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ANALYTIC APPROXIMATIONS FOR APPLICATION TO SPACECRAFT OPTICAL NAVIGATION ON SHORT DATA ARCS

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SUMMARY

The standard least-squares orbit determination procedures are simplified for application to a navigation system where position fixes are taken over a short trajectory arc. Basic assumptions used are as follows: (1) for small perturbations in spacecraft state, velocity deviations remain constant and position deviations increase linearly with time and (2) the sensitivity of the measurements to small spacecraft-position changes remains constant over the trajectory arc. Simple analytic expressions are then derived for estimating the accuracy of the spacecraft position and velocity as a function of the number of fixes and the time span over which the fixes are made, or inversely the number of fixes and the time span over which they must be made for a prescribed position and velocity accuracy. For a large number of measurements the standard deviation of the position error was shown to vary inversely as the square root of the number of measurements, as expected, and the standard deviation of the velocity error was shown to vary inversely as the product of the square root of the number of measurements and the time span over which the measurements are made. Simplified equations are also derived for determining the orbit (state) of the spacecraft.

INTRODUCTION

Spacecraft navigation is concerned with both the problem of the single fix, effectively taking three simultaneous measurements to determine the spacecraft position, and with the problem of determining the orbit from numerous measurements taken along the trajectory. The use of a single position fix in spacecraft navigation has been the subject of several papers (for example, refs. 1 and 2). The development of navigational procedures for combining multiple fixes or measurements to obtain an estimate of the orbit has also been the subject of many papers (for example, refs. 3, 4, and 5). These latter procedures are essentially based on modified least-squares techniques and are applicable to combining measurements made over a trajectory arc of arbitrary length. If the trajectory arc is short as in orbit verification calculations or rapid orbit estimation techniques, simplifying assumptions may be made in the usual least-squares procedures.

In the present paper, onboard navigation procedures are developed for measurements made on a short trajectory arc. The following two simplifying assumptions are made in the least-squares procedures: (1) the gravity field is a constant in a small region around any point on the nominal trajectory and (2) the magnitude and direction of the gradient of the measurement remain constant over a short arc of the trajectory. Using these assumptions, equations are established for predicting the number of fixes as well as the time span, over which they are to be made, for a prescribed position and velocity accuracy. Simplified least-squares equations are also derived for determining the orbit and estimating its accuracy.

The procedures presented are useful for the design of onboard orbit determination systems using multiple position fixes. The simplified equations presented for determination of the orbit from short data arcs are adaptable for either manual or machine computation. The equations are illustrated by an example and the results compare favorably to those obtained from the complete least-squares solution of reference 6.

SYMBOLS

$[G]$	matrix relating measurement deviations to perturbations in spacecraft position vector
h	magnitude of measurement gradient $\left[\left(\frac{\partial \hat{y}}{\partial x}\right)^2 + \left(\frac{\partial \hat{y}}{\partial y}\right)^2 + \left(\frac{\partial \hat{y}}{\partial z}\right)^2\right]^{1/2}$
I_3	unit matrix of rank 3
$[m]$	matrix relating measurement deviations to perturbations in the spacecraft state vector
N	number of time points when measurements are made, that is, number of fixes
0_3	null matrix of 3×3 dimensions
$[P]$	local position information matrix; inverse of position covariance matrix
t	time from epoch
T	length of data arc
$[W]$	measurement weighting matrix, a diagonal of $1/\sigma_m{}^2$

x,y,z	perturbations in spacecraft geocentric equatorial coordinates
$\hat{y}$	deviation in optical measurement
Y	vector of coefficients in least-squares normal equations
ϵ	random error in optical measurement
σ	standard deviation
σ_m	standard deviation of optical measurement
σ_r	square root of trace of position covariance matrix, $\left(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2\right)^{1/2}$
σ_v	square root of trace of velocity covariance matrix, $\left(\sigma_{\dot{x}}{}^2 + \sigma_{\dot{y}}{}^2 + \sigma_{\dot{z}}{}^2\right)^{1/2}$
$[\phi]$	transition matrix
$\{x\}$	column matrix of the state vector
$\{\hat{y}\}$	column matrix of observations

Subscripts:

i	summation index
o	at epoch
p	position
t	time from epoch
t_o	at time of the epoch
x,y,z	derivative with respect to x,y,z, respectively
$\dot{x},\dot{y},\dot{z}$	derivative with respect to $\dot{x},\dot{y},\dot{z}$, respectively

Matrix notations:

{ }	column matrix
[ ]	square or rectangular matrix
[cov]	covariance matrix
[Info]	information matrix (inverse of covariance matrix)
$[\ ]^T$	transpose of matrix [ ]

Optical measurement symbols:

☽▷	subtended angle of Moon
∗Ō	observation of an angle from a star to Earth horizon
∗ L☽	observation of an angle from a star to a Moon landmark
∗ ☽	observation of an angle from a star to Moon horizon
∗ Ⓛ	observation of an angle from a star to an Earth landmark
Ⓐ	observation of Earth azimuth
Ⓔ	observation of Earth elevation
○▷	angular diameter of Earth
$*_1, *_2, *_3$	observation to star 1, star 2, and star 3, respectively

Dots over symbols denote derivatives with respect to time.

WEIGHTED LEAST-SQUARES PROCEDURES

The weighted least-squares procedures for spacecraft navigation are based on the assumption that small trajectory perturbations are transitioned linearly along the trajectory. This concept may be expressed mathematically as

$$\{x_t\} = [\phi]\{x_{to}\} \tag{1}$$

where the column matrices $\{x_t\}$ and $\{x_{to}\}$ are the trajectory perturbations at time t and t_o, respectively.

The elements of the so-called transition matrix $[\phi]$ are partial derivatives which relate perturbations in spacecraft state at time t due to perturbations in state at time t_o. The perturbations in the observations made on the trajectory are also assumed to be linearly related to the trajectory perturbations; this leads to the expression

$$\{\hat{y}\} = [m]\{x_t\} + \{\epsilon\} \tag{2}$$

The column matrix $\{\hat{y}\}$ is the difference between the measurement and the values calculated for the estimated trajectory. The state vector $\{x_t\}$ represents the deviation of the actual trajectory from the estimated trajectory.

Combining equations (1) and (2) gives equations of condition, which in matrix form are

$$\{\hat{y}\} = [m][\phi]\{x_{to}\} + \{\epsilon\} \tag{3}$$

It is the usual procedure to multiply each observation equation by the reciprocal of the measuring accuracy; this gives large multiplying or weighting numbers to those equations associated with high measuring accuracy and small weighting numbers to those equations with low measuring accuracy. Thus, the observation equations become

$$\left[\frac{1}{\sigma_m}\right]\{\hat{y}\} = \left[\frac{1}{\sigma_m}\right][m][\phi]\{x_{to}\} + \frac{1}{\sigma_m}\{\epsilon\} \tag{4}$$

These are a set of linear equations expressing the deviations in the observations at time t due to perturbations in the spacecraft state at time t_o and the random error introduced when the observations were made. Given a set of observational data and estimated values of the observations, the weighted least-squares technique may be used to determine the trajectory perturbation $\{x_{to}\}$. This involves the fitting of equation (4) minus the error term to the data. Multiplying the modified equation (4) by $[\phi]^T[m]^T\left[\frac{1}{\sigma_m}\right]^T$ and solving the resulting equation by matrix inversion yield the weighted least-squares solution as

$$\{x_{to}\} = \left[[\phi]^T[m]^T[W][m][\phi]\right]^{-1}[\phi]^T[m]^T[W]\{\hat{y}\} \tag{5}$$

For normally distributed, uncorrelated measurement errors, the matrix

$$\left[[\phi]^T[m]^T[W][m][\phi]\right]^{-1} \tag{6}$$

is the covariance matrix of the estimated $\{x_{to}\}$ and expresses the accuracy of this estimate from the knowledge of the measurement accuracy as expressed by the matrix $[W]$ and by using the linear relation expressed by $[m]$ and $[\phi]$.

In subsequent sections of this paper, it is shown that the matrix $[m]$ can be assumed to be constant over a short trajectory arc and that the matrix $[\phi]$ can be simply approximated by assuming linear perturbative motion. This, in addition to simplifying the equations, yields insight into the behavior of onboard navigation procedures.

APPROXIMATION TO THE TRANSITION MATRIX

For the weak gravitational field in most of translunar and interplanetary space, the approximation can be made that $\ddot{x}_x = \ddot{x}_y = \ddot{x}_z = \ddot{y}_y = \ddot{y}_z = \ddot{z}_x = \ddot{z}_y = \ddot{z}_z = 0$ in a small region about any point on the nominal trajectory. Then the equations, for the perturbative motion, reduce to

$$\left.\begin{aligned} \dot{x}_t &= \dot{x}_o \\ \dot{y}_t &= \dot{y}_o \\ \dot{z}_t &= \dot{z}_o \end{aligned}\right\} \tag{7}$$

From which it follows that

$$\begin{aligned} x_t &= x_o + \dot{x}_o t \\ y_t &= y_o + \dot{y}_o t \\ z_t &= z_o + \dot{z}_o t \end{aligned}$$

and the perturbative motions are linear. In this case, the transition matrix $[\phi]$ can then be expressed as:

$$
[\phi] = \begin{bmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}
$$

The transition matrices obtained from integration of the complete equations of motion, that is, with $\ddot{x}_x$ and $\ddot{y}_y$ and so forth not equal to zero, are shown in table I for a translunar trajectory. Selected elements of the transition matrices obtained from integrating the complete equations of motion are also plotted in figure 1. By examining equation (8), figure 1, and table I, it can be seen that the approximation of linear perturbative motion is good except in the near-earth region (flight time = 0). When comparing the various curves of figure 1, note that the linear position variation, for 10, 20, and 52 hours of flight time, is masked by the highly sensitive vertical scale. In any near-body region a more complete transition matrix could be employed such as the expressions for the transition matrix given in reference 7. The diagonal elements of the transition matrix for values of t up to 8 days on a Mars approach trajectory are shown in figure 2.

APPROXIMATION TO THE GRADIENTS OF THE MEASUREMENTS

The relation between the local measurements and the spacecraft local position is given by

$$
\begin{Bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{Bmatrix} = [G] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix} \tag{9}
$$

where

$$[G] = \begin{bmatrix} \frac{\partial \hat{y}_1}{\partial x} & \frac{\partial \hat{y}_1}{\partial y} & \frac{\partial \hat{y}_1}{\partial z} \\ \frac{\partial \hat{y}_2}{\partial x} & \frac{\partial \hat{y}_2}{\partial y} & \frac{\partial \hat{y}_2}{\partial z} \\ \frac{\partial \hat{y}_3}{\partial x} & \frac{\partial \hat{y}_3}{\partial y} & \frac{\partial \hat{y}_3}{\partial z} \end{bmatrix}$$

For optical measurements, the 3×3 matrix $[G]$ expresses the gradients of the measurements with spacecraft position. For short data arcs, say, 4 hours of a translunar orbit, the direction and magnitude of the gradient vector should not change considerably, except in the regions where the spacecraft is near the body on which the measurements are being made. A plot of the magnitude of the gradient of the measurements over a 4-hour interval for flight in various portions of translunar space is shown in figure 3 which indicates that the magnitudes are fairly constant except for a measurement made near the earth at 10 hours and measurements made near the moon at 52 hours. Thus, if one avoids the near-body regions, the gradients should remain approximately constant over a 4-hour translunar flight time or, in order to maintain approximately constant gradient over the arc, the length of the data arc has to be shortened in the near-body region.

The results for the Mars mission are shown in figure 4. The plot shows that the magnitudes of the gradients are fairly constant over 1-day segments in the near-body regions of the Mars trajectory. The magnitude of gradients would be expected to be more nearly constant in the far-body region.

DETERMINATION OF THE STATISTICS OF THE ORBIT

Accuracy estimates of the spacecraft state are a necessary part of navigational computations. These estimates are usually in the form of a covariance matrix whose diagonal elements are the square of the 1-sigma uncertainty of the spacecraft state vector. A simplified expression for the covariance matrix is derived by using the previously mentioned assumptions of a short data arc, linear propagation of small trajectory perturbations, and constant sensitivities of the optical measurements with respect to small deviations in spacecraft position.

The equation relating the spacecraft position to the measurements is given by equation (9) and in referring to the reasoning in equations (4) to (6) the associated covariance matrix of position is

$$\left[\mathrm{cov}\ x_p\right] = \left[G^T W G\right]^{-1} \tag{10}$$

where W is the measurement weighting matrix. The local state information matrix due to the local measurement is thus given by

$$[\mathrm{Info}]_t = \left[\begin{array}{c|c} G^T W G & 0_3 \\ \hline 0_3 & 0_3 \end{array}\right] = \left[\begin{array}{c|c} P & 0_3 \\ \hline 0_3 & 0_3 \end{array}\right] \tag{11}$$

where $G^T W G$ is the upper 3×3 of a 6×6 matrix and is designated as P. The information matrix of the state at time t_o due to the local fix (three simultaneous observations) is given by

$$[\mathrm{Info}]_{to} = [\phi]^T [\mathrm{Info}]_t [\phi]$$

which, by using equation (8), becomes

$$[\mathrm{Info}]_{to} = \left[\begin{array}{c|c} 1I_3 & tI_3 \\ \hline tI_3 & t^2 I_3 \end{array}\right] \left[\begin{array}{c|c} P & 0_3 \\ \hline 0_3 & P \end{array}\right] \tag{12}$$

and the information matrix at time t_o due to all fixes over the data interval is given by the summation process

$$[\mathrm{Info}]_{to} = \left[\begin{array}{c|c} NI_3 & \sum_{i=1}^{N} t_i I_3 \\ \hline \sum_{i=1}^{N} t_1 I_3 & \sum_{i=1}^{N} t_i^2 I_3 \end{array}\right] \left[\begin{array}{c|c} P & 0_3 \\ \hline 0_3 & P \end{array}\right] \tag{13}$$

where, as in equation (13), P is assumed constant for all data points and N, $\sum_{i=1}^{N} t_i$, and $\sum_{i=1}^{N} t_i^2$ are scalar coefficients of 3×3 identity matrices. This expression may be written in closed form as

$$[\mathrm{Info}]_{to} = \left[\begin{array}{c:c} NI_3 & \frac{NT}{2} I_3 \\ \hdashline \frac{NT}{2} I_3 & \frac{N(2N-1)T^2}{6(N-1)} I_3 \end{array}\right] \left[\begin{array}{c:c} P & 0_3 \\ \hdashline 0_3 & P \end{array}\right] \tag{14}$$

where the assumptions have been made that the first data point is obtained at $t = t_o = 0.0$ and the spacing between data points is constant.

The covariance matrix associated with the estimate of x_{to} is given by

$$[\mathrm{cov}\ x_o] = [\mathrm{Info}]_{to}^{-1} = \left[\begin{array}{c:c} \frac{2(2N-1)}{N(N+1)} I_3 & \frac{-6(N-1)}{N(N+1)T} I_3 \\ \hdashline \frac{-6(N-1)}{N(N+1)T} I_3 & \frac{12(N-1)}{N(N+1)T^2} I_3 \end{array}\right] \left[\begin{array}{c:c} P^{-1} & 0_3 \\ \hdashline 0_3 & P^{-1} \end{array}\right] \tag{15}$$

A comparison of the results of this equation and the results of reference 6 for a navigation system utilizing 4 hours of data beginning at 20 hours of flight time is shown in table II. It is seen that the results from the approximate solution compare favorably to those obtained from the complete least-squares solution.

Examination of equation (15) shows that the relation between the position and velocity uncertainty at any time t is given by

$$\frac{\sigma_r}{\sigma_v} = T\sqrt{\frac{(2N-1)}{6(N-1)}} \approx \frac{T}{\sqrt{3}} \quad \text{for large } N \tag{16}$$

The comparison of this equation with the complete least-squares results of reference 6 is given in figure 5. In this figure the data span is varied from 0 to 4 hours in various translunar regions. As can be seen, the agreement is good from 10 to 40 hours of flight time. It should be stated here that the ratio of the position to the velocity uncertainty is not a function of the type of measurements being made.

This same position-to-velocity uncertainty relation is shown in figure 6 for a 4-hour data span at various distances from earth for numerous navigation systems.

VARIATION OF POSITION AND VELOCITY ACCURACY WITH NUMBER OF FIXES

From equation (15), it is seen that

$$\sigma_r{}^2 = \frac{2(2N - 1)}{N(N + 1)} \cdot \text{Trace } \left[P\right]^{-1}$$

From which it follows that

$$\sigma_r = \sqrt{\frac{2(2N - 1)}{N(N + 1)}}\, \sigma_{r,o} \approx \frac{2\sigma_{r,o}}{\sqrt{N}} \quad \text{for large } N \tag{17}$$

Also from equation (15)

$$\sigma_v{}^2 = \frac{12(N - 1)}{N(N + 1)T^2} \cdot \text{Trace } \left[P\right]^{-1}$$

or

$$\sigma_v = 2\sqrt{3}\sqrt{\frac{N - 1}{N(N + 1)}}\, \frac{\sigma_{r,o}}{T} \approx \frac{2\sqrt{3}}{\sqrt{N}T}\, \sigma_{r,o} \quad \text{for large } N \tag{18}$$

Plots showing a comparison of results from equations (17) and (18) with the complete least-squares results of reference 6 are given in figures 7 to 10. The values used for $\sigma_{r,o}$ in figures 7, 8, and 9 are shown in figure 10. This figure shows a comparison of the values of a single position fix uncertainty $\sigma_{r,o}$ obtained from equation (17) and those obtained from diagonal elements of the equation

$$\left[\text{cov } x_o\right] = \left[\left[G\right]^T \left[W\right] \left[G\right]\right]^{-1} \tag{19}$$

DETERMINATION OF NUMBER OF FIXES AND TIME SPAN TO OBTAIN A PRESCRIBED POSITION AND VELOCITY ACCURACY

The number of measurements to obtain a prescribed σ_r, by using a navigation system with accuracy $\sigma_{r,o}$ for a single fix, is given from equation (17) as

$$N = \frac{4\left(\frac{\sigma_{r,o}}{\sigma_r}\right)^2 - 1 + \sqrt{\left[1 - 4\left(\frac{\sigma_{r,o}}{\sigma_r}\right)^2\right]^2 - 8\left(\frac{\sigma_{r,o}}{\sigma_r}\right)^2}}{2} \tag{20}$$

For large values of N, equation (17) reduces to

$$\left(\frac{\sigma_r}{\sigma_{r,o}}\right)^2 \approx \frac{4}{N}$$

Thus

$$N \approx 4\left(\frac{\sigma_{r,o}}{\sigma_r}\right)^2 \quad \text{for large } N$$

The time span over which the measurements must be made to obtain a specified velocity uncertainty σ_v along with this σ_r is given from equation (16) as

$$T = \sqrt{\frac{6(N - 1)}{2N - 1}}\,\frac{\sigma_r}{\sigma_v} \approx \frac{\sigma_r}{\sigma_v}\sqrt{3} \quad \text{for large } N \tag{21}$$

USE OF LESS THAN THREE OBSERVATIONS PER TIME POINT

From equation (13), the information matrix is approximated by the equation

$$\left[\text{Info}\right]_{to} = \left[\begin{array}{c:c} NI_3 & \sum_{i=1}^{N} t_i I_3 \\ \hdashline \sum_{i=1}^{N} t_i I_3 & \sum_{i=1}^{N} t_i^2 I_3 \end{array}\right]\left[\begin{array}{c:c} P & 0_3 \\ \hdashline 0_3 & P \end{array}\right] \tag{22}$$

This equation does not require that three measurements (one fix) be made per time point. The G matrix could have the second or second and third row zeros. This would correspond to making one or two measurements per time point. However, in this case, the information matrix would not have an inverse since the P matrix of the above product has no inverse. Thus the simplified expressions relating σ_r and σ_v in terms of $\sigma_{r,o}$, N, and T developed in this paper are only applicable when three or more measurements are made per time point. Nevertheless, if a priori information is available, it is possible to obtain an improved estimate of the orbit with less than three observations per time point while retaining the simplifying assumptions about the gradients of the measurements and the time variation of the transition matrices. A comparison of some results from reference 6 shows that in the unrestricted orbit determination process, the three observations per time point cases give considerably better results than the two observations per time point cases; however, the orbit was determined in both cases. These comparisons are shown in figure 11.

DETERMINATION OF THE SPACECRAFT STATE USING THE APPROXIMATE TRANSITION MATRIX

The weighted least-squares normal equations (eq. (5)) for the spacecraft state are of the form

$$\left\{x_{to}\right\} = \left[\mathrm{cov}\ x_o\right]^{-1}\left\{Y\right\} \tag{23}$$

where the column $\{Y\}$ is given by

$$\left\{Y\right\} = \left[\phi\right]^T\left[m\right]^T\left[W\right]\left\{\hat{y}\right\} = \begin{Bmatrix} \sum_{i=1}^{N} \frac{\partial y_i}{\partial x_o}\frac{\hat{y}_i}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\partial y_i}{\partial y_o}\frac{\hat{y}_i}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\partial y_i}{\partial z_o}\frac{\hat{y}_i}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\partial y_i}{\partial \dot{x}_o}\frac{\hat{y}_i}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\partial y_i}{\partial \dot{y}_o}\frac{\hat{y}_i}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\partial y_i}{\partial \dot{z}_o}\frac{\hat{y}_i}{\sigma_i^2} \end{Bmatrix} \tag{24}$$

From equations (8) and (9), $\frac{\partial y_i}{\partial x_o}$ is assumed constant over the data interval and $\frac{\partial y_i}{\partial \dot{x}_o}$ is assumed given by

$$\frac{\partial y_i}{\partial \dot{x}_o} = t_i \frac{\partial y_i}{\partial x_o} \tag{25}$$

thus, the expression for $\{Y\}$ can be written from equations (24), (25), and (9) as

$$\{Y\} = \begin{bmatrix} G^T & 0_3 \\ 0_3 & G^T \end{bmatrix} \begin{Bmatrix} \sum_{i=1}^{N} \frac{\hat{y}_{1i}}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\hat{y}_{2i}}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\hat{y}_{3i}}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\hat{y}_{1i} t_i}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\hat{y}_{2i} t_i}{\sigma_i^2} \\ \sum_{i=1}^{N} \frac{\hat{y}_{3i} t_i}{\sigma_i^2} \end{Bmatrix} \tag{26}$$

Combining this equation with equation (15) yields the solution for spacecraft state as

$$\begin{Bmatrix} x \\ \hline \dot{x} \end{Bmatrix} = \left[\begin{array}{c:c} \frac{2(2N - 1)}{N(N + 1)} I_3 & -\frac{6(N - 1)}{N(N + 1)T} I_3 \\ \hdashline -\frac{6(N - 1)}{N(N + 1)T} I_3 & \frac{12(N - 1)}{N(N + 1)T^2} I_3 \end{array}\right] \left[\begin{array}{c:c} P^{-1} & 0_3 \\ \hdashline 0_3 & P^{-1} \end{array}\right] \{Y\} \tag{27}$$

A comparison of approximate navigational results obtained from equation (27) with those from the complete least-squares solution of reference 6 is shown in table III. The data in table III represent 40 fixes made from 20 to 24 hours of flight time.

CONCLUDING REMARKS

Optical navigational techniques applicable to position fixing on short trajectory arcs were developed. The gradients of the optical measurements over a short arc were assumed to be constant in magnitude and direction. Also, it was shown that the gradient of the gravity field in a limited region about the nominal trajectory could be assumed to be zero. By combining these two assumptions, simplified least-squares normal equations for the determination of the orbit and its accuracy were developed. These equations show that the standard deviation of the position error determined from a large number of measurements varies inversely as the square root of the number of measurements, as expected, and the standard deviation of the velocity error varies inversely as the product of the square root of the number of measurements and the time span over which the measurements are made. From these relations, the time span of the measurements and the number of measurements which must be made for a prescribed spacecraft position and velocity accuracy was readily established. The approximate least-squares equations for determining the orbit were applied to an example and the results compared favorably to those obtained from the complete least-squares solutions.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 11, 1968,
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TABLE I.- TRANSITION MATRICES AT 20 AND 52 HOURS OF FLIGHT TIME

IN A TRANSLUNAR TRAJECTORY

Transition matrix $[\phi]$ from 20 to 23.9 hours of flight time:

$$\begin{bmatrix} 0.9945 & 0.29 \times 10^{-3} & 0.19 \times 10^{-3} & 14015 & 1.1 & 0.58 \\ .98 \times 10^{-3} & 1.0078 & .68 \times 10^{-2} & 1.0 & 14070 & 31 \\ .0 & .63 \times 10^{-2} & .9980 & .52 & 31 & 14032 \\ -.74 \times 10^{-6} & .37 \times 10^{-7} & .20 \times 10^{-7} & .9950 & .18 \times 10^{-3} & .93 \times 10^{-4} \\ .37 \times 10^{-7} & .98 \times 10^{-6} & .96 \times 10^{-6} & .18 \times 10^{-3} & 1.0064 & .63 \times 10^{-2} \\ .19 \times 10^{-7} & .95 \times 10^{-6} & -.22 \times 10^{-6} & .92 \times 10^{-4} & .63 \times 10^{-2} & .9985 \end{bmatrix}$$

Transition matrix $[\phi]$ from 52 to 55.9 hours of flight time:

$$\begin{bmatrix} 0.9997 & 0.15 \times 10^{-2} & 0.24 \times 10^{-3} & 14040 & 7.1 & 3.2 \\ .0 & .9999 & .39 \times 10^{-2} & 6.2 & 14045 & 9.3 \\ .97 \times 10^{-3} & .20 \times 10^{-2} & .9990 & 3.6 & 9.8 & 14033 \\ .75 \times 10^{-8} & .22 \times 10^{-6} & .10 \times 10^{-6} & 1.0002 & .17 \times 10^{-2} & .83 \times 10^{-3} \\ .22 \times 10^{-6} & .19 \times 10^{-6} & .29 \times 10^{-6} & .17 \times 10^{-2} & 1.0013 & .21 \times 10^{-2} \\ .10 \times 10^{-6} & .29 \times 10^{-6} & -.20 \times 10^{-6} & .83 \times 10^{-3} & .21 \times 10^{-2} & .9984 \end{bmatrix}$$

TABLE II.- COMPARISON OF COMPUTED COVARIANCE MATRIX WITH ANALYTIC APPROXIMATION TO COVARIANCE MATRIX

[Similar to flight time of 20 hours of ref. 6; 100 fixes at 1 per 0.04 hour; star to earth horizon, two stars to moon horizon]

Computed least-squares values:

$$\text{cov } x_0 = \left[[\phi]^T [m]^T [W][m][\phi]\right]^{-1}$$

$$\begin{bmatrix}
5.2876 & 5.8115 & 4.0730 & -0.5675 \times 10^{-3} & -0.6031 \times 10^{-3} & -0.4211 \times 10^{-3} \\
5.8115 & 22.1179 & 12.2335 & -.6077 \times 10^{-3} & -2.2777 \times 10^{-3} & -1.2597 \times 10^{-3} \\
4.0730 & 12.2335 & 10.0330 & -.4232 \times 10^{-3} & -1.2573 \times 10^{-3} & -1.0323 \times 10^{-3} \\
-.5675 \times 10^{-3} & -.6031 \times 10^{-3} & -.4211 \times 10^{-3} & .8088 \times 10^{-7} & .8379 \times 10^{-7} & .5792 \times 10^{-7} \\
-.6077 \times 10^{-3} & -2.2777 \times 10^{-3} & -1.2596 \times 10^{-3} & .8379 \times 10^{-7} & .3063 \times 10^{-6} & 1.6888 \times 10^{-7} \\
-.4232 \times 10^{-3} & -1.2573 \times 10^{-3} & -1.0323 \times 10^{-3} & .5792 \times 10^{-7} & 1.6888 \times 10^{-7} & 1.3913 \times 10^{-7}
\end{bmatrix}$$

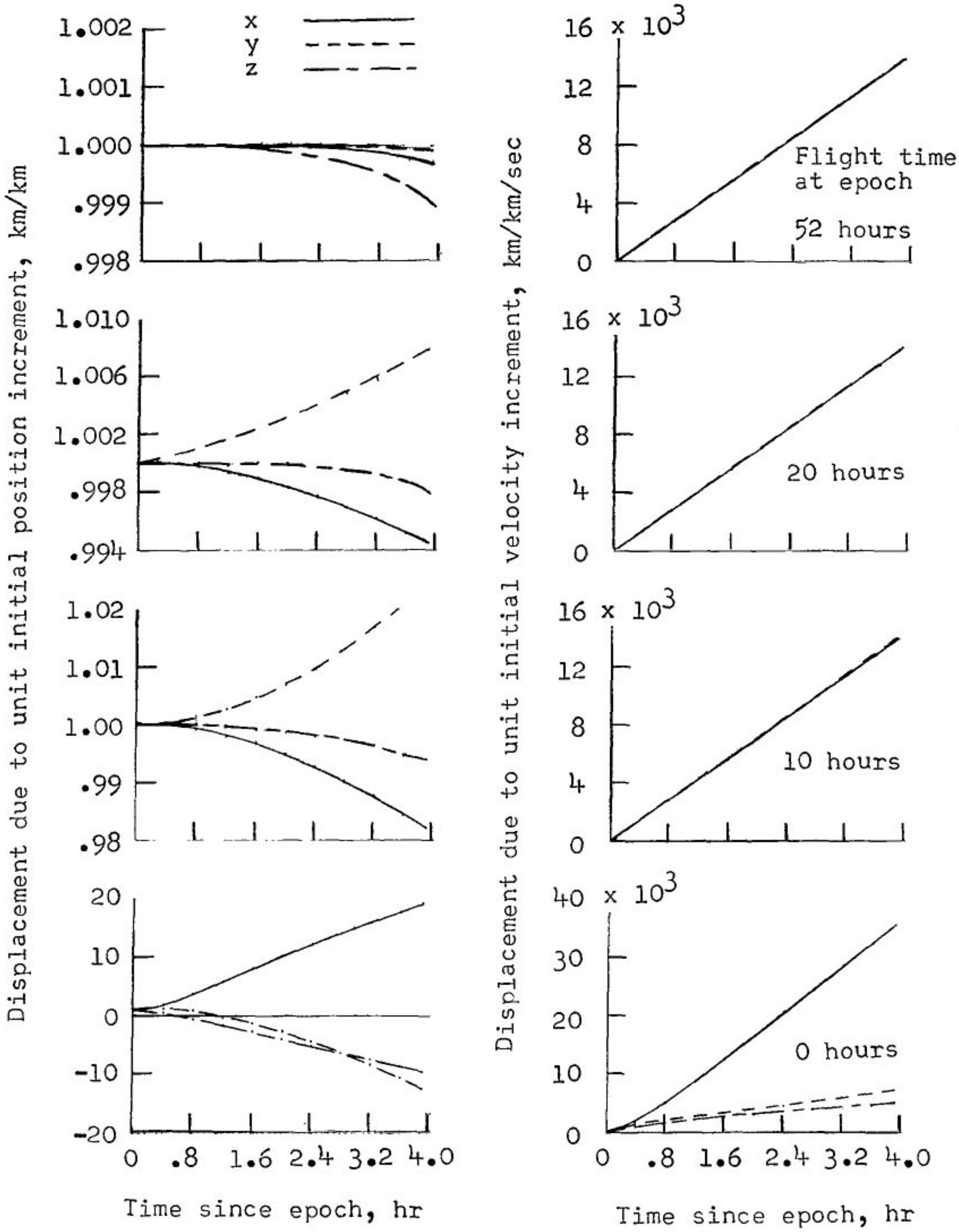
Analytic approximate values:

$$\text{cov } x_0 = \left[\begin{array}{c:c}
\dfrac{2(2N - 1)I_3}{N(N + 1)} & \dfrac{-6(N - 1)I_3}{N(N + 1)T} \\
\hdashline
\dfrac{-6(N - 1)I_3}{N(N + 1)T} & \dfrac{12(N - 1)I_3}{N(N + 1)T^2}
\end{array}\right]
\left[\begin{array}{c:c}
P_0^{-1} & 0_3 \\
\hdashline
0_3 & P_0^{-1}
\end{array}\right]$$

$$\begin{bmatrix}
5.3431 & 6.0529 & 4.2825 & -0.5565 \times 10^{-3} & -0.6305 \times 10^{-3} & -0.4461 \times 10^{-3} \\
6.0529 & 24.0818 & 13.3539 & -.6305 \times 10^{-3} & -2.5085 \times 10^{-3} & -1.3910 \times 10^{-3} \\
4.2825 & 13.3539 & 10.8570 & -.4461 \times 10^{-3} & -1.3910 \times 10^{-3} & -1.1309 \times 10^{-3} \\
-.5565 \times 10^{-3} & -.6305 \times 10^{-3} & -.4461 \times 10^{-3} & .7703 \times 10^{-7} & .8757 \times 10^{-7} & .6195 \times 10^{-7} \\
-.6305 \times 10^{-3} & -2.5085 \times 10^{-3} & -1.3910 \times 10^{-3} & .8757 \times 10^{-7} & .3484 \times 10^{-6} & 1.9319 \times 10^{-7} \\
-.4461 \times 10^{-3} & -1.3910 \times 10^{-3} & -1.1309 \times 10^{-3} & .6195 \times 10^{-7} & 1.9319 \times 10^{-7} & 1.5707 \times 10^{-7}
\end{bmatrix}$$

TABLE III.- EXAMPLE OF APPROXIMATE AND COMPLETE LEAST-SQUARES SOLUTION FOR SPACECRAFT STATE ON A TRANSLUNAR TRAJECTORY

Parameter	True deviation of state from nominal value	Least-squares solution of reference 6	Approximate solution of this paper
x, km	9.656	11.4	8.5
y, km	9.656	9.6	5.5
z, km	9.656	15.4	10.8
$\dot{x}$, km/sec	0.0061	0.0062	0.0066
$\dot{y}$, km/sec	0.0061	0.0062	0.0072
$\dot{z}$, km/sec	0.0061	0.0055	0.0067



(a) Position.

(b) Velocity.

Figure 1.- Trajectory displacements from nominal due to unit position and velocity perturbations on a translunar trajectory.

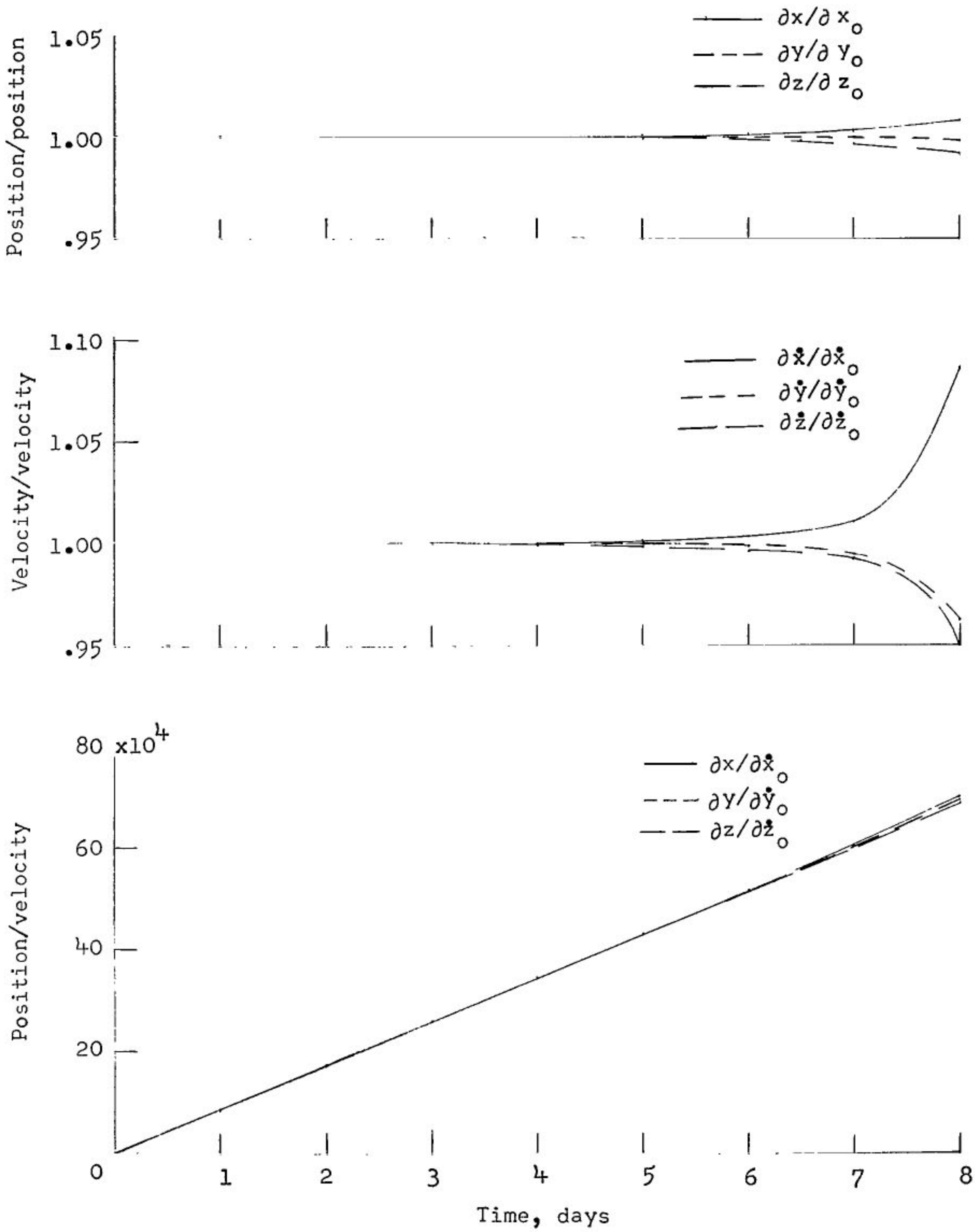


Figure 2.- Variation of elements of state transition matrix over an 8-day span near Mars. Range to Mars from 3 270 000 km to 289 000 km.

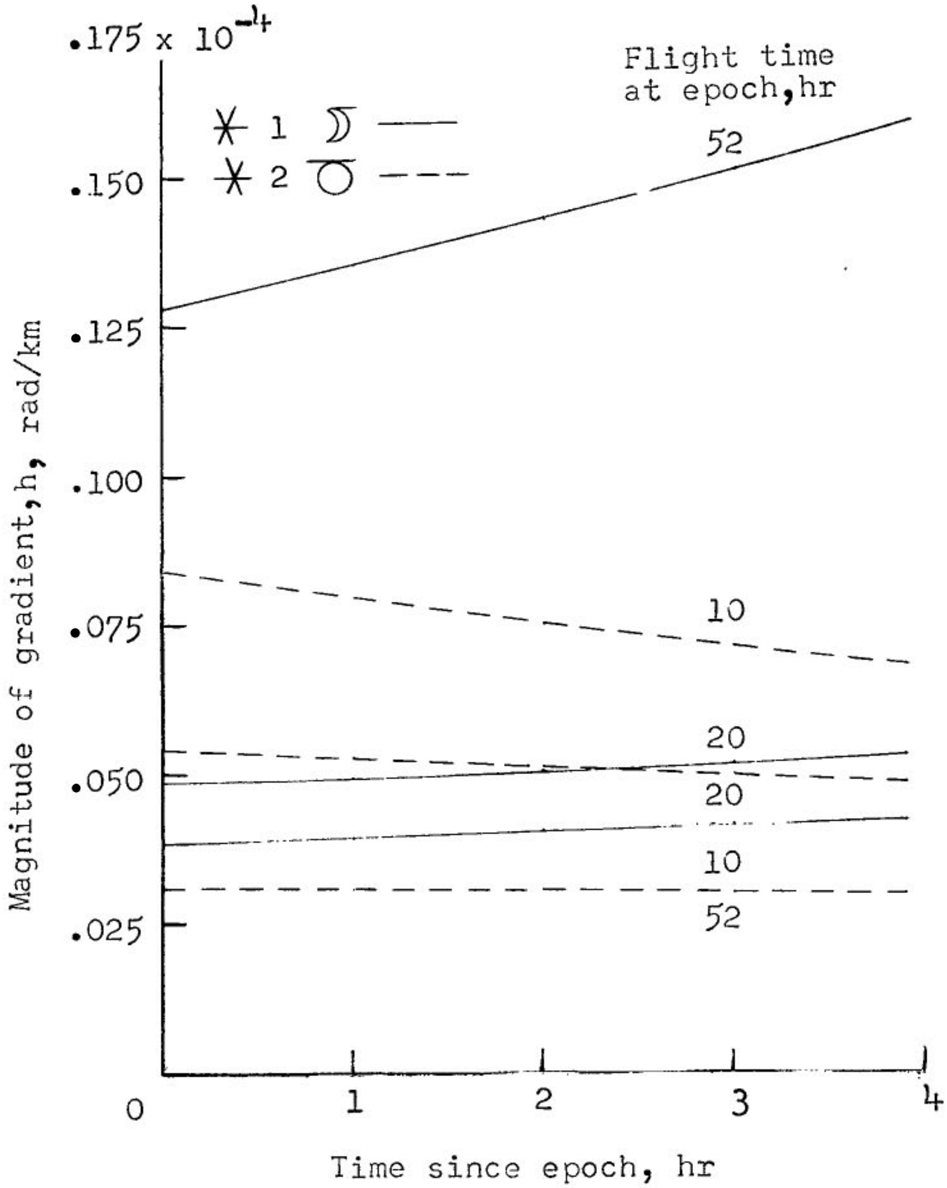


Figure 3.- Magnitude of measurement gradient over four intervals of a translunar trajectory beginning at various flight times.

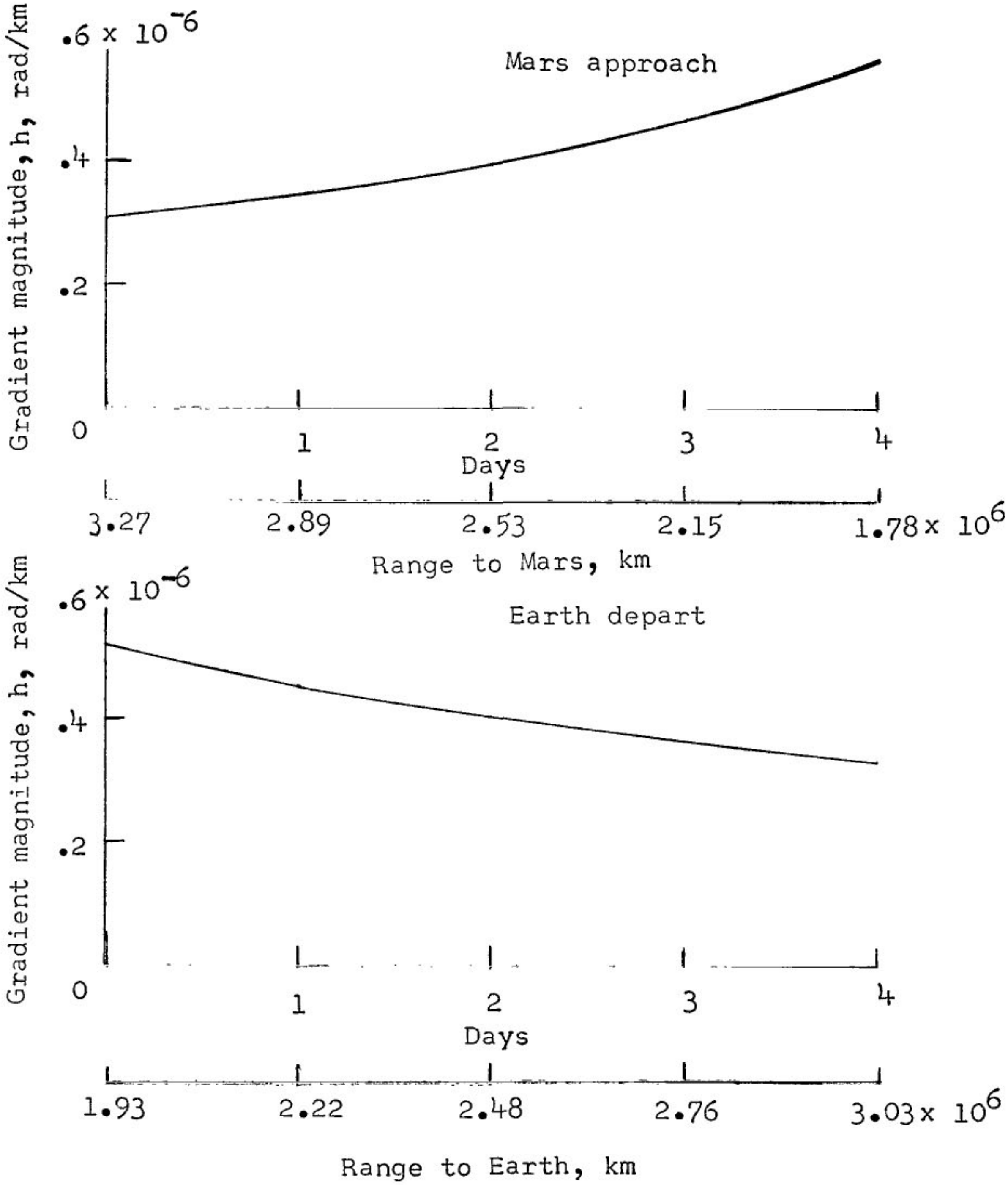


Figure 4.- Magnitude of star to body center measurement gradient on a Mars trajectory.

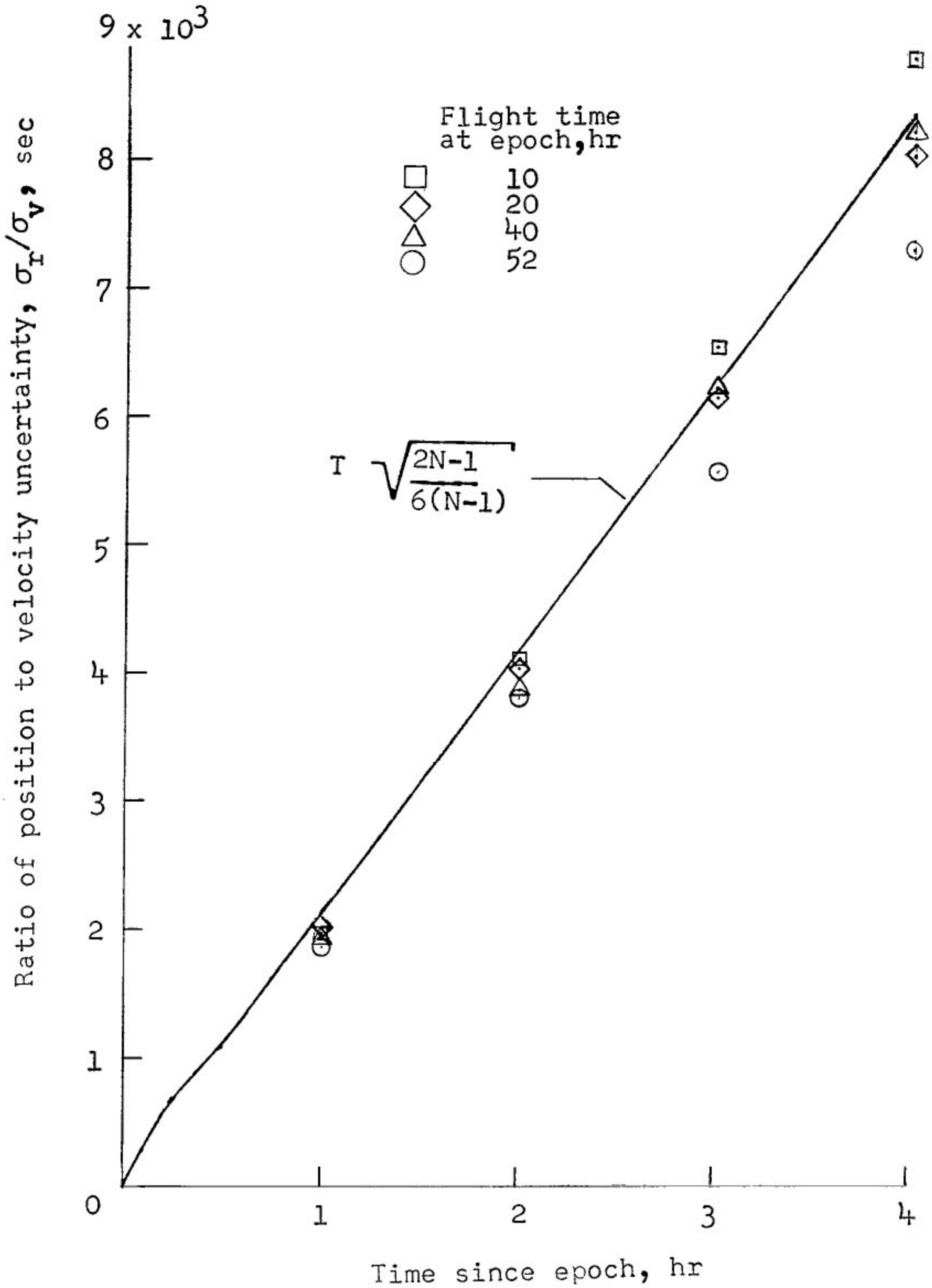


Figure 5.- Comparison of simplified expressions for σ_r/σ_v with translunar data from reference 6. Fixes taken at 10 per hour.

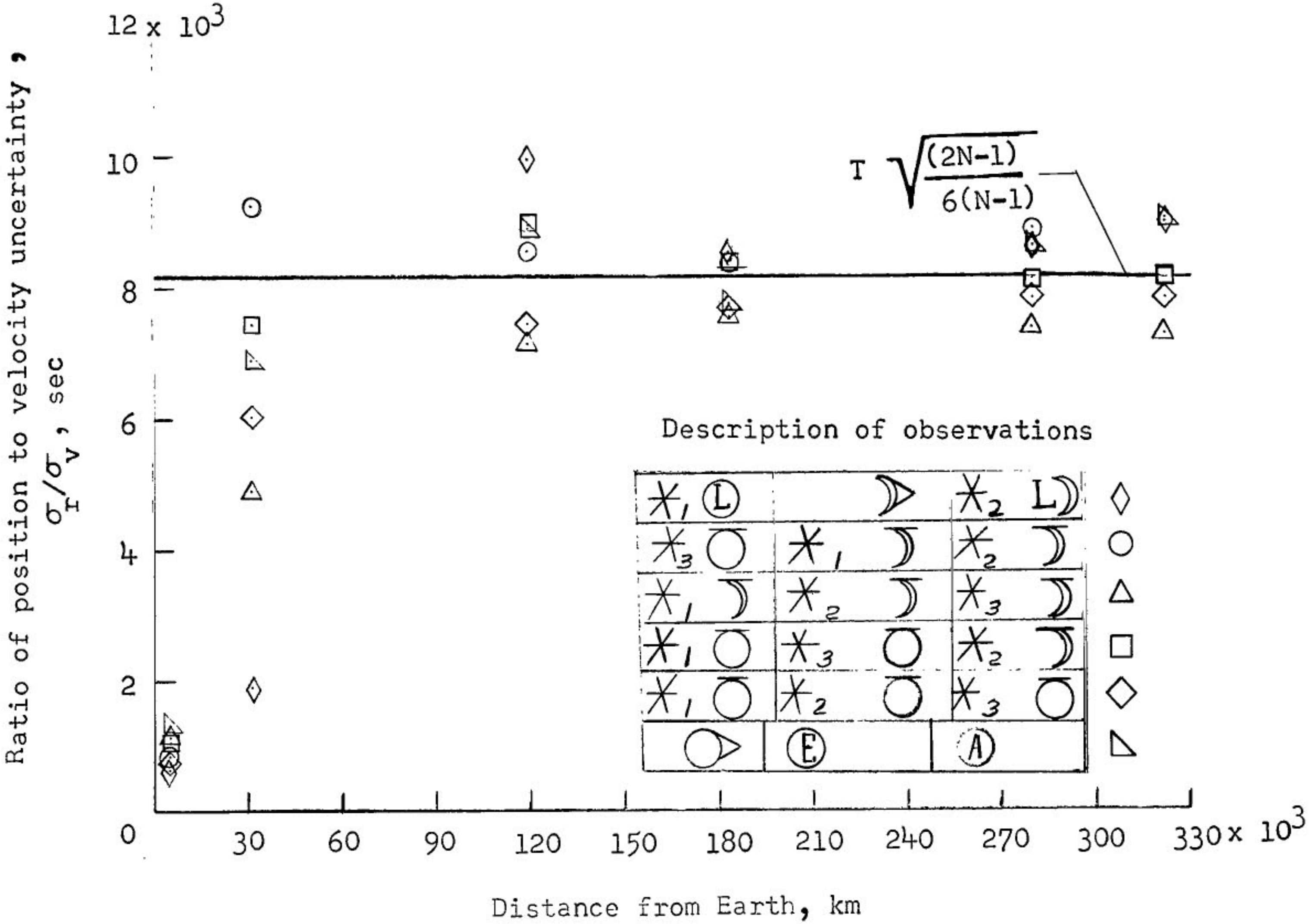
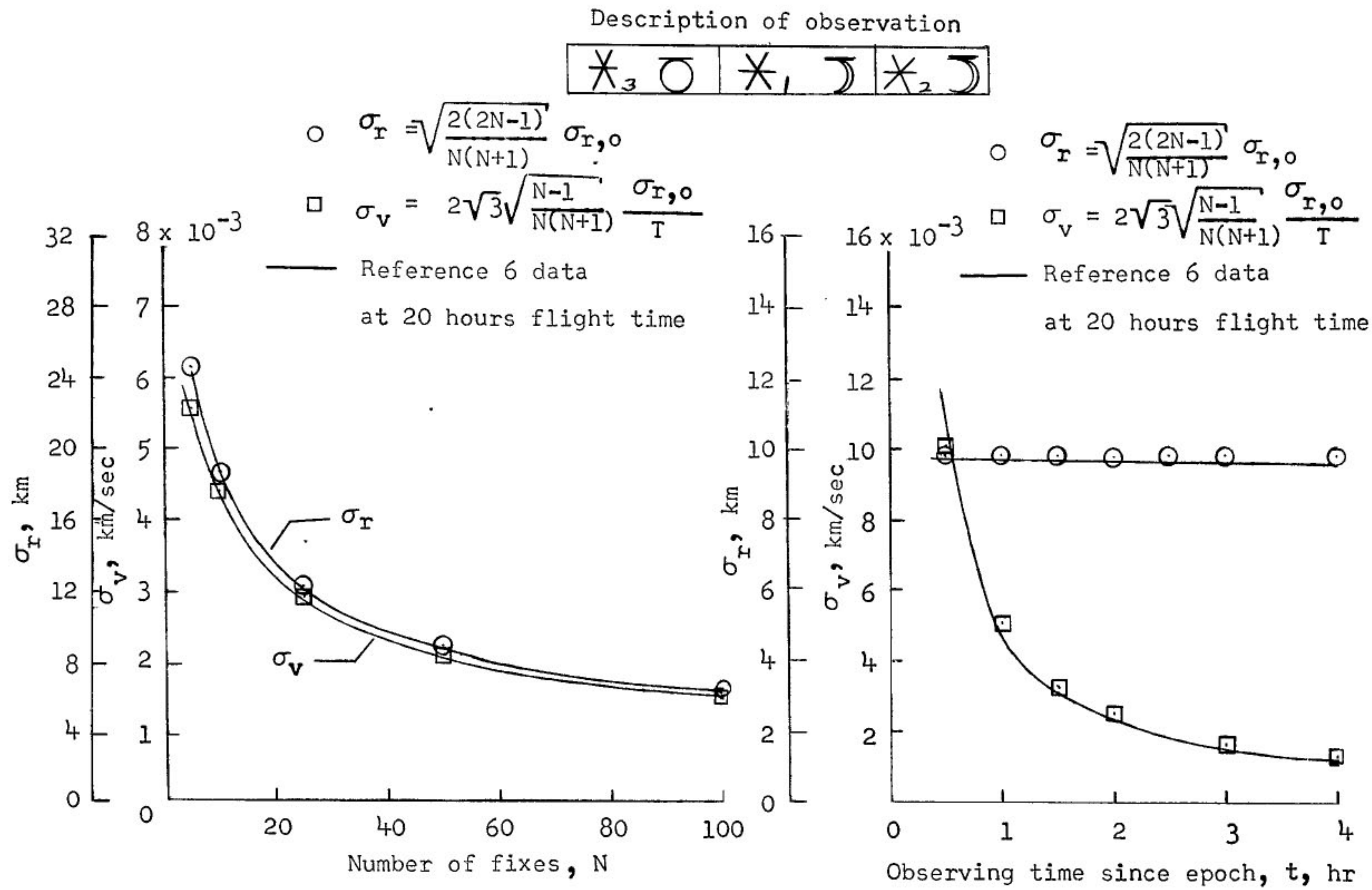


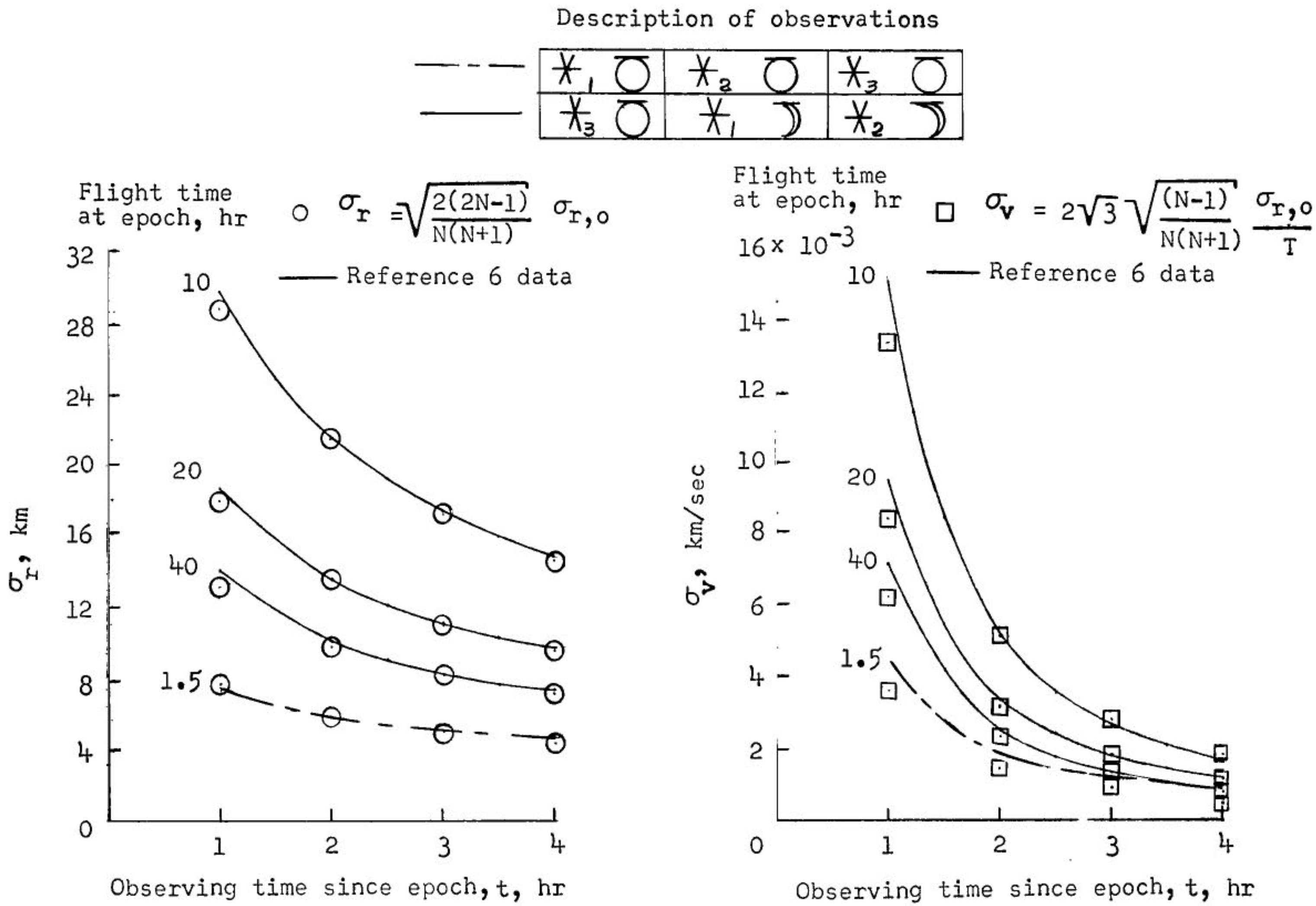
Figure 6.- Comparison of simplified expressions for σ_r/σ_v with translunar data from reference 6. N = 40; T = 14 040 sec.



(a) Constant observing time (2 hr).

(b) Constant number of fixes (40).

Figure 7.- Comparison of simplified expression for navigation accuracy with numerical results of reference 6.



(a) Position.

(b) Velocity.

Figure 8.- Comparison of simplified expressions for navigation accuracy with numerical results of reference 6. Fixes taken at 40 per hour.

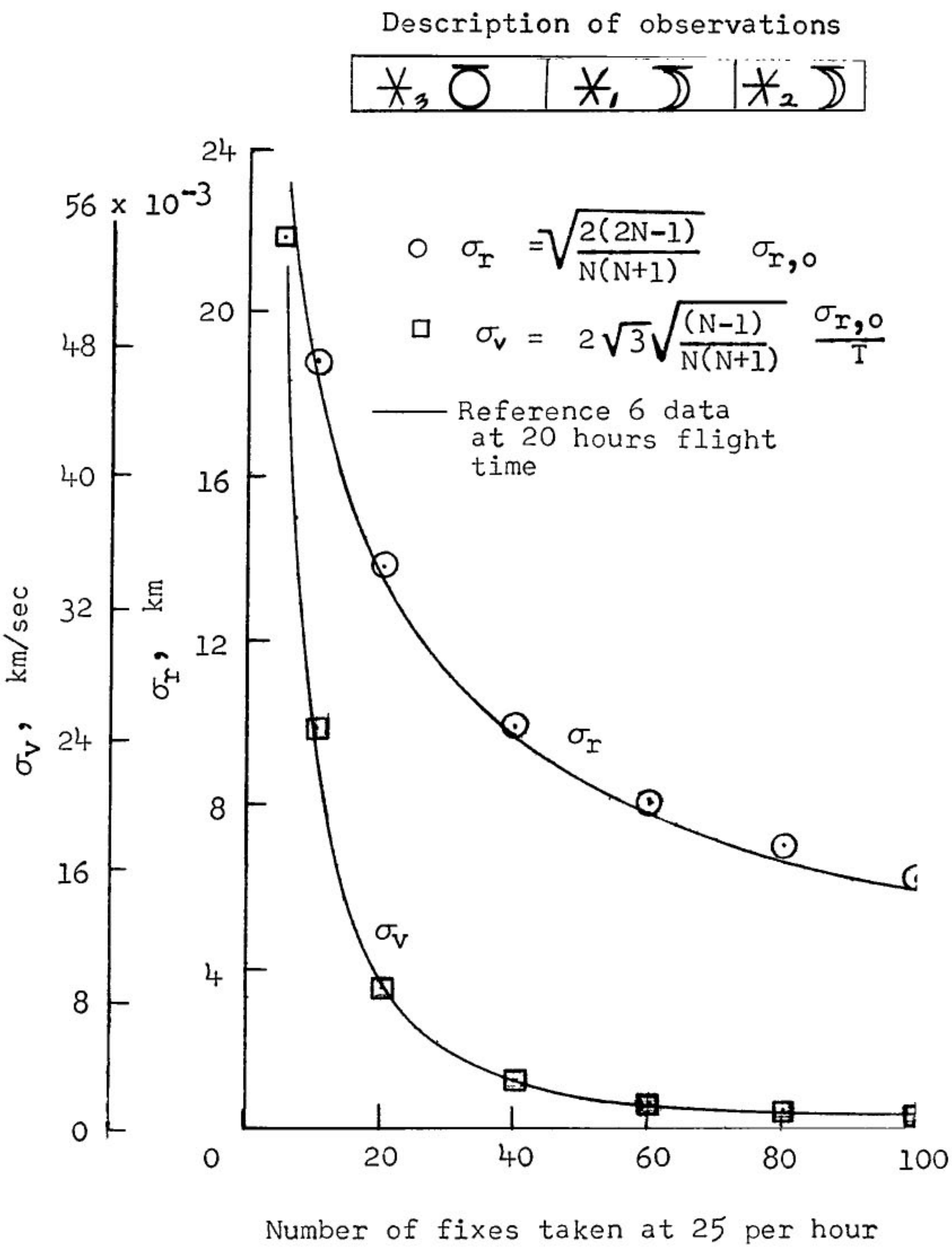


Figure 9.- Root sum square of standard deviations of errors in position and velocity (20 hours after translunar injection) as a function of the number of fixes.

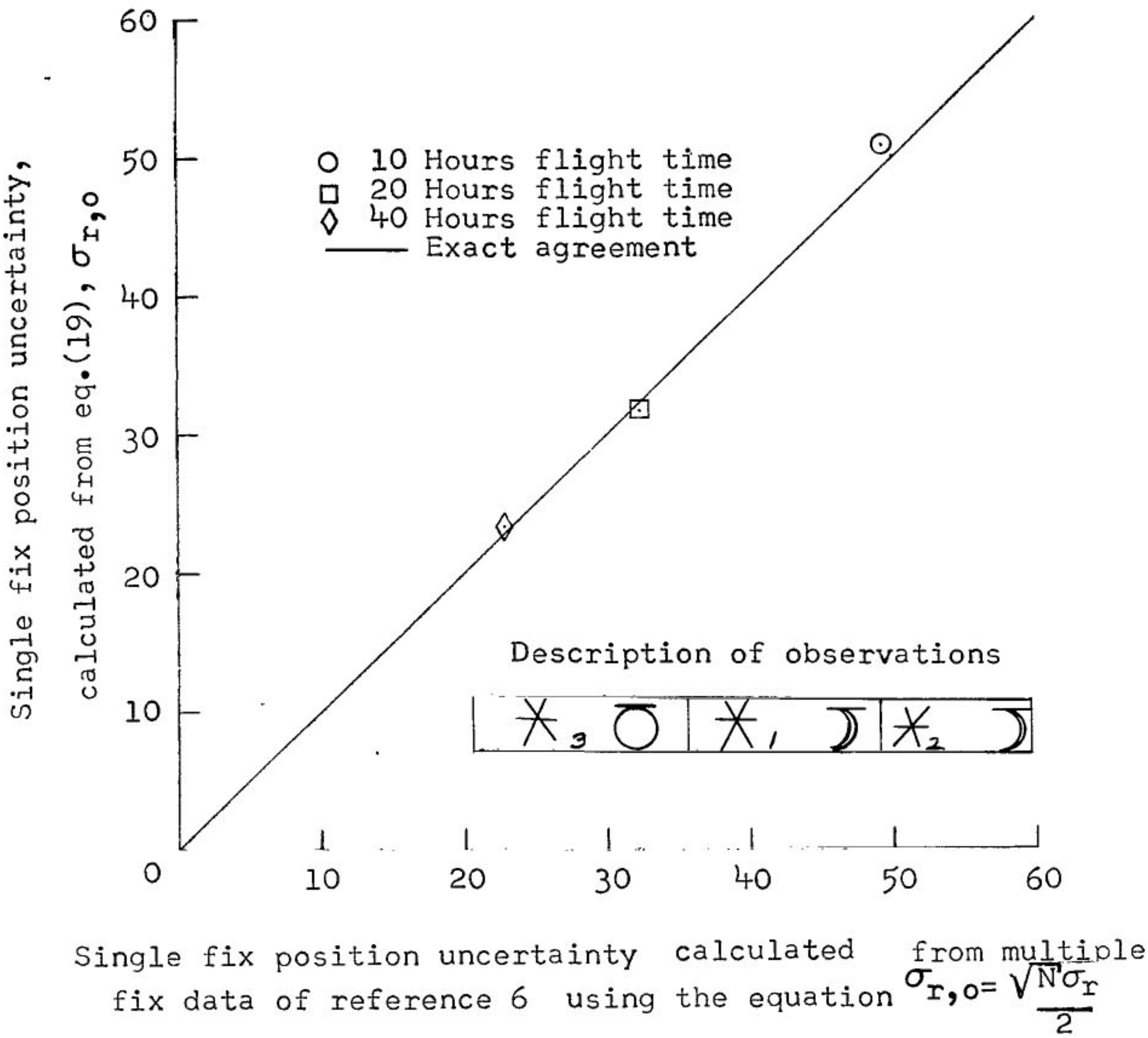


Figure 10.- Comparison of single fix position uncertainty obtained from equation (19) with that obtained from extrapolating the results of 40 fixes taken evenly over a 4-hour period to a single fix.

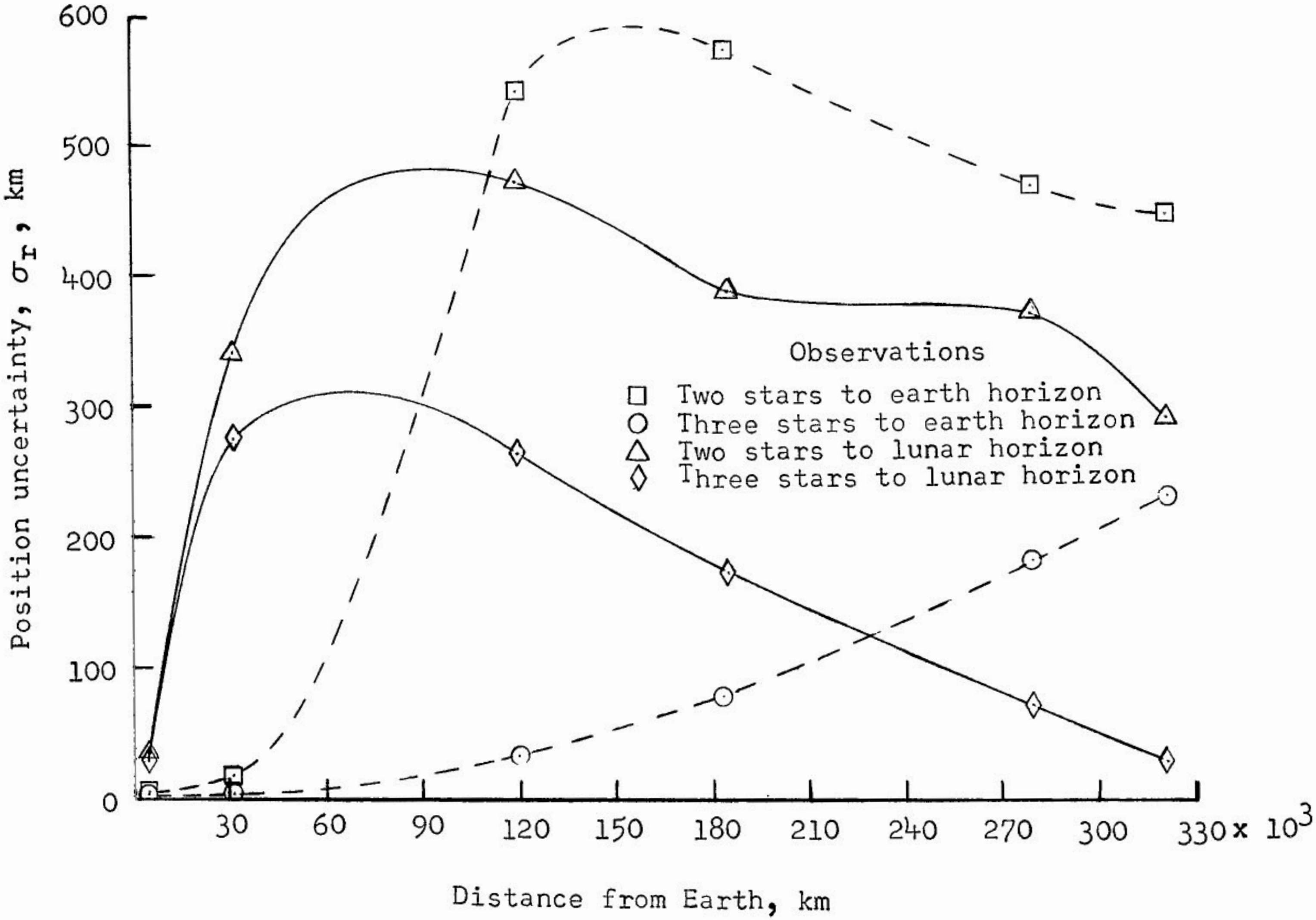


Figure 11.- Comparison of navigational accuracy obtained with three observations per data time with that using only two observations. Data are from reference 6.

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