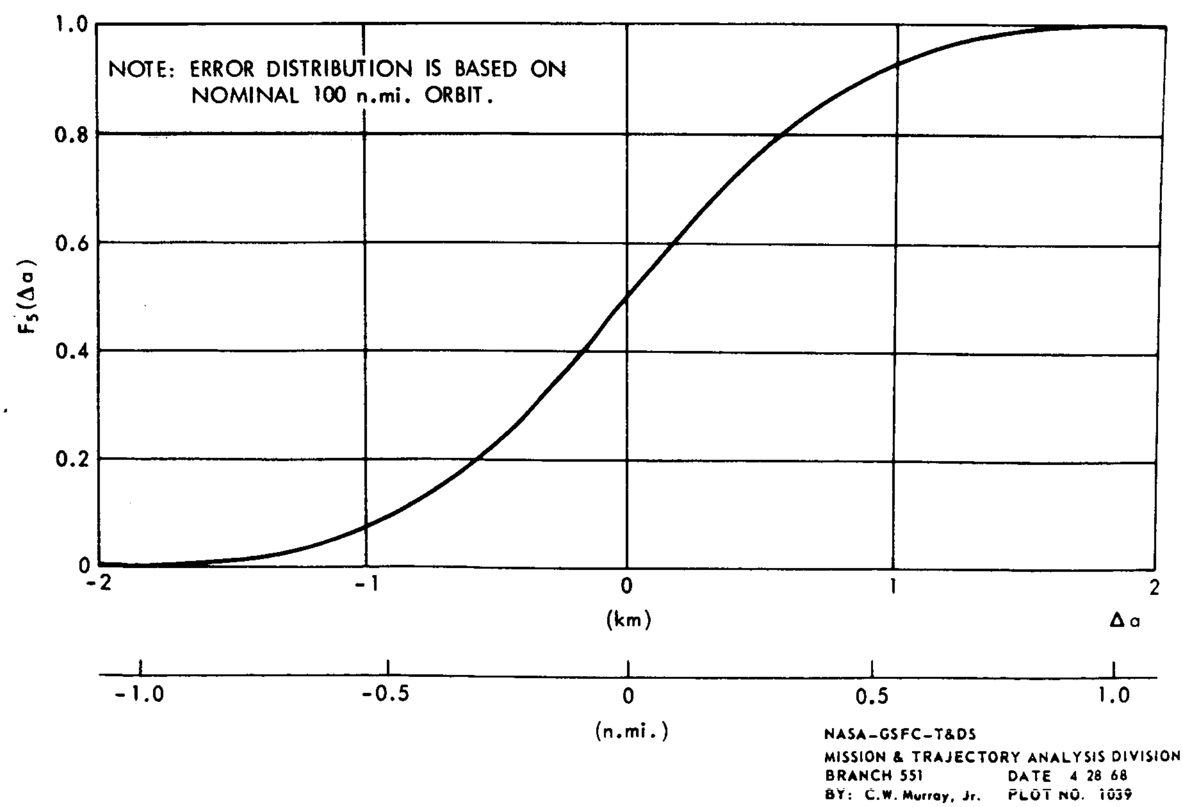

Figure 6. The Cumulative Distribution Function of the Error in Semi-major Axis ΔA

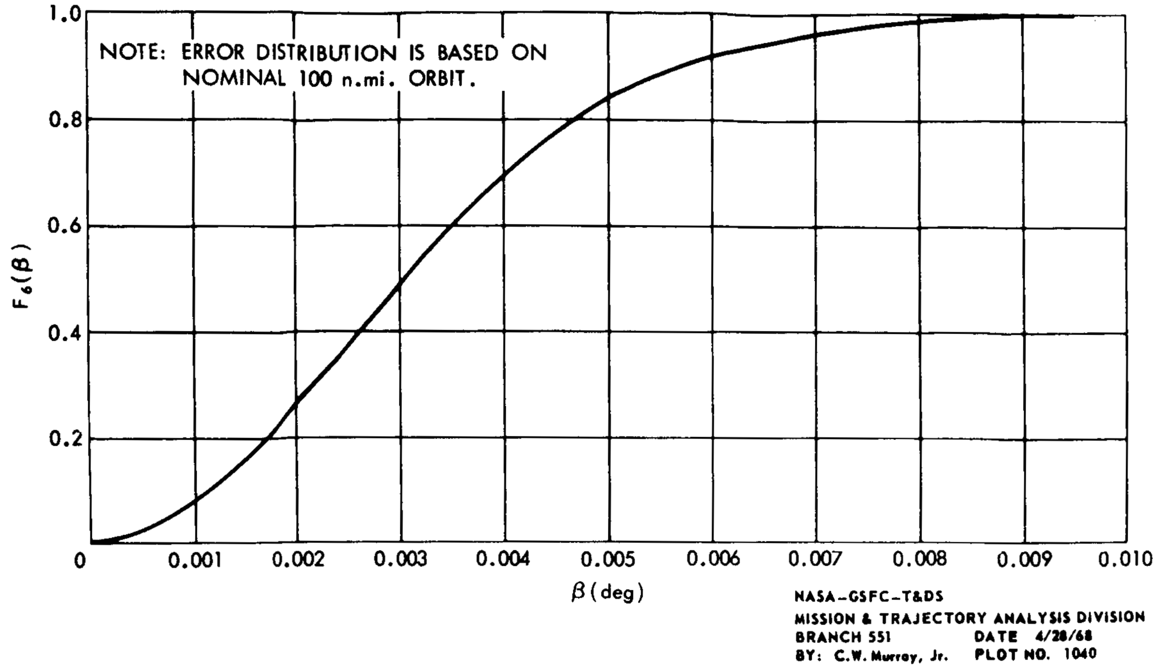

Figure 7. The Cumulative Distribution Function of the Angle Between the Actual Position Vector and the Nominal Position Vector at Insertion B

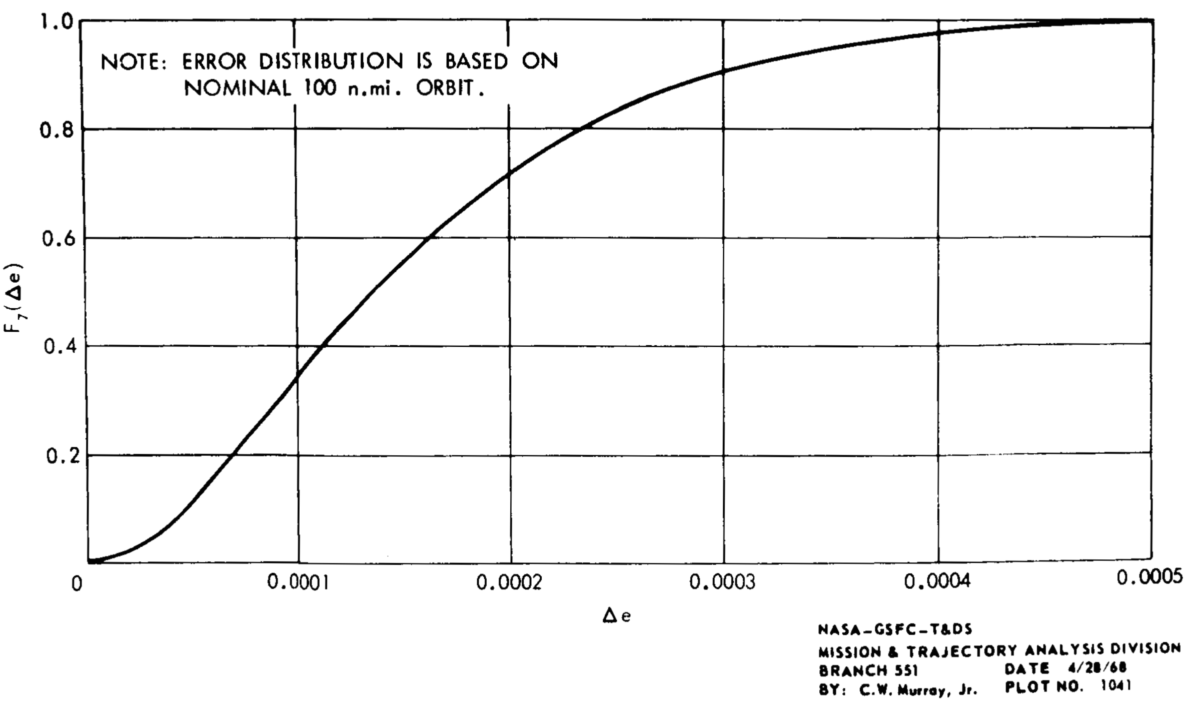

Figure 8. The Cumulative Distribution Function of the Error in Eccentricy ΔE

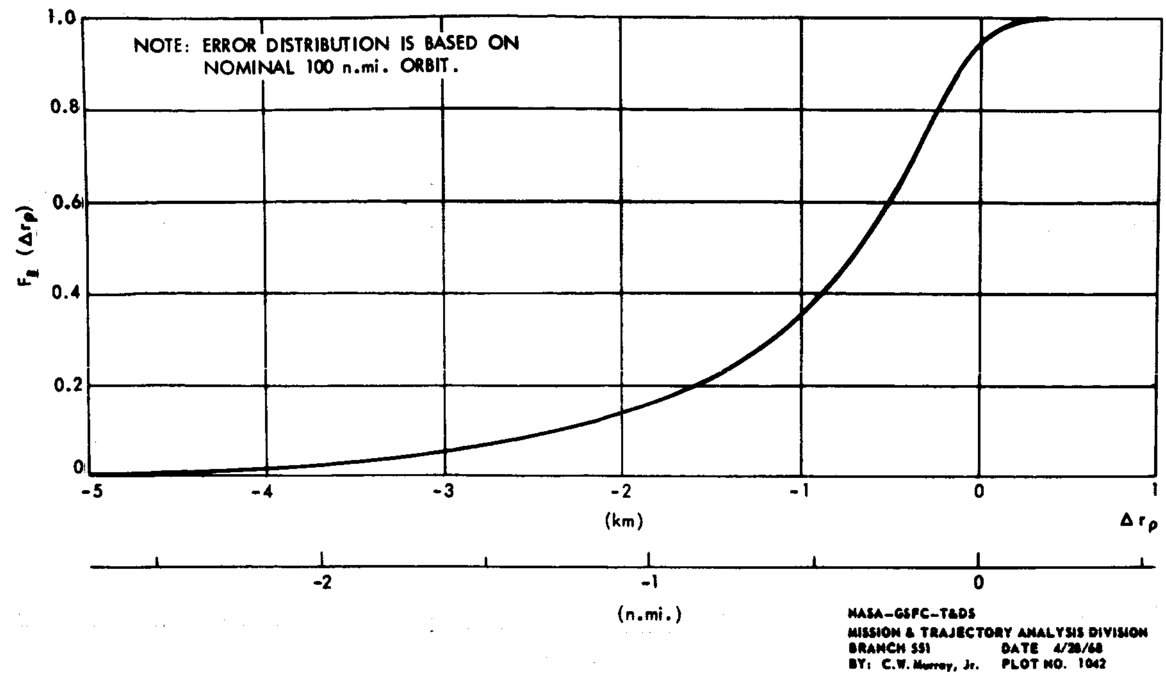

Figure 9. The Cumulative Distribution Function of the Error in Perigee Radius ΔR_p

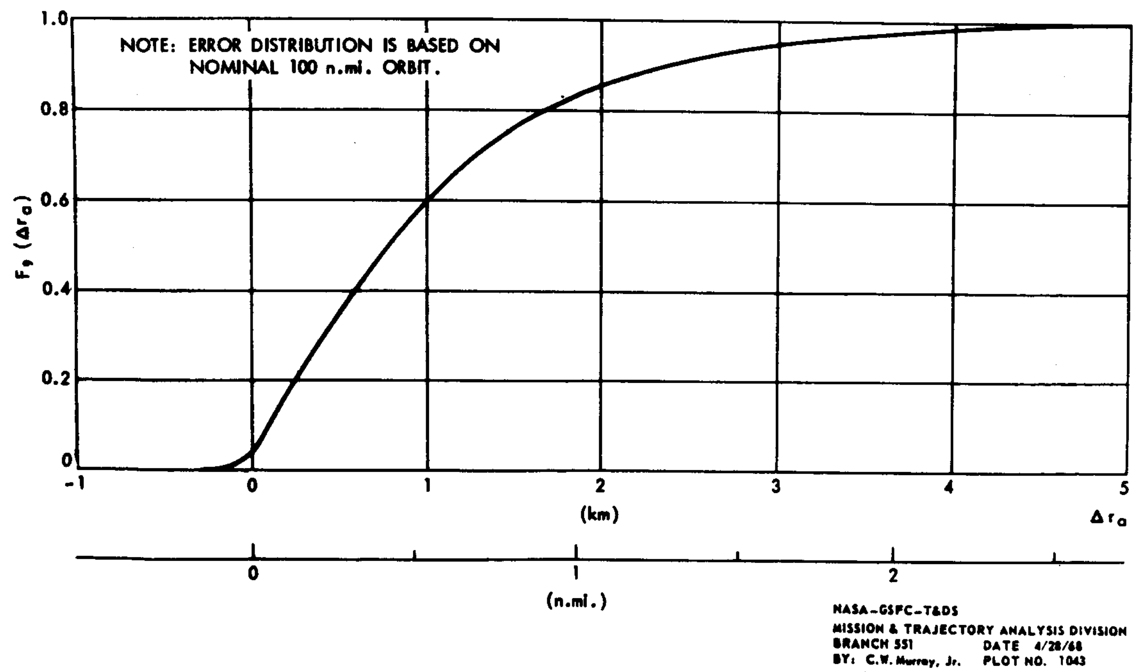

Figure 10. The Cumulative Distribution Function of the Error in Apogee Radius ΔR_a

Table 5

Lower and Upper 99.5% Probability Points for the Errors in the Parameters of the Orbit

Error	Lower 99.5% Point	Upper 99.5% Point
ΔR_0	-0.7 km (-0.4 n.mi.)	+0.7 km (+0.4 n.mi.)
ΔV_0	-1.8 m/sec (-6.5 ft/sec)	+1.8 m/sec (+6.5 ft/sec)
$\Delta\Gamma_0$	-0.018° (-0.3 mrad)	+0.018° (+0.3 mrad)
ΔC_3	-0.016 $(km/sec)^2$	+0.016 $(km/sec)^2$
ΔA	-1.7 km (-0.9 n.mi.)	+1.7 km (+0.9 n.mi.)
B	+0.0002°	+0.0085°
ΔE	+0.0001 (dimensionless)	+0.00048 (dimensionless)
ΔR_p	-4.7 km (-2.5 n.mi.)	+0.2 km (+0.1 n.mi.)
ΔR_a	-0.2 km (-0.1 n.mi.)	+4.5 km (2.4 n.mi.)

Table 5 summarizes the results of this analysis in tabular form, showing both lower and upper 99.5% probability points for the errors in the parameters. Thus, there is a 99% probability that the errors in these parameters will lie between these indicated values.

It is significant to note that there is a 99.5% probability that perigee error will exceed -2.5 n.mi. or that the actual perigee height will exceed 97.5 n.mi. for the nominal 100 n.mi. circular parking orbit. Likewise, there is a 99.5% probability that the actual apogee height will be less than 102.4 n.mi.

From the above we see that the actual parking orbit should be very close to the nominal circular orbit of 100 n.mi. This is also evident from inspection of Figure 8 where we see there is a 99.5% probability that the error in eccentricity, and hence the eccentricity, will be less than 0.00048.

Figure 11 shows a comparison between the cumulative distribution function of perigee error using Equation (15) and a cumulative normal distribution function having the same mean and standard deviation. The difference between the curves is significant. For example, using the normal distribution, there is a 99.5% probability that the perigee error will exceed -1.7 n.mi. and a 99.5%

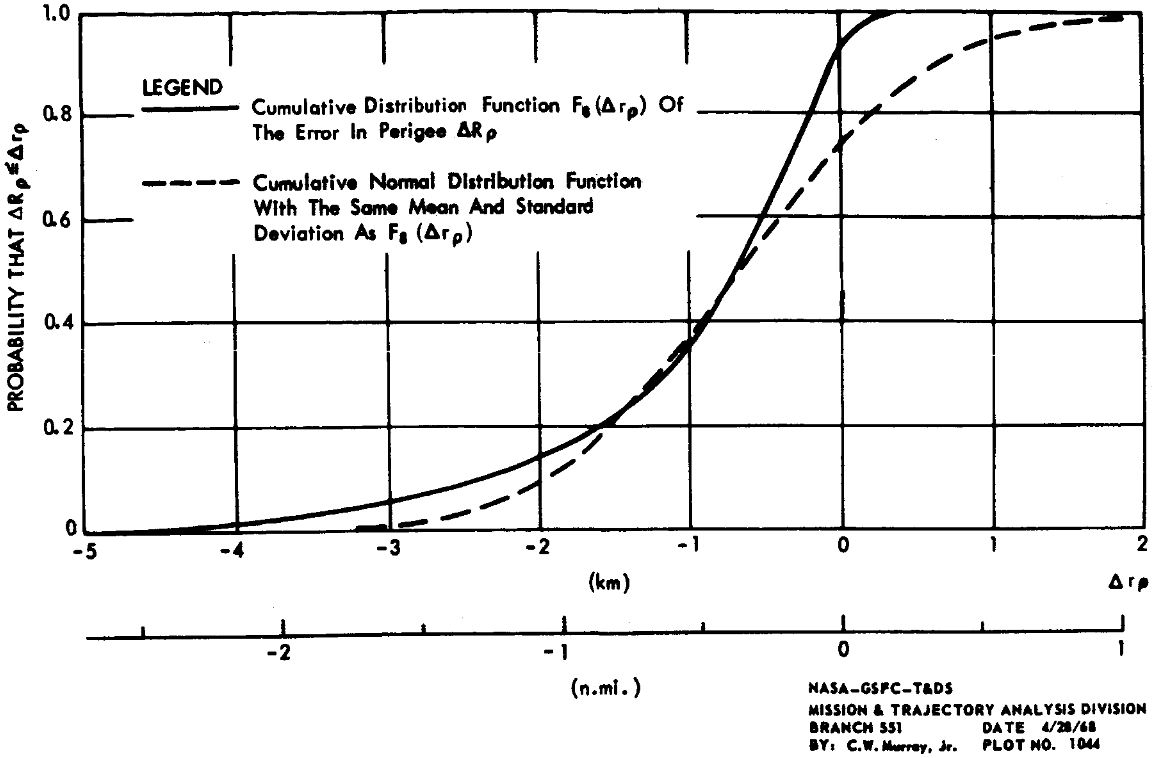

Figure 11. Comparison of the Cumulative Distribution Function of the Error in Perigee with a Cumulative Normal Distribution Having the Same Mean and Standard Deviation

probability that the error will be less than 1.1 n.mi. From the calculated curve the corresponding points are -2.5 n.mi and 0.1 n.mi.

Combining Tracking Errors and Insertion Errors

As a further example, we will show how insertion ship tracking errors (calculated minus actual) can be combined with insertion errors (actual minus nominal) to answer such questions as, "For a nominally circular parking orbit of 100 n.mi., what is the probability that the calculated perigee height will exceed a given amount?"

At insertion of the Apollo spacecraft, the following parameters:

radius	r_1
speed	v_1
flight path angle	γ_1

will be determined from one minute of ship's tracking data. In Reference 1 it is required that these parameters be determined to the following accuracies:

$$3\sigma_{r_1} = 2.4 \text{ n.mi. } (4.44 \text{ km})$$

$$3\sigma_{v_1} = 16 \text{ ft/sec } (4.87 \text{ m/sec})$$

$$3\sigma_{\gamma_1} = 0.16° \ (2.79 \text{ mrad})$$

Let

$$\Delta \mathbf{Z}_0 = \begin{pmatrix} \Delta R_0 \\ \Delta V_0 \\ \Delta \Gamma_0 \end{pmatrix} \tag{27}$$

be an error vector representing the error or deviation of the actual insertion radius, speed, and flight path angle from the nominal, and let

$$\Delta \mathbf{Z}_1 = \begin{pmatrix} \Delta R_1 \\ \Delta V_1 \\ \Delta \Gamma_1 \end{pmatrix} \tag{28}$$

be an error vector representing the deviation of the measured or calculated insertion radius, speed, and flight path angle from the actual.

Then

$$\Delta \mathbf{Z} = \Delta \mathbf{Z}_0 + \Delta \mathbf{Z}_1 \tag{29}$$

is an error vector representing the deviation of the measured or calculated insertion radius, speed, and flight path angle from the nominal.

For this example, we will assume that $\Delta \mathbf{Z}_0$ and $\Delta \mathbf{Z}_1$ are independent error vectors each with a trivariate normal distribution, having zero mean vectors and covariance matrices Σ_0 and Σ_1 respectively. Then $\Delta \mathbf{Z}$ has a trivariate normal distribution with a zero mean vector and covariance matrix Σ

$$\Sigma = \Sigma_0 + \Sigma_1 \tag{30}$$

Since it is practically impossible to determine the actual coefficients of correlation between the errors ΔR_1, ΔV_1, and $\Delta\Gamma_1$, the analysis has been carried out for coefficients of correlation of 0.0 (uncorrelated) and ±0.9. That is, all possible combinations of signs have been considered with the expection of those for which the covariance matrix Σ_1 becomes singular. The possible combinations are:

$\rho_{\Delta r_1 \Delta v_1}$	$\rho_{\Delta r_1 \Delta \gamma_1}$	$\rho_{\Delta v_1 \Delta \gamma_1}$
+ 0.9	+ 0.9	+ 0.9
+ 0.9	− 0.9	− 0.9
− 0.9	+ 0.9	− 0.9
− 0.9	− 0.9	+ 0.9

(31)

The matrix Σ_1 is shown in Table 6 for the case $\rho_{\Delta r_1 \Delta v_1} = \rho_{\Delta r_1 \Delta \gamma_1} = \rho_{\Delta v_1 \Delta \gamma_1} = 0.9$.

Table 6

Covariance Matrix Σ_1 of ΔR_1, ΔV_1, and $\Delta\Gamma_1$*

$$\begin{bmatrix} 0.64 & 3.84 & 0.038400 \\ & 28.4444 & 0.25600 \\ \text{Symmetric} & & 0.002844 \end{bmatrix}$$

*The elements down the main diagonal are the variances of ΔR_1, ΔV_1, and $\Delta\Gamma_1$ expressed in units of $(\text{n.mi.})^2$, $(\text{ft/sec})^2$, and $(\text{deg})^2$. The off-diagonal elements are the covariances between ΔR_1, ΔV_1, and $\Delta\Gamma_1$, expressed in units of (n.mi.)(ft)/sec, (n.mi.)(deg), and (ft)(deg)/sec.

The matrix Σ obtained by adding Σ_0 (Table 3) and Σ_1 (Table 6) is shown in Table 7. Similar matrices can be obtained using other coefficients of correlation indicated in (31) and $\rho_{\Delta r_1 \Delta v_1} = \rho_{\Delta r_1 \Delta \gamma_1} = \rho_{\Delta v_1 \Delta \gamma_1} = 0.0$.

Using these matrices and the method outlined in Section 5, the probability distribution of the perigee error can be calculated.

Table 7

Covariance Matrix Σ of $(\Delta R_0 + \Delta R_1)$, $(\Delta V_0 + \Delta V_1)$, and $(\Delta\Gamma_0 + \Delta\Gamma_1)$*

$$\begin{bmatrix} 0.66644932 & 3.50108947 & 0.03739799 \\ & 33.73524000 & 0.27188871 \\ \text{Symmetric} & & 0.00289330 \end{bmatrix}$$

*The elements along the main diagonal are the variances of $(\Delta R_0 + \Delta R_1)$, $(\Delta V_0 + \Delta V_1)$, and $(\Delta\Gamma_0 + \Delta\Gamma_1)$, respectively, expressed in units of $(\text{n.mi.})^2$, $(\text{ft/sec})^2$, and $(\text{deg})^2$. The off-diagonal elements are the covariances between these random variables, expressed in units of (n.mi.)(ft)/sec, (n.mi.) deg, and (ft)(deg)/sec.

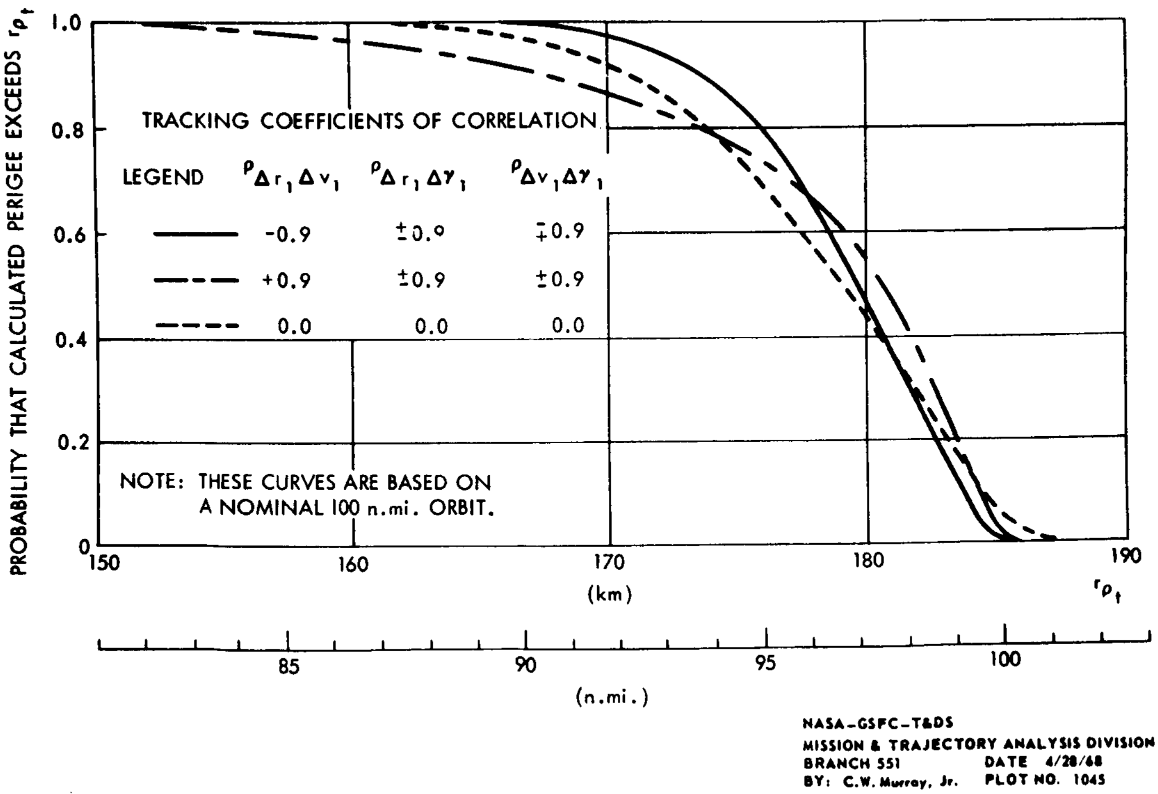

Figure 12. Probability That Calculated Perigee Height Exceeds r_{p_t}

Figure 12 shows the cumulative distribution function of the calculated perigee height for a 100 n.mi. parking orbit and several combinations of coefficients of correlation between ΔR_1, ΔV_1, and $\Delta\Gamma_1$.

In order to compare these curves, we note that if $\rho_{\Delta r_1 \Delta v_1} = +0.9$, and $\rho_{\Delta r_1 \Delta\gamma_1} = \rho_{\Delta v_1 \Delta\gamma_1} = \pm 0.9$, there is a 90% probability that the calculated perigee height is greater than 90.8 n.mi. If the errors are uncorrelated, there is a 90% probability that the calculated perigee will exceed 92.3 n.mi. If $\rho_{\Delta r_1 \Delta v_1} = -0.9$, $\rho_{\Delta r_1 \Delta\gamma_1} = \pm 0.9$, and $\rho_{\Delta v_1 \Delta\gamma_1} = \mp 0.9$, there is a 90% probability that the calculated perigee height will exceed 93.7 n.mi.

CONCLUSIONS

Distributions

A statistical technique has been described for analyzing the effects of insertion errors (deviations of actual values from nominal values) on the parameters of near-earth circular orbits. The technique assumes that the error in the state vector (position and velocity) of the vehicle at insertion has a multivariate normal distribution with zero mean and a given covariance matrix. Probability distributions for some of the errors in the parameters are obtained numerically using Keplerian two-body equations of motion. Results of the analysis indicate:

(1) Perigee error, apogee error, the error in eccentricity, and the angular error in the vehicle's position vector at insertion are non-Gaussian with non-zero means.

The Apollo Parking Orbit

The technique has been applied to a nominal Apollo earth parking orbit using an expected covariance matrix typical of the performance of the Saturn V Launch Vehicle at insertion into a near-earth nominally circular parking orbit of 100 n.mi. Results of the analysis indicate that:

(2) The actual near-earth parking orbit will be close to the nominally circular 100 n.mi. orbit. For example, there is a 99.5% probability that the actual perigee height will exceed 97.5 n.mi., and a 99.5% probability that the actual apogee height will be less than 102.4 n.mi.

Combining Tracking Errors and Insertion Errors

As a further example, insertion ship tracking errors and insertion errors have been combined. The analysis indicates that for 3 sigma values of

$$3\sigma_{\Delta r_1} = 2.4 \text{ n.mi. } (4.44 \text{ km})$$

$$3\sigma_{\Delta v_1} = 16 \text{ ft/sec } (4.87 \text{ m/sec})$$

$$3\sigma_{\Delta \gamma_1} = 0.16° \ (2.79 \text{ mrad})$$

for the insertion ship's tracking errors (calculated minus actual) in insertion radius r_1, speed v_1, and flight path angle γ_1, and for coefficients of correlation of +0.9 between these errors:

(3) There is a 90% probability that the calculated perigee height will exceed 91 n.mi. for the nominal 100 n.mi. parking orbit.

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