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DESIGN PROCEDURES FOR DOMINANT TYPE SYSTEMS
WITH LARGE PARAMETER VARIATIONS

By Don Eugene Olson

November 12, 1968

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Prepared under research grant NGR 06-003-083 by
Department of Electrical Engineering
University of Colorado
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for
Flight Research Center
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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DESIGN PROCEDURES FOR DOMINANT TYPE SYSTEMS
WITH LARGE PARAMETER VARIATIONS

by

Don Eugene Olson

This research was sponsored by the
National Aeronautics and Space Administration
under research grant NGR 06-003-083

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November 12, 1968
DESIGN PROCEDURES FOR DOMINANT TYPE SYSTEMS WITH LARGE PARAMETER VARIATIONS

Abstract—This report presents design procedures for fourth-order dominant type systems with large plant parameter variations. The s-domain specifications of the system are assumed to be in the form of an acceptable dominant closed loop pole region and bounds on the location of the "far-off" closed loop poles. The design philosophy is to place compensation zeros within the acceptable dominant closed loop pole region such that the dominant closed loop poles remain within their prescribed region despite the large variations in the plant parameters. Design procedures are presented for variation in the plant gain factor only and for simultaneous variation in the plant gain factor and the plant poles. Finally, an approximate procedure is presented which considers simultaneous variation in the plant gain factor, the plant poles and a plant zero located on the real axis in the s-plane. In all cases, the design procedures are such as to minimize the sensitivity of the system to internal noise.
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CHAPTER I

PROBLEM STATEMENT AND DESIGN PHILOSOPHY

1.1 Statement of the Problem

The problem considered in this document may be stated as follows: 1). A single input-single output plant* \( P(s) \) has parameters (gain factor, poles and zeroes) which may lie or "slowly" vary within a given region in the s-plane. 2). The acceptable region of closed loop poles is specified in the s-plane. This acceptable region is determined by transforming the typical type domain specifications such as rise time, over-shoot and settling time into specifications on the location of the dominant poles of the system. This transformation of these specifications from the time domain to the pole-zero domain is considered by Barber.¹ 3). Linear time invariant compensation is to be chosen to satisfy the above specifications such that the effect of internal noise at the plant input

*The term "plant" is used here in the commonly accepted sense in the control literature to denote the constrained part of the system whose output is the system output.
is minimized. The pictorial representation of a typical problem encountered in flight control is shown in Fig. 1.1.

1.2 State of the Art

This problem has been treated in the literature, however, the design techniques presented have several deficiencies. The mapping of the region of plant parameter variation into the acceptable closed loop pole region is approximate and is only valid if the acceptable closed loop pole region is relatively small and well removed from the plant parameter variation in the s-plane. An additional deficiency is that the unavoidable large loop transmission bandwidth involved in using these design techniques results in a very unfavorable high frequency response to internal noise at the plant input.

A design procedure which greatly alleviates these deficiencies is the topic of a recent paper by Horowitz. The design procedure presented in the present document is essentially that developed in the paper by Horowitz, expanded to take into account the effect of one of the "far-off" poles. Thus Horowitz uses a third order representation in deriving the dominant poles of the system whereas this treatment uses a fourth order representation. Chapter IV also
FIG. 1.1 STATEMENT OF THE PROBLEM

REGION OF PLANT POLE VARIATION
ACCEPTABLE DOMINANT CLOSED LOOP POLE REGION
COMPENSATION ZERO
DOMINANT CLOSED LOOP POLE

NEAREST "FAR-OFF" OPEN LOOP POLE
OPEN LOOP POLE USED TO PARTIALLY CANCEL THE EFFECT OF THE DRIFTING ZERO
"FAR-OFF" CLOSED LOOP POLES MUST LIE TO THE LEFT OF THIS BOUNDARY
considers the additional effect of a drifting zero on the real axis.

1.3 Design Philosophy

The design procedure presented in this paper is based on the dominant pole concept. Compensation zeroes are strategically placed near or within the acceptable closed loop pole region (See Figure 1.1). The gain of the system is then determined such that the closed loop poles remain within the acceptable region in the s-plane, despite the large plant parameter variations. "Far-off" poles (and zeroes) are then assigned in such a manner as to allow the loop transmission to decrease as fast as possible without violating the system time domain specifications. The problem of the placement of these "far-off" poles and zeroes, excluding the nearest "far-off" pole placed on the real axis, is not considered in this paper. Horowitz has presented a method for the placement of these "far-off" poles and zeroes.

1.4 Scope of Work and Terminology

The plant parameter variation and acceptable dominant closed loop pole region are somewhat similar to those encountered in flight control.
Chapter II considers the problem of variation in the plant gain only. Plant pole cancellation and replacement is anticipated. The plant transfer function $P(s)$ is assumed to be of the form

$$P(s) = \frac{k}{s(s^2 + S_p s + P_p)}$$  \hspace{1cm} (1.1)

where: $k$ = gain factor of the plant which may vary between $k_{\text{min}}$ and $k_{\text{max}}$;

$S_p$ and $P_p$ are fixed parameters that determine the position of the plant poles.

For complex plant poles located at $\sigma_p \pm j\omega_p$, $S_p$ and $P_p$ are given by

$$S_p = -2\sigma_p$$  \hspace{1cm} (1.2)

$$P_p = \sigma_p^2 + \omega_p^2$$  \hspace{1cm} (1.3)

The loop transmission $L_d(s)$ is assumed to have the form

$$L_d(s) = \frac{kh(s^2 + S_0 s + P_0)}{s(s^2 + S_0 s + P_0)(s + P_1)} = \frac{knh_d(s)}{d_d(s)}$$  \hspace{1cm} (1.4)

where: $K$ = fixed gain added to the system;

$S_0$ and $P_0$ are fixed parameters that determine the position of the compensation zeroes;
$S_t$ and $P_t$ are fixed parameters that determine the position of the plant replacement poles; $-P_1$ is the location of the nearest "far-off" open loop pole on the real axis.

For complex compensation zeroes located at $\sigma_z \pm j\omega_z$, $S_0$ and $P_0$ are given by

$$S_0 = -2\sigma_z \quad (1.5)$$
$$P_0 = \frac{\sigma_z^2 + \omega_z^2}{2} \quad (1.6)$$

For plant replacement poles located on the real axis at $r_1$ and $r_2$, $S_t$ and $P_t$ are given by

$$S_t = -(r_1 + r_2) \quad (1.7)$$
$$P_t = r_1 r_2 \quad (1.8)$$

The closed-loop transfer function $T_d(s)$ is assumed to have the form

$$T_d(s) = \frac{P_r P_f_1 P_f_2}{(s^2 + S_r s + P_r)(s + P_f_1)(s + P_f_2)} \Delta = \frac{P_r P_f_1 P_f_2}{D_d(s)} \quad (1.9)$$
where: \(-P_{f_1}, -P_{f_2}\) are the positions of the non-dominant closed loop poles which may be real or complex conjugate; 
\(S_r\) and \(P_r\) are parameters that determine the position of the dominant closed loop poles.

For complex dominant closed loop poles located at \(\sigma_d + j\omega_d\), \(S_r\) and \(P_r\) are given by

\[
S_r = -2\sigma_d \quad (1.10)
\]
\[
P_r = \sigma_d^2 + \omega_d^2 \quad (1.11)
\]

Chapter III considers the problem of simultaneous plant gain and plant pole variation. The plant transfer function has the same form as in Eq. 1.4 but \(S\) and \(P\) are now slowly varying or unknown parameters. Since plant pole cancellation is not practical here, the loop transmission has the form

\[
L_d(s) = \frac{kh(s^2+S_o+P_n)}{s(s^2+S_1+P)(s+P_1)} = \frac{\Delta}{d_d(s)} \quad (1.12)
\]

The closed loop transfer function is of the same form as in Eq. 1.9.

Chapter IV considers the problem of simultaneous plant gain, plant pole and plant zero variation.
The plant transfer function is assumed to be of the form

\[ p(s) = \frac{k(s+z)}{s(s^2+p_s+p_p)} \]  

(1.13)

where: \(-z\) is the position of the drifting zero on the real axis.

The loop transmission has the form

\[ L_d(s) = \frac{kh(s^2+p_s+p_p)(s+z)}{s(s^2+p_s+p_p)(s+p_z)} = \frac{k\eta_d(s)}{d_d(s)} \]  

(1.14)

where: \(-p_z\) is the position of the fixed pole used to partially cancel the effect of the drifting zero.

The closed loop transfer function is

\[ T_d(s) = \frac{p_r p_{f_1} p_{c_z}}{D_d(s)} \]  

(1.15)

where: \(-p_{c_z}\) is the position of the closed loop pole near the drifting zero \(z\) on the real axis;

\(-p_{f_1}\) is the position of the "far-off" closed loop pole on the real axis.
Chapters II and III are extensions of work by Horowitz in that the effect of the "far-off" pole $P_1$ has been included while Chapter IV is completely original.

Appendix A presents various convergence procedures for factoring polynomials on a digital computer. Appendix B presents a geometric proof of a design procedure used in Chapter II.
CHAPTER 11

DESIGN FOR PLANT GAIN VARIATION ONLY

2.1 Plant Pole Cancellation

This chapter treats the case when the plant pole variation is "sufficiently small" such that cancellation of the plant poles is valid providing, of course, that they are not located in the right half plane. A discussion on what constitutes "sufficiently small" variation in the plant poles is given in (4). The saving in the gain-bandwidth product of the loop transmission, obtained by cancelling the plant poles and replacing them with poles nearer the desired closed loop pole region, may be quite substantial, especially if the acceptable closed loop pole region is far removed from the original plant poles.

2.2 Design Philosophy

The design philosophy in the case of plant gain variation only is to first cancel the existing plant poles and replace them with poles nearer the desired closed loop pole region. These poles will be near or on the boundary of the acceptable closed loop pole region at minimum plant gain, $k = k_{\text{min}}$. Compensation
zeroes are then located so that the dominant closed loop poles lie within the acceptable region despite the variations in plant gain factor \( k \). The problem is depicted pictorially in Fig. 2.1.

### 2.3 Design Equations

The expression for the loop transmission \( L_d(s) \) is from Eq. 1.4

\[
L_d(s) = \frac{kh(s^2 + s_0s + p_0)}{s(s^2 + s_1s + p_1)(s + p_1)} \quad \Delta \frac{khn_d(s)}{d_d(s)} \tag{2.1}
\]

and the expression for the system transmission \( T_d(s) \) is from Eq. 1.9

\[
T_d(s) = \frac{\rho r_p f_1 f_2}{(s^2 + s + p_r)(s + p_{f_1})(s + p_{f_2})} \quad \Delta \frac{\rho r_p f_1 f_2}{d_d(s)} \tag{2.2}
\]

The characteristic equation of the system is then

\[
D_d(s) = d_d(s) + kh_d(s) \tag{2.3}
\]

Equating the zero degree coefficients in Eq. 2.3 gives

\[
kh\rho_0 = \rho f_1 f_2 f_r \tag{2.4}
\]
FIG. 2.1 PROBLEM OF PLANT GAIN VARIATION ONLY

\[ L_d(s) = \frac{AK(s^2 + S_{1}s + P_0)}{s(s^2 + S_1s + P_0)(s + P_1)} \]

\[ T_d(s) = \frac{P_0P_1P_2}{(s^2 + S_1s + P_0)(s + P_1)(s + P_2)} \]
Let $P_r$, $P_{f_1}$, and $P_{f_2}$ denote $P_r$, $P_{f_1}$ and $P_{f_2}$ at $k = k_{min}$.

Also define

$$K^G \triangleq k_{min} = \frac{P_r \cdot P_{f_1} \cdot P_{f_2}}{P_o} \quad (2.5)$$

It is very desirable to minimize the value of $K^G$, which is the system gain necessary to bring the root locii to the acceptable region of the dominant closed loop poles. Also, the high frequency asymptote of $L_d(s)$ (Eq. 2.1) is $h_1/s^2$, which is an important factor in determining the effect of internal noise at the plant input. Large $h_1$ increases the possibility of plant saturation by internal high frequency noise. The next section deals with the choice of the parameters in Eq. 2.5 in which the minimization of $h_1$ is the prime objective.

2.4 Choice of Design Parameters to Minimize System Gain

The choice of the position of the dominant closed loop poles at $k = k_{min}$ depends somewhat on the shape of the acceptable region for the closed loop poles. Let the position of the dominant closed loop poles at $k = k_{min}$ be denoted by

$$P_d^*, \quad P_d = \sigma_d \pm j\omega_d \quad (2.6)$$

Note that
Pr = \left| p_d^* \right|^2 = (\sigma_d^*)^2 + (\omega_d^*)^2 \quad \text{(See Fig. 2.2)} \quad (2.7)

It is desirable to minimize \( \left| p_d^* \right| \) and thereby \( P_r^* \), since, from Eq. 2.5 this tends to minimize \( h_1 \). This implies that \( p_d^* \) should be located on or very close to the boundary of the acceptable region for the dominant closed loop poles.

The next problem is to choose the values for \( p_k^* \) and \( p_k^* \), the non-dominant closed loop poles at \( k = k_{\text{min}} \). These closed loop poles will lie on the real axis for small values of gain since \( P_1 \) is assumed to be on the real axis. The values of \( p_{f_1}^* \) and \( p_{f_2}^* \) will have to be determined from considerations of the time domain specifications of the system transmission. They should be chosen as close in as possible, since from Eq. 2.5, this will tend to minimize \( h_1 \) but if they are too close to the origin, the system response can no longer be characterized by the dominant pole pair. The latter consideration determines the minimum distance (from the origin) of these poles. Henceforth, it is assumed that \( p_{f_1}^* \) and \( p_{f_2}^* \) are known.

The position of the compensation zeroes \( Z \) and \( \overline{Z} \) is now considered. Denote the position of the zeroes \( Z \) as

\[ Z, \overline{Z} = \sigma \pm j\omega \] \quad (2.8)
FIG. 2.2 THE EFFECT OF THE DOMINANT CLOSED LOOP POLE AND ZERO LOCATION ON $K_2$

$$K_2 = k_{\text{MIN}} K = \frac{P_0 P_2^* P_3^*}{P_0}$$

Acceptable closed loop pole region

Dominant closed loop pole at $\phi = \phi_{\text{MIN}}$

"Far-off" closed loop poles at $\phi = \phi_{\text{MIN}}$

FIG. 2.3 LOCATION OF THE COMPENSATION ZEROES FROM THE ANGLE OF DEPARTURE OF THE ROOT LOCUS FROM $P_2^*$

Acceptable closed loop pole region

$$\psi_2 = 180 - (\psi_1 + \phi_{\text{MIN}} + 90 - \theta_2)$$

$$\theta_2 = \theta_1 + \phi_{\text{MAX}}$$
Note that

\[ P_0 = |Z|^2 = \sigma_z^2 + \omega_z^2 \]  
(See Fig. 2.2)  \hspace{1cm} (2.9)

Therefore \(|Z|\) should be made as large as possible to maximize \(P_0\) which, from Eq. 2.5, will tend to minimize \(K_1\). If the variation in plant gain is very large, the dominant closed loop poles will, at \(k = k_{\text{max}}\), be very close to the compensation zeroes. This means that these zeroes must be located near the boundary of the acceptable closed loop pole region.

2.5 Positioning of Dominant Closed Loop Poles and Compensation Zeroes

The method used in this paper to fix the position of the compensation zeroes is the same as in reference (4). This method is to demand that the angle of departure of the root locus from the dominant pole \(p_d^*\) for \(k > k_{\text{min}}\) be within a prescribed sector, given in Fig. 2.3 as \(H_1p_d^*H_2\). The choice of this sector is somewhat arbitrary but should be fairly general and easily applied to different acceptable regions of dominant closed loop poles.

In Fig. 2.3, the angle of departure, \(\psi_d\), of the root locus from \(p_d^*\) is given by

\[ \psi_d = 180^\circ - (\Psi_1 + \Phi_2 + 90^\circ - \angle Z) \]  \hspace{1cm} (2.10)
where:  

\[ \Phi_1 = \angle - p_{2}^* p_{d}^* \]

\[ \Phi_2 = \angle - p_{1}^* p_{d}^* \]

\[ 90^\circ = \angle \bar{p}_{d}^* p_{d}^* \]

\[ \theta_1 = \angle z_{p_{d}^*} \]

\[ \theta_2 = \angle z_{p_{d}^*} \]

\[ \theta_Z = \theta_1 + \theta_2 \]

Solving Eq. 2.10 for \( \theta_Z \) gives

\[ \theta_Z = \phi_d + (\Phi_1 + \Phi_2 + 90^\circ) - 180^\circ \]  \hspace{1cm} (2.11)

Denote the extreme values of the departure angle \( \phi_d \), which lie within the sector \( H_1 p_{d}^* H_2 \), as

\[ \psi_{\text{dmin}} \leq \psi_d \leq \psi_{\text{dmax}} \]  \hspace{1cm} (2.12)

The extreme values of \( \theta_Z \) are then

\[ \theta_{Z\text{min}} = 90^\circ + \phi_1 + \phi_2 + \psi_{\text{dmin}} - 180^\circ \]  \hspace{1cm} (2.13)

\[ \theta_{Z\text{max}} = 90^\circ + \phi_1 + \phi_2 + \psi_{\text{dmax}} - 180^\circ \]  \hspace{1cm} (2.14)

The locus of zero positions, \( Z, \bar{Z} \), such that \( \theta_Z \) is a constant, is an arc of a circle drawn through the
points \( p_d^* \), \( \overline{p_d}^* \), and a third point \( X \) on the real axis defined by the equation

\[
\angle Xp_d^* = \frac{\beta}{2}
\]

(2.15)

The proof of this statement is given in Appendix B.

The design procedure is, then, to locate two points \( X_1 \) and \( X_2 \) on the real axis corresponding to \( \beta_{Z_{\text{min}}} \) and \( \beta_{Z_{\text{max}}} \) in Eq. 2.15. Circular arcs \( c_1 \) and \( c_2 \) are drawn through the points \( p_d^*X_1p_d \) and \( p_d^*X_2p_d \) as shown in Fig. 2.3. Locating the compensation zeroes between the arcs \( c_1 \) and \( c_2 \) will then insure that the angle of departure of the root locus from the dominant pole \( p_d^* \) is within the specified sector \( H_1p_d^*H_2 \).

When \( k_{\text{max}} \) is much greater than \( k_{\text{min}} \), the closed loop poles will, at \( k = k_{\text{max}} \), be very close to the compensation zeroes. Therefore, in order to insure that the root locus remains in the acceptable region as \( k \) approaches \( k_{\text{max}} \), the angle of entry of the locus into the complex zero should be checked. The angle of entry \( \psi_{c^*} \) is given by

\[
\psi_c = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 - 90^\circ - 180^\circ
\]

(2.16)

where:

\[
\gamma_1 = \angle -p_{d_1}^*Z
\]

\[
\gamma_2 = \angle -p_{d_2}^*Z
\]
\[ \alpha_3 = \angle \beta_4 \]
\[ \alpha_4 = \angle \beta_4 \]
\[ 90^\circ = \angle \beta_4 \]

These angles are shown in Fig. 2.4.

The value \( k_1 \) may be computed from Eq. 2.5 once the zeroes have been located. It should be noted that if \( k_{\text{max}} \) is not much greater than \( k_{\text{min}} \), the zeroes may not have to be located within the acceptable closed loop pole region.

2.6 Open Loop Poles of \( L_d(s) \)

The last step in the design procedure is to locate the open loop poles of \( L_d(s) \). From Eq. 2.3, the expression for \( D_d^*(s) \) is

\[ D_d^*(s) = \frac{d_d(s)}{D_d(s)} + h_1 n_d(s) \]

Therefore

\[ d_d(s) = D_d^*(s) - h_1 n_d(s) \quad (2.17) \]

or

\[ d_d(s) = (s^2 + S_0 s + P_0)(s + p_{12})(s + p_{13}) - h_1 (s^2 + S_0 s + P_0) \]

\[ (2.18) \]
Fig. 2.4 Calculation of the angle of entry of the root locus into the compensation zero $z$

$\gamma_c = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 - 90^\circ - 180^\circ$

Acceptable closed loop pole region

Fig. 2.5 Anticipated root locus
\[ s^3 + (p_{f1}^* + p_{f2}^* + S_{r}^*) s^2 + (p_{f1}^* + p_{f2}^*) S_r (p_{f1}^* + p_{f2}^*) + P_r^* - k_1 s \]

\[ + S_r^* (p_{f1}^* + p_{f2}^*) - k_1 S_0 \]  

\[ = s(s^2 + S_{r}^* s + P_r^*)(s + P_r) \]  

The roots of \( s^2 + S_{r}^* s + P_r^* \) may be complex or real depending on the location of the acceptable closed loop pole region and the value of \( k_1 \). The anticipated root locus, for a design of this type, for the given acceptable closed loop pole region, is shown in Fig. 2.5.

2.7 Design Example

The following is an application of the design procedure for gain variation only.

The design equations are

\[ T_d^*(s) = \frac{p_{f1}^* p_{f2}^* p_r^*}{(s^2 + S_{r}^* s + P_r^*)(s + p_{f1}^*)(s + p_{f2}^*)} \]

\[ L_d^*(s) = \frac{k_{\min} K(s^2 + S_{r}^* s + P_r^*)}{s(s^2 + S_{r}^* s + P_r^*)(s + P_r)} \]

where \( T_d^*(s) \) and \( L_d^*(s) \) are the system transmission and loop transmission respectively, at \( k = k_{\min} \). The plant transfer function is assumed to be of the form
\[ P(s) = \frac{k}{s(s^2 + s + p)} \]

where \( k \), the plant gain factor, may vary from \( k_{\text{min}} = 1 \) to \( k_{\text{max}} = 1000 \). The complex plant poles are assumed to be fixed and have been cancelled by zeroes placed near them.

The closed loop pole values \(-p_{f1}^* \) and \(-p_{f2}^* \) are assumed to be -10 and -15 respectively. These choices for \( p_{f1}^* \) and \( p_{f2}^* \) are purely arbitrary other than that they must lie to the left of the boundary shown in Fig. 2.6. This report does not show how these values of \( p_{f1}^* \) and \( p_{f2}^* \) are obtained but how the design proceeds once they are known. The acceptable region for the dominant closed loop poles, as well as other pertinent quantities for this design example, are shown in Fig. 2.6.

As a first choice for \( p_d^* \), the dominant closed loop pole at \( k = k_{\text{min}} \), let

\[ p_d^* = -3 + j3 \]

since this is close to the minimum value of \( P_r \) for this acceptable dominant closed loop pole region. The angle of departure of the root locus from the dominant pole \( p_d^* \) must be within the sector \( H_{1} p_d^* H_{2} \), i.e.

\[ 140^\circ \leq \psi_d \leq 230^\circ \]
FIG. 2.6 DESIGN EXAMPLE FOR PLANT GAIN VARIATION ONLY

ACCEPTABLE CLOSED LOOP POLE REGION

BOUNDARY FOR THE "FAR-OFF" CLOSED LOOP POLES

FIG. 2.7 APPROXIMATE ROOT LOCUS FOR THE DESIGN EXAMPLE

ACCEPTABLE CLOSED LOOP POLE REGION
The angle of departure of the root locus from the dominant pole $p_d^*$ is given by

$$\Phi_d = 180^\circ - (\Phi_1 + \Phi_2 + 90^\circ - \theta_Z).$$

where:

$$\Phi_1 = \angle p_{f1}^* p_d^* = \tan^{-1} \frac{2}{3} = 25.4^\circ$$

$$\Phi_2 = \angle p_{f2}^* p_d^* = \tan^{-1} \frac{3}{12} = 14.0^\circ$$

$$\theta_Z = \angle p_{d}^* + \angle p_{d}^*$$

From Eqs. 2.13 and 2.14, the values of $\theta_{Z_{\text{max}}}$ and $\theta_{Z_{\text{min}}}$ are given by

$$\theta_{Z_{\text{max}}} = \Phi_{d_{\text{max}}} - 90^\circ + \Phi_1 + \Phi_2$$

$$= 230^\circ - 90^\circ + 25.4^\circ + 14.0^\circ$$

$$= 179.4^\circ$$

$$\theta_{Z_{\text{min}}} = \Phi_{d_{\text{min}}} - 90^\circ + \Phi_1 + \Phi_2$$

$$= 140^\circ - 90^\circ + 25.4^\circ + 14.0^\circ$$

$$= 89.4^\circ$$

Next, two points on the real axis $X_1$, $X_2$ are determined such that

$$\angle X_1 p_d^* = \frac{\theta_{Z_{\text{max}}}}{2} \approx 90^\circ$$

$$\angle X_2 p_d^* = \frac{\theta_{Z_{\text{min}}}}{2} \approx 45^\circ$$
Two arcs are now drawn through the points $p_d^*x_1p_d^*$ and $p_d^*x_2p_d^*$ (See Fig. 2.6). The construction begins by locating a point on the real axis that is equidistant from the points $p_d^*$, $x$ and $p_d^*$. Using this point on the real axis as the center of a circle, circular arcs are drawn through the points $p_d^*$, $x$ and $p_d^*$ with a compass.

As a first choice for the position of the compensation zeroes $Z$ and $\overline{Z}$, let

$$Z, \overline{Z} = -6 \pm j0.5$$

This position for the compensation zeroes will tend to maximize $P_o$ for the circular arcs $c_1$ and $c_2$ shown in Fig. 2.6. Large $P_o$ from Eq. 2.5 will mean a smaller value for $k_1$ which is the object of this design procedure.

The expression for $n_d(s)$ is

$$n_d(s) = (s-Z)(s+\overline{Z})$$

$$= (s+6+j0.5)(s+6-j0.5)$$

$$= s^2+12s+36.25$$

The value of $k_1$ may be computed from Eq. 2.5, i.e.

$$k_1 = k_{min} \frac{P^*p^*_1p^*_2}{P_o}$$

$$= \frac{(18)(10)(15)}{36.25} = 74.5$$
From Eqs. 2.17 and 2.19, the expression for \( d_d(s) \) is

\[
d_d(s) = u^*_d(s) - h_1 n_d(s)
\]

or

\[
d_d(s) = s\left( s^3 + (p^*_1 - p^*_2 + S^*_t) s^2 + (p^*_1 + p^*_2) s + p^*_1 - h_1 \right) + S_r \left( p^*_1 + p^*_2 \right) - h_1 S_o \}
\]

\[
= s\left( s^3 (10 + 15) s^2 + (150 + 6(25)) + 18 - 74.5 \right) s
\]

\[
+(6(150) + 18(25) - 12(74.5))
\]

\[
= s(s^3 + 31s^2 + 243.5s + 456)
\]

To obtain the open loop poles of \( L_d(s) \), this equation must be factored. Using the methods in Appendix A, this equation is factored into the following open loop poles

\[
d_d(s) = s(s^2 + 28.4s + 168)(s + 2.65)
\]

\[
= s(s + 2.65)(s + 9.00)(s + 19.44)
\]

from which

\[
P_1 = 19.44
\]

\[
(s^2 + S_t s + P_t) = (s + 2.65)(s + 9.00)
\]

\[
= s^2 + 11.65s + 23.85
\]

The angle of departure \( \phi_d \) of the root locus from the dominant pole \( p_d^* \) is
\[
\psi_d = 180^\circ - (\Phi_1 + \Phi_2 + 90^\circ - \theta_Z)
\]
\[
\theta_Z = \angle Zp_d^* + \angle \bar{Z}p_d^*
\]
\[
= \tan^{-1} \left( \frac{2.5}{3} \right) + \tan^{-1} \left( \frac{2.5}{3} \right) = 39.8^\circ + 49.3^\circ
\]
\[
= 89.1^\circ
\]
Therefore \( \psi_d = 180^\circ - (25.4^\circ + 14.0^\circ + 90^\circ - 89.1^\circ) \)
\[
= 139.7^\circ
\]

which is satisfactory since it is very close to the minimum value of 140°. The angle of entry of the root locus into the zeroes is given by Eq. 2.15, i.e.
\[
\psi_e = a_1 + a_2 + a_3 + a_4 - 90^\circ - 180^\circ
\]
\[
= \angle -p_{1}^* Z + \angle -p_{2}^* Z + \angle p_d^* Z + \angle \bar{p}_d^* Z - 270^\circ
\]
\[
= \tan^{-1} \left( \frac{5}{4} \right) + \tan^{-1} \left( \frac{5}{4} \right) + \tan^{-1} \left( \frac{2.5}{3} \right) + \tan^{-1} \left( \frac{2.5}{3} \right) - 270^\circ
\]
\[
= 7.1^\circ + 3.2^\circ + 219.8^\circ + 130.7^\circ - 270^\circ
\]
\[
= 90.8^\circ
\]
which is satisfactory for this design (see Fig. 2.6).

For this particular placement of the zeroes, an angle of entry between 20° and 160° would probably be satisfactory. The approximate root locus for this design is shown in Fig. 2.7.

2.8 Improvement in Design Example

The first design could be improved. If the compensation zeroes could be moved further to the left, it would, from Eq. 2.5, decrease the value of \( K_1 \).

With this as the objective, let the second choice for
If \( p_d^* \) be 

\[ p_d^* = -3.5 + j2.0 \]

The angle of departure \( \psi_d \) is constrained to be within the sector \( H_1 p_d^* H_2 \) defined by the equation 

\[ 110^\circ \leq \psi_d \leq 230^\circ \]

Quantities pertinent to this design are shown in Fig. 2.8.

The angles \( \psi_1 \) and \( \psi_2 \) are 

\[ \psi_1 = \tan^{-1}\frac{2}{6.5} = \tan^{-1}0.308 = 17.2^\circ \]

\[ \psi_2 = \tan^{-1}\frac{2}{11.5} = \tan^{-1}0.174 = 9.85^\circ \]

The same method as before will be used to fix the position of the compensation zeroes. The angle of departure of the root locus from \( p_d^* \) is 

\[ \psi_d = 180^\circ - (\psi_1 + \psi_2 + 90^\circ - \theta_z) \]

\[ = 90^\circ - 17.2^\circ - 9.85^\circ - \theta_z \]

or 

\[ \theta_z = \psi_d - 62.95^\circ \]

Therefore 

\[ \theta_{z_{\text{max}}} \approx 230^\circ - 63^\circ = 167^\circ \]

\[ \theta_{z_{\text{min}}} \approx 110^\circ - 63^\circ = 47^\circ \]

Again two points on the real axis \( X_1, X_2 \) are determined such that 

\[ \angle X_1 p_d^* = 83.5^\circ \]

\[ \angle X_2 p_d^* = 23.5^\circ \]
FIG. 2.8 IMPROVEMENT IN THE DESIGN FOR PLANT GAIN VARIATION ONLY

ACCEPTABLE CLOSED LOOP POLE REGION

FIG. 2.9 APPROXIMATE ROOT LOCUS FOR THE SECOND DESIGN

ACCEPTABLE CLOSED LOOP POLE REGION
Two circular arcs are now drawn through the points $p_d^* p_d^*$ and $p_d^* p_d^*$ as shown in Fig. 2.8.

The compensation zero positions are chosen as

$$z, \bar{z} = -7 \pm j1$$

The expression for $n_d(s)$ is then

$$c_d(s) = s^2 + 14s + 50$$

The value of $k_1$ from Eq. 2.5 is

$$k_1 = \frac{p_1 p_2^*}{p_0} = \frac{(16,22)(10,15)}{50} = 48.69$$

This is a reduction of about 4 db. from the $k_1$ of the previous design.

The angle of departure $\phi_d$ of the root locus from $p_d^*$, for this position of compensation zeroes, is $120^\circ$. The angle of entry $\phi_e$ of the root locus into the compensation zero $z$ is $84^\circ$. Both of these values are satisfactory for the given acceptable closed loop pole region (see Fig. 2.8). The reason that the angle of entry should be checked is as follows: If the compensation zeroes were placed in the extreme left hand corner of the acceptable closed loop pole region, an angle of entry greater than approximately $80^\circ$ would be unsatisfactory. An angle of entry larger than $80^\circ$ would probably indicate that the root locus would be
outside the acceptable closed loop pole region for some value of $k$ between $k = k_{\text{min}}$ and $k = k_{\text{max}}$.

The expression for $d_d(s)$ is found from Eq. 2.19

$$d_d(s) = s\{s^3+(10+15+7)s^2+(15c+7(25)+16.23-48.69)s$$
$$+(7(150)+16.23(25)-14(48.69))\}$$

$$= s\{s^3+32s^2+292.5s+773\}$$

Factoring this equation results in the following expression for $d_d(s)$

$$d_d(s) = (s+4.46)(s^2+27.7s+173)$$

$$= (s+4.46)(s+9.49)(s+18.21)$$

The root locus for this design is of the same form as the previous design and is shown in Fig. 2.9.

2.9 Summary of Design Procedure

The design procedure for variation in plant gain factor only is summarized below.

1. Cancel the plant poles.
2. Fix the position of the dominant closed loop poles at $k = k_{\text{min}}$.
3. Determine the values of the "far off" closed loop poles at $k = k_{\text{min}}$.
4. Determine the position of the compensation zeroes so that the root locus from $k = k_{\text{min}}$
to \( k = k_{\text{max}} \) remains within the acceptable closed loop pole region.

5. Solve for the open loop poles of \( L_d(s) \) from Eq. 2.20.

The next chapter in this paper considers variation in both the plant poles and plant gain factor.
CHAPTER III

PROBLEM OF SIMULTANEOUS PLANT GAIN
AND PLANT POLE VARIATION

3.1 Problem Definition

In the case where the plant poles vary as well as the plant gain factor, plant pole cancellation is not feasible. In this case the system gain must be sufficiently high so that the dominant closed loop poles remain within their acceptable region despite the variations in the plant poles. The problem resolves into locating the compensation zeroes such that the system gain necessary to accomplish this is minimized.

3.2 Design Equations

The expressions for the dominant part of the loop transmission and system transmission given by Eqs. 1.12 and 1.9 are repeated below.

\[
L_d(s) = \frac{kh(s^2 + S_0 s + P_0)}{s(s^2 + S_p s + P_p)(s + P_1)} = \frac{k \cdot \kappa}{\kappa_d(s)}
\]

\[
T_d(s) = \frac{P_1 P_2 P_r}{(s^2 + S_r s + P_r)(s + P_{f_1})(s + P_{f_2})} = \frac{P_1 P_2 P_r}{D_d(s)}
\]
The plant transfer function is of the form

\[ P(s) = \frac{k}{s(s^2+S_p s+P_p)} \]

The plant gain factor \( k \) varies from \( k = k_{\text{min}} \) to \( k = k_{\text{max}} \) and the plant poles may lie or "slowly vary" within the region shown in Fig. 3.1. The characteristic equation of the system \( D_d(s) \) from Eq. 2.3 is

\[ D_d(s) = d_d(s) + k h n_d(s) \]

or

\[ (s^2+S_p s+P_p)(s+p_{f_1}^2)(s+p_{f_2}^2) = s(s^2+S_p s+P_p)(s+P_1) \]

\[ + k h (s^2+S_0 s+P_0) \]

(3.1)

\[ s^4+(S+p_{f_1}^2 s+p_{f_2}^2) s^3+(P_{f_1}+P_{f_2}+S_p (p_{f_1}+p_{f_2})+P_r) s^2+(S_r p_{f_1} p_{f_2}+P_{f_1}+P_{f_2}) s+P_r s^2 \]

\[ + k h (S_{f_1} p_{f_2}+S_{f_2} p_{f_1}) s^2+P_r p_{f_1} p_{f_2} = s^4+(S_p+P_1) s^3+(S_p P_1+P_p+k h) s^2 \]

\[ +(P_p P_1+k h S_0) s+k h P_0 \]

(3.2)

Equating the coefficients of Eq. 3.2 yields the following set of equations

\[ S_r p_{f_1} p_{f_2} = S_p P_1 \]

(3.3)

\[ p_{f_1} p_{f_2} S_r (p_{f_1}+p_{f_2})+P_r = S_p P_1 P_p+k h \]

(3.4)
**Fig. 3.1 Problem of Simultaneous Plant Gain and Plant Pole Variation**

\[ P(s) = \frac{k}{s(s^2 + S_p s + P_p)} \]

\[ L_d(s) = \frac{kk(s^2 + S_s s + P_s)}{s(s^2 + S_p s + P_p)(s + P_p)} \]

\[ T_d(s) = \frac{P_1 P_2 P_3}{(s^2 + S_m s + P_m)(s + P_1)(s + P_2)} \]

- \( P_1 \), nearest far-off open loop pole
- Dominant closed loop pole
- Compensation zero
- Boundary for the "far-off" closed loop poles
- Region of plant pole variation

Acceptable closed loop pole region

Plant pole

- Nearest far-off open loop pole

\( P(s) \) and \( L_d(s) \) equations illustrate the simultaneous effects of gain and pole variation on system stability.
\[ S_r P_{f1} P_{f2} + P_r (P_{f1} + P_{f2}) = P_{p1} + k h S_0 \]  
(3.5)

\[ P_r P_{f1} P_{f2} = k h P_0 \]  
(3.6)

The following substitutions in Eqs. 3.4 and 3.5 are made:

\[ P_{f1} + P_{f2} = S_p + P_1 - S \]  
(3.7)

\[ p_{f1} P_{f2} = \frac{k h P_0}{P_r} \]  
(3.8)

\[ \frac{k h P_0}{P_r} + S_r (S_p + P_1 - S_r) + P_r = S_p P_1 + P_p + k h P \]  
(3.9)

\[ \frac{k h P_0}{P_r} + P_r (S_p + P_1 - S_r) = P_p P_1 + k h S_0 \]  
(3.10)

Define the following quantities:

\[ \gamma \overset{\Delta}{=} k h P_0 \]  
(3.11)

\[ X \overset{\Delta}{=} \frac{S_p P_1 + P_p + k h P}{P_r} \]  
(3.12)

\[ Y \overset{\Delta}{=} \frac{P_p P_1 + k h S_0}{P_r} \]  
(3.13)

Equations 3.9 and 3.10 are then

\[ \frac{\gamma}{P_r} + S_r (S_p + P_1 - S_r) + P_r = X \]  
(3.14)

\[ \frac{\gamma S_r}{P_r} + P_r (S_p + P_1 - S_r) = Y \]  
(3.15)
Considering the equations for $X$ and $Y$, (Eqs. 3.12, 3.13), the variation in $X$, $\Delta X$, and the variation in $Y$, $\Delta Y$, due to the variation in plant poles only, i.e., parameters $S_p$ and $P_p$, can be expressed as

$$\Delta X = P_1 \Delta S_p + \Delta P_p \quad (3.16)$$

$$\Delta Y = P_1 \Delta P_p \quad (3.17)$$

The variation in plant gain factor will be considered later in the design.

### 3.3 Design Procedure

An outline of the design procedure that will be followed in this problem is as follows:

1.) Map the acceptable region for the dominant closed loop poles into the $X,Y$ plane using Eqs. 3.14 and 3.15 for fixed values of the parameters $\gamma$, $P_1$, and $S_p$.

2.) Map the plant pole variation into the $\Delta X$, $\Delta Y$ plane using Eqs. 3.16 and 3.17 for fixed values of the parameter $P_1$.

3.) Compare the two mappings in (1,2) above. If the mapping of the plant pole variation in the $\Delta X$, $\Delta Y$ plane does not fit into the interior of the mapping of the acceptable dominant closed loop pole region, the mapping of the latter will have to be repeated,
using a larger value of $\gamma$.

4.) Solve for the values of $k_k$, $S_o$ and $P_o$ using Eqs. 3.11, 3.12 and 3.13, where the values of $X$ and $Y$ are obtained from the positioning of the mapping of the plant pole variation in the interior of the mapping of the dominant closed loop pole region in the $X$, $Y$ plane.

The next four sections in this chapter elaborate on these four steps in the design procedure.

3.4 Mapping of the Dominant Closed Loop Pole Region

The mapping of the acceptable dominant closed loop pole region into the $X$, $Y$ plane involves the parameters $\gamma$, $P_1$ and $S_p$. Large $\gamma$ implies large gain, i.e., large $k_k$, since $P_o$ does not have a large range of values (See. Eq. 3.11). An approximation for the value of $\gamma$ can be obtained from Eq. 3.6, i.e.

$$\gamma = P_1 P_2 P_r$$  \hspace{1cm} (3.18)

The maximum value of $P_r$ can be found from the acceptable region for the dominant closed loop poles. Denoting the value of the dominant closed loop pole by $p_d = \sigma_d + j\omega_d$, the value of $P_r$ is

$$P_r = |p_d|^2 = \sigma_d^2 + \omega_d^2$$  \hspace{1cm} (3.19)
The values of $p_{f_1}$ and $p_{f_2}$ can be roughly approximated by considering the boundary for these closed loop poles shown in Fig. 3.1. These poles will be complex for large values of system gain.

$P_1$, the nearest "far off" open loop pole located on the real axis, should be chosen as close in as possible, since from Eq. 3.3, this will decrease the values of $p_{f_1}$ and $p_{f_2}$ which, from Eq. 3.8, will tend to decrease the value of fixed gain that must be added to the system. If $P_1$ is chosen too close in, though, the closed loop poles $p_{f_1}$ and $p_{f_2}$ may lie to the right of the vertical boundary shown in Fig. 3.1 violating the specifications of the problem.

The major problem in this mapping operation is the parameter $S_p$. Denoting the value of the plant pole as $p_{p'} = \sigma_p + j\omega_p$, the value of $S_p$ is

$$S_p = -2\sigma_p \quad (3.20)$$

In the point by point mapping of the acceptable region for the dominant closed loop poles, there is no criteria for determining what value of $S_p$ to associate with a particular point on the boundary of the acceptable dominant closed loop pole region. This dilemma is resolved by considering the following argument: If the dominant closed loop poles are to lie within their
acceptable region despite the variation in the plant poles, the mapping of the dominant closed loop pole region can not be highly sensitive to the position of the plant poles and hence the value of $S_p$. The actual point by point mapping of the dominant closed loop pole region into the X,Y plane is performed using several different values of $S_p$ for each value of $\gamma$ and $P_1$. These different values of $S_p$ should include the minimum and maximum values of $S_p$ for the given region of plant pole variation as well as values in between. A median value of $S_p$ is denoted by $S_{p\text{med}}$ in Fig. 3.2. The mapping of the acceptable closed loop pole region into the X,Y plane is depicted graphically in Fig. 3.2.

The values of $p_{f1}$ and $p_{f2}$, the "far off" closed loop poles, are also of interest in the mapping of the dominant closed loop pole region. The values of $p_{f1}$ and $p_{f2}$ must lie to the left of the boundary shown in Fig. 3.1. For each point on the boundary of the dominant closed loop pole region and given values of the parameters $\gamma$, $P_1$ and $S_p$, the values of $p_{f1}$ and $p_{f2}$ may be obtained as follows: From Eqs. 3.3 and 3.6

\[ p_{f1} + p_{f2} = S_{p} + P_{1} - S_r \]  
\[ p_{f1} p_{f2} = \gamma/P_r \] 
Define $p_{f1} + p_{f2} = S_f$ 

(3.21) 
(3.22) 
(3.23)
Fig. 3.2 Mapping of the dominant closed loop pole region in the S-plane into the X-Y plane

Mapping equations:
\[ X = \frac{y}{p_a} + \frac{S_n}{p_a} (S_p + P_2 - S_n) + P_a \]
\[ Y = \frac{y S_n}{p_a} + \frac{P_a (S_p + P_2 - S_n)}{p_a} \]

Parameters: \( y, p_2, S_p \)

Acceptable closed loop pole region
\[ P_{f_1} P_{f_2} = P_f \]  

Solving Eqs. 3.23 and 3.24 for \( P_{f_1} \) and \( P_{f_2} \): yields

\[ P_{f_1} = \frac{S_f}{2} + \sqrt{(S_f/2)^2 - P_f} \]  \hspace{1cm} (3.25)

\[ P_{f_2} = \frac{S_f}{2} - \sqrt{(S_f/2)^2 - P_f} \]  \hspace{1cm} (3.26)

If the values of \( P_{f_1} \) and \( P_{f_2} \) fall to the right of the vertical boundary shown in Fig. 3.1, \( P_1 \) has been placed too far in. The values of \( P_{f_1} \) and \( P_{f_2} \) obtained during this mapping operation will not correspond exactly with those in the final design, since the boundary of the plant pole variation will not, in general, map exactly onto the boundary of the acceptable dominant closed loop pole region. Additional features of this mapping operation are covered in Section 3.8.

3.5 Mapping of the Plant Pole Variation

The only parameter in the mapping of the plant pole variation into the \( \Delta X, \Delta Y \) plane is \( P_1 \) (See Eqs. 3.16 and 3.17). The value of \( P_1 \) used in the mapping must be the same as that used in the mapping of the dominant closed loop pole region (Eqs. 3.14, 3.15). The mapping of the plant pole variation is implemented by defining any point \( (q_{po}, p_{po}) \) on the boundary of the plant pole variation. The nominal values of \( q \) and \( p \)
are then
\[ S_{po} = -2\zeta_{po} \]
\[ P_{po} = \frac{2}{\omega_{po}} + u_{po} \]

Equations 3.16 and 3.17 may then be written as

\[ \Delta X = P_1 (S_{po} - S_{po}) + (P_{po} - P_{po}) \]
\[ \Delta Y = P_1 (P_{po} - P_{po}) \]

The region of plant pole variation is then mapped point by point into the AX, AY plane, as shown in Fig. 3.3. It should be noted at this point that the shape and size of the mapping of the plant pole variation in the AX, AY plane is not dependent on the choice of the point \((o_{po} , w_{po})\) and hence on the values of \(S_{po}\) and \(P_{po}\). Different choices for this point will only alter the position of the mapping in the AX, AY plane. The units on the AX, AY axes in the AX, AY plane must be the same as those on the X,Y axes in the mapping of the dominant closed loop pole region in the X,Y plane. When the mapping of the plant pole variation in the AX, AY plane is transferred to the X,Y plane, its angular position with respect to the AX, AY axes must be preserved, i.e., the mapping of the plant pole variation may not be rotated in the X,Y plane. Additional features of this mapping operation are covered in Section 9 of this chapter.
Fig. 3.3 Mapping of the plant pole variation region in the $s$-plane into the $\Delta x-\Delta y$ plane.

Mapping equations:

\[ \Delta x = p_2 (s_p - s_P) + p_P - p_p \]

\[ \Delta y = p_2 (P_P - P_p) \]

Parameter $p_2$.
The problem now is to fit the mapping of the plant pole variation in the $\Delta x, \Delta y$ plane into the interior of the mapping of the dominant closed loop pole region in the $X,Y$ plane as shown in Fig. 3.4. If this fit is not possible, the mapping of the dominant closed loop pole region will have to be performed for larger values of $\gamma$.

3.6 Calculation of the System Gain and the Compensation Zero Location

Once the mapping of the plant pole variation fits inside the mapping of the dominant closed loop pole region in the $X,Y$ plane, the value of the system gain $k_k$ and the compensation zero positions, given by the parameters $S_0$ and $P_0$, may be computed. For a linear, time invariant, minimum-phase system, a value of system gain $k_k$ can always be found such that the mapping of the plant pole variation will fit inside the mapping of the dominant closed loop pole region in the $X,Y$ plane.$^7$

Figure 3.4 shows the mapping of the plant pole variation fitted inside the mapping of the dominant closed loop pole region in the $X,Y$ plane. To solve for the values of $k_k$, $S_0$ and $P_0$, a point on the boundary of the mapping of the plant pole variation is chosen where the values of $S_p$ and $P_p$ are known. In Fig. 3.4, this point is denoted by A. The values of $S_p$ and $P_p$ at point A are, from Fig. 3.3 and Eqs. 3.27 and 3.28,
FIG. 3.4  CALCULATION OF THE SYSTEM GAIN AND
THE ZERO LOCATION FROM THE MAPPINGS OF THE
DOMINANT CLOSED LOOP POLE REGION AND THE
PLANT POLE VARIATION

PARAMETERS $\delta, S_p, \rho$

MAPPING OF THE
DOMINANT CLOSED LOOP
POLE REGION

MAPPING OF THE
PLANT POLE
VARIATION IN THE
$\Delta x-\Delta y$ PLANE

AT POINT $B$:
$S_p = S_{p_{\text{MAX}}}$
$S_p = S_{p_{\text{MED}}}$
$S_p = S_{p_{\text{MIN}}}$
$X = X_b$
$Y = Y_b$

AT POINT $A$:
$S_p = S_{p_{\text{MAX}}}$
$S_p = S_{p_{\text{MED}}}$
$X = X_a$
$Y = Y_a$
respectively, \( S_{pa} \) and \( P_{pa} \). Denote the coordinates of point \( A \) in the \( X,Y \) plane as \( X_a \) and \( Y_a \). Since \( y \) and \( P_1 \) are known for this particular mapping, \( k_h, S_o \) and \( P_o \) may be obtained from Eqs. 3.11, 3.12 and 3.13, as follows:

\[
kh = X_a - S_{pa} P_1 - P_{pa} \quad (3.31)
\]

\[
P_o = \frac{Y}{kh} \quad (3.32)
\]

\[
S_o = \frac{Y_a - P_{pa} P_1}{kh} \quad (3.33)
\]

The value of \( kh \) should be interpreted as the necessary value of system gain, i.e. \( k_{min} \). The actual value of added gain to the system is, from Eq. 3.31

\[
k = \frac{X_a - S_{pa} P_1 - P_{pa}}{k_{min}} \quad (3.34)
\]

The choice of the point used to compute the values of \( kh, S_o \) and \( P_o \) has no effect on the values obtained for these quantities, so long as the point is on or within the mapping of the plant pole variation in the \( X,Y \) plane. To prove this, a second point \( B \) is chosen, as shown in Fig. 3.4. The value of system gain using point \( B \) is, from Eq. 3.31

\[
k_h = X_b - S_{pb} P_1 - P_{pb} \quad (3.35)
\]

Now \( S_b \) and \( P_b \) are not known but they can be computed using Eqs. 3.29 and 3.30, i.e.
\[ \Delta X = X_b - X_a = P_1(S_p - s_{p_a})(P_{p_b} - P_{p_a}) \]  
\[ \Delta Y = Y_b - Y_a = P_1(P_{p_b} - P_{p_a}) \]  

Solving Eqs. 3.36 and 3.37 for \( s_{p_b} \) and \( P_{p_b} \) results in

\[ P_{p_b} = \frac{Y_b - Y_a}{P_1} + P_{p_a} \]  
\[ S_{p_b} = \frac{X_b - X_a - (Y_b - Y_a)/P_1}{P_1} + S_{p_a} \]

Substitution of Eqs. 3.38 and 3.39 into Eq. 3.35 yields

\[ k_k = X_b - X_a + X_a + \frac{Y_b - Y_a}{P_1} - S_{p_a} P_1 - \frac{Y_b - Y_a}{P_1} P_{p_a} \]

\[ = X_a - S_{p_a} P_1 - P_{p_a} \]

This equation is identical with equation 3.31, which implies that the value of \( k_k \) is not dependent on the point used to compute it. From Eqs. 3.32 and 3.33, the same statement can be seen to hold for \( s_o \) and \( P_0 \).

3.7 Mathematical Explanation of the Design Procedure

This section presents the mathematical justification for the design procedure presented in Section 3.3. Referring to Fig. 3.5, the dominant closed loop pole regions in the s-plane are denoted by \( C \) and \( \overline{C} \). The mapping function, which maps the dominant closed loop pole region into the X,Y plane, is denoted by \( T \).
FIG. 3.5 MAPPING OF THE DOMINANT CLOSED LOOP

POLE REGION, $T_f : C, \tilde{C} \rightarrow S_f$ FOR $s_p = s_p^f \in [s_{p_{\text{MIN}}}, s_{p_{\text{MAX}}}]$

$S$-PLANE

FIG. 3.6 MAPPING OF THE DOMINANT CLOSED LOOP

POLE REGION FOR ALL $s_p \in [s_{p_{\text{MIN}}}, s_{p_{\text{MAX}}}]$

$X$-$Y$ PLANE
Denote the mapping of the sets $C, \overline{C}$ for

$$S_p = S_{p_i} \in [S_{p_{\min}}, S_{p_{\max}}]$$

by means of Eqs. 3.14 and 3.15, as $S_i$ i.e. $T_i: C, \overline{C} \rightarrow S_i$ (See Fig. 3.5). Now from Eqs. 3.14 and 3.15, it is evident that no two pair of points in $C, \overline{C}$ map into the same point in $S_i$, therefore $T_i$ is a one-to-one mapping function. Since every element of $S_i$ appears as the image of at least one pair of points in $C, \overline{C}$, $T_i$ maps $C$ and $\overline{C}$ onto $S_i$. Now, since $T_i$ is a one-to-one mapping function and also maps $C$ and $\overline{C}$ onto $S_i$, then the inverse mapping function $T_i^{-1}$ exists and maps $S_i$ onto $C, \overline{C}$ in a one-to-one fashion.

The mapping of $C$ and $\overline{C}$ into the $X, Y$ plane for an infinite number of values of $S_p$ between $S_{p_{\min}}$ and $S_{p_{\max}}$ will result in the set shown in Fig. 3.6. This set may be represented as

$$S = \bigcap_{i=1}^{\infty} S_i$$

(3.36)

Under the assertion that $T_i^{-1}$ exists and maps $S_i$ onto $C, \overline{C}$ in a one-to-one fashion, i.e.,

$$T_i^{-1}: S_i \rightarrow C, \overline{C} \text{ for } S_p = S_{p_i} \in [S_{p_{\min}}, S_{p_{\max}}]$$

(3.37)

it follows that

$$T_i^{-1}: S_i \rightarrow C, \overline{C} \text{ for all } S_p = S_{p_i} \in [S_{p_{\min}}, S_{p_{\max}}]$$

(3.38)

Referring to Fig. 3.7, this means that if the given variation in $X$ and $Y$, denoted by the set $P$, is a
FIG. 3.7 INVERSE MAPPING OF THE SET P IN THE X-Y PLANE TO THE S-PLANE
subset of the set \( S = \bigcap_{i=1}^{n} S_i \), then the inverse mapping of \( P \) into the \( s \)-plane will be contained in the sets \( C, \overline{C} \). Letting the set \( P \) represent the mapping of the plant pole variation, then the solution of Eqs. 3.31, 3.32 and 3.33 yield the necessary values of \( k_h, S_0 \) and \( P_0 \) to position the set \( P \) within the set \( S \) which represents the mapping of the dominant closed loop pole region. In this manner the difficult problem of choosing \( k_h, S_0 \) and \( P_0 \) so as to insure that the dominant closed loop poles lie within their acceptable region in the \( s \)-plane, is transformed into the less difficult problem of determining \( k_h, S_0 \) and \( P_0 \) such that the mapping of the plant pole variation in the \( \Delta X, \Delta Y \) plane may be fitted inside the mapping of the acceptable dominant pole region in the \( X,Y \) plane.

3.8 Analytic Aspects of the Mapping of the Dominant Closed Loop Pole Region

The relative complexity of Eqs. 3.14 and 3.15 require that the actual mapping of the dominant closed loop pole region be done point by point using a digital computer. In this section, the mapping of curves of constant \( P_r \) and curves of constant \( S_r \) in the \( s \)-plane into the \( X,Y \) plane is examined. Curves of constant \( P_r \) in the \( s \)-plane are shown in Fig. 3.8. The mapping equations are
**Fig. 3.8 Curves of Constant Parameter $p_c$ in the S-Plane**

- $s + j\omega$
- $p_c = \sigma^2 + \omega^2$

**Fig. 3.9 Mapping of Curves of Constant Parameter $p_c$ in the S-Plane into the X-Y Plane**

Parabola vertex at

\[
X = \frac{1}{p_c} + p_c + \left( \frac{s_0 + p_c}{2} \right)^2
\]

\[
Y = \frac{1}{2} \left( s_0 + p_c \right) \left( \frac{1}{p_c} + p_c \right)
\]
\[ \frac{Y_{r}}{P_{r}} + S_{r}(S_{p} + P_{1} - S_{r}) + P_{r} = X \]

\[ \frac{S_{r}Y_{r}}{P_{r}} + P_{c}(S_{p} + P_{1} - S_{r}) = Y \]

Since \( P_{r} \) is to be held constant, \( S_{r} \) must be eliminated in these two equations. The parameters \( Y, P_{1} \) and \( S_{p} \) will also be held constant in these two equations.

Solving the second equation for \( S_{r} \) yields

\[ S_{r} = \frac{Y - P_{r}P_{1} - P_{r}S_{p}}{Y/P_{r} - P_{r}} \] (3.39)

Substitution of Eq. 3.39 into the first equation gives

\[ \frac{Y - P_{r}P_{1} - P_{r}S_{p}}{P_{r}} - \frac{Y/P_{r} - P_{r}}{P_{r}}(S_{p} + P_{1}) - \frac{Y/P_{r} - P_{r}}{P_{r}} + P_{r} = X \] (3.40)

After considerable algebraic manipulation, Eq. 3.40 may be placed in the form of

\[ -(\gamma/P_{r} - P_{r})^{2}(X - \gamma/P_{r} - P_{r} - \frac{S_{p} + P_{1}}{4}) = \left[ Y - \frac{1}{2}(S_{p} + P_{1})(\gamma/P_{r} + P_{r}) \right]^{2} \] (3.41)

Equation 3.41 is in the form

\[ -4a(X - h) = (Y - k)^{2} \] (3.42)

which is the equation of a parabola opening to the left, whose vertex is at \((h, k)\) with a focal length equal to \(a\). Relating these quantities to Eq. 3.41, the vertex
of the parabola, represented by Eq. 3.41, has the following coordinates in the X,Y plane

\[ X = \frac{\gamma}{P_r} + \frac{(S_p + P_1)^2}{4} \quad (3.43) \]

\[ Y = \frac{\dot{z}(S_p + P_1)(\gamma/P_r + P_r)}{2} \quad (3.44) \]

The focal length of this parabola is

\[ a = \frac{(\gamma/P_r - P_r)^2}{4} \quad (3.45) \]

The position of a parabola defined by Eq. 3.41 is shown in the X,Y plane in Fig. 3.9. It should be noted that Eq. 3.41 represents a parabola whose axis is parallel with the X axis in the X,Y plane. This is indicated by the absence of any terms of the form CXY in Eq. 3.41 with C = constant.

Curves of constant \( S_r \) in the s-plane are shown in Fig. 3.10. Since \( S_r \) is to be held constant, \( P_r \) must be eliminated in the two mapping equations.

Defining

\[ A \triangleq S_p + P_1 - S_r \quad (3.46) \]

and solving the second mapping equation for \( P_r \) yields

\[ P_r = \frac{\gamma}{2A} \pm \sqrt{\frac{\gamma^2}{4A^2} - \frac{S_r \gamma}{A}} \quad (3.47) \]

Substitution of Eq. 3.47 into the first mapping equation gives
Fig. 3.10 Curves of constant $S_n$ in the $s$-plane

$S_n = -2\lambda$

$S_{n1} > S_{n2} > S_{n3} > S_{n4}$

Fig. 3.11 Mapping of curves of constant $S_n$ in the $s$-plane into the $x$-$y$ plane

Major axis of the hyperbola

Minor axis of the hyperbola
After much algebraic manipulation, this equation may be put in the following form:

\[ X^2 - XY\left(\frac{S + P_1}{S_\gamma A}\right) + \frac{Y^2}{S_\gamma A} - 2AS_\gamma X + Y(S_\gamma + P_1) + A^2\left(\frac{\gamma(A - S_\gamma)}{S_\gamma A}\right) = 0 \]  

(3.49)

This equation now is in the form of

\[ ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \]  

(3.50)

with

\[ a = 1 \]
\[ b = -\frac{1}{2} \left(\frac{S + P_1}{S_\gamma A}\right) \]
\[ c = \frac{1}{4} \left(\frac{1}{S_\gamma A}\right) \]

which is of the form of a second degree equation in a rotated system of coordinates. The type of second degree equation which Eq. 3.50 represents can be determined by examining the coefficients \(a\), \(b\) and \(c\). From Eq. 3.50

\[ b^2 - ac = \left[-\frac{1}{2}\left(\frac{S + P_1}{S_\gamma A}\right) \right]^2 - \frac{1}{S_\gamma A} \]

\[ = 4 S_\gamma A^2 (S + P_1)^2 - S_\gamma A \]  

(3.51)
It can be shown that the following relationships exist between the coefficients of Eq. 3.50 and the type of second degree equation.

\[ b^2 - ac < 0 \rightarrow \text{an ellipse} \]

\[ b^2 - ac = 0 \rightarrow \text{a parabola} \]

\[ b^2 - ac > 0 \rightarrow \text{a hyperbola} \]

Equation 3.51 can be put in the form

\[ 4S_r^2A^2[(S_r+A)^2-S_rA] \]

or

\[ 4S_r^2A^2[S_r^2+S_rA+A^2] \quad (3.52) \]

Now if \( A \) is greater than zero, so is Eq. 3.52, i.e.,

\[ 4S_r^2A^2(S_r^2+S_rA+A^2) > 0 \quad (3.53) \]

since \( S_r \) is always greater than zero (closed loop poles in left hand plane). For \( A \) to be greater than zero, \( S_p, P_1 \) and \( S_r \) must satisfy the following

\[ P_1 > S_r - S_p \quad (3.54) \]

The parameter \( S_p \) is negative if the plant poles lie in the right half plane. Therefore Eq. 3.54 can be written as

\[ P_1 > S_r + |S_p| \quad \text{for } S_p < 0 \quad (3.55) \]

This equation is certainly satisfied for most feedback control systems. Curves of constant \( S_r \) in the \( s \)-plane therefore map as hyperbolas in the \( X,Y \) plane under the
mapping equations 3.14 and 3.15. The position of a
hyperbola defined by Eq. 3.49 in the X,Y plane is
shown in Fig. 3.11.

The angle which the major axis of the hyperbola is rotated from the X axis in the X,Y plane is
given by

\[
d = \tan^{-1}\left(\frac{c-a}{2b}\right)
\]

\[
= \tan^{-1}\left\{\left[\left(\frac{c-a}{2b}\right) \pm \sqrt{\left(\frac{c-a}{2b}\right)^2 + \frac{4b^2}{2b} - 2}\right] \right\}
\]

(3.56)

To obtain an insight on the magnitude of the angle \(d\),
the following values are assigned to the parameters in
Eq. 3.57 which are typical for the design example in
Chapter II and the design example that will be con-
sidered in this chapter

\[S_r = 10\]
\[S_p = 2\]
\[P_1 = 30\]
\[\Lambda = 22\]

Substitution of these quantities into Eq. 3.57 gives

\[
d = \tan^{-1}\left\{\left[\frac{\sqrt{\left(\frac{1}{880}\right)^2 \pm \left(\frac{1}{880} \right)^2 + \left(\frac{32}{220}\right)^2}{2}} - \left(\frac{32}{220}\right)\right] \right\}
\]

\[= \tan^{-1} 13.81 = 85.86^\circ\]
For the range of parameters considered in this paper, a good approximation to Eq. 3.57 is

\[ \theta \approx \tan^{-1} \frac{2s_y A}{s_{f} + A} \]  

(3.58)

The sign in Eq. 3.57 may be arbitrarily chosen to make \( \theta \) positive.

The only other two parameters in the mapping of the dominant closed loop pole region are \( p_1 \), the "far-off" pole on the real axis, and \( y \) defined by Eq. 3.11. The qualitative effects of these parameters on the mapping of the dominant closed loop pole region are shown in Figs. 3.13 and 3.14. The dominant closed loop pole region used for this investigation is shown in Fig. 3.12. From inspection of Figs. 3.13 and 3.14, it is seen that increasing the value of \( y \) both increases the relative size of the mapping of the dominant closed loop pole region and its coordinates in the X,Y plane. Increasing the value of \( p_1 \) merely increases the coordinates of the mapping in the X,Y plane. Figure 3.15 illustrates the variation in the "far-off" closed loop poles, \( p_{f_1} \) and \( p_{f_2} \), during the mapping of dominant closed loop region into the X,Y plane. These "far-off" poles are obtained using Eqs. 3.25 and 3.26 and the procedure outlined in Section 4 during the mapping operation.
FIG. 3.12 DOMINANT CLOSED LOOP POLE REGION USED TO ILLUSTRATE THE EFFECT OF $\alpha$ AND $p_1$ ON THE MAPPING INTO THE X-Y PLANE.

CENTER OF CIRCLE AT $-5 + j3$

RADIUS = 2.5
FIG. 3.13 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION AS A FUNCTION OF THE PARAMETER $\xi$

$P_1 = 30$

$S_\omega = 2$

$\xi = 20000$

$\xi = 16000$

$\xi = 10000$
FIG. 3.14 MAPPING OF THE CLOSED LOOP POLE REGION AS A FUNCTION OF THE OPEN LOOP POLE \( \phi \)

\[ \phi = 20000 \quad S_\rho = 2 \]
FIG. 3.15 VARIATION IN THE FAR-OFF CLOSED LOOP POLE $\gamma_k$ FOR THE MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION

$\gamma = 20000 \quad \rho_i = 30$
3.9 Analytic Aspects of the Mapping of the Plant Pole Variation

As with mapping of the dominant closed loop pole region, the mapping of the plant pole variation (Eqs. 3.29, 3.30) is most easily accomplished by mapping point by point using a digital computer. In this section, the mapping of curves of constant $P_p$ constant $S_p$ and constant $\omega$ in the $s$-plane into the $\Delta X$, $\Delta Y$ plane is investigated.

The mapping equations are

\[
\begin{align*}
\Delta X &= P_1 (S_p - S_{p_o}) + P_p (P_p - P_{p_o}) \\
\Delta Y &= P_1 (P_p - P_{p_o})
\end{align*}
\]

where $S_p$, $P_p$ define an arbitrary point $S_{p_o} + j\omega P_{p_o}$ on the boundary of the plant pole variation in the $s$-plane.

Curves of constant $P_p$ in the $s$-plane are shown in Fig. 3.16. The mapping equations with $P_p$ equal to a constant are

\[
\begin{align*}
\Delta X &= P_1 (S_p - S_{p_o}) + P_p (P_{p_o} - P_{p_o}) \quad (3.58) \\
\Delta Y &= P_1 (P_p - P_{p_o}) = \text{constant} \quad (3.59)
\end{align*}
\]

The mapping of curves of constant $P_p$ in the $s$-plane into the $X,Y$ plane is shown in Fig. 3.17.
Fig. 3.16 Curves of constant $p_0$ in the $s$-plane.

Fig. 3.17 Mapping of curves of constant $p_0$ in the $s$-plane into the $\Delta x$-$\Delta y$ plane. The $\Delta y$ axis intercept is $p_0 (p_0 - p_n)$. 
Curves of constant $S_p$ in the $s$-plane are shown in Fig. 3.18. Eliminating $P_p$ in the two mapping equations yields

$$
\Delta Y = P_1 \Delta X - P_1^2 (S_p - S_p) \tag{3.60}
$$

Equation 3.60 is in the form

$$
\Delta Y = m \Delta X + b \tag{3.61}
$$

which is the equation of a straight line in the $\Delta X, \Delta Y$ plane with slope $m$ and $\Delta Y$ intercept $b$ where

$$
m = P_1 ; \quad b = -P_1^2 (S_p - S_p)
$$

The mapping of curves of constant $S_p$ in the $s$-plane into the $\Delta X, \Delta Y$ plane is shown in Fig. 3.19.

Curves of constant $\omega$ are shown in Fig. 3.20. A point on the $\omega = \omega_1$ line is defined as $\sigma + j \omega_1$. The mapping equations are

$$
\Delta X = P_1 \Delta S_p + \Delta P_p = P_1 (-2 \sigma + 2 \sigma) + (\sigma^2 + \omega_1^2 - \sigma^2 - \sigma^2) \tag{3.62}
$$

$$
\Delta Y = P_1 \Delta P_p = P_1 (\sigma^2 + \omega_1^2 - \omega^2) \tag{3.63}
$$

Since $\omega_1$ is to be held constant, $\sigma$ must be eliminated in Eqs. 3.62 and 3.63. Solving equation 3.63 for $\sigma$ yields

$$
\sigma = \pm \sqrt{\frac{\Delta Y}{P_1} + \frac{\sigma^2}{\omega^2} + \frac{\omega_1^2}{\omega^2} - \frac{\omega^2}{\omega_1^2}} \tag{3.64}
$$

Substitution of Eq. 3.64 into Eq. 3.62 gives
FIG. 3.18 CURVES OF CONSTANT $s_p$

IN THE S-PLANE

$\{p_0 + j\omega_p, s_p = -2\omega_p\}

\{p_0 + j\omega_p, s_p = -2\omega_p = s_1\}

FIG. 3.19 MAPPING OF CURVES OF CONSTANT $s_p$

IN THE S-PLANE INTO THE $\Delta x, \Delta y$ PLANE

$\Delta y = \Delta x$, $s_1, s_2, s_3, s_4$

$\Delta y$ AXIS INTERCEPT

$15 - \frac{p_1}{2}(s_p - s_0)$

SLOPE $= p_1$
FIG. 3.20 CURVES OF CONSTANT \( \omega \) IN THE

\[ S - \text{PLANE} \]

\[ \omega = \omega_3 \]

\[ \omega = \omega_2 \]

\[ \omega = \omega_1 \]

\[ \omega_3 > \omega_2 > \omega_1 \]

\[ \delta + j \omega_0 \]

\[ \delta + j \omega_0 + j \omega_0 \]

FIG. 3.21 MAPPING OF CURVES OF CONSTANT \( \omega \)

IN THE S-PLANE INTO THE \( \Delta x - \Delta y \) PLANE

\[ \Delta y \]

\[ \Delta x' \]

ROTATED \( \Delta x' - \Delta y' \) SYSTEM OF COORDINATES

\[ \delta + j \omega_0 \]

\[ \delta + j \omega_0 + j \omega_0 \]
\[ \Delta X = -2P_1 \left[ z \sqrt{\frac{\Delta Y}{P_1}} + \sigma \frac{w^2}{P_0} - \sigma \frac{w^2}{P_0} \right] + \frac{\Delta Y}{P_1} \] (3.65)

Rearranging Eq. 3.65 gives

\[ (\Delta X)^2 - \frac{2\Delta X \Delta Y}{P_1} \left( \frac{\Delta Y}{P_1} \right)^2 - 4P_1 \sigma \frac{\Delta X + 4\Delta Y (\sigma_p - P_1)}{P_0} - 4P_1 (\sigma_w^2 - w_0^2) = 0 \]

(3.66)

This equation is now in the form of Eq. 3.50, the form of a second degree equation in a rotated system of coordinates. The form of this second degree equation may be determined by examining the first three coefficients, a, b, and c of Eq. 3.66. For Eq. 3.66

\[ a = 1 \quad b = -\frac{\Delta Y}{P_1} \quad c = \frac{\Delta Y}{P_1} \]

and

\[ b^2 - ac = \frac{\sigma_w^2}{P_1} - \frac{\sigma_w^2}{P_1} = 0 \]

According to the conditions previously stated, equation 3.66 represents a parabola in a rotated system of coordinates.

The angle which the axis of the parabola makes with the \( \Delta X \) axis in the \( \Delta X, \Delta Y \) plane is given by Eq. 3.56 which is

\[ \theta = \tan^{-1} \left( \frac{\sigma_w^2}{\sqrt{\left( \frac{\sigma_w^2}{P_1} - 1 \right)^2 + \frac{4}{P_1^2}}} \right) \]

\[ = \tan^{-1} \left[ \frac{\left( \frac{\sigma_w^2}{P_1} - 1 \right) \frac{4}{P_1^2}}{-2/P_1} \right] \]

\[ = \tan^{-1} P_1 \] (3.67)
where the sign on the radical has been chosen to make $\delta$ positive.

Unlike the mapping equations developed for the dominant closed loop pole region, equation 3.66 is not overly complex and a transformation may be made that will eliminate the $\Delta X \Delta Y$ term in Eq. 3.66. The transformation equations which will transform Eq. 3.66 into the following form

$$ a'(AX')^2 + c'(AY')^2 + 2d'AX' + 2e'AY' + f = 0 \quad (3.68) $$

where the primes indicate quantities referred to the rotated system of coordinates are

$$ a' = a \cos^2 \theta + 2b \sin \theta \cos \theta + c \sin^2 \theta \quad (3.69) $$

$$ c' = a \sin^2 \theta - 2b \sin \theta \cos \theta + c \sin^2 \theta \quad (3.70) $$

$$ d' = d \cos \theta + e \sin \theta \quad (3.71) $$

$$ e' = -d \sin \theta + e \cos \theta \quad (3.72) $$

The trigonometric functions may be expressed as follows

$$ \cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + P^2_1}} \quad (3.73) $$

$$ \sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + P^2_1}} \quad (3.74) $$

The new coefficients $a'$, $c'$, $d'$ and $e'$ are
\[ a' = \frac{1}{1+P_1^2} - \frac{(2/P_1)P_1^2}{1+P_1^2} + \frac{(4/P_1^2)P_1^2}{1+P_1^2} = 0 \] (3.75)

\[ c' = \frac{p_1^2}{1+P_1^2} + \frac{(2/P_1)P_1^2}{1+P_1^2} + \frac{1/P_1^2}{1+P_1^2} = \frac{p_1^2 + 1}{p_1^2} \] (3.76)

\[ d' = \frac{-2P_1p_0^2}{\sqrt{1+P_1^2}} + \frac{2P_1(\sigma p_0 - P_1)}{\sqrt{1+P_1^2}} = \frac{-2p_1^2}{\sqrt{1+P_1^2}} \] (3.77)

\[ e' = \frac{2P_1\sigma p_0^2}{\sqrt{1+P_1^2}} + \frac{2(\sigma p_0 - P_1)}{\sqrt{1+P_1^2}} = \frac{2\sigma p_0 (P_1^2 + 1) - 2P_1}{\sqrt{1+P_1^2}} \] (3.78)

Equation 3.68 with \( a' = 0 \) is of the following form

\[ c'(\Delta Y')^2 + 2d'\Delta X' + 2e'\Delta Y' + f = 0 \] (3.79)

which may be put in the form of

\[ (\Delta Y' + e'/c')^2 = \frac{2d'}{c^2}(\Delta X' + \frac{f}{2d'} - \frac{(e')^2}{2d'c'}) \] (3.80)

From Eq. 3.42, this equation represents a parabola in \( \Delta X', \Delta Y' \) system of coordinates with its center at

\[ \Delta X = -\frac{f}{2d'} + \frac{(e')^2}{2d'c'} \] (3.81)

\[ \Delta Y' = -\frac{e'}{c'} \] (3.82)

and focal length \( a \) of

\[ a = -\frac{d'}{2c'} \] (3.83)
In most feedback control systems, $P_1$ is much greater than unity. Therefore the following approximations can be used for the coefficients $c'$, $d'$ and $e'$:

$$c' = \frac{P_1^2 + 1}{P_1^2} \approx 1 \quad (3.84)$$

$$d' = -\frac{2P_1}{\sqrt{1+P_1^2}} \approx -2P_1 \quad (3.85)$$

$$e' = \frac{2\sqrt{P_0 P_2}}{\sqrt{1+P_1^2}} \approx 2\left(\sqrt{P_0 P_1^{-1}}\right) \quad (3.86)$$

The approximation to Eq. 3.80 is then

$$[\Delta Y' + 2(\sqrt{P_0 P_1^{-1}})]^2 = 4P_1[\Delta X' + P_1(w_2 - \omega_2) + \frac{(\sqrt{P_0 P_1^{-1}})^2}{P_1}]$$

(3.87)

The position of the parabolas defined by Eq. 3.87 in the $\Delta X, \Delta Y$ plane is shown in Fig. 3.21. From Eq. 3.87 and Fig. 3.21, it is seen that the $\Delta Y'$ coordinate of the center of the parabola in the $\Delta X', \Delta Y'$ rotated coordinate system is not dependent on the parameter $\omega_1$.

The only other parameter in the mapping of the plant pole variation is $P_1$. The qualitative effect of $P_1$ on the mapping of the plant pole variation is shown in Fig. 3.23. The plant pole variation used in the
FIG. 3.22 PLANT POLE VARIATION REGION
USED TO ILLUSTRATE THE EFFECT OF $P_1$ ON
THE MAPPING INTO THE $\Delta X - \Delta Y$ PLANE

CENTER OF CIRCLE
AT $-1 + j6$

RADIUS $= 4$
FIG. 3.23 MAPPING OF THE PLANT POLE VARIATION REGION AS A FUNCTION OF THE OPEN LOOP POLE $p_i$.
mapping is shown in Fig. 3.22. From Fig. 3.23, it is observed that the size of the mapping of the plant pole variation increases with increasing $P_1$. Since the size of the mapping of the dominant closed loop pole region is not highly sensitive to $P_1$ (see Fig. 3.14), it is very important to bring $P_1$ in as far as possible to minimize the needed system gain $kh$.

3.10 The Effect of Plant Gain Variation on the Design

It is possible that the dominant closed loop poles may lie within their acceptable region for $k=k_{\text{min}}$ and yet move outside the acceptable region for $k>k_{\text{min}}$. Figure 3.24 illustrates this possibility. This undesirable possibility can be predicted by considering the angle of departure of the root locus from the dominant closed loop pole at $k=k_{\text{min}}$ and the angle of entry of the root locus into the compensation zeroes. This is the procedure used by Korowitz in his paper.

Figure 3.25 illustrates the dominant closed loop poles within their acceptable region at $k=k_{\text{min}}$. For a given plant pole variation, the dominant closed loop poles for $k=k_{\text{min}}$ may lie anywhere on a boundary such as the boundary $ABCD$ shown in Fig. 3.25. It is usually sufficient to check the angle of departure of the root locus for a few points around the boundary at $k=k_{\text{min}}$. 
FIG. 3.24 POSSIBLE EFFECT OF GAIN VARIATION ON
THE DOMINANT
CLOSED LOOP POLES

ACCEPTABLE
DOMINANT
CLOSED LOOP
POLE REGION

PLANT POLE
COMPENSATION
ZERO

DOMINANT CLOSED LOOP POLE AT \( k = k_{\text{MAX}} \)
ROOT LOCUS

DOMINANT CLOSED LOOP POLE AT \( k = k_{\text{MIN}} \)

FIG. 3.25 COMPUTATION OF THE ANGLE OF DEPARTURE
OF THE ROOT LOCUS FROM THE DOMINANT
CLOSED LOOP POLE AT \( k = k_{\text{MIN}} \)

LINE SEGMENT
FROM \(-p_1\)

ACCEPTABLE
DOMINANT
CLOSED LOOP
POLE REGION

COMPENSATION
ZERO

BOUNDARY OF
THE DOMINANT
CLOSED LOOP
POLES AT \( k = k_{\text{MIN}} \)

LINE SEGMENT
FROM \(-p_2\)

\( 90^\circ \)

LINE SEGMENT
FROM \( \bar{z} \)
To outline the procedure, consider dominant closed loop poles at $B$ and $\bar{B}$ for $k = k_{\text{min}}$ and "far-off" closed loop poles at $-p_{f_1}$ and $-p_{f_2}$ for $k = k_{\text{min}}$. Define the following angles:

$$\angle \overline{BB} = 90^\circ$$ \hspace{1cm} (3.90)

$$\angle -p_{f_1}B = \alpha_1$$ \hspace{1cm} (3.91)

$$\angle -p_{f_2}B = \alpha_2$$ \hspace{1cm} (3.92)

$$\angle ZZB = \phi_1$$ \hspace{1cm} (3.93)

$$\angle \overline{ZB} = \psi_2$$ \hspace{1cm} (3.94)

The angle of departure of the root locus $\phi_d$ from the dominant closed loop pole located at $B$ is then

$$\phi_d = 180^\circ - (\alpha_1 + \alpha_2 + 90^\circ - \phi_1 - \psi_2)$$ \hspace{1cm} (3.95)

It is relatively easy to ascertain for a particular mapping what angles of departure may lead to an unsatisfactory design. For example, for a dominant closed loop pole located at point $B$ in Fig. 3.25, an angle of departure $\phi_d$ of $-10^\circ$ would probably be unsatisfactory.

If there is still some doubt whether the root locus remains within the acceptable region for a particular dominant closed loop pole location, the angle of entry of the root locus into the compensation zeroes
can be checked. As before consider dominant closed loop poles located at B and \( \overline{B} \) and define the following angles

\[
\angle \overline{Z} Z = 90^\circ \quad (3.96) \\
\angle B Z = \theta_1 \quad (3.97) \\
\angle \overline{B} Z = \theta_2 \quad (3.98) \\
\angle -p_1 Z = \theta_3 \quad (3.99) \\
\angle -p_2 Z = \theta_4 \quad (3.100)
\]

The angle of entry of the root locus \( \psi_e \) to the complex zero \( Z \) for dominant closed loop poles located at B and \( \overline{B} \) is then

\[
\psi_e = \theta_1 + \theta_2 + \theta_3 + \theta_4 - 90^\circ - 180^\circ \quad (3.101)
\]

As stated for the angle of departure criteria, it should be relatively easy to determine for a given mapping what angles of entry are acceptable. Considering the case again for dominant closed loop poles located at B and \( \overline{B} \), if the root locus had an angle of departure from B of \(-10^\circ\) and an angle of entry to the compensation zero Z of \(+10^\circ\), this would probably confirm that the root locus is outside the dominant closed loop pole region for some value of \( k > k_{\text{min}} \).
If such a situation arises in the design, the procedure is to increase the value of added gain $h$ until all angles of departure and entry are satisfactory. The object, of course, is to obtain a design with the least amount of system gain $kh$.

### 3.11 Design Example

A design example is presented in this section based on the previous design procedure. The region of plant pole variation, acceptable dominant closed loop pole region and boundary for the "far-off" closed loop poles are shown in Fig. 3.26. The acceptable dominant closed loop pole region is the same as used in the design example of Chapter II. The plant pole variation is essentially the same as used by Horowitz. The boundary for the "far-off" closed loop poles has been arbitrarily chosen as shown in Fig. 3.26.

Following the design procedure outlined in Section 3 of this Chapter, the dominant closed loop pole region is mapped into the X,Y plane with parameter $P_1$, $\gamma$ and $S_P$.

After several computer runs, values for $P_1$ and $\gamma$ of 10 and 20,000 respectively were used for the first design. The value of 10 for $P_1$ was chosen because this value places the "far-off" closed loop poles very near the boundary shown in Fig. 3.26. Using Eq. 3.18 with
FIG. 3.24 S-DOMAIN SPECIFICATIONS FOR THE DESIGN EXAMPLE

BOUNDARY FOR THE "FAR-OFF" CLOSED LOOP POLES

ACCEPTABLE CLOSED LOOP POLE REGION (DOMINANT)

BOUNDARY OF PLANT POLE VARIATION

Pole Variation

-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
\( P_r = 60 \) and \( p_{f_1} = p_{f_2} = +15-j10 \), a value of approximately 19,000 for \( \gamma \) is obtained which is quite close to the value used in the first design. The value of \( P_r \) for this approximation was found from the acceptable dominant closed loop pole region while the values of \( p_{f_1} \) and \( p_{f_2} \) were estimated from the knowledge of the anticipated root locus. The values of \( S_p \) used were -5, 2 and 10 which correspond to the minimum, median and maximum possible values of \( S_p \) for the plant pole variation shown in Fig. 3.26. As anticipated the mapping of dominant closed loop pole region is not highly sensitive to the value of \( S_p \) at this large value of \( \gamma \). The mapping of the dominant closed loop pole region for these parameter values is shown in Fig. 3.27.

The mapping of the plant pole variation for various values of \( P_1 \) including \( P_1=30 \) is shown in Fig. 3.28.

It should be emphasized that the only practical means of performing these mapping operations is on a digital computer. On such a machine, the mapping of the dominant closed loop pole region may be easily performed for many different combinations of \( \gamma \) and \( P_1 \). The same values of \( P_1 \) are then used in the mapping of the plant pole variation. Once the mappings have been plotted, it is not difficult to choose a minimum value
FIG. 3.27 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION FOR THE DESIGN EXAMPLE

$y = 20000$

$P_1 = 30$
FIG. 3.28 MAPPING OF THE PLANT POLE VARIATION FOR THE DESIGN EXAMPLE
of \( P_1 \) such that the "far-off" closed loop poles are at the vertical boundary shown in Fig. 3.26 and a value of \( \gamma \) such that the mapping of the plant pole variation may be fitted inside the mapping of the dominant closed loop pole region for the same value of \( P_1 \).

The mapping of the dominant closed loop pole region with the mapping of the plant pole variation placed inside it is shown in Fig. 3.29. Since the mapping of the plant pole variation does not quite fit inside the mapping of the dominant closed loop pole region, a slightly larger value of \( \gamma \) may have to be used.

The values of \( k_K, S_o \) and \( P_o \) are solved for using Eqs. 3.31, 3.32 and 3.33 and point A in Fig. 3.29. At point A

\[
X_a = 800 \quad Y_a = 5450
\]

and from Fig. 3.26, at point A

\[
S_p = -6 \quad P_p = 10
\]

For \( \gamma = 2000 \) and \( P_1 = 30 \), \( k_K, S_o \) and \( P_o \) are

\[
k_K = \frac{X_a - S_p P_1 - P_p}{P_p} = \frac{800 - (-6)(30) - 10}{10} = 970
\]

\[
S_o = \frac{Y_a - P_p}{k_K} = \frac{5450 - 10(30)}{970} = 5.32
\]
FIG. 3.29 CALCULATION OF SYSTEM GAIN AND COMPENSATION ZERO POSITION FOR $\delta = 20000$
AND $p_1 = 30$

MAPPINGS OF THE DOMINANT CLOSED LOOP POLE REGION

MAPPING OF THE PLANT POLE VARIATION

1st Position

2nd Position

$\delta_p = 6, \delta_p = 2, \delta_p = 10$
\[ P_0 = \frac{\gamma}{kk} = \frac{20000}{970} = 20.61 \]

Solving for the position of the compensation zeroes yields

\[ S_0 = -2a_z \]
\[ a_z = \frac{S_0}{-2} = -2.66 \]
\[ P_0 = a_z^2 + \omega_z^2 \]
\[ \omega_z = \sqrt{P_0 - a_z^2} = \sqrt{20.61 - 2.66^2} \]
\[ = 3.60 \]

The position of the compensation zeroes \( Z, \bar{Z} \) is then

\[ Z, \bar{Z} = -2.66 \pm j3.60 \]

The actual closed loop poles for this choice of system gain and compensation zero position are found by determining the roots of the characteristic equation \( 1 + L_d(s) = 0 \) where \( L_d(s) \) is given by Eq. 1.12 for points around the boundary of the plant pole variation. The convergence procedures in Appendix A were used to factor the resulting fourth order polynomial. The results of this operation are shown in Fig. 3.30 (Trial 1). The points A, B, C, and D correspond with those in Fig. 3.30. From Fig. 3.30, it is seen that the dominant closed loop poles lie outside
Fig. 3.30 Dominant Root Test for $s = 20000$ and $P_1 = 30$

Trial 1: $R_k = 970 \quad Z = -2.66 + j3.60$

Trial 2: $R_k = 1045 \quad Z = -2.46 + j3.62$
their acceptable region for plant poles located at points $B, \bar{B}$. The closest approach of the "far-off" closed loop poles also occurs when the plant poles are at points $B, \bar{B}$ which is $-6.498 \pm j26.395$. For all other points used in the dominant root test, the "far-off" closed loop poles were well to the left of the established boundary shown in Fig. 3.26.

In an attempt to place the dominant closed loop poles within their acceptable region, the position of the mapping of the plant pole variation within the mapping of the dominant closed loop pole region was slightly altered as shown in Fig. 3.29. For this new position and using the new coordinates for point $A$, the values of system gain and compensation zero position are

$$kK = 1045 \quad z, \bar{z} = -2.46 \pm j3.62$$

The results of this trial are shown in Fig. 3.30 (Trial 2). The dominant closed loop poles are still outside their acceptable region when the plant poles lie at points $B, \bar{B}$. The closest approach of the "far-off" closed loop poles is $-7.019 \pm j27.571$ when the plant poles are at points $B, \bar{B}$. Therefore the dominant closed loop pole region must be mapped into the $X,Y$ plane using a larger value of $\gamma$. The mapping of the plant
pole variation for $P_1 = 30$ in the $\Delta X$, $\Delta Y$ plane remains unchanged.

Figure 3.31 shows the mapping of the dominant closed loop pole region for $\gamma = 22000$ with the mapping of the plant pole variation fitted within the interior. Using point A again gives the following

\[ k_h = 1165 \quad Z, \bar{E} = -2.432 \pm j3.608 \]

From Fig. 3.32, this value of gain and compensation zero position place the dominant closed loop poles within their acceptable region for all plant pole positions. The closest approach of the "far-off" closed loop poles occurs again when the plant poles are at $B, \bar{E}$ and is $-7.513 \pm j29.661$.

From inspection of Figs. 3.22 and 3.23, it is noted that the mapping of the plant pole variation is quite sensitive to the value of $P_1$ and that the size of the mapping of the plant pole variation decreases with decreasing $P_1$. Therefore it may be possible to place the dominant closed loop poles within their acceptable region at a slightly smaller value of system gain $k_h$ if the value of $P_1$ is decreased. This will also place the "far-off" closed loop poles closer in. Figure 3.33 shows the results of using a value of $P_1 = 28$ and the same of gain and zero position as for Trial 2 with $\gamma = 20000$. The saving in gain between
FIG. 3.31 CALCULATION OF SYSTEM GAIN AND COMPENSATION ZERO LOCATION FOR $\zeta = 22000$ AND $\rho_1 = 30$

MAPPINGS OF THE DOMINANT CLOSED LOOP POLE REGION

MAPPINGS OF THE PLANT POLE VARIATION
FIG. 3.32 DOMINANT ROOT TEST FOR $s=22000$

AND $P_r=30$

$R_K=1165 \quad \Xi=-2.50+j3.56$
FIG. 3.33 DOMINANT ROOT TEST FOR $\gamma = 20000$ AND $P_1 = 2.8$

$RK = 1045 \quad Z = -2.46 + j3.62$

DOMINANT CLOSED LOOP POLE REGION
\( kK = 1045 \) and \( kK = 1165 \) is approximately 1 dB. The closest approach of the "far-off" closed loop poles is \(-6.324 \pm j28.355\) with the plant poles at \( B, \overline{B}\). This design is probably satisfactory since the closed loop poles lie to the left of the boundary shown in Fig. 3.26 for all but three other plant pole positions near point B used in the root test.

From Figs. 3.29 and 3.31, it is noted that the mapping of the plant pole variation need not fit completely inside the mapping of the dominant closed loop pole region for all values of \( S_p \) in order for the dominant closed loop poles to lie within their acceptable region. The requirement that must be satisfied is that the mapping of \( S = S_p \) in the \( \Delta X, \Delta Y \) plane must fit inside the mapping of the dominant closed loop pole region for \( S_p = S_p \). From Fig. 3.31, the line segment \( AB \) corresponds to \( S_p = -6 \) which is completely within the mapping of the dominant closed loop pole region for \( S_p = -6 \). The line segment \( CD \) corresponds to \( S_p = 10 \) which is completely within the mapping of the dominant closed loop pole region for \( S_p = 10 \).

So for the design example, the problem of variation in the plant gain factor has not been considered. The actual value of gain which must be added to the system is given by Eq. 3.34. For the
design of $P_1 = 28, Z, \bar{Z} = -2.46 \pm j3.62$ and $kK = 1045$, the value is $1045/k_{\text{min}}$. If the plant gain variation is assumed to be

$$k_{\text{min}} = 1 \leq k \leq 1000 = k_{\text{max}}$$

the compensation networks must have a gain of at least 1045. At $k = k_{\text{max}}$, the system gain is $1.045 \times 10^6$ and for this large value of gain, the dominant closed loop poles are essentially at the position of the compensation zeroes. Using the method outlined in Section 10 of this chapter, the angles of departure of the root locus for dominant closed loop poles located at points A, B, C and D in Fig. 3.33 were computed. The results were as follows:

Point A: $\phi_d = -14^\circ$
B: $\phi_d = 69^\circ$
C: $\phi_d = 64^\circ$
D: $\phi_d = 133^\circ$

These values are certainly satisfactory for the acceptable dominant closed loop pole region shown in Fig. 3.33. Therefore the design is complete.

### 3.12 Summary

The design procedure for gain and plant pole variation is summarized below.

1.) Map the acceptable dominant closed loop pole region into the X,Y plane using
Eqs. 3.14 and 3.15 with parameters $\gamma$, $P_1$ and $S_p$. Map the plant pole variation into the $\Delta X, \Delta Y$ plane using Eqs. 3.16 and 3.17 with parameter $P_1$.

Determine the value of $\gamma$ and $P_1$ such that the "far-off" closed loop poles just satisfy the minimum damping factor specifications for the problem and the mapping of the plant pole variation in the $\Delta X, \Delta Y$ plane can be fitted into the interior of the mapping of the dominant closed loop pole region in the $X,Y$ plane.

2.) Solve for the values of $k\alpha$, $S_o$ and $P_o$ using Eqs. 3.11, 3.12 and 3.13 which determine the value of system gain and the compensation zero position.

3.) Check the design by determining the actual closed loop poles for plant poles lying on the boundary of the plant pole variation. This is accomplished by determining the roots of $1+L_d(s) = 0$. Check the final design for plant gain variation by using the method outlined in Section 10 of this chapter.
This design procedure which includes the effect of the "far-off" pole $P_1$ has several advantages over the third order system considered by Horowitz. Using the third order approximation, the "far-off" closed loop poles can not be positioned so that the damping factor specifications are just satisfied. This results in a waste of system gain since the "far-off" open loop pole must then be placed sufficiently "far-off" so that its effect on the dominant closed loop poles is negligible. In the fourth order approximation, the effect of this "far-off" pole is considered on both the mapping of the dominant closed loop pole region and plant pole variation and can be used to determine a more economical design.

The next chapter considers the additional effect of a drifting zero on the real axis on the design procedure.
CHAPTER IV

PROBLEM OF SIMULTANEOUS PLANT GAIN, POLE AND ZERO VARIATION

4.1 Problem Definition

This chapter presents an approximate design procedure for handling the added problem of a drifting zero on the real axis. This zero can be considered to be part of a plant with a transfer function \( P(s) \), given by

\[
P(s) = \frac{k(s+z)}{s(s^2+S_p s+P_p)}
\]

(4.1)

where \(-z\) is the position of the drifting zero on the real axis. Typically this zero on the real axis is close to the origin and hence to the dominant closed loop poles, and has an appreciable effect on the position of the dominant closed loop poles. The problem is then to choose the compensation zero position and the value of system gain such that the dominant closed loop poles lie within their acceptable region despite variations in the plant zero, parameter \( z \), and in the plant poles, parameters \( S_p \) and \( P_p \). The problem is shown in Fig. 4.1.
FIG. 4.1  PROBLEM OF PLANT GAIN, POLE AND 

ZERO VARIATION

\[ P(s) = \frac{k(s + y)}{s(s^2 + s_0s + P_0)} \]

\[ L_d(s) = \frac{AK(s^2 + s_0s + P_0)(s + y)}{s(s^2 + s_0s + P_0)(s + P_d)} \]

\[ T_d(s) = \frac{P_0P_1/P_2(s + s)}{(s^2 + s_0s + P_0)(s + P_1)(s + P_2)} \]

REGION OF PLANT POLE VARIATION

DOMINANT CLOSED LOOP POLE REGION

BOUNDARY FOR \( T_d \) MIN

"CLOSED LOOP POLE NEAR THE DRIFTING ZERO"

"FAR-OFF" CLOSED LOOP POLE

"DOMINANT CLOSED LOOP POLE"

"COMPENSATION ZERO"

"OPEN LOOP POLE USED TO PARTIALLY CANCEL THE EFFECT OF THE DRIFTING ZERO"
4.2 Design Philosophy

Any drifting zero of \( T(s) \) will appear as a drifting zero of \( T(s) \) (the system transmission), since it is impossible to precisely cancel such zeroes. To minimize the effect of a drifting zero on \( T(s) \), a pole of \( L_d(s) \) is placed near the zero in an attempt to at least partially cancel the effect of this zero. If the system gain is sufficiently high, a closed loop pole will be very close to the zero despite its drift. The resulting dipole will have a negligible effect on the system response, if the system gain is large enough.

4.3 The Effect of Zero Variation on the Dominant Closed Loop Poles

The effect of a pole-zero pair on the real axis, near the dominant pole region, is shown in Figs. 4.2 and 4.3. These figures are for a system designed according to the procedure in Chapter III, with the added pole-zero pair on the real axis. The mappings were obtained by factoring the roots of \( 1 + L_d(s) = 0 \), where \( L_d(s) \) is given by

\[
L_d(s) = \frac{k(s^2 + s + P_z)}{s(s^2 + s + P_z)(s + P_1)(s + P_2)} \tag{4.2}
\]

and \( z \) and \( P_z \) represent the pole-zero pair on the real axis. The region of plant pole variation is the same as was used in Chapter III, as is the acceptable dominant closed loop pole region.
Fig. 4.2 The effect of zero variation on the dominant closed loop poles (Case 1)

Parameters: \( R_k = 115 \), \( P_1 = 30 \)
\( Z = -2.50 + j3.56 \), \( \gamma = 3 \), \( P_2 = 2 \)

![Diagram showing the variation of the closed loop pole region with a zero indicated]
FIG. 4.5 THE EFFECT OF ZERO VARIATION ON THE DOMINANT CLOSED LOOP POLES (CASE 2)

PARAMETERS: \( kK = 1156 \), \( \rho = 30 \)

\[ z = -2.50 + j3.56 \]

\( z = 1 \), \( \rho_z = 2 \)
Figure 4.2 illustrates the effect on the dominant closed loop poles for \( P_z=2 \) and \( z=3 \); it is similar to the effect of lag compensation. Figure 4.3 is for \( P_z=2 \) and \( z=1 \) which has an effect similar to lead compensation. These two figures, then, illustrate the effect on the dominant closed loop poles of a zero drifting from -1 to -3 when an open loop pole of \( L_d(s) \) is placed at -2. As seen from Fig. 4.2, the system designed according to the procedure in Chapter III is not adequate to handle this zero variation on the real axis, since the dominant closed loop poles lie outside their acceptable region. The dominant closed loop poles could be forced into their acceptable region by using a larger value of system gain. However, this may result in a waste of system gain. Horowitz presents a method\(^9\) to determine whether a design for a given region of plant pole variation is adequate to handle the effect of a drifting zero on the real axis for a specified dipole (closed loop pole-zero) separation. The presence of the pole-zero pair on the real axis may influence the choice of the compensation zero position. The design procedure presented in this chapter takes into account the zero variation in determining the position of the compensation zeroes.
4.4 Design Equations

In considering this problem, it does not take long to determine that the drifting zero added to the plant pole variation, complicates the design procedure to a considerable extent. The dominant loop transmission must now be designed to take into account two independent types of variation, i.e., zero variation along the real axis and plant pole variation in the complex plane. Because of this, the nearest "far-off" open loop \( L(s) \) is omitted from \( L_d(s) \) to retain a fourth order representation for the system. The expression for \( L_d(s) \) is from Eq. 1.14

\[
L_d(s) = \frac{k_h(s^2 + S_0 s + P_0)(s+z)}{s(s^2 + S_p s + P_p)(s+P_s)}\Delta = \frac{k_h n_d(s)}{d_d(s)}
\]

(4.3)

The expression for \( T_d(s) \) is from Eq. 1.15

\[
T_d(s) = \frac{P_r P_f P_c / z(s+z)}{(s^2 + S_r s + P_r)(s+P_c z)(s+P_f)} = \frac{P_r P_f P_c / z(s+z)}{d_d(s)}
\]

(4.4)

The real axis root locus for this system for the extreme positions of the drifting zero is shown in Fig. 4.4.

The mapping equations for this type of system are derived in the same manner as in Chapter III. The
FIG. 4.4 REAL AXIS ROOT LOCUS FOR A FOURTH ORDER SYSTEM WITH A DRIFTING ZERO ON THE REAL AXIS

\[ L_d(s) = \frac{kk(s^2 + s_0^2 + P_o)(s+\gamma)}{s(s^2 + s_p^2 + P_p)(s+P_z^2)}, \quad kk > 0 \]

CASE 1

CASE 2
characteristic equation of the system is, from Eq. 2.3

\[ D_d(s) = d_d(s) + kK n_d(s) \]

\[ (s^2 + S_r s + P_r)(s + P_{f_1}) (s + P_{c_z}) = s(s^2 + s + P_p)(s + P_z) + kK (s^2 + S_p s + P_p)(s + P_z) \]

or

\[ s^4 + (S_r + P_{f_1} + P_{c_z}) s^3 + (P_{f_1} P_{c_z} + S_r (P_{f_1} + P_{c_z}) + P_r) s^2 + \]

\[ (P_r (P_{f_1} + P_{c_z}) + S_r P_{f_1} P_{c_z}) s + P_r P_{f_1} P_{c_z} = \]

\[ s^4 + (S_r + P_{f_1} + P_{c_z}) s^3 + (P_{f_1} P_{c_z} + S_r (P_{f_1} + P_{c_z}) + P_r) s^2 + \]

\[ (P_r P_{f_1} P_{c_z} + kK (S_o + z)) s^2 + (P_r P_{f_1} P_{c_z} + kK (P_o + S_o) + z)) s + kK P_o z \]

Equating the coefficients in Eq. 4.5 yields the following set of equations

\[ S_r + P_{f_1} + P_{c_z} = P_z + S_p + kK \]  

(4.6)

\[ P_{f_1} P_{c_z} + S_r (P_{f_1} + P_{c_z}) + P_r = P_z S_p + P_p + kK (S_o + z) \]  

(4.7)

\[ P_r (P_{f_1} + P_{c_z}) + S_r P_{f_1} P_{c_z} = P_z P_p + kK (P_o + S_o) \]  

(4.8)

\[ P_r P_{f_1} P_{c_z} = kK P_o z \]  

(4.9)
For the problem considered in Chapter III, the parameters relating to the non-dominant closed loop poles on the left hand side of these equations were eliminated by substitution. In this case a slightly different approach is used. Define

\[ U = P_z + S_p + kh \]  
\[ V = k h P_o z \]  
\[ X = P_z P + P + kh (S_o + z) \]  
\[ Y = P_z P + kh (P_o + S_o z) \]

The mapping equations for the dominant closed loop pole region are then

\[ U = S_r + P_{r1} + P_{c \ z} \]  
\[ V = P_{r1} P_{c \ z} \]  
\[ X = P_{r1} P_{c \ z} + S_r (P_{r1} + P_{c \ z}) + P_r \]  
\[ Y = P_r (P_{r1} + P_{c \ z}) + S_r P_{r1} P_{c \ z} \]

The total variation in \( \lambda \), \( \Delta \lambda \), and the total variation in \( Y \), \( \Delta Y \), due to the variation in plant poles, i.e. parameters \( S_p \) and \( P_p \) and the variation in the zero \( z \) are given by
\[ \Delta X = P_z \Delta S_p + \Delta P_p + kh \Delta z \quad (4.18) \]
\[ \Delta Y = P_z \Delta S_p + kh S_p \Delta z \quad (4.19) \]

Since \( kh \) and \( S_p \) are not apriori known, the first approximations to \( \Delta X \) and \( \Delta Y \) are taken to be

\[ \Delta X \approx P_z \Delta S_p + \Delta P_p \quad (4.20) \]
\[ \Delta Y \approx P_z \Delta P_p \quad (4.21) \]

These equations are now identical to the mapping equations 3.16 and 3.17 in Chapter III.

### 4.5 Design Procedure

An outline of the design procedure that will be followed in this problem is as follows:

1.) Map the acceptable dominant closed loop pole region into the \( U,V \) plane using Eqs. 4.14 and 4.15 for fixed values of \( p_{f_1} \) and \( p_{c_z} \).

2.) Map the acceptable dominant closed loop pole region into the \( X,Y \) plane using Eqs. 4.16 and 4.17 for the same fixed values of \( p_{f_1} \) and \( p_{c_z} \).

3.) Map the plant pole variation into the \( \Delta X, \Delta Y \) plane using Eqs. 4.20 and 4.21 for fixed values of \( P_z \).

4.) Compare the two mappings in (2.3) above. If the plant pole variation in the \( \Delta X, \Delta Y \)
plane does not fit within the interior of the mapping of the dominant closed loop pole region in the $X,Y$ plane, then the dominant closed loop pole region will have to be mapped into both the $U,V$ plane and $X,Y$ plane using a larger value of $p_{11}$.

5.) Solve for the values of $kh$, $S_0$ and $P_0$ using Eqs. 4.10-4.13 where the values of $U,V$ are obtained from the mapping of the dominant closed loop pole region in the $U,V$ plane and the values of $X$ and $Y$ are obtained from the positioning of the mapping of the plant pole variation within the interior of the mapping of the dominant closed loop pole region in the $X,Y$ plane.

6.) Determine the additional variation of the mapping of the plant pole variation in the $X,Y$ plane by using Eqs. 4.18 and 4.19 and the values of $kh$ and $S_0$ obtained in (5). If this additional variation cannot be accommodated within the interior of the mapping of the dominant closed loop pole region in the $X,Y$ plane, then the dominant closed loop pole region must be mapped into both the $U,V$ plane and $X,Y$ plane using a larger value of $p_{11}$. 
The next five sections of this chapter elaborate on these steps in the design procedure.

4.6 Mapping of the Dominant Closed Loop Pole Region Into the U,V Plane

The design procedure begins by mapping the dominant closed loop pole region into the U,V plane with parameters $p_{f_1}$ and $p_{c_z}$ using Eqs. 4.14 and 4.15.

The parameter $p_{f_1}$, which is the "far-off" closed loop pole on the real axis, plays the same part as the parameter $y = khP_o$ did in the design procedure of Chapter III, i.e., large $p_{f_1}$ implies large $kh$.

This can be ascertained from Eq. 4.6 since the parameters $S_r$, $P_{c_z}$, $P_z$ and $S_p$ do not have a large variation.

The other parameter in this mapping operation is $p_{c_z}$, the closed loop pole near the drifting zero.

As an approximation to the value of $p_{c_z}$ in the mapping operation, it is assigned values that include the minimum and maximum values of the zero $z$ as well as intermediate values. In practice, this approximation can be improved by performing the mapping operations for $p_{c_z} = z_{max} - \delta_1$ and $p_{c_z} = z_{min} + \delta_2$ where $\delta_1$ and $\delta_2$ are the estimated dipole separations when the zero is at its maximum and minimum positions respectively. The mapping of the dominant closed loop pole region into the U,V plane using Eqs. 4.14 and 4.15 is shown in Fig. 4.5.
FIG. 4.5 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION IN THE S-PLANE INTO THE U-V PLANE

MAPPING EQUATIONS

\[ U = S_n + P_b + P_c y \]
\[ V = P_n P_b + P_c y \]

PARAMETERS \( P_b, P_c \)

---

**S-PLANE**

**U-V PLANE**

DOMINANT CLOSED LOOP POLE REGION
From Fig. 4.5 denote the mapping of the dominant closed loop pole region into the \( s \)-plane into the \( U,V \) plane by

\[ Q \text{ for } p_c = p_c \in \{ z_{\text{min}}, z_{\text{max}} \}. \]

The mapping of the dominant closed loop pole region into the \( U,V \) plane for all values of \( p_c = p_c \in \{ z_{\text{min}}, z_{\text{max}} \} \) is shown in Fig. 4.6. The unshaded region shown in Fig. 4.6, denoted by the set \( Q \), can be written as

\[ Q = \{ \text{Figure 4.6} \}. \tag{4.22} \]

Now any point in \( Q \) will map into the acceptable dominant closed loop pole region in the \( s \)-plane for all values of \( p_c = p_c \in \{ z_{\text{min}}, z_{\text{max}} \} \). This is the same type of argument used in Section 7 of Chapter III. An arbitrary point in \( Q \) will be used later to solve for the required values of system gain \( K_h \) and \( S'' \), \( P_u \) which determine the compensation zero position.

In order to insure that the set \( Q \) exists, i.e., \( Q \neq \emptyset \) where \( \emptyset \) is the null set, restrictions must be placed on the magnitude of the zero variation on the real axis and hence \( p_c \).

Consider the mapping of a point in the dominant closed loop pole region characterized by \( S_{1,i} \) and \( P_{r_1} \) for \( p_c = p_c_{\text{zmin}} \). Equations 4.14 and 4.15 map this point into the \( U,V \) plane for a fixed value of \( p_c \) as follows:

\[ S_{1,i} = P_{r_1} + P_{c_{\text{zmin}}} \]

\[ \text{(4.23)} \]
FIG. 4.6 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION FOR ALL

\[ P_s \in [P_{\min}, P_{\max}] \]
\[ P_{f_1} P_{c_{z_{\min}}} P_{r_1} = V_1 \]  

(4.24)

Now consider another point defined by \( S_{r_2} \) and \( P_{r_2} \) mapped into the \( U,V \) plane for \( P_c = P_{c_{z_{\max}}} \) for the same fixed value of \( P_f \), i.e.

\[ S_{r_2} + P_{r_2} + P_{c_{z_{\max}}} = U_2 \]  

(4.25)

\[ P_{f_1} P_{c_{z_{\max}}} P_{r_2} = V_2 \]  

(4.26)

If the mapping defined by Eqs. 4.23, 4.24 and Eqs. 4.25, 4.26 are to have at least one point in common, the following conditions must be satisfied:

\[ U_1 = U_2; \quad V_1 = V_2 \]

or

\[ S_{r_1} - S_{r_2} = P_{c_{z_{\max}}} - P_{c_{z_{\min}}} \]  

(4.27)

\[ \frac{P_{r_1}}{P_{r_2}} = \frac{P_{c_{z_{\max}}}}{P_{c_{z_{\min}}}} \]  

(4.28)

Therefore in order to ensure that the set \( \mathcal{Q} \) exists, there must exist two points contained in the acceptable dominant closed loop pole region defined by \( S_{r_1}, P_{r_1} \), and \( S_{r_2}, P_{r_2} \) respectively such that Eqs. 4.27 and 4.28 are satisfied. These equations set a limit on the maximum allowable variation in \( P_c \), and hence \( z \) in the mapping operation. These restrictions are not overly stringent as will be illustrated in the design example in this chapter.
If the zero variation is large compared to the acceptable dominant closed loop pole region and Eqs. 4.27 and 4.28 are not satisfied, the design procedure presented in this chapter would have to be modified to obtain a design.

The effect of the parameter $p_{cz}$ for fixed $p_{f_1}$ on the mapping of the dominant closed loop pole region into the U,V plane is shown in Fig. 4.7. The acceptable dominant closed loop pole region used in this and subsequent mappings in this chapter is the same as that used in the design examples in Chapters II and III and is shown in Fig. 3.26 and in the root tests of Chapter III.

The effect of the parameter $p_{f_1}$ at fixed $p_{cz}$ on the mapping of the dominant closed loop pole region into the U,V plane is shown in Fig. 4.8.

4.7 Mapping of the Dominant Closed Loop Pole Region Into X,Y Plane

The mapping of the dominant closed loop pole region into the X,Y plane is accomplished using Eqs. 4.16 and 4.17 with the same values of $p_{f_1}$ and $p_{cz}$ that were used in the mapping in the previous section. This mapping operation is shown in Fig. 4.9. As in Chapter III, the mapping of the plant pole variation will be fitted inside this mapping to solve for the values of $k_k$, $S_0$, and $P_0$. 
Fig. 4.7 The effect of the parameter $\tau_{32}$ on the mapping of the dominant closed loop pole region into the U-V plane

$\tau_{31} = 40$

- $\tau_{32} = 3$
- $\tau_{32} = 2$
- $\tau_{32} = 1$
Fig. 4.8 The effect of the parameter $p_3$ on the mapping of the dominant closed loop pole region into the $u$-$v$ plane.

$\gamma_3 = 2$

$\gamma_3 = 60^\circ$

$\gamma_3 = 70^\circ$
FIG. 4.9 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION IN THE S-PLANE INTO THE X-Y PLANE

MAPPING EQUATIONS

\[ X = P_3, P_2 + S_n (P_3 + P_2) + P_n \]
\[ Y = P_n (P_3 + P_2) + S_n P_3, P_2 \]

PARAMETERS \( P_3, P_2 \)

S-PLANE

X-Y PLANE

DOMINANT CLOSED LOOP POLE REGION
The effect of the parameter $p_z$ for fixed $p_{f_1}$ on the mapping of the dominant closed loop pole region into the $X,Y$ plane is shown in Fig. 4.10. The effect of the parameter $p_{f_1}$ for fixed $p_z$ on this same mapping is shown in Fig. 4.11.

4.8 Mapping of the Plant Pole Variation Into the $\Delta X,\Delta Y$ Plane

The mapping of the plant pole variation into the $\Delta X,\Delta Y$ plane is accomplished using Eqs. 4.20 and 4.21. Since these equations are identical in form as those in Chapter III, they will not be investigated in detail here. This mapping operation only considers the variation in the plant poles. The variation in the zero on the real axis will be taken into account later in the design. The variation in plant gain may be handled in the same manner as outlined in Chapter III.

The parameter $P_z$, the open loop pole placed near the drifting zero, occurs in the same manner in Eqs. 4.20 and 4.21 as $P_1$ did in the mapping of the plant pole variation in Chapter III.

The mapping of the plant pole variation into the $\Delta X,\Delta Y$ plane using Eqs. 4.20 and 4.21 is shown in Fig. 4.12 for various values of $P_z$. The plant pole variation used in this mapping is identical to that used in Chapter III and is shown in Fig. 3.26. As in Chapter III, the units on this mapping in the $\Delta X,\Delta Y$
Fig. 4.10 The effect of the parameter $p_3$ on the mapping of the dominant closed loop pole region into the $X-Y$ plane.

$p_3 = 40$
Fig. 4.11 The effect of the parameter $\beta_3$ on the mapping of the dominant closed loop pole region into the $x$-$y$ plane.
FIG. 4.12 MAPPING OF THE PLANT POLE VARIATION INTO THE $\Delta X$-$\Delta Y$ PLANE WITH $P_3$ AS A PARAMETER
plane must be the same as those used in the mapping of the dominant closed loop pole region in the X,Y plane. The mappings of the plant pole variation for \( P_z = 2 \) and comparable units as those used in the mapping of the dominant closed loop pole region into the X,Y plane are also shown in Fig. 4.12.

4.9 Calculation of System Gain and Compensation Zero Location

Once the mappings of the dominant closed loop pole region are obtained, an attempt is made to fit the mapping of the plant pole variation in the \( \Delta X, \Delta Y \) plane into the interior of the mapping of the dominant closed loop pole region in the X,Y plane as shown in Fig. 4.13. If this is not possible, then the mapping of the dominant closed loop pole region must be performed at higher values of \( P_z \). The mapping of the plant pole variation in the \( \Delta X, \Delta Y \) plane is unchanged since \( P_z \) is assumed to be fixed at the beginning of the design. Once the mapping of the plant pole variation can be fitted inside the mapping of the dominant closed loop pole region in the X,Y plane, the value of system gain \( k_k \) and \( S_0, P_0 \), the parameters that determine the compensation zero position, can be determined.

The first step in solving for \( k_k, S_0 \) and \( P_0 \) is to choose an arbitrary point within the set \( Q \) of the mapping of the dominant closed loop pole region in the

MAPPINGS OF THE DOMINANT CLOSED LOOP POLE REGION

MAPPING OF THE PLANT POLE VARIATION WHICH INCLUDES THE EFFECT OF THE DRIFTING ZERO

AT POINT A:
X = X₀
Y = Y₀
S₆ = S₆₀
P₀ = P₀₀
U = U₀
Y = Y₀

APPROXIMATE MAPPING OF THE PLANT POLE VARIATION FROM EQUATIONS 4.20 & 4.21
U, V plane. Denote this point by "0" and the values of U and V at this point as $U_0$ and $V_0$.

At point A in Fig. 4.13, denote the values of $S_p$, $P_p$, $X$ and $Y$ as $S_{p_a}$, $P_{p_a}$, $X_a$ and $Y_a$ respectively. Since $P_z$ is also known, $kK$ can be found from Eq. 4.10 i.e.,

$$kK = U_0 - P_z - S_p$$  \hspace{1cm} (4.29)

This leaves Eqs. 4.11, 4.12 and 4.13 to solve for $P_0$ and $S_0$. The value of $z$ is also an unknown. This value of $z$ is needed to compute the added variation in the mapping of the plant pole variation in the X, Y plane (Eqs. 4.18, 4.19). The effect of the zero variation on the design is considered in the next section.

Using Eq. 4.11 to eliminate $z$ in Eq. 4.12 and 4.13 yields

$$X_a = P_z P_{p_a} + P_{p_a} + kK(S_0 + V_0/kK P_0)$$  \hspace{1cm} (4.30)

$$Y_a = P_z P_{p_a} + kK(P_0 + S_0 V_0/kK P_0)$$  \hspace{1cm} (4.31)

Solving for $P_0$ in these two equations yields the following third order equation for $P_0$

$$P_0^3 + \frac{(P_{p_a} - Y_a)P_0^2}{kK} + \frac{V_0 (X_a - P_z S_{p_a} - P_{p_a})P_0}{(kK)^2} - \frac{V_0^2}{(kK)^2} = 0$$  \hspace{1cm} (4.32)
Now Eq. 4.32 must have at least one real root which is the one of interest. Note that $P_0$ can not be negative or complex since $P_0$ is related to the magnitude of the compensation zero. Once $P_0$ has been obtained, $S_o$ can be found by solving Eq. 4.30 for $S_o$, i.e.,

$$S_o = \frac{X_a - P_Z S_p - P_v}{k_{kh} - k_{KP_o}}$$

(4.33)

4.10 Effect of the Zero Variation on the Design

The design is not complete once these values have been obtained since the zero variation $\Delta z$ in Eqs. 4.18 and 4.19 was neglected in the mapping of the plant pole variation into the $\Delta X, \Delta Y$ plane. This added variation in the mapping of the plant pole variation is taken into account in the $X, Y$ plane by considering that the coordinates of any point on the mapping of the plant pole variation in the $X, Y$ plane could change by as much as

$$\Delta X_z = k_{kh} \Delta z$$

(4.34)

$$\Delta Y_z = k_{kS} \Delta z$$

(4.35)

where $\Delta X_z$ and $\Delta Y_z$ is the change in any point on the mapping of the plant pole variation in the $X, Y$ plane due to the variation in the drifting zero only. The zero variation $\Delta z$ is computed as follows: From Eq. 4.11, the nominal value of $z$ denoted by $z_0$ is
\[ z_0 = \frac{V_0}{khP_0} \quad (4.36) \]

This is the value of \( z \) when the mapping of the plant pole variation in the \( X,Y \) plane is in its original position. (The position used in the previous section to compute \( k_k, S_0 \) and \( P_0 \)). The nominal value of \( z, z_0 \), will lie somewhere in the interval \([z_{\text{min}}, z_{\text{max}}]\) because of the values of \( p_c \) used in the mapping operations. The positive change in the coordinates of any point on the mapping of the plant pole variation in the \( X,Y \) plane is

\[ \Delta X^+ = kk(z_{\text{max}} - z_0) = +kkz \quad (4.37) \]
\[ \Delta Y^+ = khS_0(z_{\text{max}} - z_0) = +khS_0 z \quad (4.38) \]

whereas the negative change is

\[ \Delta X^- = kk(z_{\text{min}} - z_0) = -kkz \quad (4.39) \]
\[ \Delta Y^- = khS_0(z_{\text{min}} - z_0) = -khS_0 z \quad (4.40) \]

This variation is shown in Fig. 4.13. Note that \( \Delta X^+_z \) is not necessarily equal to \( \Delta X^-_z \) since \( z_0 \) may lie anywhere in the interval \([z_{\text{min}}, z_{\text{max}}]\). This added variation is taken into account by considering the changes in the points \( A, B, C \) and \( D \) on the boundary of the mapping of the plant pole variation in the \( X,Y \) plane.
plane as shown in Fig. 4.13. The maximum possible plant variation is the figure \( A'A'B'C'C'D' \) where the primes indicate the new positions of the points \( A, B, \) etc. If this added variation can not be fitted within the interior of the mapping of the dominant closed loop pole region in the \( X,Y \) plane, then, the dominant closed loop pole region must be mapped into the \( U,V \) and \( X,Y \) planes using a larger value of \( p_{f_1} \). The mapping of the plant pole variation into the \( \Delta X, \Delta Y \) plane remains unchanged as long as \( P_z \) is unchanged.

The design should be checked when the total variation can nearly be fitted within the mapping of the dominant closed loop pole region in the \( X,Y \) plane since this is an approximate design procedure. The assumption that \( p_{f_1} \) remains constant as the dominant closed loop poles vary is only approximately true. The mapping of the dominant closed loop poles at fixed \( p_{f_1} \) essentially determines the minimum value that \( p_{f_1} \) attains when the actual closed loop poles are obtained by factoring \( 1 + L_d(s) = 0 \). The mapping of the dominant closed loop pole region in Chapter III at constant \( \gamma = kKP \) was completely valid since \( kK \) and \( P \) are fixed if the gain variation is neglected.

4.11 Summary

Unlike the design procedure developed in Chapter III, the design procedure developed here is approximate.
The assumption that \( p_{f1} \) remains fixed as the dominant closed loop poles vary is not strictly valid. For systems with large plant parameter variations, this approximation should lead to an acceptable design since large gain implies a large value of \( p_{f1} \) and thus its effect on the dominant closed loop poles will be slight.

The major difficulty encountered in this problem was determining what effect the zero variation had on the mapping of the plant pole variation (Eqs. 4.18 and 4.19). From these equations, the values of \( kK \) and \( S_0 \) were needed to map the plant pole variation into the \( \Delta X, \Delta Y \) plane exactly but this mapping itself was needed to solve for \( kK \) and \( S_0 \). This difficulty was overcome by first considering the effect of the plant pole variation alone and then checking to determine if this design was adequate to handle the added zero variation. The effect of the zero variation on the mapping of the dominant closed loop pole region was taken into account by the parameter \( p_{cz} \).

4.12 Design Example

The design example presented here has the same s-domain specifications as the design example in Chapter III (Fig. 3.26) with these two exceptions: The added zero variation along the real axis, i.e. \( z_{min} = z_{max} \), where \( z_{min} = 1 \) and \( z_{max} = 3 \). The effect of the "far-off"
pole $p_1$ is neglected in order to retain a fourth order representation for the system.

The first step in the design is to choose a fixed value of $p_{f_1}$. In this design procedure, it is difficult to obtain an approximation for the value of $p_{f_1}$ that should be used. Obviously $p_{z_1}$ is going to be relatively far-removed from the acceptable dominant closed loop pole region for large parameter variations. Also the time domain specifications for the problem will probably dictate some minimum value of $p_{f_1}$ (See Fig. 4.1). A value of 40 for $p_{f_1}$ was chosen as the first estimate. Figures 4.14 and 4.15 illustrate the mapping of the dominant closed loop pole region in the $U,V$ and $X,Y$ plane respectively for $p_{f_1}=40$. In addition, the dipole separations for $z_{\max}, \delta_1$, and $z_{\min}, \delta_2$, have been estimated at 0.3 and 0.05 respectively (See Section 4.6). Only the mappings for the maximum and minimum values of $p_{cz}$ are shown in Fig. 4.14 since the set $Q$ is completely defined by these two mappings.

A value of $p_z=2$ will be used since this is midway between the extreme zero positions and is the value used to illustrate the effect of zero variation on the dominant closed loop poles (Figs. 4.2 and 4.3). The mapping of the plant pole variation for $p_z=2$ into the $\Delta X, \Delta Y$ plane is available in Fig. 4.12.
FIG. 4.14 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE U-V PLANE

$\gamma_3 = 40$

$\gamma_3 = 3.3$

POINT "O" IN SET $\Omega$

$\gamma_2 = 0.95$
FIG. 4.15  CALCULATION OF THE SYSTEM AND
COMPENSATION ZERO POSITION FOR $\gamma_2 = 40$
AND $p_j = 2$

MAPPING OF THE
PLANT POLE VARIATION
INCLUDING THE
EFFECT OF THE
DRIFTING ZERO

MAPPING OF THE
PLANT POLE
VARIATION FROM
FIG. 4.12

$\gamma_3 = 3.3$
$\gamma_3 = 0.95$
$\gamma_3 = 2$
Figure 4.15 shows the mapping of the plant pole variation for $P_z = 2$ within the mapping of the dominant closed loop pole region for $p_{f_1} = 40$ and $p_{cz} = 0.95, 2$ and $3.3$. At point A in Fig. 4.15

\[ X_a = 420 \quad Y_a = 2025 \]

From Fig. 3.26 at point A

\[ S_{p_a} = -6 \quad \frac{P_{p_a}}{P_a} = 10 \]

From Fig. 4.14, point "0" in the set $O$ is arbitrarily chosen to have the coordinates

\[ U_0 = 51 \quad V_0 = 2100 \]

Using $P_z = 2$, the value of $kh$ can be obtained from Eq. 4.29, i.e.

\[ kh = U_0 - P_z - \frac{S_{p_a}}{P_a} = 51 - 2 + 6 \]

\[ = 55 \]

Using Eq. 4.32, the third order equation for $P_0$ is

\[ p_0^3 + \frac{(P_z P_a - Y_a)}{kh} p_0^2 + \frac{V_0 (X_a - P_z S_{p_a} - P)}{P_a} p_0 + \frac{V_0^2}{(kh)^2} = 0 \]

\[ p_0^3 + \frac{(2(10) - 20(2))}{55} p_0^2 + \frac{2100 (420 - 2(51) - 10)}{(55)^2} p_0 - \frac{(2100)^2}{(55)^2} = 0 \]

\[ p_0^3 - 36.5 p_0^2 + 293 p_0 - 1460 = 0 \]
Factoring the one real positive root of interest using the convergence procedures in Appendix A yields

\[ P_0 = 27.85 \]

\[ S_0 \] is obtained from Eq. 4.33, i.e.

\[ S_0 = \frac{X_a - P_z S_{Pa} - P_Pa}{kk} - \frac{V_o}{kkP_o} = \frac{420 - 2(-6) - 10}{55} - \frac{2100}{55(27.85)} = 6.31 \]

The position of the compensation zeroes is

\[ z_z = -\frac{S_0}{2} = -3.105 \]

\[ w_z = \pm \sqrt{p_z^2 - \frac{2}{0 \alpha_z}} = \pm \sqrt{18.20} = \pm 4.266 \]

The design can now be checked to determine if it is adequate for the zero variation. The nominal value of \( z \), \( z_0 \), is found from Eq. 4.36, i.e.

\[ z_0 = \frac{V_o}{kkP_o} = \frac{2100}{55(27.85)} = 1.370 \]

The positive variation in the mapping of the plant pole variation in the \( X \), \( Y \) plane is found from Eqs. 4.37 and 4.38, i.e.

\[ \delta \chi^+ = k\alpha (z_{\text{max}} - z_0) = 55(3.00 - 1.37) = +89.7 \]
\[ \Delta Y^+ = k h S_o (z_{\text{max}} - z_o) = 55(6.31)(1.63) = +566 \]

The negative variation is found from Eqs. 4.39 and 4.40, i.e.

\[ \Delta X^- = k h (z_{\text{min}} - z_o) = 55(1.00 - 1.37) = -20.4 \]

\[ \Delta Y^- = k h S_o (z_{\text{min}} - z_o) = 55(6.31)(-0.37) = -128.5 \]

This added variation is taken into account by considering the change in the coordinates of points A, B, C and D in the X,Y plane as shown in Fig. 4.15. Since this added variation cannot be completely accommodated within the mapping of the dominant closed loop pole region, the design is probably not adequate. Using these values of \( k h, S_o \) and \( P_o \), the actual closed loop poles were found for the boundary of the plant pole variation shown in Fig. 3.26 by determining the roots of \( 1 + L_d(s) = 0 \). The results are shown in Fig. 4.16 which confirms that the design is inadequate.

As a check on the design procedure, the position of the compensation zeroes was arbitrarily moved to

\[ z, \bar{z} = -3.75 \pm j 4.25 \]
FIG. 4.16 DOMINANT ROOT TEST FOR $\theta_3 = 40$

AND $\theta_3 = 2$

$\alpha = 5.5 \quad Z = -3.105 + j4.264$
in an attempt to force the dominant closed loop poles into their acceptable region. The result using the same value of gain as before is shown in Fig. 4.17 which indicates that this compensation zero position does not yield a satisfactory design either. This implies that the mappings must be performed using a larger value of \( p_{f1} \).

The next value chosen for \( p_{f1} \) was 50. The mapping of the dominant closed loop pole region into the \( U,V \) plane and \( X,Y \) plane for this value of \( p_{f1} \) is shown in Figs. 4.18 and 4.19 respectively. The same value of \( P_z=2 \) was used so the mapping of the plant pole variation in the \( \Delta X,\Delta Y \) plane is unchanged. The mapping of the plant pole variation fitted within the interior of the mapping of the dominant closed loop pole region in the \( X,Y \) plane is also shown in Fig. 4.19. At point A in Fig. 4.19

\[
\lambda_a = 525 \quad Y_a = 2500 \\
S_{Pa} = -6 \quad P_a = 10
\]

The point "0" in the set \( \mathcal{Q} \) from Fig. 4.18 is arbitrarily chosen as

\[
U_0 = 61 \quad V_0 = 2650
\]

Using \( P_z=2 \), the value of \( k_0 \) from Eq. 4.29 is 65. Using Eq. 4.32, the third order equation for \( P_0 \) is
FIG. 4.17 DOMINANT ROOT TEST FOR $\rho_2 = 4.0$
AND $\rho_3 = 2$

$\text{AK} = 55 \quad \omega = -3.750 + j4.250$

DOMINANT CLOSED LOOP POLE REGION
FIG. 4.18 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE U-V PLANE

$\nu_2 = 50$

$\nu_3 = 3.3$

$\nu_3 = 0.95$

POINT "O" IN SET $Q$
FIG. 4.19 CALCULATION OF THE SYSTEM GAIN AND COMPENSATION ZERO POSITION FOR $\gamma_z = 50$ AND $\rho_3 = 2$.

MAPPING OF THE PLANT POLE VARIATION INCLUDING THE EFFECT OF THE DRIFTING ZERO.

MAPPING OF THE PLANT POLE VARIATION FROM FIG. 4.12.
\[ p^3_o - 38.18 p^2_o + 331 p_o - 1663 = 0 \]

The real root of interest is

\[ p_o = 28.53 \]

\[ s = 6.68 \]

The position of the compensation zeroes is

\[ \alpha_z = -3.34 \]

\[ \omega_z = \pm 4.17 \]

The nominal value of \( z, z_o \), from Eq. 4.36 is

\[ z_o = 1.43 \]

The added variation in the mapping of the plant pole variation is found using Eqs. 4.37 - 4.40, i.e.

\[ \Delta X^+_Z = +102 \quad \Delta X^-_Z = -28 \]

\[ \Delta Y^+_Z = +682 \quad \Delta Y^-_Z = -186.5 \]

This additional variation is also shown in Fig. 4.19. The additional variation can be easily accommodated within the mapping of the dominant closed loop pole region. The actual closed loop poles for this value of system gain and compensation zero position are shown
in Fig. 4.20. The design is more than adequate to handle the plant pole and zero variation. The difference between the gain of this design and the previous design is approximately 1.7 db.

For this particular design, a system gain of less than 65 would probably be adequate. This could be verified by using a value of $p_f = 45$ for the design procedure. This would result in a system gain of 60. Further reduction in system gain may be possible by varying the value of $P_Z$, the open loop pole used to partially cancel the effect of the drifting zero.

As can be anticipated from the root test shown in Fig. 4.20, the variation in plant gain will not result in the dominant closed loop poles leaving their acceptable region. Using the procedure presented in Chapter III to check the angle of departure of the dominant closed loop poles for the root test shown in Fig. 4.20 confirms this.
FIG. 4.20 DOMINANT ROOT TEST FOR $P_3 = 50$
AND $P_3 = 2$

$K_K = 65$ \hspace{1em} $Z = 3.34 + j4.17$

DOMINANT CLOSED LOOP POLE REGION
CHAPTER V

CONCLUSIONS

The design procedures presented in this paper are for fourth order systems with large variations in the plant parameters (gain factor, poles and zeroes). Extension of these design procedures to a fifth order system would be difficult since the mapping equations would be very cumbersome.

Two possibilities for additional work in this area are presented in this chapter which would be a significant improvement over the design procedures developed in this paper.

The first of these is a computer design routine. An initial guess would be made as to the compensation zero location at a fixed value of system gain. By examining the resulting dominant closed loop poles and possibly using a gradient technique, one could determine in what direction the compensation zeroes should be moved to place the dominant closed loop poles within their acceptable region. This is complicated by the fact that the necessary value of system gain is also unknown since for small values of system gain, a design is impossible for any compensation zero location.
Analytic expressions for the boundary of the plant pole variation and the acceptable dominant closed loop pole region would probably be required.

The second possibility in this problem, would be an attempt to obtain an analytic solution for the compensation zero location and necessary value of system gain to place the dominant closed loop poles within a given region in the s-plane for a given region of plant pole variation. Intuitively, one would think that there is a unique value of system gain and compensation zero location such that the closed loop poles lie within their acceptable region and the system gain is minimized. Unfortunately, a solution in closed form appears to be very difficult to obtain in even trivial cases. Analytic expressions for the boundary of the plant pole variation and the acceptable dominant closed loop pole region would certainly be required in this design procedure.
BIBLIOGRAPHY


Appendix A


Appendix B

APPENDIX A

POLYNOMIAL FACTORING

A.1 Statement of Problem

In many investigations of feedback control systems, polynomials of rather high degree must be factored. In the analysis of feedback control systems, the approximate location of the roots in the complex plane are often known. The problem usually is one of determining if the roots lie within an acceptable region despite variations in gain and/or plant parameters. The polynomial considered in this appendix is of fifth order. A procedure is first developed for extracting one real root from this polynomial. The resulting fourth order polynomial is then factored into the product of two quadratics using Lin's method. This method is applicable whether the fourth order polynomial has real or complex roots. A method is then presented for extracting two real roots from a fourth order polynomial. The nomenclature is chosen to aid in the programming of these procedures on a digital computer. The convergence of any of these methods is not guaranteed.
A.2 Extraction of One Real Root From A Fifth Order Polynomial

The fifth order polynomial is

\[ s^5 + W_1 s^4 + W_2 s^3 + W_3 s^2 + W_4 s + W_5 \]  \hspace{1cm} (A.1) \]

The coefficients \( W_1, W_2, \) etc., are considered to be real in all cases.

This procedure is based on the fact that an approximation to the real root is given by the quotient \( W_5/W_4. \) An improvement on this approximation can be made after one long division trial by considering the binomial term in the last subtraction process. With this in mind define

\[ V(J) = \frac{W_5}{W_4} \quad J=1 \text{ only} \hspace{1cm} (A.2) \]

where: \( V(J) = J^{th} \) approximation to the real root. The long division operation is shown below.

\[
\begin{align*}
\frac{s^4 + A_1(J) s^3 + A_2(J) s^2 + A_3(J) s + A_4(J)}{s + V(J)} & \div \frac{s^5 + W_1 s^4 + W_2 s^3 + W_3 s^2 + W_4 s + W_5}{s^5 + V(J)s^4 + \left( W_1 - V(J) \right) s^3 + W_2 s^2 + W_3 s + W_4 s + W_5} \\
\end{align*}
\]

\[
\begin{align*}
\frac{\left( W_1 - V(J) \right) s^4 + \left( W_1 V(J) - V^2(J) \right) s^3}{\left( W_2 - W_1 V(J) + V^2(J) \right) s^3 + W_3 s^2} & \div \frac{\left[ W_2 - W_1 V(J) + V^2(J) \right] s^3 + \left( W_2 V(J) - W_1 V^2(J) + V^3(J) \right) s^2}{\left( W_3 - W_2 V(J) + W_1 V^2(J) - V^3(J) \right) s^2 + W_4 s} \\
\end{align*}
\]
\[
\begin{align*}
\frac{\{W_3 - W_2 V(J) + W_1 V^2(J) - V^3(J)\} s^2 + \{W_3 V(J) - W_2 V^2(J) + W_1 V^3(J) - V^4(J)\} s}{\{W_4 - W_3 V(J) + W_2 V^2(J) - W_1 V^3(J) + V^4(J)\} s + W_5} \\
\frac{\{W_4 - W_3 V(J) + W_2 V^2(J) - W_1 V^3(J) + V^4(J)\} s}{\{W_4 V(J) - W_3 V^2(J) + W_2 V^3(J) - W_1 V^4(J) + V^5(J)\}}
\end{align*}
\]

The remainder term is the test for convergence. Let

\[X = W_5 - W_4 V(J) + W_3 V^2(J) - W_2 V^3(J) + W_1 V^4(J) - V^5(J) \quad (A.3)\]

If \(X\) is not sufficiently small, then the next trial divisor \(V(J+1)\) is given by

\[V(J+1) = \frac{W_5}{W_4 - W_3 V(J) + W_2 V^2(J) - W_1 V^3(J) + V^4(J)} \quad (A.4)\]

The coefficients of the fourth order polynomial are given by

\[A_1(J) = W_1 - V(J) \quad (A.5)\]

\[A_2(J) = W_2 - W_1 V(J) + V^2(J) \quad (A.6)\]

\[A_3(J) = W_3 - W_2 V(J) + W_1 V^2(J) - V^3(J) \quad (A.7)\]

\[A_4(J) = W_4 - W_3 V(J) + W_2 V^2(J) - W_1 V^3(J) + V^4(J) \quad (A.8)\]

The iteration procedure is continued until the value of \(X\) is sufficiently small or until a sufficient number of trials have been made.
A.3 Lin's Method

This method is essentially the same as the previous procedure except the trial divisor is a quadratic term. Let the fourth order polynomial be given by

\[ s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4 \]  

(A.9)

Define

\[ B_1(I) = \frac{A_3}{A_2}; \quad B_2(I) = \frac{A_4}{A_2} \quad I = 1 \text{ only} \]  

(A.10a, 10b)

where: \( B_1(I), B_2(I) = \text{coefficients of } I^{\text{th}} \text{ trial quadratic} \)

The trial divisor in the long division operation is given by

\[ s^2 + B_1(I)s + B_2(I) \]  

(A.11)

Performing the long division operation results in the following

\[ \frac{s^2 + C_1(I)s + C_2(I)}{s^2 + B_1(I)s + B_2(I)} \]

\[ \frac{s^4 + B_1(I)s^3 + B_2(I)s^2}{A_1 - B_1(I)} s^3 + \left\{ \frac{A_2 - B_2(I) + B_1^2(I)}{A_2 - B_2(I) - A_1 B_1(I) + B_1^2(I)} \right\} s^2 + \{ A_3 - A_1 B_2(I) + B_1(I) B_2(I) \} s + A_4 \]
The two remainder terms are the test for convergence.

Let

\[ X = A_3 - A_1B_2(I) + 2B_1(I)B_2(I) - A_2B_1(I) + A_1B_1^2(I) - B_1^3(I) \]  
(A.12)

\[ Y = A_4 - A_2B_2(I) + B_2^2(I) + A_1B_1(I)B_2(I) - B_2(I)B_1^2(I) \]  
(A.13)

If \( X \) and \( Y \) are not sufficiently small, then the coefficients of the next trial quadratic divisor are given by

\[ B_1(I+1) = \frac{A_3 - A_1B_2(I) + B_1(I)B_2(I)}{A_2 - B_2(I) - A_1B_1(I) + B_1^2(I)} \]  
(A.14)

\[ B_2(I+1) = \frac{A_4}{A_2 - B_2(I) - A_1B_1(I) + B_1^2(I)} \]  
(A.15)

The coefficients of the remaining quadratic are

\[ C_1(I) = A_1 - B_1(I) \]  
(A.16)

\[ C_2(I) = A_2 - B_2(I) - A_1B_1(I) + B_1^2(I) \]  
(A.17)
The iteration procedure is continued until the values of both X and Y are sufficiently small or a sufficient number of trials have been performed.

A.4 Extraction of Two Real Roots From a Fourth Order Polynomial

This method is applicable when it is known that the fourth order polynomial has at least two real roots.

The fourth order polynomial is given by

\[ s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4 \]

As with the fifth order polynomial, define

\[ V(J) = \frac{A_4}{A_3} \quad J = 1 \text{ only} \quad (A.18) \]

where: \( V(J) = J^{th} \) approximation to the real root.

The long division operation results in

\[
\begin{align*}
\frac{s^3 + D_1(J)s^2 + D_2(J)s + D_3(J)}{s + V(J)} &= \frac{s^4 + V(J)s^3}{[A_1 - V(J)]s^3 + A_2 s^2} \\
&= \frac{[A_1 - V(J)]s^3 + [A_1 V(J) - V^2(J)]s^2}{[A_2 - A_1 V(J) + V^2(J)]s^2 + A_3 s} \\
&= \frac{[A_2 - A_1 V(J) + V^2(J)]s^2 + [A_2 V(J) - A_1 V^2(J) + V^3(J)]s}{[A_3 - A_2 V(J) + A_1 V^2(J) - V^3(J)]s + A_4} \\
&= \frac{[A_3 - A_2 V(J) + A_1 V^2(J) - V^3(J)]s + [A_3 V(J) - A_2 V^2(J) + V^4(J)]}{[A_4 - A_3 V(J) + A_2 V^2(J) - A_1 V^3(J) + V^4(J)]}
\end{align*}
\]
The remainder term is again the test for convergence, i.e., let

\[ X = A^4 - A^3 V(J) + A^2 V^2(J) - A^1 V^3(J) + V^4(J) \]  \hspace{1cm} (A.19)

If \( X \) is not sufficiently small, the next trial divisor is

\[ V(J+1) = \frac{A^4}{A^3 - A^2 V(J) + A^1 V^2(J) - V^3(J)} \]  \hspace{1cm} (A.20)

The coefficients of the third order polynomial are

\[ D_1(J) = A^1 - V(J) \]  \hspace{1cm} (A.21)

\[ D_2(J) = A^2 - A^1 V(J) + V^2(J) \]  \hspace{1cm} (A.22)

\[ D_3(J) = A^3 - A^2 V(J) + A^1 V^2(J) - V^3(J) \]  \hspace{1cm} (A.23)

After the first real root is obtained with a sufficient degree of accuracy, the second real root is extracted from the remaining third degree equation. Let the third order polynomial be defined by

\[ s^3 + C_1 s^2 + C_2 s + C_3 \]  \hspace{1cm} (A.24)

Define

\[ U(I) = C_3 / C_2 \quad I = 1 \text{ only} \]  \hspace{1cm} (A.25)

where: \( U(I) = I^{th} \) approximation to real root.

The long division operation results in
The test for convergence is

\[ X = C_3 - 2C_2U(I) + C_1U^2(I) \] (A.26)

The next trial divisor, if \( X \) is not sufficiently small, is

\[ U(I+1) = \frac{C_3}{C_2 - C_1U(I) + U^2(I)} \] (A.27)

The coefficients of the quadratic equation are

\[ B_1(I) = C_1 - U(I) \] (A.28)

\[ B_2(I) = C_2 - C_1U(I) + U^2(I) \] (A.29)

In some cases convergence fails in the case of extracting one real root from a third order polynomial. In this case convergence can sometimes be obtained by extracting a quadratic from the third order polynomial.

Define the third order polynomial as
Also define

\[ D_1(I) = \frac{C_2}{C_1}; \quad D_2(I) = \frac{C_3}{C_1} \quad \text{I} = 1 \text{ only} \quad (A.31a, A.31b) \]

where: \( D_1(I), D_2(I) \) = coefficients of the \( I \text{th} \) trial quadratic.

The trial divisor in the long division operation is given by

\[ s^2 + D_1(I)s + D_2(I) \quad (A.32) \]

The long division operation results in the following:

\[
\frac{s + V(I)}{s^2 + D_1(I)s + D_2(I)} \]

\[
\frac{s^3 + D_1(I)s^2 + D_2(I)s}{[C_1 - D_1(I)]s^2 + [C_2 - D_2(I)]s + C_3}
\]

\[
\frac{[C_1 - D_1(I)]s^2 + [C_1D_1(I) - D_1^2(I)]s + [C_1D_1(I) - D_1(I)D_2(I)]}{[C_2 - D_2(I) - C_1D_1(I) + D_1^2(I)]s + [C_3 - C_1D_1(I) + D_1(I)D_2(I)]}
\]

The two remainder terms are a test for convergence. Let

\[ X = C_2 - D_2(I) - C_1D_1(I) + D_1^2(I) \quad (A.33) \]

\[ Y = C_3 - C_1D_1(I) + D_1(I)D_2(I) \quad (A.34) \]

If \( X \) and \( Y \) are not sufficiently small, then the coefficients of the next trial quadratic are given by
The real root is

\[ s = -\sqrt{d(I)} = -(C_1 - D_1(I)) \]  \hspace{1cm} (A.37)

A.5 Conclusion

The factoring procedures presented in this appendix may be used for fifth, fourth or third order polynomials possessing either real or complex roots or both. It should be stated again that the convergence of none of these procedures is guaranteed.
APPENDIX B

ANGLE CONTRIBUTION THEOREM

B.1 Statement of Problem

This appendix presents a geometric proof of the statement in Chapter II regarding the angle contribution of two zeroes to a complex pole located in the s-plane. The statement to be proved is as follows: The angle contribution due to two conjugate zeroes to a complex pole is a constant if the zeroes are located on a circular arc drawn through the complex pole and its conjugate and a third point X on the real axis defined by the equation

$$X_p = \frac{\theta}{2}$$

where: $p$ is the complex pole

$\theta$ is the angle contribution due to the zeroes

B.2 Geometric Proof

The geometric proof essentially shows that if the compensation zeroes are located on a circular arc through the points $pX_p$, the angle contribution $\theta$ is a constant independent of the position of the zeroes on the circular arc and that Eq. B.1 is satisfied. The proof is shown in Fig. B.1.
FIG. B-1 GEOMETRIC PROOF OF THE ANGLE OF CONTRIBUTION THEOREM
The proof begins by defining

\[ a = \angle \overrightarrow{P_d} \]

\[ \Phi_1 = \angle \overrightarrow{P_d} \]

\[ \Phi_2 = \angle \overrightarrow{P_d} \]

for which

\[ \Phi_1 + \Phi_2 = 2\alpha \]  \hspace{1cm} (B.5)

From Fig. B.1, it is noted that the inscribed angles \( \angle \overrightarrow{P_d} \), \( \angle \overrightarrow{P_d} \), and \( \angle \overrightarrow{P_d} \) all intercept the same circular arc \( \overrightarrow{P_d} \). Therefore from a basic theorem in plane geometry, the following can be stated

\[ \angle \overrightarrow{P_d} = \frac{1}{2} \overrightarrow{P_d} \]

\[ \angle \overrightarrow{P_d} = \frac{1}{2} \overrightarrow{P_d} \]

\[ \angle \overrightarrow{P_d} = \frac{1}{2} \overrightarrow{P_d} \]

where: \( \frac{1}{2} \overrightarrow{P_d} \) = one-half of the measure of the circular arc \( \overrightarrow{P_d} \).

Also from Fig. B.1,

\[ \angle \overrightarrow{P_d} = \Phi_1 + \Phi_2 \]  \hspace{1cm} (B.9)

\[ \angle \overrightarrow{P_d} = 2\alpha \]  \hspace{1cm} (B.10)

\[ \angle \overrightarrow{P_d} = \Phi_1 + \Phi_2 \]  \hspace{1cm} (B.11)

Therefore

\[ \Phi_1 + \Phi_2 = 2\alpha \]