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by

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Dynamical systems and stability

by Jack K. Hale

1. Introduction. Of basic importance in the theory of a dynamical system on a Banach space $\mathscr{B}$ is the concept of a limit set $\omega(\gamma)$ of an orbit γ through a point φ in $\mathscr{B}$. One can be assured that $\omega(\gamma)$ is nonempty and invariant if γ belongs to a compact subset of $\mathscr{B}$. In applications it is much easier to show that an orbit belongs to a bounded set than it is to show it belongs to a compact set. However, if the dynamical system arises from an ordinary differential equation and $\mathscr{B}$ is therefore finite dimensional, the local compactness of $\mathscr{B}$ insures that a bounded orbit belongs to a compact set of $\mathscr{B}$. If the dynamical system arises from a functional differential equation of retarded type, then $\mathscr{B}$ is infinite dimensional and not locally compact. However, for a certain class of such equations, it is easily shown that bounded orbits do belong to compact subsets. The basic reason for this nice property in retarded functional differential equations is that the trajectory becomes "smoother" with the evolution of the system.

If the dynamical system arises from a system of functional differential equations of neutral type or from hyperbolic partial differential equations, then trajectories do not in general become smoother as time evolves. The basic space $\mathscr{B}$ in such situations is usually a Sobolev space and the well known Sobolev imbedding theorems implies in general the existence of a Banach space $\mathscr{C}$ such that the unit ball in $\mathscr{B}$ belongs to a compact set in $\mathscr{C}$. Therefore, any bounded orbit of the dynamical system on $\mathscr{B}$ would have a nonempty limit set in $\mathscr{C}$. The limit set in $\mathscr{C}$ should then enjoy an invariance property.

It is the purpose of this paper to exploit these remarks in some detail. The basic ideas were announced in [1], but certain aspects of that paper are unsatisfactory for the applications. In section 2 of this paper we discuss the different types of topologies that may be introduced on the state space for differential equations (ordinary, functional and partial) in order to obtain dynamical systems. Section 3 is devoted to a discussion of limit sets and stability to be applied to the limit dynamical systems introduced in section 4. In section 5 the theory is applied to specific dynamical systems and section 6 is devoted to a discussion of the relationship of limit dynamical systems to the extended system introduced in [1].

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2. Examples of dynamical systems. Let R^n denote an n-dimensional vector space with norm $|\cdot|$, R^+ denote the interval $[0,\infty)$ and if $\mathscr{A}$ is a Banach space let $\|\varphi\|_{\mathscr{A}}$ be the norm of an element φ in $\mathscr{A}$.

Definition 1. A dynamical system on a Banach space $\mathscr{B}$ is a function $u: R^+ \times \mathscr{B} \to \mathscr{B}$ such that u is continuous, $u(0,\varphi) = \varphi$, $u(t+\tau,\varphi) = u(t,u(\tau,\varphi))$ for all $t,\tau \geq 0$ and φ in $\mathscr{B}$. An orbit (positive orbit) $\gamma^+ = \gamma^+(\varphi)$ through φ in $\mathscr{B}$ is defined to be $\gamma^+(\varphi) = \cup_{t\geq 0} u(t,\varphi)$. It is sometimes convenient to have the concept of a dynamical system on a subset S of a Banach space $\mathscr{B}$ and this will signify a function $u: R^+ \times S \to S$ which satisfies the properties listed above.

This definition coincides with the term generalized dynamical system used by Zubov [2]. Zubov introduced the adjective "generalized"

to distinguish between his definition and that of a dynamical system defined on $(-\infty,\infty) \times \mathscr{B}$ rather than $R^+ \times \mathscr{B}$. One could also discuss dynamical systems on metric spaces but except for one example this will not be needed here.

Let us give some examples of dynamical systems.

Example 1. Ordinary differential equations. Suppose $f: R^n \to R^n$ is continuous and for any ξ in R^n the solution $u(t,\xi)$, $u(0,\xi) = \xi$, of the equation

$$\dot{x} = f(x) \tag{1}$$

exists for all $t \geq 0$, is unique and depends continuously upon t,ξ. Uniqueness of the solution implies $u(t+\tau,\xi) = u(t,u(\tau,\xi))$ for all $t,\tau \geq 0$. Therefore, u is a dynamical system on R^n.

Example 2. Functional differential equations with finite retardation. Let $C = C([-r,0],R^n)$, $r \geq 0$, be the space of continuous functions mapping $[-r,0]$ into R^n with the topology of uniform convergence. For any continuous function x defined on $[-r,A)$, $A > 0$, and any t in $[0,A)$, let x_t in C be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. Suppose $f: C \to R^n$ is continuous and maps bounded sets into bounded sets. A function $x = x(\varphi)$ defined and continuous on $[-r,A)$, $A > 0$, is said to be a solution of the functional differential equation

$$\dot{x}(t) = f(x_t) \tag{2}$$

with initial value φ at 0 if $x_0 = \varphi$ and $x(\varphi)(t)$ satisfies (2) for t in $[0,A)$. For any φ in C, assume that a solution $x(\varphi)$

exists on $[-r,\infty)$, is unique and $x(\varphi)(t)$ is jointly continuous in t,φ. With $u(t,\varphi) = x_t(\varphi)$ one easily sees that u is a dynamical system on C. Local existence, uniqueness and continuous dependence is easily proved if f is assumed to be continuous and locally lipschitzian.

If $\gamma^+ = \cup_{t\geq 0} x_t(\varphi)$ is a bounded orbit of (2), then γ^+ belongs to a compact subset of C. In fact, if there is a constant M such that $\|x_t(\varphi)\|_C \leq M$, $t \geq 0$, then there is constant N such that $|\dot{x}(t)| = |f(x_t)| \leq N$ for all $t \geq 0$. This clearly shows that γ^+ belongs to a compact subset of C.

<u>Example 3. Functional differential equations with infinite retardations.</u> Consider the complete locally convex linear topological space $\mathscr{M}$ consisting of all bounded continuous functions mapping $(-\infty,0]$ into R^n with the topology of uniform convergence on compact subsets of $(-\infty,0]$. Define the metric ρ in $\mathscr{M}$ by

$$\rho(\varphi,\psi) = \sum_{N=0}^{\infty} m_N$$

$$m_N = \min(2^{-N}, \sup_{-N-1\leq\theta\leq -N}|\varphi(\theta)-\psi(\varphi)|)$$

A set U in $\mathscr{M}$ is bounded if there exists an M such that $\sup_{\theta\in(-\infty,0]}|\varphi(\theta)| < M$ for all φ in U. Suppose $f:\mathscr{M} \to R^n$ is continuous and maps bounded sets into bounded sets and let x_t, t in $[0,A)$, be defined by $x_t(\theta) = x(t+\theta)$, θ in $(-\infty,0]$, for any function x defined on $[-\infty,A)$, $A > 0$. For any φ in $\mathscr{M}$, suppose that the solution $x(\varphi)$ of (2) exists on $(-\infty,\infty)$, is unique and $x(\varphi)(t)$ is jointly continuous in t,φ. The function $u(t,\varphi) = x_t(\varphi)$ is thus a dynamical system on $\mathscr{M}$. Using the triangularization procedure together with arguments similar to those in

Example 2 (see [3]), one shows that any bounded orbit of (2) in $\mathscr{M}$ belongs to a compact subset of $\mathscr{M}$. Local existence and uniqueness theorems may be found in [4].

A specification of a different metric space of functions on $(-\infty,0]$ leads to a different class of functional differential equations. A very interesting class has been discussed by Coleman and Mizel [5]. A special case of their results concerns the Banach space V of functions mapping $(-\infty,0]$ into R^n with

$$\|\varphi\|_V = |\varphi(0)| + \int_{-\infty}^{0} k(\theta)|\varphi(\theta)|\,d\theta$$

where $k(\theta) > 0$, $\int_{-\infty}^{0} k(\theta)d\theta < \infty$. Suppose $f: V \to R^n$ is continuous and takes bounded sets into bounded sets and for any φ in V the solution $x(\varphi)$ of (2) exists on $(-\infty,\infty)$, is unique and $x(\varphi)(t)$ is jointly continuous in t,φ. Then $u(t,\varphi) = x_t(\varphi)$ is a dynamical system on V and Coleman and Mizel [5] show that any bounded orbit of (2) in V belongs to a compact subset of V.

The above norm of Coleman and Mizel is quite natural for the problem they were discussing, but, in other applications, such a norm is unsatisfactory since the right hand sides of even simple differential difference equations will not be continuous in this norm. It is therefore necessary to discuss a more general class of Banach spaces and the associated functional differential equations. For some problems it is essential to have the property that every bounded orbit belongs to a compact subset. We now specify a class of Banach spaces with this property.

Let $\mathscr{B} = \mathscr{B}((-\infty,0],R^n)$ be a Banach space of functions mapping $(-\infty,0]$ into R^n with norm $\|\cdot\|_{\mathscr{B}}$. For any φ in $\mathscr{B}$ and any β in $[0,\infty)$, let φ^β be the restriction of φ to the interval $(-\infty,-\beta]$. This is a function mapping $(-\infty,-\beta]$ into R^n. Denote the space of such functions by $\mathscr{B}_\beta$ and for any η in $\mathscr{B}_\beta$, define

$$\|\eta\|_{\mathscr{B}_\beta} = \inf_\varphi\{\|\varphi\|_{\mathscr{B}},\ \varphi^\beta = \eta\}.$$

The space $\mathscr{B}_\beta$ is then a Banach space with norm $\|\cdot\|_{\mathscr{B}_\beta}$.

If x is any function defined on $(-\infty,A)$, $A > 0$, then for each t in $[0,A)$ define the function x_t by the relation $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leqq 0$.

Let $\mathscr{F}_A = \mathscr{F}((-\infty,A),R^n)$, $A > 0$, be the class of functions taking $(-\infty,A)$ into R^n such that each x in $\mathscr{F}_A$ is a continuous function on $[0,A)$ and x_o belongs to $\mathscr{B}$. We make the following hypotheses concerning the space $\mathscr{B}$.

h_1) If x is in $\mathscr{F}_\infty$, then x_t is in $\mathscr{B}$ for all t in $[0,\infty)$ and x_t is a continuous function of t.

h_2) All bounded continuous functions mapping $(-\infty,0]$ into R^n are in $\mathscr{B}$.

h_3) For any $\delta \geqq 0$, $\beta \geqq 0$, and φ in $\mathscr{B}$, there is a continuous function $a(\delta,\beta,\varphi)$ nondecreasing in δ, $a(\delta,\beta,\varphi) \to 0$ as $\beta \to \infty$, $a(0,\beta,0) = 0$, such that

$$\|x_t^\beta\|_{\mathscr{B}_\beta} \leqq a(\delta,\beta,\varphi)$$

for all $t \geqq 0$, $\beta \geqq 0$, provided that $x_o = \varphi$ and $|x(t)| \leqq \delta$, $t \geqq 0$.

h_4) There are continuous, nondecreasing functions $b(\gamma)$, $c(\gamma)$, $\gamma \geqq 0$, $b(0) = c(0) = 0$, such that for any φ in $\mathscr{B}$,

$$\|\varphi\|_{\mathscr{B}} \leqq Kb[\sup_{-\beta \leqq \theta \leqq 0} |\varphi(\theta)|] + c(\|\varphi^{\beta}\|_{\mathscr{B}_{\beta}})$$

for any $\beta \geqq 0$.

h_5) There is a nondecreasing positive definite function $d(\gamma)$, $\gamma \geqq 0$, such that

$$d(|\varphi(0)|) \leqq \|\varphi\|_{\mathscr{B}}.$$

Some Banach spaces $\mathscr{B}$ that will satisfy the above properties are those consisting of all functions mapping $(-\infty,0]$ into R^n for which

$$\|\varphi\|^2_{\mathscr{B}} = [\sup_{-\alpha \leqq \theta \leqq 0} |\varphi(\theta)|]^2 + \int_{-\infty}^{0} |\varphi(\theta)|^2 dp(\theta)$$

where $\alpha \geqq 0$ and p is a function of bounded variation. For $\alpha = 0$, these are Hilbert spaces. If this norm is applied to continuous functions defined on $[-r,0]$ and $\alpha = r$, $p = 0$, then this is $C([-r,0],R^n)$.

<u>Lemma 1.</u> For any $\delta \geqq 0$ and any φ in $\mathscr{B}$, there is a continuous function $e(\delta,\varphi)$, $e(0,0) = 0$ such that for any x in $\mathscr{F}_{\infty}$ with $x_o = \varphi$ and $|x(s)| \leqq \delta$, $s \geqq 0$ and any t in $[0,\infty)$,

$$\|x_t\|_{\mathscr{B}} \leqq e(\delta,\varphi)$$

Proof: Hypotheses h_3 and h_4) imply that

$$\|x_t\|_{\mathscr{B}} \leq Kb[\sup_{-t\leq\theta\leq 0}|x(t+\theta)|] + c(\|x_t^t\|_{\mathscr{B}_t})$$
$$\leq Kb(\delta) + c(a(\delta,t,\varphi))$$

Since $a(\delta,t,\varphi) \to 0$ as $t \to \infty$ and is continuous in t, it has a maximum $\mu(\delta,\varphi)$. The desired result is obtained by letting $e(\delta,\varphi) = Kb(\delta) + c(\mu(\delta,\varphi))$.

Suppose $f: \mathscr{B} \to R^n$ is continuous and maps bounded sets into bounded sets. These hypotheses are sufficient to guarantee a local existence and continuation theorem for solutions of (2). For each φ in $\mathscr{B}$, assume that a solution $x(\varphi)$ of (2) exists for $t \geq 0$, is unique and $x(\varphi)(t)$ is jointly continuous in t,φ. Then $u(t,\varphi) = x_t(\varphi)$ is a dynamical system on $\mathscr{B}$.

Lemma 2. Suppose $\mathscr{B}$ is the space satisfying h_1)-h_5) and (2) defines a dynamical system on $\mathscr{B}$. Then every bounded orbit of $\mathscr{B}$ is relatively compact in $\mathscr{B}$.

Proof: Suppose $x = x(\varphi)$ is a bounded orbit in $\mathscr{B}$. Hypothesis h_5) implies there is a constant M such that $|x(t)| \leq M$ for all $t \geq 0$. Since f maps bounded sets into bounded sets, it follows that there is a constant N such that $|\dot{x}(\varphi)(t)| \leq N$, $t \geq 0$. Compactness in $\mathscr{B}$ is equivalent to sequential compactness. Take any sequence $\{x_{t_k}\}$ and the continuity of x_t in t guaranteed by h_1) implies that we may as well assume that $t_k \to \infty$ as $k \to \infty$ monotonically. For any α in $[0,\infty)$ choose $k(\alpha)$ so that $t_{k(\alpha)} - \alpha \geq 0$. Then the sequence $\{x_{t_k}\}$ is such

that $x_{t_k}(\theta) = x(t_k+\theta)$, θ in $[-\alpha,0]$ is continuous and bounded together with its first derivative for all $k \geqq k(\alpha)$. Therefore, this sequence of functions is uniformly bounded and equicontinuous on $[-\alpha,0]$ and one can choose a subsequence which converges uniformly on $[-\alpha,0]$. Choosing $\alpha = 1,2,\ldots$ and using the familiar triangularization procedure, one can get a convergent subsequence that will be uniformly convergent on all compact subsets of $(-\infty,0]$ to a function ψ. The limit function ψ is continuous and bounded and by h_2) belongs to $\mathscr{B}$. For any $\epsilon > 0$, choose β so that $a(M,\beta,\varphi) < \epsilon/4$ where a is the function given in h_3). If b is the function given in h_4), choose $\delta = \delta(\epsilon)$ so that $Kb(\delta) < \epsilon/2$ and choose $n = n(\epsilon)$ so large that

$$\sup_{-\beta\leqq\theta\leqq 0}|x(t_k+\theta) - x(t_j+\theta)| < \delta$$

for $k,j \geqq n(\epsilon)$. From h_3) and h_4) it follows that

$$\begin{aligned}\|x_{t_k}-x_{t_j}\|_{\mathscr{B}} &\leqq Kb(\sup_{-\beta\leqq\theta\leqq 0}|x(t_k+\theta) - x(t_j+\theta)|) \\ &\quad + \|x^{\beta}_{t_k}-x^{\beta}_{t_j}\|_{\mathscr{B}_\beta} \\ &\leqq Kb(\delta) + 2a(M,\beta,\varphi) \\ &< \epsilon\end{aligned}$$

if $k,j \geqq n(\epsilon)$. Therefore, the sequence $\{x_{t_k}\}$ is a Cauchy sequence and the lemma is proved.

It may be more convenient in some applications to show that

a solution $x(\varphi)$ of (2) satisfies $|x(\varphi)(t)| \leq \delta$, $t \geq 0$, for some constant δ. Lemma 1 then implies that the orbit through φ is bounded and one can assert from Lemma 2 that this orbit belongs to a compact set of $\mathscr{B}$.

Example 4. Functional differential equations of neutral type. Consider the special equation

$$\dot{x}(t) = B\dot{x}(t-r) + f(x_t), \quad t \geq 0, \tag{3}$$

where $r \geq 0$ is a constant and B is a constant $n \times n$ matrix.

Equation (3) is a special case of a system considered by Driver [6] in which the initial value φ was assumed to belong to the class AC of absolutely continuous functions with $\|\varphi\|$ defined by

$$\|\varphi\|_{AC} = |\varphi(0)| + \int_{-r}^{0} |\dot{\varphi}(\theta)|\, d\theta.$$

If f is lipschitz continuous in φ and if φ belongs to AC, then Driver has shown that a solution $x(\varphi)$ of (3) with initial value φ at zero exists over some t-interval and the function $u(t,\varphi) = x_t(\varphi)$ is continuous in t,φ over its domain of definition. Therefore, if solutions are assumed to exist for all $t \geq 0$, then $u(t,\varphi)$ defines a dynamical system on AC.

If equation (3) is considered in its integrated form; namely,

$$(4)\quad \begin{aligned} x(t) &= x(0) - B\varphi(-r) + Bx(t-r) + \int_0^t f(x_s)ds, \quad t \geqq 0, \\ x(t) &= \varphi(t), \quad t \text{ in } [-r,0], \end{aligned}$$

then the equation can be considered as a dynamical system on another space. In fact, Hale and Meyer [7] have shown that if f is continuous and locally lipschitzian on C, then a solution $x(\varphi)$ of (4) with initial value φ in C at $t = 0$ exists over some t-interval and $u(t,\varphi) = x_t(\varphi)$ is jointly continuous in t,φ on its domain of definition. Therefore, if solutions are assumed to exist for all $t \geqq 0$, then $u(t,\varphi)$ defines a dynamical system on C.

Equations (3) and (4) have the undesirable property that the solution $x(\varphi)(t)$, $t \geqq 0$, is in general no "smoother" than the initial value φ. Therefore, one cannot expect a bounded orbit necessarily to possess a limit set. In retarded functional differential equations, this smoothing property was precisely what made a bounded orbit have a limit set. Is it possible in some way to obtain a reasonable theory of dynamical systems which will enable one to conclude more about a bounded orbit of (3) or (4) than just the fact that it is bounded? One possible approach is to try to prove that (4) is also a dynamical system on a space $\mathscr{B}$ which has the property that it can be imbedded in C (or AC) and such that the unit ball in $\mathscr{B}$ embedded in C (or AC) is relatively compact. Any bounded orbit in $\mathscr{B}$ will then necessarily have a limit set in C (or AC) and one should be able to use this fact to great advantage. This is the basic idea used so often in the modern theory of partial differential equations.

Let $L_2[-r,0]$ designate the square integrable functions on $[-r,0]$. We now show that (3) and thus (4) is a dynamical system on the Sobolev space W_2^1 consisting of all functions on $[-r,0]$ which together with their generalized first derivatives are in $L_2[-r,0]$. A norm which is equivalent to the usual one on this space is

$$\|\varphi\|^2_{W_2^1} = |\varphi(0)|^2 + \int_{-r}^{0}|\dot{\varphi}(\theta)|^2 d\theta.$$

Each element of W_2^1 is continuous and the unit ball in W_2^1 belongs to a compact subset of C.

A function $x = x(\varphi)$ defined on an interval $[-r,A]$, $A > 0$, is said to be a <u>generalized solution of (3)</u> for an initial function φ in W_2^1 if $x(t) = \varphi(t)$, $-r \leqq t \leqq 0$, and

$$\int_0^A \dot{u}(t)[x(t)-Bx(t-r) - \int_0^t f(x_s)ds]dt = 0 \tag{5}$$

for all continuously differentiable functions u which have compact support in $[0,A]$.

Suppose $f: C \to C$ is continuous and locally lipschitzian. Since φ in W_2^1 implies φ in C we know from [7] that there is an $A > 0$ which we suppose $< r$ such that a solution $x(\varphi)$ of (4) exists on $[-r,A]$, is unique and $u(t,\varphi) = x_t(\varphi)$ as a function in C is continuous in t,φ for t in $[0,T]$ and φ in some open set in W_2^1, $u(0,\varphi) = \varphi$, $u(t+\tau,\varphi) = u(t,u(\tau,\varphi))$. We wish to show that $u(t,\varphi)$ actually belongs to W_2^1 and considered as an element of this space W_2^1 is continuous

in t,φ. Since $x = x(\varphi)$ satisfies (4) it obviously satisfies (5). Since $A < r$, it follows that

$$0 = \int_0^A \dot{u}(t)[x(t) - B\varphi(t-r) - \int_0^t f(x_s)ds]dt$$
$$= \int_0^A \dot{u}(t)x(t)dt + \int_0^A u(t)[B\dot{\varphi}(t-r) + f(x_t)]dt$$

and thus $x(t)$ has a generalized first derivative on $[0,A]$ given by

$$\dot{x}(t) = B\dot{\varphi}(t-r) + f(x_t), \quad 0 \leqq t \leqq A. \tag{6}$$

For $-r \leqq t \leqq 0$, we also know that $x(t)$ has a generalized derivative given by $\dot{\varphi}(t)$. The function $B\dot{\varphi}(t-r) + f(x_t)$ is obviously square integrable and thus $\dot{x}(t)$ is square integrable. This proves that $u(t,\varphi) = x_t(\varphi)$ belongs to W_2^1 for $0 \leqq t \leqq A$.

It remains only to show that the function $\dot{x}(\varphi)(t+\theta) = du(t,\varphi)(\theta)/d\theta$, $-r \leqq \theta \leqq 0$, as an element of $L_2[-r,0]$ is continuous in t,φ. From (6), we have

$$\begin{aligned} \dot{x}(\varphi)(t+\theta) - \dot{x}(\psi)(t+\theta) &= B[\dot{\varphi}(t+\theta-r) - \dot{\psi}(t+\theta-r)] + f(x_{t+\theta}(\varphi)) - f(x_{t+\theta}(\psi)) \\ &\qquad t+\theta \geqq 0 \\ \dot{x}(\varphi)(t+\theta) - \dot{x}(\psi)(t+\theta) &= \dot{\varphi}(t+\theta) - \dot{\psi}(t+\theta), \quad t+\theta \leqq 0, \\ &\qquad -r \leqq \theta \leqq 0. \end{aligned} \tag{7}$$

Since f is locally lipschitzian, there is a constant L such that $|f(x_t(\varphi))-f(x_t(\psi))| \leqq L|x_t(\varphi)-x_t(\psi)|_C$, $0 \leqq t \leqq A$ and thus relation (7) obviously implies there is a constant K such that

$$\|\dot{x}_t(\varphi)-\dot{x}_t(\psi)\|_{L_2} \leqq K[\|\dot{\varphi}-\dot{\psi}\|_{L_2} + \|x_t(\varphi)-x_t(\psi)\|_C].$$

In [7], it has been shown there is an $L_1 > 0$ such that $\|x_t(\varphi)-x_t(\psi)\|_C \leqq L_1\|\varphi-\psi\|_C$, $0 \leqq t \leqq T$, and, therefore, $\dot{x}_t(\varphi)$ is a continuous function of φ uniformly with respect to t in $[0,A]$. To show the continuity with respect to t observe that it is sufficient to show this for $t = 0$ since $u(t+\tau,\varphi) = u(t,u(\tau,\varphi))$ and $u(t,\varphi)$ is continuous in φ. Let $z(t+\theta) = \dot{x}(t+\theta) - \dot{x}(\theta)$, $-r \leqq \theta \leqq 0$. From (6),

$$\dot{z}(t) = B[\dot{\varphi}(t-r) - \dot{\varphi}(-r)] + f(x_t) - f(\varphi), \quad t \geqq 0,$$
$$\dot{z}(t+\theta) = \dot{\varphi}(t+\theta) - \dot{\varphi}(\theta), \; t + \theta \leqq 0, \quad -r \leqq \theta \leqq 0.$$

It is clear that $\|\dot{z}_t\|_{L_2}$ approaches zero as $t \to 0$ and this proves continuity with respect to t. Since the continuity in φ is uniform with respect to t, it follows that $\dot{x}_t(\varphi)$ is jointly continuous in t,φ and, thus, $u(t,\varphi)$ is jointly continuous in t,φ.

The above remarks show that $u(t,\varphi)$ is a dynamical system on W_2^1 if we assume global existence of the solutions.

One can obviously generalize this example to the system

$$\dot{x}(t) = \sum_{k=1}^{n} B_k\dot{x}(t-\tau_k) + f(x_t)$$

where $0 < \tau_k \leqq r$.

Example 5. Quasilinear hyperbolic equations. Consider the equation

$$
\begin{aligned}
&v_{tt} - v_{xx} = f(v, v_t, v_x), \\
&v(0,x) = \varphi(x), \quad v_t(0,x) = \psi(x), \quad 0 \leq x \leq 1, \\
&v(t,0) = 0, \quad v_t(t,1) = 0, \quad t \geq 0,
\end{aligned} \tag{8}
$$

where $f(v_1, v_2, v_3)$ is an analytic function of v_1, v_2, v_3 in the whole space. Let W_{20}^k be the space of functions mapping $[0,1]$ into R which have generalized derivatives of order $\leq k$ in $L_2[0,1]$ and these derivatives vanish at $x = 0$ and $x = 1$. If φ is in W_{20}^k, the norm $\|\varphi\|_{W_{20}^k}$ is defined by

$$\|\varphi\|^2_{W_{20}^k} = \int_0^1 [\varphi^2 + (\varphi')^2 + \ldots + (\varphi^{(k)})^2] dx$$

where $\varphi^{(j)}$ is the jth derivative of φ.

For any φ in W_{20}^k and any ψ in W_{20}^{k-1}, $k \geq 1$, it follows from the work of Sobolev [8] that (8) has a unique generalized solution $v(t,x,\varphi,\psi)$ existing on an interval $0 \leq t < \eta$ and the pair $[v(t,\cdot,\varphi,\psi), v_t(t,\cdot,\varphi,\psi)]$ belongs to $W_{20}^k \times W_{20}^{k-1}$ and is a continuous function of t, φ, ψ. Therefore, if we assume that solutions exist for all $t \geq 0$, then the function $u(t,\Phi) = [v(t,\cdot,\Phi), v_t(t,\cdot,\Phi)]$, $\Phi = (\varphi,\psi)$, is a dynamical system on the Banach space $\mathscr{B}^k = W_{20}^k \times W_{20}^{k-1}$ for any $k \geq 1$.

The famous Sobolev embedding theorem asserts that the unit ball in W_{20}^k belongs to a compact subset of W_{20}^ℓ for $k > \ell$. Therefore, any orbit $\gamma^+(\Phi)$ of (8) which is bounded when considered as a subset of

$\mathscr{B}_k$ will belong to a compact subset of $\mathscr{B}_\ell$ if $k > \ell$ and, therefore, it is meaningful to speak of the limit of this orbit in $\mathscr{B}_\ell$.

3. Limit sets, Lyapunov functions and stability.

Definition 2. Let u be a dynamical system on $\mathscr{B}$. For any φ in $\mathscr{B}$, the ω-limit set $\omega(\varphi)$ of the orbit through φ is the set of ψ in $\mathscr{B}$ such that there is a nondecreasing sequence $\{t_n\}$, $t_n > 0$, $t_n \to \infty$ as $n \to \infty$ such that $\|u(t_n,\varphi)-\psi\|_{\mathscr{B}} \to 0$ as $n \to \infty$. This definition is equivalent to

$$\omega(\varphi) = \cap_{\tau \geq 0} \mathrm{Cl}(\cup_{t \geq \tau} u(t,\varphi)).$$

Definition 3. Let u be a dynamical system on $\mathscr{B}$. A set M in $\mathscr{B}$ is an invariant set of the dynamical system if for each φ in M there is a function $U(t,\varphi)$ defined and in M for t in $(-\infty,\infty)$ such that $U(0,\varphi) = \varphi$ and for any σ in $(-\infty,\infty)$,

$$u(t,U(\sigma,\varphi)) = U(t+\sigma,\varphi)$$

for all t in R^+.

It should be noted that sets are invariant according to the above definition relative to the interval $(-\infty,\infty)$ and not relative to $[0,\infty)$. We now prove the simple but very basic

Lemma 3. Let u be a dynamical system on $\mathscr{B}$ and suppose the orbit $\gamma^+(\varphi)$ through φ belongs to a compact subset of $\mathscr{B}$. Then the ω-limit set $\omega(\varphi)$ of $\gamma^+(\varphi)$ is a nonempty, compact, connected invariant set.

Proof: Since $\gamma^+(\varphi)$ belongs to a compact subset of $\mathscr{B}$, it is clear that $\omega(\varphi)$ is nonempty and compact since it belongs to a compact subset and is closed. Suppose ψ is in $\mathscr{B}$ and the nondecreasing, unbounded, nonnegative sequence $\{t_n\}$ satisfies $\|u(t_n,\varphi)-\psi\| \to 0$ as $n \to \infty$ (the subscript on the norm is dropped in this proof). For a given τ in $[0,\infty)$, there is an $n_o(\tau)$ such that $t_n-\tau \geqq 0$ for $n \geqq n_o(\tau)$ and it is therefore meaningful to consider the sequence $u_n(t,\varphi) \overset{\text{def}}{=} u(t+t_n,\varphi)$, $n \geqq n_o(\tau)$, t in $[-\tau,\tau]$. By hypothesis, there is an M such that $\|u(t,\varphi)\| \leqq M$ for all t in R^+. Therefore, the sequence $u_n(t,\varphi)$, $n \geqq n_o(\tau)$, t in

$[-\tau,\tau]$, is uniformly bounded. Since $u_n(t+s,\varphi) = u(s,u_n(t,\varphi))$ and $\gamma^+(\varphi)$ is assumed to belong to a compact set, it follows that for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|u_n(t+s,\varphi) - u_n(t,\varphi)\| < \epsilon$$

for $0 \leqq s \leqq \delta$, $n \geqq n_o(\tau)$, t in $[-\tau,\tau]$; that is, the sequence $u_n(t,\varphi)$ is equicontinuous. Since this sequence by hypothesis belongs to a compact subset of $\mathscr{B}$, the Ascoli-Arzela theorem implies the existence of a subsequence which we again label as t_n such that it converges uniformly on $[-\tau,\tau]$; that is, there exists a function $U(t,\psi)$, $-\tau \leqq t \leqq \tau$, continuous in t and such that $\lim_{n\to\infty}\|u_n(t,\varphi)-U(t,\varphi)\| = 0$ uniformly on $[-\tau,\tau]$. Obviously, $U(0,\psi) = \psi$. Letting now $\tau = 1,2,\ldots$ successively and using the familiar triangularization procedure, we determine a subsequence which is again labeled by t_n and a function $U(t,\psi)$ defined and continuous on $-\infty < t < \infty$ such that $\lim_{n\to\infty}\|u_n(t,\varphi) - U(t,\psi)\| = 0$ uniformly on compact subsets of $(-\infty,\infty)$. It is clear that $U(t,\psi)$ is in $\omega(\varphi)$.

Let σ be an arbitrary real number in $(-\infty,\infty)$. For this σ and any $t \geqq 0$, we have

$$\begin{aligned}\|u(t,U(\sigma,\varphi)) &- U(t+\sigma,\psi)\| \\ &\leqq \|u(t,U(\sigma,\psi)) - u(t,u_n(\sigma,\varphi))\| \\ &\quad + \|u_n(t+\sigma,\varphi) - U(t,\psi)\|.\end{aligned}$$

Since the right hand side of this expression approaches zero as $n \to \infty$,

it follows that $u(t,U(\sigma,\psi)) = U(t+\sigma,\psi)$ or $\omega(\varphi)$ is invariant. It is clear that $\omega(\varphi)$ is connected. The fact that $\mathrm{dist}(u(t,\varphi),\omega(\varphi)) \to 0$ as $t \to \infty$ is obvious and the lemma is proved.

If u is a dynamical system on $\mathscr{B}$ and V is a continuous scalar function defined on $\mathscr{B}$, define the function $\dot{V}(\varphi) = \dot{V}_{\mathscr{B}}(\varphi)$ by

$$\dot{V}(\varphi) = \overline{\lim}_{t\to 0^+} \frac{1}{t}\,[V(u(t,\varphi))-V(\varphi)].$$

Following LaSalle [10], we say a function $V: \mathscr{B} \to R$ is a <u>Lyapunov function on a set G in $\mathscr{B}$</u> if V is continuous on $\overline{G}$, the closure of G,

and $\dot{V}(\varphi) \leq 0$ for φ in G. Let S be the set defined by

$$S = \{\varphi \text{ in } \overline{G}: \dot{V}(\varphi) = 0\}$$

and let M be the largest invariant set in S of the dynamical system.

<u>Theorem 1.</u> Suppose u is a dynamical system on $\mathcal{B}$. If V is a Lyapunov function on G and an orbit $\gamma^+(\varphi)$ belongs to G and is in a compact set of $\mathcal{B}$, then $u(t,\varphi) \to M$ as $t \to \infty$.

In the applications of Theorem 1, one can be assured that an orbit $\gamma^+(\varphi)$ remains in G if φ belongs to G provided that the conditions of the theorem are satisfied for G a component of the set $U_\rho = \{\varphi \text{ in } \mathcal{B} : V(\varphi) < \rho\}$.

<u>Theorem 2.</u> Suppose u is a dynamical system on $\mathcal{B}$. Let $u(t,0) = 0$ for all t (i.e. $\varphi = 0$ is an equilibrium point) and suppose zero belongs to the closure of some open set U and N is an open neighborhood of zero. Assume that

i) V is a Lyapunov function on $G = N \cap U$,

ii) $M \cap G$ is either the empty set or zero,

iii) $V(\varphi) < \eta$ on G when $\varphi \neq 0$,

iv) $V(0) = \eta$ and $V(\varphi) = \eta$ when φ is in that part of the boundary of G in N.

If N_o is a bounded neighborhood of zero properly contained in N, then $\varphi \neq 0$ in $G \cap N_o$ implies either there exists a $\tau > 0$ such that $u(\tau,\varphi)$ belongs to the boundary of N_o or $u(t,\varphi)$ remains in $G \cap N_o$ but does not belong to a compact set of $G \cap N_o$.

The proofs of Theorems 1 and 2 are not given since they are essentially the same as the ones in [10] for ordinary differential equations.

4. Limit dynamical systems. In the previous section, we stated a result for determining the limiting behavior of an orbit of a dynamical system provided the orbit remains in a compact subset of the space. The problem remaining is to give a procedure for determining when such a situation prevails. In Section 2, we have given illustrations of dynamical systems (examples 1-3) such that any bounded orbit necessarily belongs to a compact subset of the space. It is in general much easier to show that an orbit is bounded. In fact, this is usually the immediate consequence of the existence of a Lyapunov function. In examples 4 and 5 of Section 2, there is no inherent smoothing effect in the dynamical system and thus bounded orbits may not lie in compact sets. On the other hand, the equations in examples 4 and 5 define dynamical systems on different Banach spaces $\mathscr{B}, \mathscr{L}$ such that, when $\mathscr{B}$ is considered as imbedded in $\mathscr{L}$, the unit ball in $\mathscr{B}$ belongs to a compact subset of $\mathscr{L}$. Therefore, in all of the examples it is possible to assert that a bounded orbit does have a nonempty limit set if the convergence is interpreted in the appropriate space. These ideas will now be formalized and exploited in more detail.

Let $\mathscr{B}$ and $\mathscr{C}$ be Banach spaces. If there is a continuous linear injection $i: \mathscr{B} \to \mathscr{C}$ we shall say $\mathscr{B} \subset \mathscr{C}$. If $\mathscr{B} \subset \mathscr{C}$, there is thus a constant $K > 0$ such that $\|i(\varphi)\|_{\mathscr{C}} \leq K\|\varphi\|_{\mathscr{B}}$ for all φ in $\mathscr{B}$. When it is clear from the context we shall think of $\mathscr{B}$ as contained in $\mathscr{C}$, consider φ in $\mathscr{C}$ as well as in $\mathscr{B}$ and therefore omit the connotation $i(\varphi)$.

<u>Definition 4.</u> Suppose $\mathscr{B} \subset \mathscr{C}$ and u is a dynamical system on $\mathscr{B}$ <u>and</u> $\mathscr{C}$. Let $\mathscr{B}^* \subset \mathscr{C}$ be the set consisting of the union of $\mathscr{B}$ and any φ in $\mathscr{C}$ for which there is a ψ in $\mathscr{B}$ such that φ belongs to $\omega_{\mathscr{C}}(\psi)$, the ω-limit set in $\mathscr{C}$ of the orbit $\gamma^+(\psi)$ in $\mathscr{C}$; that is,

$$\omega_{\mathscr{C}}(\psi) = \cap_{\tau \geq 0} \mathrm{Cl}_{\mathscr{C}} \cup_{t \geq \tau} u(t, \psi).$$

Then $u: R^+ \times \mathscr{B}^* \to \mathscr{B}^*$ is a dynamical system and we refer to this dynamical system as the <u>limit dynamical system</u> of $\mathscr{B}$ in $\mathscr{C}$.

Roughly speaking, the limit dynamical system of $\mathscr{B}$ in $\mathscr{C}$ is an extension of the dynamical system on $\mathscr{B}$ to a larger set $\mathscr{B}^*$ in $\mathscr{C}$ where $\mathscr{B}^*$ is obtained by taking orbits in $\mathscr{B}$, considering them as embedded in $\mathscr{C}$ and adding their limit points in $\mathscr{C}$. The limit sets of a dynamical system divide the space into equivalence classes in which two points belong to the same class if their limit sets have common points. By taking the limit of an orbit even in a larger space, one can still obtain these equivalence classes.

If $\mathscr{B}$ is a Hilbert space, then the Banach-Saks theorem [9] implies that $\mathscr{B}^* = \mathscr{B}$. In spite of this fact, there is an advantage to looking at the dynamical system u in the above manner.

Lemma 4. Suppose $\mathscr{B} \subset \mathscr{L}$ and u is a dynamical system on $\mathscr{B}$ and $\mathscr{L}$. If φ in $\mathscr{B}$ is such that $\gamma^+(\varphi)$ belongs to a bounded set of $\mathscr{B}$ and a compact set of $\mathscr{L}$, then the ω-limit set $\omega(\varphi)$ of the orbit through φ is a nonempty, compact, connected set in $\mathscr{B}^*$, an invariant set of the limit dynamical system and $\operatorname{dist}_{\mathscr{L}}(u(t,\varphi),\omega(\varphi)) \to 0$ as $t \to \infty$.

This lemma does not require proof since, using Definition 4, it is a restatement of Lemma 3 with $\mathscr{B}$ replaced by $\mathscr{B}^*$.

The following result is very useful for the applications.

Theorem 3. Suppose $\mathscr{B} \subset \mathscr{L}$, u is a dynamical system on $\mathscr{B}$ and $\mathscr{L}$ and each bounded orbit of $\mathscr{B}$ belongs to a compact set of $\mathscr{L}$. Also suppose the function $V_{\mathscr{B}}$ is a Lyapunov function on $G_{\mathscr{B}} = \{\varphi \text{ in } \mathscr{B}: V_{\mathscr{B}}(\varphi) < \eta\}$, $V_{\mathscr{L}}$ is a Lyapunov function on $G_{\mathscr{L}} = \{\varphi \text{ in } \mathscr{L}: V_{\mathscr{L}}(\varphi) < \eta\}$, $G_{\mathscr{B}} \subset G_{\mathscr{L}}$,

$$R = \{\varphi \text{ in } \overline{G}_{\mathscr{L}}: \dot{V}_{\mathscr{L}}(\varphi) = 0\}$$

and N is the largest invariant set in R of the limit dynamical system. If $G_{\mathscr{B}}$ is bounded and φ is in $G_{\mathscr{B}}$, then $u(t,\varphi) \to N$ in $\mathscr{L}$ as $t \to \infty$.

Proof: Since $V_{\mathscr{B}}$ is a Lypunov function on $G_{\mathscr{B}}$ it follows that $u(t,\varphi)$ remains in $G_{\mathscr{B}}$ for all $t \geqq 0$. The hypotheses imply that it belongs to a compact set of $G_{\mathscr{C}}$. Theorem 1 completes the proof.

At first glance, the hypotheses in Theorem 3 may look artificial but in some respects are very natural. In fact, to show that a differential equation defines a dynamical system, one often proceeds as follows. From the general theory, one obtains local (in t) existence, uniqueness and continuous dependence on the initial data. To obtain global existence, one constructs a Lyapunov function and invokes the continuation theorem to obtain a dynamical system of a subset of the space. Therefore, the Lyapunov functions have been constructed in the process of showing the existence of a dynamical system.

5. Examples.

5.1. A functional differential equation with infinte lag.

In this section, we consider an equation which generalizes in some respects an equation considered by Levin and Nohel [11] for a finite lag. Suppose $a: R^+ \to R$ is a continuous function with $\dot{a}, \ddot{a}$ continuous and

$$
\begin{gathered}
a(t) > 0,\ \dot{a}(t) < 0,\ \ddot{a}(t) \geqq 0,\ t \geqq 0 \\
t^2\dot{a}(t) \to 0 \quad \text{as} \quad t \to \infty. \\
\int^{\infty} t^2 \dot{a}(t)\,dt < \infty.
\end{gathered} \tag{9}
$$

For any integrable function $k(\theta) > 0$, $-\infty < \theta \leqq 0$, $\int_{-\infty}^{0} k < \infty$, let $\mathscr{B}$ be the Banach space consisting of all functions $\varphi: (-\infty, 0] \to R^n$ for which

$$
\|\varphi\|^2 = |\varphi(0)| + \int_{-\infty}^{0} k(\theta)|\varphi(\theta)|\,d\theta
$$

is finite. Suppose $g: R^n \to R^n$ is a continuously differentiable function such that

$$(10) \qquad G(x) = \int_0^x g(s)ds \to \infty \quad \text{as} \quad |x| \to \infty.$$

For any φ in $\mathcal{B}$ for which it is meaningful, consider the function

$$(11) \qquad f(\varphi) = -\int_{-\infty}^{0} a(-\theta)g(\varphi(\theta))d\theta$$

and the functional differential equation

$$(12) \qquad \dot{x}(t) = f(x_t) = -\int_{-\infty}^{t} a(t-u)g(x(u))du.$$

The domain of f in general is not the whole space $\mathcal{B}$ but the hypotheses on a certainly imply that all bounded functions φ belong to the domain of f. If $g(x) = x$ and $k(\theta) \geq a(-\theta)$, then the domain of f is $\mathcal{B}$.

Let $\mathcal{B}_b$ be the subset of $\mathcal{B}$ consisting of all functions in $\mathcal{B}$ such that $\sup_\theta|\varphi(\theta)| < b$. One can show that the basic local existence, uniqueness and continuous dependence theorem for (12) holds in $\mathcal{B}_b$. More specifically there is an $A = A(b) > 0$ and an $r = r(b) \geq b$ such that the function $x_t(\varphi)$ defined by the solution of (12) through φ in $\mathcal{B}_b$ belongs to $\mathcal{B}_r$ for t in $[0,A]$ and is continuous in t,φ. Furthermore, if x is a solution of (12), then a few simple calculations show that

$$(13) \qquad \ddot{x}(t) + a(0)g(x(t)) = \int_{-\infty}^{0} \ddot{a}(-\theta)\left(\int_{\theta}^{0} g(x(t+u))du\right)d\theta$$

<u>Lemma 5</u>. Suppose $g(x)$ has a finite number of zeros. Then every solution of (12) with φ in $\mathcal{B}_b$ approaches a zero of g; that is, an equilibrium point of (12).

<u>Proof:</u> For any b and any φ in $\mathcal{B}_b$, let $V(\varphi)$ be defined by

$$(14) \qquad V(\varphi) = G(\varphi(0)) - \frac{1}{2}\int_{-\infty}^{0} \dot{a}(-\theta)\left[\int_{\theta}^{0} g(\varphi(s))ds\right]^2 d\theta.$$

The hypothesis (9) on a implies V is defined on $\mathcal{B}_b$ and is continuous. A few simple calculations yield $\dot{V}(\varphi)$ along the solutions of (12) as

$$(15) \qquad \dot{V}(\varphi) = -\frac{1}{2}\int_{-\infty}^{0} \ddot{a}(-\theta)\left[\int_{\theta}^{0} g(\varphi(s))ds\right]^2 d\theta \leq 0$$

and this implies $V(x_t(\varphi)) \leq V(\varphi)$, $t \geq 0$. In particular, there is a constant $M = M(b)$ such that for any φ in $\mathcal{B}_b$, $|x(\varphi)(t)| \leq M$, $t \geq 0$, φ in $\mathcal{B}_b$. Since our norm in $\mathcal{B}$ satisfies hypothesis $h_1)$-$h_5)$ of Example 3 of Section 2, this implies there is an $N = N(b)$ such that $\|x_t(\varphi)\|_{\mathcal{B}} \leq N(b)$, $t \geq 0$, φ in $\mathcal{B}_b$. Since the orbit through φ in $\mathcal{B}_b$ is bounded, Lemma 2 implies the orbit is relatively compact. One can now apply Theorem 1 directly to this system taking the two Banach spaces in that theorem to be the same, namely the closure of the subset in our Banach space $\mathcal{B}$ consisting of all orbits of (12) which have initial value in $\mathcal{B}_b$ for a given b. This theorem implies from (15) and (13) that the ω-limit set of any

solution of (12) with φ in $\mathcal{B}_b$ exists and is the union of orbits of

$$\ddot{y} + a(0)g(y) = 0 \tag{16}$$

which satisfy

$$\int_{-s}^{0} g(y(t+\theta))d\theta = 0, \ -\infty < t < \infty, \text{ if } \ddot{a}(s) > 0. \tag{17}$$

Since $a(t) > 0$, $\dot{a}(t) < 0$, $t \geqq 0$, there is an s_o in $(0,\infty)$ such that $\ddot{a}(s_o) > 0$. Also, continuity of $\ddot{a}$ implies there is an $\epsilon > 0$ such that $\ddot{a}(s) > 0$ for s in $[s_o-\epsilon, s_o+\epsilon]$. Since (17) must be satisfied, this implies the ω-limit set of any solution of (12) with φ in $\mathcal{B}_b$ must be generated by a solution of (16) satisfying $\dot{y}(t) = \dot{y}(t-s)$, $-\infty < t < \infty$, s in $[s_o-\epsilon, s_o+\epsilon]$. This implies $\dot{y}$ = constant. But this clearly implies $g(y) = 0$. Therefore, the lemma is proved.

<u>5.2. A stability theorem for neutral equations.</u> Consider the equation

$$\dot{x}(t) = \sum_{k=1}^{N} B_k \dot{x}(t-\tau_k) + f(x_t), \ f(0) = 0 \tag{18}$$

where $0 < \tau_1 < \tau_2 < \ldots < \tau_N \leqq r$ are rational and f is continuous and locally lipschitzian on C. We will need to consider the equation

$$\det[I - \sum_{k=1}^{N} B_k \rho^{-\tau_k}] = 0 \tag{19}$$

<u>Theorem 4</u>. If $a: R \to R$, $b: R \to R$, $c: R^n \to R$ are positive definite scalar functions and there is a scalar function $V: W_2^1 \to R$ such that

$$a(\|\varphi\|_{W_2^1}) \leqq V(\varphi) \leqq b(\|\varphi\|_{W_2^1}) \tag{20}$$

and $\dot{V}_{(18)}(\varphi) \leqq -c(\varphi(0))$ for all φ in $U_\eta = \{\varphi: V(\varphi) < \eta\}$, then the solution $x = 0$ (18) is stable. If, in addition, no root of the equation (19) has modulus equal to one, then every solution x of (18) with initial value φ in U_η and $\text{ess sup}|\dot{\varphi}(\theta)|$ bounded satisfies

$$\text{ess sup}_{\tau \geqq \tau}[|x(t)| + |\dot{x}(\tau)|] \to 0 \quad \text{as} \quad t \to \infty.$$

<u>Proof</u>: If φ is in U_η, $\dot{V}_{(18)} \leqq 0$ implies $V(x_t(\varphi)) \leqq V(\varphi) < \eta$ for all $t \geqq 0$ and (20) implies $\|x_t(\varphi)\|_{W_2^1} \leqq a^{-1}(V(\varphi))$, $t \geqq 0$. Therefore, the solution $x = 0$ of (18) is stable. If no root of equation (19) has modulus equal to one, there is an essentially bounded solution w^* on $[-r,\infty)$ of the equation

$$w(t) = \sum_{k=1}^{N} B_k w(t-\tau_k) + g(t) \tag{21}$$

if $g(t)$ is bounded on $[0,\infty)$ and the initial function ψ for w is essentially bounded. Furthermore, any other solution approaches w^* exponentially as $t \to \infty$ or becomes exponentially unbounded as $t \to \infty$. In addition, $\text{ess sup}_{\tau \geqq t}|w(\tau)| \to 0$ as $t \to \infty$ if $g(t) \to 0$ as $t \to \infty$.

Let $x(\varphi)$ be a solution of (18) with initial value φ in U_η and ess $\sup|\dot{\varphi}(\theta)|$ bounded. Then the first part of the theorem implies $f(x_t(\varphi))$ and $\int_{t-r}^{t}\dot{x}^2(\varphi)(s)ds$ are bounded on $[0,\infty)$. Since $\dot{x}(\varphi)(t)$ must correspond to a solution w of (21) for $g(t) = f(x_t(\varphi))$, it follows that $\dot{x}(\varphi)(t)$ is essentially bounded if $\dot{\varphi}(\theta)$ is essentially bounded.

Since $\dot{V}(\varphi) \leqq -c(|\varphi(0)|) \leqq 0$ and $V(x_t(\varphi))$ is bounded below, $V(x_t(\varphi)) \to$ a constant as $t \to \infty$ and, thus, $\int_0^\infty c(x(\varphi)(t))dt$ exists. Suppose $x(\varphi)(t)$ does not approach zero in C as $t \to \infty$ and let $p \neq 0$ be any n-vector such that there is a sequence $t_n \to \infty$ as $n \to \infty$ with $x(\varphi)(t_n) \to p$ as $n \to \infty$. Such a p exists since $|x(\varphi)(t)|$ is bounded for $t \geqq -r$. There is an $\epsilon > 0$ such that $c(y) > \delta$ for y in $S_\epsilon(p) = \{y: |y-p| < \epsilon\}$. If $x(\varphi)(t)$ remains in $S_\epsilon(p)$ for all $t \geqq t_1 \geqq 0$, then $\int_0^\infty c(x(\varphi)(t))dt = +\infty$ which is a contradiction. The other possibility is that $x(\varphi)(t)$ leaves $S_\epsilon(p)$ an infinite number of times. Since $|\dot{x}(\varphi)(t)|$ is essentially bounded, each time $x(\varphi)(t)$ returns to N_ϵ it must remain a positive time τ. Again, this implies $\int_0^\infty c(x(\varphi)(t))dt = \infty$ and a contradiction. Therefore $\|x_t(\varphi)\| \to 0$ as $t \to \infty$ and, consequently; $f(x_t(\varphi)) \to 0$ as $t \to \infty$. From the previous remarks, one finally obtains ess $\sup_{\tau \geqq t}|\dot{x}(\varphi)(\tau)| \to 0$ as $t \to \infty$ to complete the proof of the theorem.

As a particular application, consider the equation

$$\dot{x}(t) + ax(t) + b\dot{x}(t-r) = 0 \tag{22}$$

where $a > 0$, $b^2 < 1$. The condition $b^2 < 1$ implies the hypothesis of the theorem on the corresponding equation (19) is satisfied. If

$$V(\varphi) = \varphi^2(0) + \frac{1}{a}\int_{-r}^{0}\dot{\varphi}^2(\theta)d\theta$$

then $V^{1/2}(\varphi)$ can be used as a norm in W_2^1 and

$$\dot{V}(\varphi) = -a\varphi^2(0) - \frac{1-b^2}{a}\dot{\varphi}^2(-r) \leqq -a\varphi^2(0).$$

The conditions of the theorem are satisfied and one can thus assert that any solution $x(\varphi)$ of (22) with ess $\sup_\theta|\dot{\varphi}(\theta)|$ bounded satisfies

$$\sup_\theta|x(\varphi)(t)| + \text{ess } \sup_\theta|\dot{x}(\varphi)(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Theorem 4 above does not use the concept of limit dynamical system although imbedding in another space is used in the proof. One could easily state a result based upon Theorem 3 if we had proved in Section 2 that the neutral equation (18) defines a dynamical system on W_2^2 for $f(\varphi)$ smooth enough. Rather than dwell on this point at length, we only look at equation (22) again letting $\mathscr{B} = W_2^2$, $\mathscr{C} = W_2^1$, $V_{\mathscr{C}}(\varphi)$ be the function used before and $V_{\mathscr{B}} = V_{\mathscr{C}}(\varphi) + V_{\mathscr{C}}(\dot{\varphi})$. One easily sees that

$$\dot{V}_{\mathscr{B}} \leqq -a[\varphi^2(0) + \dot{\varphi}^2(0)]$$

and, thus, the conditions of Theorem 3 are satisfied for any η. Consequently any solution of (22) with initial value in W_2^2 is bounded in W_2^2 and approaches zero in W_2^1 as $t \to \infty$.

5.3. A special hyperbolic partial differential equation. Consider the equation

$$
\begin{aligned}
& u_{xx} = u_{tt} - \epsilon(1-u^2)u_t, \quad \epsilon > 0 \\
(23) \qquad & u(0,t) = u(1,t) = 0 \\
& u(x,0) = \varphi(x), \; u_t(x,0) = \psi(x), \quad \varphi(0) = \varphi(1) = 0.
\end{aligned}
$$

The state variables for this equation are $(u(0,t), u_t(0,t))$ with $u(0,t) = u(1,t) = 0$. Consider the space $\mathscr{E}$ of all functions (φ,ψ), $\varphi(0) = \varphi(1) = 0$, $\psi(0) = \psi(1) = 0$, $\varphi \in W_2^1$, $\psi \in L^2$. The $\|\cdot\|_{\mathscr{E}}$ is given by $\|(\varphi,\psi)\|_{\mathscr{E}}^2 = \int_0^1(\varphi_x^2+\varphi^2+\psi^2)dx$. Since $\int_0^1\varphi_x^2 dx \geq \varphi^2$ if $\varphi(0) = \varphi(1) = 0$ it follows that an equivalent norm can be defined from the function

$$V_1(\varphi,\psi) = \int_0^1(\varphi_x^2+\psi^2)dx$$

The function $V_1(u,u_t)$ along the solutions of the equation is simply the energy of the free system $(\epsilon = 0)$. A few computations gives

$$\dot{V}_1(u,u_t) = -2\epsilon \int_0^1 u_t^2(1-u^2)dx$$

$$\dot{V}_1(\varphi,\psi) = -2\epsilon \int_0^1 \psi^2(1-\varphi^2)dx$$

If $u^2 \leqq 1$, then $\dot{V}_1(u,u_t) \leqq 0$ which implies $V_1(u,u_t)$ is bounded. If $V_1(\varphi,\psi) < 1$ then $\dot{V}_1(u,u_t) \leqq 0$ implies $V_1(u,u_t) \leqq 1-\delta$, $\delta > 0$, for all t and $\int_0^1 \varphi_x^2 dx > \varphi^2(x)$, $0 \leqq x \leqq 1$ implies $u^2(\cdot,t) \leqq 1-\delta$ for all t. Therefore, if we assume the initial values satisfy $V_1(\varphi,\psi) < 1$, then the solution of (23) always stays in this set and $\dot{V}_1 \leqq 0$. This gives us a dynamical system on this set and the solution $(0,0)$ is stable relative to the norm in $\mathscr{E}$. One certainly suspects that these observations imply that solutions should also approach zero as $t \to \infty$. On the other hand, it is not obvious just from the fact that the energy is bounded.

We proceed now to show more ; namely, we construct a dynamical system on a smaller space $\mathscr{B}$ and apply the preceding theory. Consider the space $\mathscr{B}$ of functions φ,ψ with $\varphi(0) = \varphi(1) = 0$, $\varphi \in W_2^2$, $\psi \in W_{20}^1$. Then $\|(\varphi,\psi)\|_{\mathscr{B}}^2 = \int_0^1(\varphi_{xx}^2+\varphi_x^2+\varphi^2+\psi_x^2+\psi^2)$. An equivalent norm is given by the function

$$V(\varphi,\psi) = V_1(\varphi,\psi) + V_2(\varphi,\psi)$$
$$V_2(\varphi,\psi) = \int_0^1(\varphi_{xx}^2 + \psi_x^2)dx$$

V_2 is somewhat like another energy function for the free linear equation $(\epsilon = 0)$. Some more computations show that

$$\dot{V}_2(u,u_t) = -2\epsilon \int_0^1(u_{xt})^2(1-u^2)dx$$
$$\dot{V}_2(\varphi,\psi) = -2\epsilon \int_0^1\psi_x^2(1-\varphi^2)dx$$

Combining the above results, we obtain

$$\dot{V}(\varphi,\psi) = -2\epsilon\int_0^1(\psi_x^2+\psi^2)(1-\varphi^2)dx$$

If $u^2 < 1$, then $\dot{V}(u,u_t) \leq 0$ and $V(u,u_t)$ is bounded. If initial values satisfy $V(\varphi,\psi) < 1$, then $V(u,u_t) \leq 1-\delta$, $\delta > 0$, for all t and the solution u of (23) satisfies $u^2(x,t) < 1-\delta$ for all t. We therefore have a dynamical system on $\mathscr{B} \cap \{(\varphi,\psi): V(\varphi,\psi) < 1\}$. Also,

$V(\varphi,\psi) < 1$ implies $V_1(\varphi,\psi) < 1$. We can now apply the above theory since the natural mapping which imbeds $\mathscr{B}$ into $\mathscr{C}$ is a compact map. Therefore, every bounded orbit in $\mathscr{B}$ is in a compact set in $\mathscr{C}$. The conditions of Theorem 3 are satisfied. The set R is given by

$$R = \{(\varphi,\psi) \in \mathscr{B}: \dot{V}(\varphi,\psi) = 0\} = \{(\varphi,0)\}$$

The largest invariant set N in R of the limit dynamical system certainly belongs to the set of generalized solutions of the equation which are defined on $(-\infty,\infty)$ and belong to $\mathscr{C} \cap \{\varphi: V(\varphi,\psi) < 1\} \cap R$. This implies the generalized solution must have $u_t = 0$; that is, the solution is a function of x alone. But this implies the solution also is a generalized solution of $u_{xx} = 0$. Therefore $u(t,x) = ax+b$ and $u(t,0) = u(t,1) = 0$ implies $u = 0$. Consequently, $M = (0,0)$ and we have that every solution in $\mathscr{B}$ approaches 0 in $\mathscr{C}$; that is, $u_x, u, u_t \to 0$ as $t \to \infty$.

6. Extended dynamical systems. The purpose of this section is to point out the relationship between the concept of limit dynamical system introduced in section 3 and the concept of extended dynamical system discussed by Hale and Infante in [1]. Throughout this section, it will be assumed $\mathcal{B}$ and $\mathcal{E}$ are Banach spaces and $\mathcal{B} \subset \mathcal{E}$.

Definition 5. Let u be a dynamical system on $\mathcal{B}$. Let $\mathcal{B}_1^*$ be the set of φ in $\mathcal{E}$ which are in the closure of $\mathcal{B}$ in $\mathcal{E}$ by bounded sequences such that to every φ in $\mathcal{B}_1^*$ there is associated a function $u^*(t,\varphi)$ in $\mathcal{E}$ for t in R^+ with the property that $\|u(t,\varphi_n)-u^*(t,\varphi)\|_{\mathcal{E}} \to 0$ as $n \to \infty$ uniformly on compact subsets of R^+ for every bounded sequence $\{\varphi_n\}$ in $\mathcal{B}$ with $\|\varphi_n-\varphi\|_{\mathcal{E}} \to 0$ as $n \to \infty$. We refer to the function $u^*: R^+ \times \mathcal{B}_1^* \to \mathcal{B}_1^*$ as the extension of the dynamical system u to $\mathcal{B}_1^*$ or simply as the extended dynamical system.

If the extended dynamical system exists, then it is an extension of u in the usual sense; that is, $u^*(t,\varphi) = u(t,\varphi)$ if φ is in $\mathcal{B}$. Also, it is easy to prove that $u^*(0,\varphi) = \varphi$, $u^*(t+\tau,\varphi) = u^*(t,u^*(\tau,\varphi))$, $t,\tau \geq 0$ and $u^*(t,\varphi)$ is continuous in t. It is not known whether $u^*(t,\varphi)$ is continuous in φ and, therefore, it is not known whether u^* is a dynamical system on $\mathcal{B}^*$.

If u is a dynamical system on $\mathcal{B}$ and $\mathcal{E}$, then the extended dynamical system u^* exists and $u^* = u$. Suppose $\mathcal{B}^*$ is defined in Definition 4 relative to the limit dynamical system and φ in $\mathcal{B}^*$ is such that φ belongs to $\omega_{\mathcal{E}}(\psi)$, the ω-limit set of the orbit $\gamma^+(\psi)$ in $\mathcal{E}$. If $\gamma^+(\psi)$ is bounded in $\mathcal{B}$, then φ clearly belongs to $\mathcal{B}_1^*$.

Therefore, if the limit dynamical system had been defined relative to the ω-limit sets of orbits which are bounded in $\mathcal{B}$, then $\mathcal{B}^*$ is a subset of $\mathcal{B}_1^*$. Even with this definition $\mathcal{B}^*$ could be a proper subset of $\mathcal{B}_1^*$.

The question of the existence of an extension of a dynamical system seems to be rather difficult. The answer to the following question is not even known: Is there a dynamical system u on $\mathcal{B}$ which has an extension to $\mathcal{B}_1^*$ and yet is not a dynamical system on $\mathcal{B}_1^*$?

The above concept of extension of a dynamical system was introduced by Hale and Infante [1] but the definition of dynamical system in [1] is stronger than the one used here. More precisely, a dynamical system on $\mathcal{B}$ in [1] is a function $u: R^+ \times \mathcal{B} \to \mathcal{B}$ with the properties listed in Definition 5 and in addition $u(t,\varphi)$ is uniformly continuous in t,φ for t,φ in bounded sets. If $u(t,\varphi)$ is linear in φ, then this last hypothesis implies that $u(t,\cdot)$ is a uniformly continuous semigroup of transformations. Therefore, a classical result in [12] implies that the infinitesimal generator of this semigroup must be a bounded linear operator. This is much too restrictive for the applications of the theory and the above definition seems to be a more appropriate concept of extension. The author is indebted to V. Mizel for pointing out this shortcoming of the definition in [1].

The results in [1] easily carry over to the situation discussed here if one always adds the hypothesis that u is a dynamical system on $\mathcal{B}$ <u>and</u> $\mathcal{C}$. The uniformity condition mentioned above seems to be necessary if one does not make this latter assumption.

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