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Electrohydrodynamic Rayleigh-Taylor Bulk Instability

by

Robert J. Turnbull and James R. Melcher

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Robert J. Turnbull

Department of Electrical Engineering
University of Illinois, Urbana, Illinois

and

James R. Melcher

Department of Electrical Engineering
Massachusetts Institute of Technology

Abstract

A stability criterion is found for an initially static, stratified fluid subject to an electric stress. The equilibrium fluid density, permittivity, viscosity and space charge distributions are functions of the vertical spatial coordinate, with gradients directed parallel with the imposed electric field intensity and gravitational acceleration. In the limit where the fluid is perfectly insulating, sufficient conditions for stability are found; the principle of exchange of stabilities is shown, and variational principles derived for the eigenfrequencies and, in the case of no space charge, for the critical field strength. An experiment demonstrates instability in the dielectrophoretic limit of no bulk free charge. The property gradients are induced thermally, and incipience of instability, as measured by the Schmidt-Milverton heat transfer technique, is successfully predicted by the theoretical criterion.

I. Introduction

A. Background

Careful observations of motions induced by electric stresses in the bulk of slightly conducting liquids make it clear that the mechanisms for electromechanical interaction are diverse. In part, this can be traced to the variety of electrical conduction processes that can dominate in highly insulating liquids.[1] For an example, unipolar ions originating at electrodes can lead to a conduction current proportional to the local net space charge. As a result, space charge can accumulate in the bulk of a supposedly homogeneous liquid between capacitor plates[2,3]. The bulk cellular convection described by Avsec and Luntz[4] is probably related to such space charge effects.

As a second example two or more species of ionized carriers originating in the bulk of the liquid can give a conduction current even with net local charge neutrality; ohmic conduction is such a case. Even in an ohmic fluid, however, electromechanically induced motions can be traced to property gradients. It has been shown that a simple ohmic conduction model, including the possibility of gradients in the conductivity, is highly successful in understanding motions resulting from the combined action of a thermal stress, which induces the property gradients, and an electric field.[5] This class of bulk interactions can be modeled by a spatially varying electrical conductivity and permittivity (polarizability). The former accounts for an accumulation of bulk free charge and an attendent free charge force density, while the latter gives rise to both free charge and dielectrophoretic force densities.[6]

Studies have been made of the effect of electric fields on convective heat

transfer.[7] Although the dielectrophoretic (gradient in permittivity) mechanism is thought by some to be the source of motions induced in thermally stressed fluids by the fields, it is usually the case that the free charge forces due to thermally induced spatial variation of electrical conductivity is the dominant mechanism.

B. Objectives

At least superficially, the role of the electric field as it induces convection is similar to that of gravity. Thus, it is not surprising that basic studies relate to situations similar to the classic Rayleigh-Taylor instability and Bénard instability.[8] The onset of convection, with its concomitant influence on the heat transfer in cases where a thermal gradient is involved, occurs as an instability.

The fluid-field configuration to be considered here is shown in Fig. 1, where electrodes are used to impose a perpendicularly directed electric field intensity $E(z)$ on an initially static liquid. The fluid is incompressible, but nevertheless arranged in horizontal strata so that the equilibrium density ρ, viscosity μ, pressure p, space charge density q, and permittivity ε are functions of z. For present purposes, the mechanism giving rise to the liquid inhomogeneity is not of interest. For example, the equilibrium can be obtained by carefully superimposing layers of liquid which have differing properties. The inhomogeneity can also be induced by a thermal gradient; the analysis given here is appropriate, provided that the thermal diffusion time[8, p.18] is long compared to dynamical times of interest, for example, the time constant for an instability.

Detailed attention is given elsewhere to the prediction of the critical

field for the onset of convection in the configuration of Fig. 1.[9] An ohmic model is used to include the effects of the electrical conduction. To include the effects of a finite electrical relaxation time, it is necessary to make a considerable number of approximations not required if the fluid is considered at the outset to be perfectly insulating.

In the following sections, development is confined to the perfectly insulating fluid. This allows the derivation of rather general conditions for stability and instability that represent the limiting case, regardless of the electrical conduction model. That is, if a mobility model is used to represent electrical conduction, then the derivations are for the limit of zero mobility, and if an ohmic conduction model is used, then they are for the limit of zero conductivity. A variational principle to describe the stability of superimposed fluids is suggested by Chandrasekhar[8, p.436] and improved by Selig.[10] Here, this variational principle for determining eigenfrequencies is generalized to include the effects of electrical forces. The principle of exchange of stabilities is shown, thus making it possible to reduce the prediction of instability onset to an eigenvalue problem in the electric field.

Included in the analysis as a limiting case is the situation in which there is no space charge; instabilities are predicted having a purely dielectrophoretic nature. These results complement those for the case where the applied field is directed perpendicularly to the gradient in permittivity.[11] A variational principle is given for the critical imposed field in this dielectrophoretic case, and experiments are described in which the gradient in permittivity is induced by a thermal stress.

II. Stability Criterion

A. Equations of Motion

The distribution of equilibrium space charge $q(z)$ is determined by the details of a given experiment. Even though the assumption that the fluid is perfectly insulating makes it possible to ignore the origins of the bulk charge, Gauss' law requires that it be consistent with the equilibrium distribution of E and ε. Thus

$$D(\varepsilon E) = q \tag{1}$$

where $d(\ )/dz \equiv D(\ )$.

Static force equilibrium is possible because the electric force density[12]

$$\overline{F} = q\overline{E} - \frac{1}{2}\overline{E}\cdot\overline{E}\nabla\varepsilon + \nabla\left(\frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}\overline{E}\cdot\overline{E}\right) \tag{2}$$

has an equilibrium distribution which is irrotational and is balanced by the pressure. The last term in Eq. (2) is the gradient of a scalar and the quantity in parentheses will be included with the pressure, p.

To investigate the stability of the static equilibrium, small perturbations are denoted by $\overline{v}$ and ϕ respectively for the velocity and electric potential, and by ρ', p', ε' and q' respectively for the mass density, pressure, permittivity, and free charge density. Thus, the linearized equation of motion becomes

$$\rho\frac{\partial\overline{v}}{\partial t} = -g\rho'\overline{i}_z - \nabla p' + \mu\nabla^2\overline{v} + 2(\nabla\mu\cdot\nabla)\overline{v}$$
$$+ \nabla\mu \times(\nabla \times \overline{v}) + q'E\overline{i}_z - q\nabla\phi - \frac{1}{2}(\overline{E}\cdot\overline{E})\nabla\varepsilon' + E\nabla\varepsilon\frac{\partial\phi}{\partial z} \tag{3}$$

where the equilibrium terms are by definition balanced out, and $\bar{i}_z$ is the unit vector in the z direction.

The fluid is incompressible, as reflected by the equation of continuity

$$\nabla \cdot \bar{v} = 0 \tag{4}$$

At the same time, it is inhomogeneous, with the density, permittivity, and free charge density of a given fluid element constant. These three conditions require

$$\frac{\partial \gamma'}{\partial t} = - v_z D\gamma \qquad (5)-(7)$$

where $(\gamma,\gamma') = (\rho,\rho'),(\varepsilon,\varepsilon')$ and (q,q')

The perturbation equations are completed by writing Gauss' law to linear terms and taking advantage of Eq. (1)

$$- \varepsilon\nabla^2\phi + D(E\varepsilon') - D\varepsilon D\phi = q' \tag{8}$$

Boundary conditions imposed by the electrodes at z = 0 and z = d, together with Eqs. (3) - (8), formulate the problem.

The physical situation remains unaltered by a rotation of coordinates about the z axis, and hence, without loss of generality, can be considered as two-dimensional. Thus, perturbation solutions are assumed to have the standard form

$$\phi = \mathrm{Re}\ \hat{\phi}(z) \exp j(\omega t - kx) \tag{9}$$

In terms of the complex amplitudes, the conditions that the velocity and perturbation potential vanish on the electrodes are

$$\hat{v}_z = D\hat{v}_z = \hat{\phi} = 0 \qquad \text{at } z = 0, d \tag{10}$$

where Eq. (4) has been used to write the condition on $\hat{v}_x$ in terms of $\hat{v}_z$. Thus, it is convenient to work with the variables $\hat{v}_z$ and $\hat{\phi}$. This is done by substituting solutions of the form of Eq. (9) into Eqs. (3) - (8) to obtain a set of ordinary differential equation in the complex amplitudes of the perturbation quantities. Equations (4) - (7) are solved for $\hat{v}_x$, $\hat{\rho}$, $\hat{\varepsilon}$, and $\hat{q}$ as functions of $\hat{v}_z$. These expressions make it possible to solve the x component of Eq. (3) for $\hat{p}$ as a function of $\hat{v}_z$ and $\hat{\phi}$. In turn, substitution of these amplitudes into the z component of Eq. (3) gives

$$\begin{aligned} j\omega[D(\rho D\hat{v}_z) - \rho k^2\hat{v}_z] &= D[\mu(D^2 - k^2)D\hat{v}_z] \\ &- \mu k^2(D^2 - k^2)\hat{v}_z + D[D\mu(D^2 + k^2)\hat{v}_z] - 2k^2 D\mu D\hat{v}_z \\ &- k^2 g\frac{D\rho}{j\omega}\hat{v}_z - k^2 Dq\hat{\phi} + k^2 E\frac{Dq}{j\omega}\hat{v}_z + k^2 E\frac{DED\varepsilon}{j\omega}\hat{v}_z - k^2 ED\varepsilon D\hat{\phi} \end{aligned} \tag{11}$$

In a similar manipulation, Eqs. (6) and (7) eliminate $\hat{\varepsilon}$ and $\hat{q}$ from Eq. (8) to give a second equation for $\hat{v}_z$ and $\hat{\phi}$

$$D[\varepsilon D\hat{\phi}] - k^2\varepsilon\hat{\phi} + \frac{DED\varepsilon}{j\omega}\hat{v}_z + \frac{ED(D\varepsilon\hat{v}_z)}{j\omega} - \frac{Dq}{j\omega}\hat{v}_z = 0 \tag{12}$$

The eigenvalue problem is summarized by Eqs. (10) - (12).

B. Exchange of Stabilities

Without restricting the equilibrium distributions, a considerable insight into the nature and conditions for incipience of the instability can be obtained by using judiciously combined integrals of Eqs. (11) and (12) and the boundary conditions. With this objective, Eq. (11) is multiplied by $\hat{v}_z^*$

(the complex conjugate of $\hat{v}_z$) and integrated over the vertical extent of the fluid to obtain

$$j\omega I_1 = - I_2 + \frac{I_3}{j\omega} + I_4 \tag{13}$$

where

$$I_1 = - \int_0^d \hat{v}_z^*[D(\rho D\hat{v}_z) - k^2\rho\hat{v}_z]dz \tag{14}$$

$$I_2 = - \int_0^d \hat{v}_z^*\Big\{- D[\mu(D^2 - k^2)D\,\hat{v}_z] + k^2\mu(D^2 - k^2)\hat{v}_z + 2k^2D\hat{v}_z D\mu - D[D\mu(D^2 + k^2)]\hat{v}_z\Big\}\,dz \tag{15}$$

$$I_3 = k^2 \int_0^d \Theta|\hat{v}_z|^2 dz; \quad \Theta = gD\rho - EDq - EDED\varepsilon \tag{16}$$

and

$$I_4 = k^2 \int_0^d \hat{v}_z^* \,[Dq\hat{\phi} + D\varepsilon ED\hat{\phi}]dz \tag{17}$$

Integration by parts of the first term in the integrand of Eq. (14) together with the condition that $\hat{v}_z = 0$ at the integration limits shows that I_1 is positive and real

$$I_1 = \int_0^d \rho[|D\hat{v}_z|^2 + k^2|\hat{v}_z|^2]dz \tag{18}$$

Similar, but more complicated, manipulations as summarized in the Appendix show that I_2 is also positive and real

$$I_2 = \int_0^d \mu[|(D^2 + k^2)\hat{v}_z|^2 + 4k^2|D\hat{v}_z|^2]dz \tag{19}$$

The electrical equation is handled in an analogous fashion. Multiplication of Eq. (12) by $k^2\hat{\phi}^*$ and integration gives

$$- I_5 + \frac{I_6}{j\omega} = 0 \tag{20}$$

where the first two terms of Eq. (12) contribute

$$I_5 = - k^2 \int_0^d \hat{\phi}^*[D(\varepsilon D\hat{\phi}) - k^2\varepsilon\hat{\phi}]dz \tag{21}$$

while the third and fourth terms combine, and together with the last term, give

$$I_6 = k^2 \int_0^d \hat{\phi}^*[D(ED\varepsilon\hat{v}_z) - Dq\hat{v}_z]dz \tag{22}$$

Integration of Eq. (21) by parts with the boundary conditions on $\hat{\phi}$ gives

$$I_5 = k^2 \int_0^d \varepsilon[\,|D\hat{\phi}|^2 + k^2|\hat{\phi}|^2]dz; \tag{23}$$

a positive, real, function. Moreover, integration of the first term in Eq. (22) taken with the boundary conditions on $\hat{\phi}$ shows that

$$I_6 = - I_4^* \tag{24}$$

Now, the complex conjugate of Eq. (20) can be expressed in terms of I_4 and then I_4 eliminated between the resulting expression and Eq. (13) to obtain

$$- \omega^2 I_1 + j\omega I_2 - I_3 + \omega\omega^* I_5 = 0 \tag{25}$$

This expression shows that if a perturbation is unstable, it is statically unstable. To see this, α and β are defined as real quantities that represent the real and imaginary parts of ω.

$$\omega = \alpha + j\beta \tag{26}$$

Substitution of this form into Eq. (25) gives an expression with real and imaginary parts, each of which must vanish. All integrals in Eq. (25) are real. The imaginary part of Eq. (25) is

$$- 2\alpha\beta I_1 + \alpha I_2 = 0 \tag{27}$$

Equations (18) and (19) show that I_1 and I_2 are positive.

In order for Eq. (27) to be satisfied, either $\beta > 0$ (and the system is stable) or $\alpha = 0$ (in which case any instability has an exponentially growing amplitude). Thus, if an electric field is applied to a stable configuration of fluid and charge, instability is incipient at zero frequency. The principle of exchange of stabilities is valid.

C. Incipience

Now that it has been established that the incipient instability is static, the real part of Eq. (25) can be solved for β with $\alpha = 0$.

$$\beta = \left\{ I_2 \pm [I_2^2 + 4I_3(I_1 + I_5)]^{1/2} \right\} / 2(I_1 + I_5) \tag{28}$$

This expression is valid so long as the equilibrium is unstable ($\beta < 0$), because in that case it is known that $\alpha = 0$. Note that if I_3 is negative, Eq. (28) does not give the negative value of β required for instability. A sufficient condition for stability is $I_3 < 0$, a condition certainly satisfied if the integrand of I_3 is everywhere negative in the fluid; $\Theta < 0$. Because of the exchange of stabilities at zero frequency, it is possible to carry this development further and be specific about the field required to make an equilibrium that is stable in the absence of electric forces ($D\rho < 0$ everywhere) verge on instability. It has been established that as the electric field is raised,

instability is incipient with $\omega = 0$. The critical condition is described by Eqs. (11) and (12) in the limit of $\omega \to 0$. In that limit, $\hat{v}_z \to 0$ but $\hat{v}_z/j\omega$ remains finite and equations (11) and (12) become

$$\cancel{\Theta}\hat{\xi} + ED\varepsilon D\hat{\phi} + Dq\hat{\phi} = 0 \tag{29}$$

$$D[\varepsilon D\hat{\phi}] - k^2\varepsilon\hat{\phi} + D[ED\varepsilon\hat{\xi}] - Dq\hat{\xi} = 0 \tag{30}$$

where $\hat{\xi} = \hat{v}_z/j\omega$ is the z-displacement of the fluid. Since the velocity approaches zero, the only boundary conditions remaining in the limit of instability onset are on the potential.

If the integral terms are written using $\hat{\xi}$ instead of $\hat{v}_z$, Eq. (25) becomes

$$-\omega^2 I_1' + j\omega I_2' - I_3' + I_5 = 0 \tag{31}$$

where I_1', I_2' and I_3' are the same as I_1, I_2, and I_3 except that $\hat{v}_z$ is replaced with $\hat{\xi}$. Eq. (31) shows that the threshold occurs as $I_3' = I_5$, that is, as

$$\int_0^d \cancel{\Theta}|\hat{\xi}|^2 dz = \int_0^d \varepsilon[|D\phi|^2 + k^2|\hat{\phi}|^2]dz \tag{32}$$

Consider the case where the interaction is purely dielectrophoretic, with the definition:

$$\hat{u} = -\frac{\varepsilon g D\rho}{ED\varepsilon}\hat{\xi} \tag{33}$$

and the equilibrium condition $\varepsilon DE = -ED\varepsilon$, Eqs. (29) and (30) combined and expressed in terms of $\hat{u}$ give

$$D\left[\frac{1}{\varepsilon} D\hat{u}\right] - \frac{k^2}{\varepsilon}[1 - B_e^{-1} f(z)]\hat{u} = 0 \tag{34}$$

where $B_e^{-1} f(z) = - \frac{E^2(D\varepsilon)^2}{\varepsilon g D\rho}$

This is the standard Sturm-Liouville form with B_e^{-1} [13] an eigenvalue parameter that must be positive.[14] Boundary conditions are $D\hat{u} = 0$ at $z = 0,d$, as can be seen from Eq. (30) if it is recognized that $\varepsilon D\hat{\phi} \propto \hat{u}$ and $\hat{\xi} \propto \hat{u}$. Thus, in the usual way, Eq. (34) can be multiplied by $\hat{u}^*$ and integrated by parts to give

$$B_e^{-1} = \int_0^d \frac{1}{\varepsilon} [|D\hat{u}|^2 + k^2|\hat{u}|^2]dz \Big/ k^2 \int_0^d \frac{1}{\varepsilon} |\hat{u}|^2 f(z)dz \tag{35}$$

If it is assumed that $B_e^{-1}f(z)$ is bounded in $0 \leq z \leq d$, $f(z)$ can be normalized so that its maximum value is 1. Then B_e^{-1} is proportional to the voltage squared. Eq. (35) can be used as the basis for a variational principle to determine B_e^{-1} more directly; using Sturm's oscillation theorem,[14] an upper bound can be found for the lowest value of B_e^{-1} for which Eq. (34) has a solution that satisfies the boundary conditions. This bound is

$$B_e^{-1} \leq \left[(\pi/d)^2(\varepsilon_{max}/\varepsilon_{min}) + k^2\right] \Big/ k^2 f_{min} \tag{36}$$

where f_{min} is the minimum value of $f(z)$ and ε_{max} and ε_{min} are respectively the maximum and minimum values of ε. Also, the sufficient conditions for stability $\Theta < 0$ is equivalent to $B_e^{-1} \leq 1$. Since in an actual system, there is no lower limit on the wavelengths, the limit of Eq. (36) can be taken as $k \to \infty$. Then the bounds on the value of B_e^{-1} at which the threshold is reached are

$$1 \leq B_e^{-1} \leq \frac{1}{f_{min}} \tag{37}$$

In the special case where $f(z)$ can be considered a constant, $B_e^{-1} = 1$ is the point of instability.

D. Variational Principle for Eigenfrequencies

Emphasis has been given to incipient instability. For purposes of finding growth rates or oscillation frequencies under conditions other than those for onset, a variational principle provides a useful starting point. The derivation is similar to that leading to Eq. (28). First, Eq. (11) is multiplied by $\hat{v}_z$ and integrated from $z = 0$ to $z = d$. Integration by parts, using the same procedure as in Sec. IIB, yields

$$j\omega I_1 = -I_2 + \frac{I_3}{j\omega} + I_4 \tag{38}$$

where the integrals I_1 through I_4 are defined by Eqs. (16) - (19) without the complex conjugation of $\hat{v}_z$. Similarly, multiplication of Eq. (12) by $\hat{\phi}$ and integration by parts gives

$$j\omega I_5 + I_4 = 0 \tag{39}$$

where I_5 is defined by Eq. (23) without the complex conjugation of $\hat{\phi}$.

The sum of Eqs. (38) and (39) is

$$j\omega I_1 = -I_2 + \frac{I_3}{j\omega} + 2I_4 + j\omega I_5 \tag{40}$$

Now, consider the effect on ω of arbitrary variations $\delta\hat{v}_z$ and $\delta\hat{\phi}$ in $\hat{v}_z$ and $\hat{\phi}$ consistent with the boundary conditions. To first order terms in the variation, Eq. (40) becomes

$$\delta\omega\left[jI_1 + \frac{I_3}{j\omega^2} - jI_5\right] = -j\omega\delta I_1 - \delta I_2 + \frac{\delta I_3}{j\omega} + 2\delta I_4 + j\omega\delta I_5 \tag{41}$$

Integration by parts of each term on the right, and a recombination of terms, gives

$$\delta\omega[jI_1 + \frac{I_3}{j\omega^2} - jI_5] = \int_0^d 2\delta\hat{v}_z \Big\{ j\omega D(\rho D\hat{v}_z) - j\omega k^2\rho\hat{v}_z - D[\mu(D^2 - k^2)D\hat{v}_z]$$

$$+ k^2\mu(D^2 - k^2)\hat{v}_z + 2k^2 D\mu D\hat{v}_z - D[D\mu(D^2 + k^2)\hat{v}_z] + \frac{k^2}{j\omega}\theta\hat{v}_z + k^2[Dq\hat{\phi} + ED\varepsilon D\hat{\phi}]\Big\} dz$$

$$+ 2k^2 \int_0^d \delta\hat{\phi}[Dq\hat{v}_z - D(ED\varepsilon\hat{v}_z) - j\omega D(\varepsilon D\hat{\phi}) + j\omega k^2\varepsilon\hat{\phi}]dz \qquad (42)$$

Thus, for arbitrary variations of $\hat{v}_z$ and $\hat{\phi}$ the right-hand side of Eq. (42) vanishes if Eqs. (11) and (12) are satisfied. Therefore, the condition for $\delta\omega = 0$ to first order for all small arbitrary variations of $\delta\hat{v}_z$ and $\delta\hat{\phi}$ compatible with the boundary conditions is that $\hat{v}_z$ and $\hat{\phi}$ be solutions to the eigenvalue problem. A variational procedure of solving for the eigenfrequencies is therefore possible.

III. Dielectrophoretic Heat Transfer Experiment

A. Thermally-Induced Instability

Examples wherein the effects of bulk free charge, q, can be ignored as a fluid is subjected to combined thermal and dc electrical stresses are unusual. By contrast, if fields of alternating polarity are used which have sufficiently high frequency that free charges do not have time to relax to the bulk of the fluid, it is possible to demonstrate dielectrophoretic bulk effects. The limiting case $q = 0$ of the theory developed in Sec. IIC is applicable to this latter physical situation, provided the fluid is sufficiently viscous that it does not react in a parametric fashion to the pulsating component of the applied field.

In the experiment shown in Fig. 2, vertically directed gradients in density and permittivity are provided by a temperature gradient. The liquid is warmer on the top, therefore $D\rho$ is negative, and the static equilibrium is stable in the absence of an electric field. In a non-polar liquid, the permittivity varies with temperature only because the mass density changes,[15] the temperature dependence of permittivity given by the Clausius-Mossotti relation[12, p. 140]

$$\frac{d\varepsilon}{d\rho} = \frac{1}{\rho}\frac{(\varepsilon - \varepsilon_o)(\varepsilon + 2\varepsilon_o)}{3\,\varepsilon_o} \tag{43}$$

Liquids used are good insulators and sufficiently viscous that a 60 Hz applied field is equivalent to a dc field having the same rms value. Moreover, the thermal diffusion time of the liquids is sufficiently long that heat conduction effects, not included in the theoretical developments, make a negligible contribution to the incipience of instability.

The instability is detected by means of the Schmidt-Milverton technique.[16] As the electric field is raised, the instability manifests itself through the increase in the heat transferred through the liquid. If the Nusselt number, defined

$$N_u = Q/k_c(\Delta T/d), \tag{44}$$

exceeds unity, convection is present due to the incipience of instability. Here, Q is the heat flux, k_c is the thermal conductivity, and ΔT is the temperature drop across the liquid.

B. Procedure and Data Reduction

Experiments consist of measuring the temperatures of the high temperature bath, lower electrode and metal base with various voltages, V, applied to the electrodes bounding the liquid. The high temperature bath is well stirred and maintained in temperature to within 1°C. The low temperature bath consists of ice and water. Temperatures are measured by means of thermistors with the two base measurements taken to within .2° C. The temperature drop across the test liquid is usually in the range 85 - 105° C, and that across the glass in the range 13 - 30° C. The voltage, which ranges from 10 to 25 Kv, is measured to within 1%, while the liquid depth is about d = 3/16 inch.

The temperature drop across the test liquid, the applied voltage, and the fluid properties comprises sufficient information to calculate B_e^{-1}. To a good approximation, E is unaffected by the variation in ε. Then, because $D\varepsilon$ and $D\rho$ are essentially constant, $f(z) = 1$ and

$$B_e^{-1} = -E^2(D\varepsilon)^2/g\varepsilon D\rho \tag{45}$$

where $E = V/d$, $D\varepsilon = (d\varepsilon/d\rho)(d\rho/dT)(\Delta T/d)$ and $D\rho = (d\rho/dT)(\Delta T/d)$. Two liquids are phenylmethyl silicone fluid (Dow Corning 550), and a dimethyl silicone fluid (Dow Corning 200, 100 cs grade). Data required to complete Eq. (45) are summarized in Table I [17] with $d\varepsilon/d\rho$ given by Eq. (42).

The value of N_u can be computed from the temperature data using the zero voltage measurements as a calibration. Thus, $N_u = (\Delta T/\Delta T_g)_o/(\Delta T/\Delta T_g)$, where ΔT_g is the temperature drop across the glass plate, and $(\Delta T/\Delta T_g)_o$ is measured at zero voltage.

C. Measurements Compared to Predictions

The experimental values of Nusselt number as a function of B_e^{-1} are summarized in Fig. 3. For the Dow Corning 550, the Nusselt number remains at unity, indicating no convection, until B_e^{-1} reaches 1.00 ± 0.05. For the Dow Corning 200 the instability occurs at $B_e^{-1} = 1.05 \pm 0.05$. Thus, the theoretical prediction of instability at $B_e^{-1} = 1$ is well verified. Note that curves for the two fluids should not be the same beyond the break point $B_e^{-1} = 1$, because the viscosities of the two differ.

IV. Concluding Remarks

The recent literature shows that bulk instabilities found in essentially the configuration of Fig. 1 are important for understanding conduction processes in highly insulating liquids[18] and for electro-optical image reproduction[19]

The effects of finite mobility or conductivity in cases involving space charge are likely to be important; nevertheless, the zero mobility (or zero conductivity) case developed here is significant as the limiting case for more complicated and hence more specialized configurations. For future work theoretical results of importance derived here include the principle of exchange of stabilities, the sufficient conditions for stability, and the variational principle for determining the eigenfrequencies. A variational principle for determining the critical conditions for instability and the experiments, as they are successfully correlated with the theory, draws specific attention to the dielectrophoretic limit.

Similar extensions of the theory are called for to provide variational principles for the critical field in the case of bulk free charge.

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V. Appendix

The several steps required to reduce the integral I_2 of Eq. (15) to a positive definite form are summarized here. First, I_2 is rewritten as

$$I_2 = \int_0^d \hat{v}_z^* \left\{ D^2[\mu(D^2 + k^2)\hat{v}_z] - 2k^2 D(\mu D\hat{v}_z) - k^2\mu(D^2 - k^2)\hat{v}_z - 2k^2 D\hat{v}_z D\mu \right\} dz \quad \text{(A1)}$$

Further manipulations of the integrand give

$$I_2 = \int_0^d \hat{v}_z^* \left\{ D^2[\mu(D^2 + k^2)\hat{v}_z] + k^2\mu(D^2 + k^2)\hat{v}_z - 4k^2 D(\mu D\hat{v}_z) \right\} dz \quad \text{(A2)}$$

Now, the first and last terms in the integrand are integrated by parts;

$$I_2 = \hat{v}_z^* D[\mu(D^2 + k^2)\hat{v}_z]\Big|_0^d - 4k^2\hat{v}_z^* \mu D\hat{v}_z\Big|_0^d$$
$$+ \int_0^d \left\{ [- D\hat{v}_z^* D[\mu(D^2 + k^2)\hat{v}_z] + \hat{v}_z^* k^2[\mu(D^2 + k^2)\hat{v}_z] + 4k^2\mu|D\hat{v}_z|^2 \right\} dz \quad \text{(A3)}$$

From the boundary conditions on $\hat{v}_z$ the first two terms vanish. Once again, the first term in the integrand of Eq. (A3) is integrated by parts. Finally, if the boundary conditions on $D\hat{v}_z$ are used, Eq. (19) follows.

Table I: Properties of Fluids Used to Obtain Data of Fig. 3

Fluid	Coefficient of Expansion cc/cc/°C	Mass Density kg/m^3	Permittivity
Dow Corning 550 (at 25°C)	7.5×10^{-4}	1.070×10^3	$2.90\ \varepsilon_0$
Dow Corning 200 (100 cs @ 23°C)	9.6×10^{-4}	0.968×10^3	$2.73\ \varepsilon_0$

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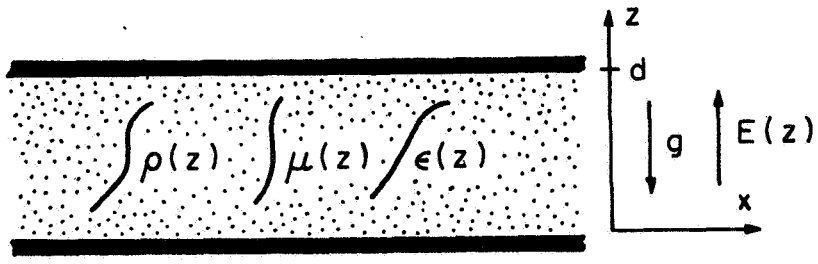
List of Figures

Fig. 1 An initially static fluid, stratified in the z direction, is stressed by an electric field intensity E(z) imposed by means of perfectly conducting electrodes (at z = 0,d) constrained to a constant potential difference. There is an equilibrium distribution of mass density, viscosity, permittivity and space charge in the vertical direction.

Fig. 2 Experimental apparatus for measuring heat transferred through liquid C as a function of the electric Bond number. A and B are, respectively, the hot and cold temperature baths, D and F are metal electrodes used to impose the electric stress, G is a glass plate to provide a thermal resistance for determining the heat transferred through C, H is a metal base at the temperature of B, and E is the plexiglas container and electrode support.

Fig. 3 Nusselt number as a function of reciprocal electric Bond number in the experiment of Fig. 2. N_u is defined by Eq. (44), while B_e^{-1} is calculated from physical parameters and measured voltages using Eq. (45). Circled data points are for 550 fluid, while points in squares are for 200 fluid. Because the average viscosity of 550 fluid is less than that of the 200 fluid, N_u increases faster for the 500 fluid.



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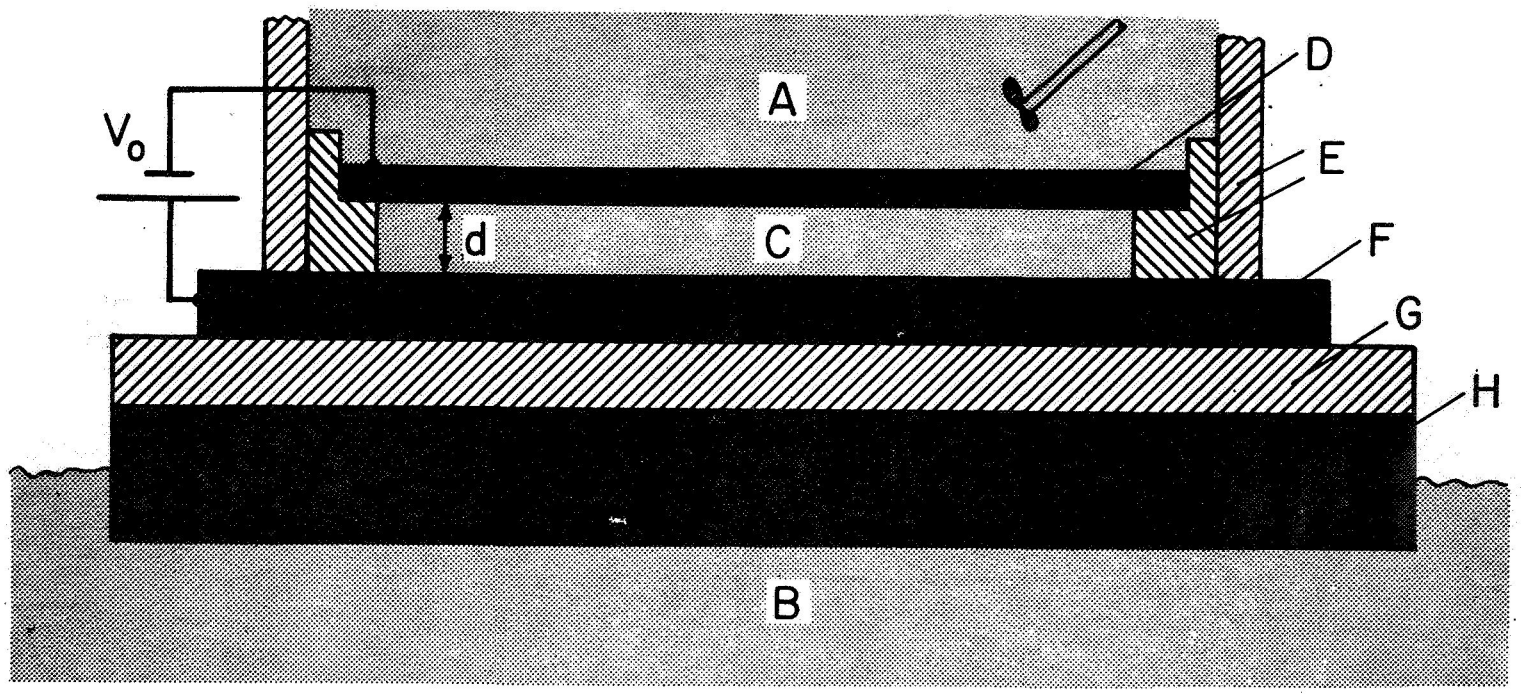

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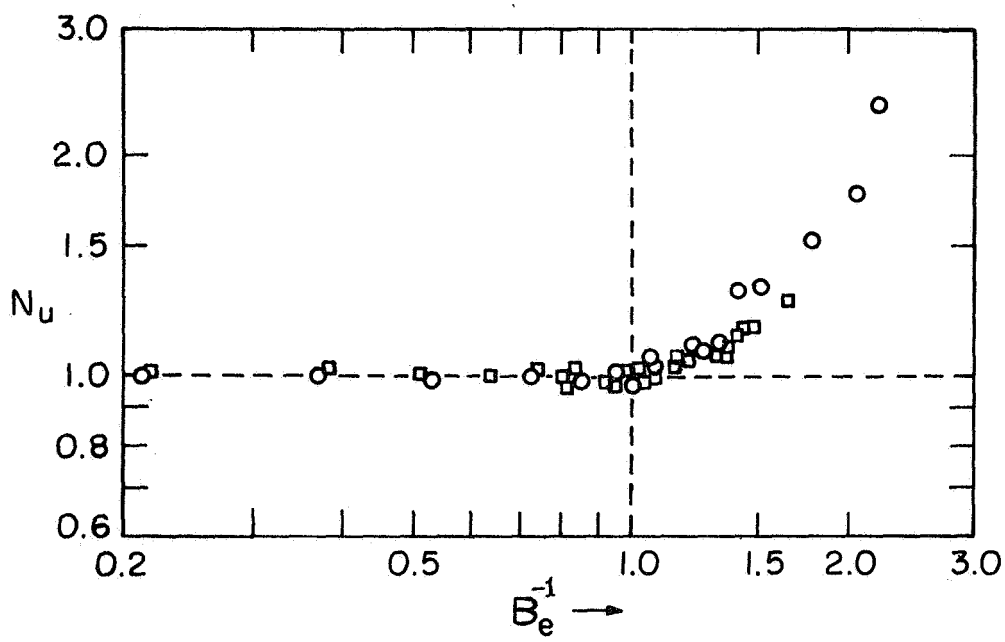

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