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Abstract

An analytical solution for the time optimal control function and optimal trajectories of a second-order linear system with a constant time delay has been obtained. The system is a simple spring mass system with a constant time delay in the position variable. In addition to expressions for the control function and trajectories, an equation for the limit cycle has been established as well as a numerical construction of a pseudo switching curve for this system. The optimal control function was deduced from the maximum principle of Pontryagin for systems with delay. The two-point boundary value problem associated with the differential-difference equation in the synthesis of the control was solved by use of a Newton-Raphson iteration scheme. The results are expressed in a form which can be easily compared to the well-known results of the corresponding system without delay.

Introduction

Recently, much interest and effort have been spent in the attempt to answer questions on the stability and control of linear systems of the hereditary type; that is, systems which are described by linear differential-difference equations whose future behavior depends upon its past and present states. Although these systems were studied by Euler in 1750, no appreciable application of them to physical situations were discovered and employed before the first quarter of the 20th century. Physical systems which contain delays occur in such fields of science as biochemistry, economics, traffic flow, control theory, and so forth.

For example, in the remote control of distance space vehicles, the communication delay can adversely affect the stability of the system. Time delays in engine response of large jet transports can seriously affect the handling qualities of the aircraft.

Many questions concerning the stability of these systems have been answered by Bellman (1953), Pontryagin (1962), Bellman and Cooke (1963), Krasovskii (1963), Kashiwagi and Flügge-Lotz (1967), Halanay (1966), and Kashiwagi and Shaughnessy (1967), just to name a few. Questions concerning the optimal control and the controllability of systems containing delays have been partially answered as well. Khratishveli (1961) has extended Pontryagin's maximum principle to systems described by differential-difference equations. Balakirev (1962) has applied the extended maximum principle to a particular linear system with delay to determine the switching line. Chyung and Lee (1965) have considered the time optimal problem for the case of a general controller restraint set. Oğuztöreli (1966) has considered the control of delay systems in general, and in particular, has extended the Neustadt method for control synthesis to the system with delay. Many questions concerning the time optimal control of linear systems with delay remain to be answered. The purpose of this paper is to examine in some detail the basic characteristics of a particular linear system containing a delay - namely, a second-order system free of friction, but which contains a time delay in its position feedback. Because of the well-developed theory and application of this problem without delay, many important analogies and differences are discussed.

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Description of the System and
the Solution Trajectories

The mathematical model considered here has the form

$$\left. \begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{x}(t - \theta) + \underline{C}u(t) & (t \geq 0) \\ \underline{x}(t) &= \underline{g}(t) & (-\theta \leq t \leq 0) \end{aligned} \right\} (1)$$

where A , B , and \underline{C} are constant matrices, and $u(t)$ is the control function which is bounded by the inequality $|u(t)| \leq 1$.

The solution of equations (1) for the state trajectory in terms of the control function is given as

$$\begin{aligned} \underline{x}(t) &= K(t)\underline{g}(0) + \int_{-\theta}^0 K(t - \tau - \theta)B\underline{g}(\tau)d\tau \\ &+ \int_0^t K(t - \tau)\underline{C}u(\tau)d\tau \end{aligned} \quad (2)$$

where $K(t)$ is the fundamental matrix for the differential-difference equation given by equations (1). The fundamental matrix $K(t)$ satisfies the following differential-difference equation and properties:

$$\left. \begin{aligned} \dot{K}(t) &= AK(t) + BK(t - \theta) \\ K(t) &= 0, \quad t < 0 \\ K(0) &= I \end{aligned} \right\} (3)$$

In this paper, one of the simplest examples is considered; that is, the case for

$$\left. \begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \underline{g}(t) &= \begin{bmatrix} at^2 + bt + c \\ 2at + b \end{bmatrix} \end{aligned} \right\} (4)$$

The initial function $\underline{g}(t)$ was assumed to be quadratic in time. This choice makes it possible to represent many functions through the proper choice of the constants a , b , and c .

For this system, the elements of the fundamental matrix $K(t)$ are

$$\left. \begin{aligned} k_{11}(t) = k_{22}(t) &= \sum_{n=0}^{N(t)} (-1)^n \frac{(t - n\theta)^{2n}}{(2n)!} \\ k_{12}(t) = -k_{21}(t + \theta) &= \sum_{n=0}^{N(t)} (-1)^n \frac{(t - n\theta)^{2n+1}}{(2n+1)!} \end{aligned} \right\} (5)$$

where $N(t) = \left[\frac{t}{\theta} \right]$ meaning the largest integer less than or equal to $\frac{t}{\theta}$. The fundamental matrix $K(t)$ for the system behaves somewhat like the fundamental matrix $\Phi(t)$ for the system without delay. It can be seen from figures 1 and 2 that for small values of the delay (in this case $\theta < 0.1$), $K(t)$ and $\Phi(t)$ are very nearly the same. However, for large delays ($\theta > 0.1$) striking differences occur between $K(t)$ and $\Phi(t)$. The most obvious of the differences is that the magnitudes of the elements of $K(t)$ increases rapidly with time.

If the results of equations (4) and (5) are substituted into equation (2), the following elements of the state vector $\underline{x}(t)$ result:

$$\left. \begin{aligned} x_1(t) &= (a\theta^2 - b\theta + c + 2a)k_{11}(t + \theta) \\ &- 2ak_{11}(t + 2\theta) + (b - 2a\theta)k_{12}(t + \theta) \\ &+ I_1[u(t), t] \\ x_2(t) &= -(a\theta^2 - b\theta + c + 2a)k_{12}(t) \\ &+ 2ak_{12}(t + \theta) + (b - 2a\theta)k_{11}(t + \theta) \\ &+ I_2[u(t), t] \end{aligned} \right\} (6)$$

where

$$\left. \begin{aligned} I_1[u(t), t] &\equiv \int_0^t u(\tau)k_{12}(t - \tau)d\tau \\ I_2[u(t), t] &\equiv \int_0^t u(\tau)k_{11}(t - \tau)d\tau \end{aligned} \right\} (7)$$

All that is needed to reduce the expressions given by equations (6) to a completely algebraic form is the integration of the integrals I_1 and I_2 . These expressions can be integrated immediately, once the optimal control function $u(t)$ is determined.

The Time Optimal Control Function
and Its Synthesis

The time optimal control function for the system given by equations (1) can be obtained from the extended maximum principle of Pontryagin for systems with delays. According to this principle, the optimal value of $u(t)$ is that value which maximizes the Hamiltonian

$$H[\psi(t), \underline{x}(t), \underline{x}(t - \theta), u(t)] = \sum_{i=1}^2 \psi_i(t) \dot{x}_i(t) \quad (8)$$

where the elements of the adjoint vector $\psi(t)$ are determined from the differential-difference equations for $0 \leq t < T - \theta$ as

$$\begin{aligned} \dot{\psi}_i(t) = & - \frac{\partial H}{\partial x_i(t)} [\psi(t), \underline{x}(t), \underline{x}(t - \theta), u(t)] \\ & - \frac{\partial H}{\partial x_i(t)} [\psi(t + \theta), \underline{x}(t + \theta), \underline{x}(t), u(t + \theta)] \\ & (i = 1, 2) \end{aligned} \quad (9)$$

and for $T - \theta \leq t \leq T$ as

$$\dot{\psi}_i(t) = - \frac{\partial H}{\partial x_i(t)} [\psi(t), \underline{x}(t), \underline{x}(t - \theta), u(t)] \quad (i = 1, 2) \quad (10)$$

For the system given by equations (1) and (4), these equations can be rewritten as

$$\begin{aligned} \dot{\psi}_i(t) = & -A' \psi(t) - B' \psi(t + \theta) \\ & (0 \leq t < T - \theta) \end{aligned} \quad (11)$$

$$\begin{aligned} \dot{\psi}(t) = & -A' \psi(t) \\ & (T - \theta \leq t \leq T) \end{aligned} \quad (12)$$

where A' and B' are the transposes of the coefficient matrices given in equations (4), and T is the optimal time. The value of $u(t)$ which makes the Hamiltonian as given by equation (8) maximum subject to the constant $|u(t)| \leq 1$ is

$$u(t) = \text{sgn}[\psi_2(t)] \quad (13)$$

where $\psi_2(t)$ can be obtained from the solution of equations (11) and (12). The solution for $\psi(t)$ is

$$\psi(t) = K'(T - t)\psi(T) \quad (0 \leq t \leq T) \quad (14)$$

The resemblance of this result to the value of the adjoint vector for systems with no delay is clear.

Substitution of $\psi_2(t)$ as determined from equation (14) into the above expression for the optimal control gives

$$u(t) = \text{sgn}[\psi_2(T)] \text{sgn}[k_{11}(T - t) + \alpha k_{12}(T - t)] \quad (15)$$

where $\alpha = \psi_1(T)/\psi_2(T)$. This form for $u(t)$ shows that both components of $\psi(T)$ need not be known; instead, all that is needed is the ratio α and the value of $\text{sgn}[\psi_2(T)]$. The fact that the optimal control function is of the bang-bang type allows an evaluation of the integral functions I_1 and I_2 as

$$\begin{aligned} I_1[u(t), t] = & \text{sgn}[\psi_2(0+)] \left[(-1)^m - k_{11}(t + \theta) \right. \\ & \left. + 2 \sum_{p=1}^m (-1)^{p+1} k_{11}(t - t_p + \theta) \right] \\ I_2[u(t), t] = & \text{sgn}[\psi_2(0+)] \left[k_{12}(t) \right. \\ & \left. + 2 \sum_{p=1}^m (-1)^p k_{12}(t - t_p) \right] \end{aligned} \quad (16)$$

The value of m occurring in the above expression is the number of switches required to control the system, and t_p is the p th switching time as determined from the p th root of the equation

$$k_{11}(T - t) + \alpha k_{12}(T - t) = 0 \quad (17)$$

where $0 < t_1 < t_2 < \dots < t_p < t_{p+1} < \dots < T$. It can be seen from equation (17) that the value $T - t_p$ is a function of α only and not of T .

The values of T and α which serve to drive the perturbed state to the origin in time T by use of a bang-bang control function can be obtained as the unique solution for T and α of the set of equations

$$\left. \begin{aligned} x_1(T, \alpha) &= 0 \\ x_2(T, \alpha) &= 0 \end{aligned} \right\} \quad (18)$$

Since the partial derivatives of $x_1(T, \alpha)$ and $x_2(T, \alpha)$ with respect to T and α can be computed analytically, the Newton-Raphson scheme of solving for the root of a system of equations can be used in equations (18), once approximate values for T and α are given. Optimal trajectories for several values of delay are shown in figure 3. The initial function for this example was taken as a linear function of time (eqs. (4)) with $b = c = 2.0$. In addition to the obvious differences in trajectory characteristics, there is also an increase in the value of the optimal time required to control the system with increasing values of the delay.

Optimal Switching Curve and Limit Cycle

An optimal switching curve can be constructed numerically for the system with delay. In the analysis, the term optimal switching curve will denote the curve that separates the parts of the trajectory for which $u = +1$ from those for which $u = -1$ for a particular family of initial functions. This curve reduces to the ordinary optimal switching curve in the case of vanishing small delay values.

There are certain points on the switching curve which will be referred to as cusp points. For the system with no delay these points correspond to the intersection of the switching curve with the x_1 axis. Initial states with these points as coordinates can be driven to the origin in the time required for a particular number of consecutive switches of the control function. The value of α corresponding to these initial states are $\alpha = +\infty$ or $-\infty$ or equivalently $\psi_2(T) = 0$. The corresponding value of T is a particular root of the equation $\phi_{12}(T) = 0$. In the case of systems with delay, the cusp points do not necessarily occur on the x_1 axis and the value of T for the synthesis is

given by a particular root of the equation $k_{12}(T) = 0$. For the family of initial functions given in equations (4), the value of $x_1(0)$ and $x_2(0)$ are c and b , respectively. This means that if T_r , satisfying $k_{12}(T_r) = 0$, is substituted into equations (6) with $x_1(T_r) = x_2(T_r) = 0$, the resulting values of c and b are the coordinates of the r th cusp point. These coordinates denoted by $x_{1,r}$ and $x_{2,r}$, respectively, are

$$\left. \begin{aligned} x_{1,r} &= 1 + a\theta^2 - 2a \left[1 + \frac{\theta k_{12}(T_r + \theta)}{k_{11}(T_r + \theta)} \right. \\ &\quad \left. - \frac{k_{12}^2(T_r + \theta)}{k_{11}^2(T_r + \theta)} - \frac{k_{11}(T_r + 2\theta)}{k_{11}(T_r + \theta)} \right] \\ &\quad + \frac{1}{k_{11}(T_r + \theta)} \left[(-1)^r \right. \\ &\quad \left. + 2 \sum_{p=1}^{r-1} (-1)^p k_{11}(T_r - T_p + \theta) \right] \\ x_{2,r} &= 2a \left[\theta - \frac{k_{12}(T_r + \theta)}{k_{11}(T_r + \theta)} \right] \end{aligned} \right\} \quad (19)$$

where T_r is the r th root of the equation $k_{12}(T) = 0$, and $0 < T_1 < T_2 < \dots < T_r < T_{r+1} < \dots$. The analytical determination of the coordinates of these points is significant to the construction of the switching curve since much work is required to locate them numerically.

An example of the switching curve associated with a system whose initial disturbance function is linear in time ($a = 0$, $b = 2.0$ in $G(t)$) is given for $\theta = 0$ and 0.3 in figure 4. Closely associated with the switching curve is the limit cycle for the system. The curve is symmetric with respect to the origin and its equation for $x_2(t) < 0$ is

$$\left. \begin{aligned} x_1(t) &= (\beta - 1)k_{11}(t + \theta) - 1 \\ x_2(t) &= -(\beta - 1)k_{12}(t) \end{aligned} \right\} \quad (20)$$

where β is the limit of the sequence $\{\beta_r\}$ whose r th element is

$$\beta_r = 1 + \frac{1}{k_{11}(\tau_r + \theta)} \left[(-1)^r + 2 \sum_{p=1}^{r-1} (-1)^p k_{11}(\tau_r - \tau_p + \theta) \right] \quad (21)$$

The limit cycle is a closed curve in state space which separates the space into two regions. Disturbed states interior to this curve can be controlled with the class of time optimal control functions used here. Disturbed states exterior to the curve cannot be controlled with these control functions. Disturbed states which lie on the curve remain on the curve indefinitely. An example of how the limit cycle of the system described by equations (20) varies with increasing values of the delay is shown in figure 5 for the case of $a = 0$. As shown, the effect of increasing the magnitude of the delay is to substantially reduce the region of state space which can be controlled optimally. The extreme cases for the limit cycles are a circle whose radius becomes infinite as θ approaches zero (the entire state space controllable) and the x_1 axis between ± 1 as θ approaches infinity.

Comments on an Approximation

An approximation that has been used in the past to study systems of the hereditary type with small delays is

$$\underline{x}(t - \theta) \cong \underline{x}(t) - \theta \dot{\underline{x}}(t) \quad (22)$$

which represents the first two terms of a Taylor's series expansion of $\underline{x}(t - \theta)$. This approximation reduces the hereditary system given by equations (1) to an ordinary linear system containing negative friction which is described by the equations

$$\left. \begin{aligned} \dot{\underline{x}}(t) &= A_0 \underline{x}(t) + B_0 u(t) \\ \underline{x}(0) &= \underline{x}_0 \end{aligned} \right\} \quad (23)$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & \theta \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \underline{x}_0 = \begin{bmatrix} c \\ b \end{bmatrix} \quad (24)$$

An indirect and simple method of comparing the results of this approximation to the exact analysis just discussed is through the fundamental matrices of the two systems. Shown in figure 6 are the elements of the matrix, plotted versus time for exact calculations from equations (1) and approximate calculations from equations (23) and (24). The figure indicates relatively good agreement between the elements of the two matrices for small delays ($\theta < 0.1$), but sizable differences for larger values of the delay. This implies that the approximation should give reasonable results for systems with small delays compared to the control time.

Concluding Remarks

A time optimal control study of a particular linear system containing a delay in its feedback position has been conducted. It was found that the optimal control function is the bang-bang type whether or not a delay is present; however, the times at which the control function switches sign and the optimal time for control are dependent upon the size of the delay present in the system. For the cases considered here, the control time increased monotonically with increasing values of the delay.

It was found that the switching curve for the system with sizable values of delay differed considerably from the switching curve of the corresponding system without delay. In fact, the switching curve for the system with delay attenuates along the x_1 axis. For large values of delay it was found that a limit cycle exists which substantially reduces the region of state space which can be controlled optimally.

The analysis further indicated that the approximation of the term containing the delay by the first few terms of a Taylor's series comparison can be useful in cases for which the delay is small compared to the control time, but should be abandoned for large values of the delay.

References

- Balakirev, V. S.: The Principle of the Maximum in the Theory of Second Order Optimal Systems. Automation and Remote Control, Vol. 23, No. 8, August, 1962, pp. 948-956 (translation published February 1963).

- Bellman, R.; and Cooke, K. L.: Differential-Difference Equations. Academic Press, New York, 1963.
- Bellman, R.; and Danskin, J. M., Jr.: The Stability Theory of Differential Difference Equations. Proceedings of the Symposium on Nonlinear Circuit Analysis, sponsored by the Polytechnic Institute of Brooklyn Microwave Research Institute, April 1953.
- Chyung, D. H.; and Lee, E. B.: In Differential Equations and Dynamical Systems (J. K. Hale and J. P. LaSalle, eds.), Academic Press, New York, 1967, pp. 103-114.
- Halanay, A.: Differential Equations - Stability, Oscillations, Time Lags. Academic Press, New York, 1966.
- Kashiwagi, Yasutada; and Flüggé-Lotz, I.: Stability Indicative Function and Its Application to Systems With Time Delay. Ingr. Arch. Bd. XXXV, Heft 5, Jan. 1967, pp. 332-333.
- Kashiwagi, Y.; and Shaughnessy, John D.: Stability of Linear Systems With Multiple Delays. Proceedings of the Canadian Congress of Applied Mechanics, Quebec, Canada, May 1967.
- Khratishveli, G. L.: The Maximum Principle in the Theory of Optimal Processes With a Delay. Doklady Akad. Nauk. U.S.S.R., 136, 1961. English Translation in Soviet Math. Doklady, 2, (1961).
- Krasoviskii, N. N.: Stability of Motion. Stanford University Press, Stanford, California, 1963.
- Oğuztöreli, M. N.: Time-Lag Control Systems. Academic Press, New York, 1966.
- Pontryagin, L. S., et al.: The Mathematical Theory of Optimal Processes. Interscience Publishers, New York, 1962.

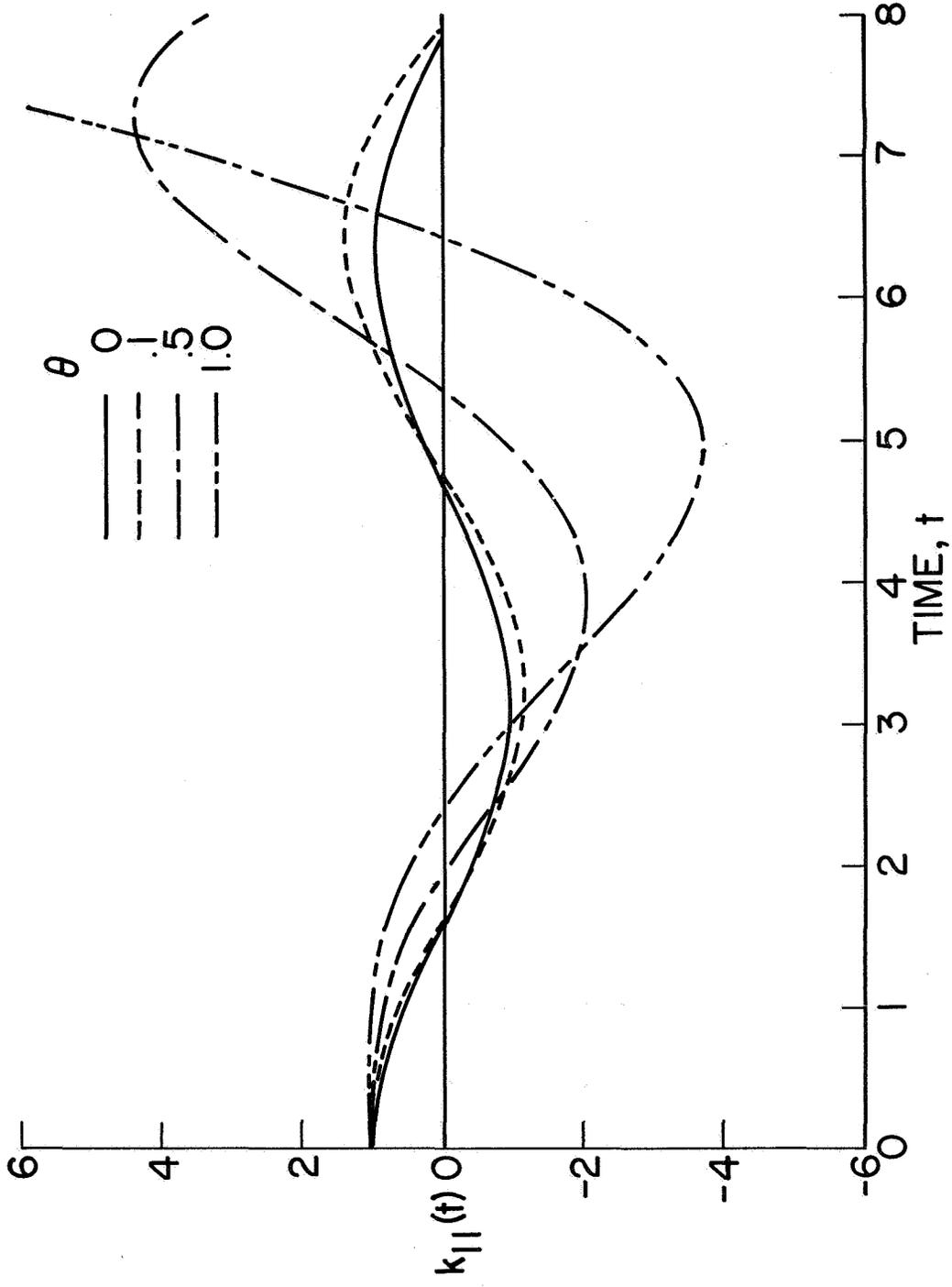


Figure 1.- Variation of the element $k_{11}(t)$ of the fundamental matrix with time.

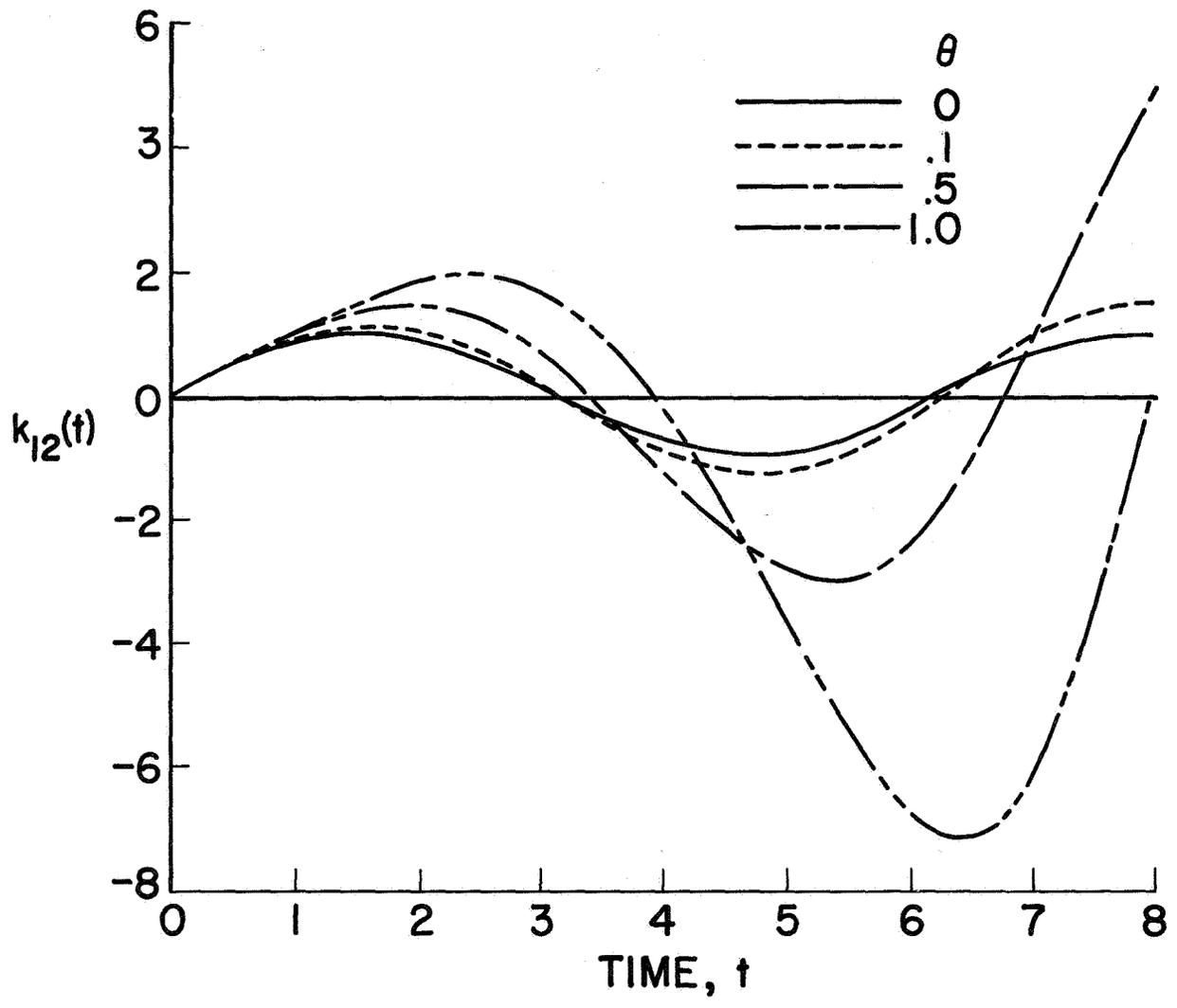


Figure 2.- Variation of the element $k_{12}(t)$ of the fundamental matrix with time.

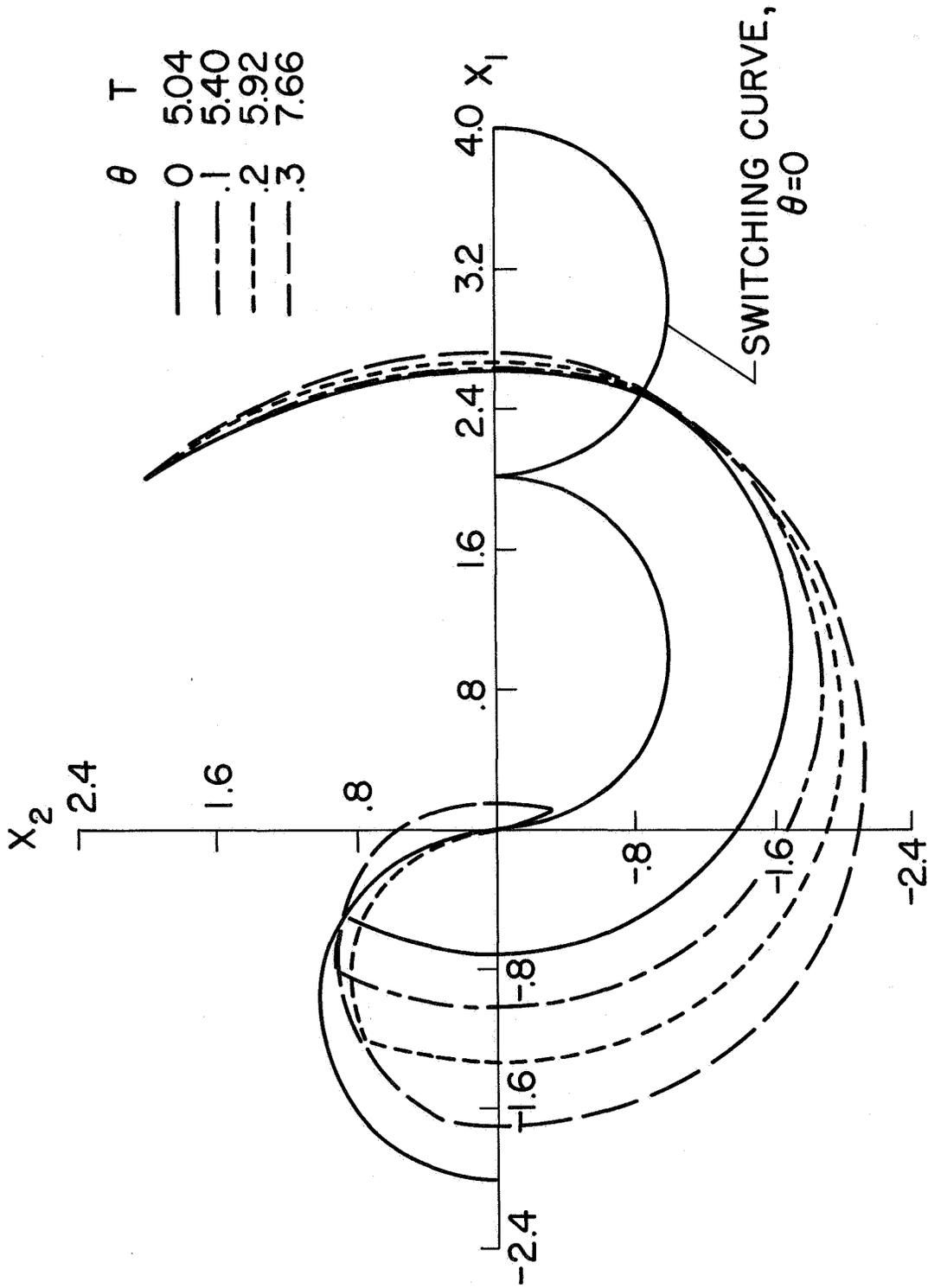


Figure 3.- Effect of time delay on the optimal trajectory in state space.

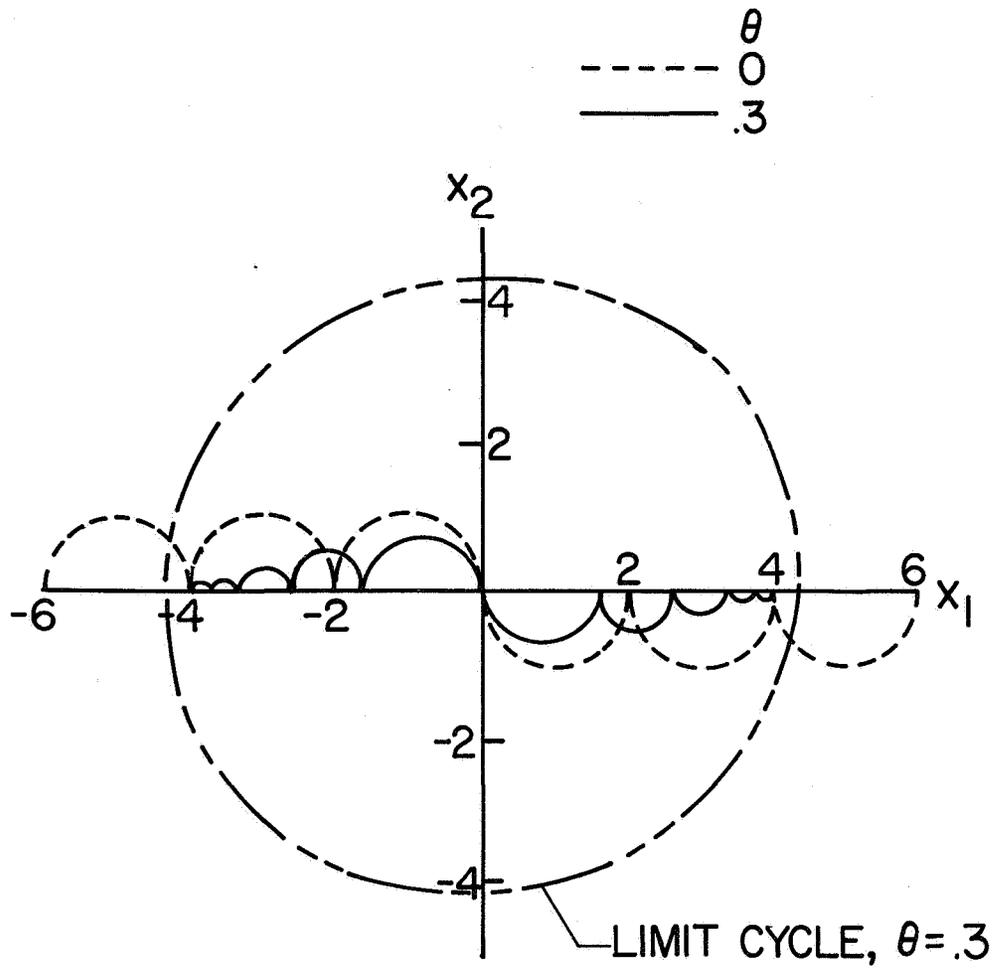


Figure 4.- Switching curve and limit cycle for $a = 0$, $b = 2.0$ and $\theta = 0$ and $.3$.

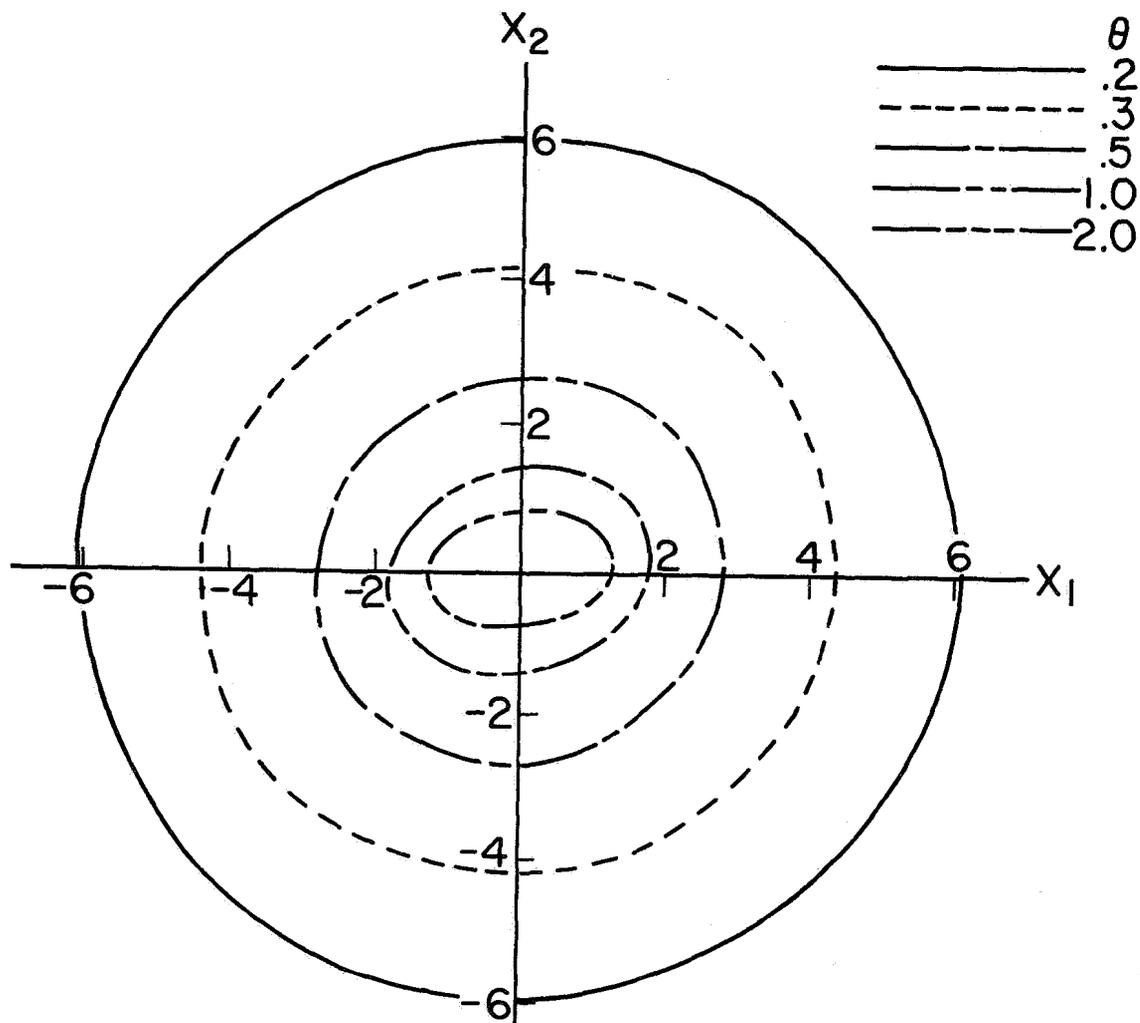


Figure 5.- Effect of increasing delay on the limit cycle.

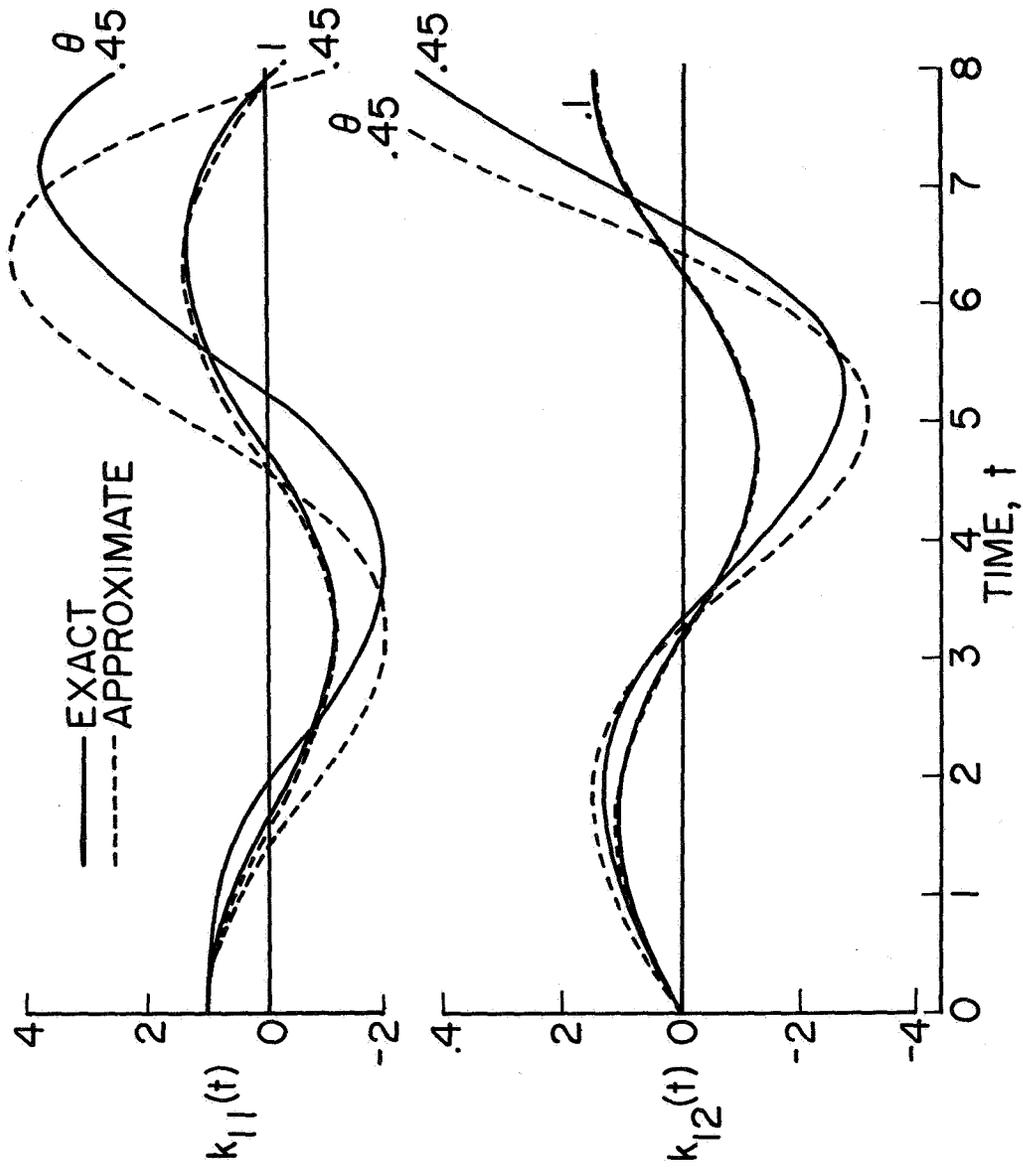


Figure 6.- Elements of the fundamental matrix obtained from exact and approximate analyses.