A COMPUTATIONAL METHOD FOR TIME-OPTIMAL SPACE RENDEZVOUS

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SUMMARY

A method is presented for obtaining space-rendezvous trajectories between a one-stage rocket vehicle and a target in a circular Keplerian orbit. The trajectories satisfy the necessary conditions of the Pontryagin maximum principle for time-optimal rendezvous in which no terminal mass constraint is placed on the rocket. The use of Pontryagin's theory leads to a two-point boundary-value problem. A digital program is given for the iterative solution of this problem. The method is successfully applied for the determination of time-optimal rendezvous trajectories between a vehicle launched from the surface of the moon and a target in an 80-nautical-mile circular orbit.

INTRODUCTION

The study of trajectories for rendezvous in space has been of interest for many years. An early summary of some aspects of the problem is given in reference 1. The application of optimization theory to trajectory computation has also received attention (refs. 2 and 3). Paiewonsky and Woodrow (ref. 4) have considered the problem of a three-dimensional time-optimal space rendezvous between a single-stage rocket vehicle, with constrained terminal mass, and a target in a circular Keplerian orbit. Linearized dynamic equations were used to describe the motion of the vehicle, and the mass limitation was of the form of a terminal inequality constraint. Linearization of the equations imposes the condition that the rocket be in proximity to the target. Reference 5 has considered the three-dimensional time-optimal rendezvous problem with unsimplified dynamic equations and fixed terminal mass through the dual problem of three-dimensional fuel-optimal rendezvous with specified final time. The present paper continues the extension of the problem of reference 4 to unsimplified dynamics by considering three-dimensional time-optimal rendezvous with unspecified terminal mass.

The mathematical model of reference 5, which treats the rocket as a point mass and takes into account rotation of the attracting center, is employed. The Pontryagin maximum principle (ref. 6) is applied to find the correct thrust magnitude and direction for time optimality. This operation leads to a two-point boundary-value problem in which certain initial conditions on a set of differential equations introduced by the maximum
principle have to be found such that certain terminal conditions are met. Following a method developed in reference 5, a digital program is written to solve the boundary-value problem by iteration. The procedure is illustrated numerically by solving the problem of finding time-optimal trajectories of a rocket vehicle launched from the moon to rendezvous with a target in an 80-nautical-mile circular orbit.

In addition to extending the work of reference 4, the digital program and accompanying analysis provide a useful method for obtaining time-optimal rendezvous trajectories and control laws. Therefore, the digital program is discussed and a listing included.

SYMBOLS

\[ A, M, N, K, L, T_1, T_2 \quad \text{constant matrices} \]

\[ A_i = \beta \left[ \frac{1}{x_7 \psi} - \frac{\psi_1^2}{x_7 (\psi')^3} \right] \]

\[ B \quad \text{seven-dimensional diagonal matrix with elements } b_i \]

\[ B_{ik} = -\frac{\beta \psi_1 \psi_k}{x_7 (\psi')^3} \]

\[ b_i \quad \text{positive weighting elements} \ (i = 1, 2, \ldots, 7) \]

\[ C_{ik} = \frac{3 \Omega^2 R_S^3 \psi_1 v_k}{(\sqrt{x})^5} \]

\[ c \quad \text{effective exhaust velocity} \]

\[ D_{ik} = -\frac{3 \Omega^2 R_S^3 (2 \psi_1 v_k + d)}{(\sqrt{x})^5} \]

\[ d = \psi_2 (x_1 + R_S x) + \psi_4 (x_3 + R_S y) + \psi_6 (x_5 + R_S z) \]

\[ E_{ik} = \frac{15 \Omega^2 R_S^3 d v_1 v_k}{(\sqrt{x})^7} \]

\[ E[\bar{\xi}(\alpha)] \quad \text{scalar measure of terminal error, } \sum_{i=1}^{7} b_i \left[ \frac{e_i(\alpha)}{2} \right]^2 \]
seven-dimensional vector with elements $e_1(\vec{a})$

error criteria ($i = 1, 2, \ldots 7$)

$$F_{ik}^{*m} = -\frac{3\Omega^2 R_S^3 (\psi e_1 + \psi m v_k)}{(\vec{r}_x)^5}$$

pseudo-Hamiltonian function of the maximum principle

identity matrix

integers ($i = 0, 1, \ldots n; \ j = 1, 2, \ldots 6$)

unit vectors

constant matrix defined in appendix A

maximum value of $H$ with respect to $\vec{u}$

mass of launch vehicle

initial mass of launch vehicle

magnitude of $\vec{R}_S$

vector from center of attracting body to target

elements of $\vec{R}_S$

first derivative of $\vec{R}_S$

first derivatives of $R_{Sx}, R_{Sy}, R_{Sz}$

magnitude of $\vec{R}_V$

vector from center of attracting body to vehicle

elements of $\vec{R}_V$
\[ \dot{\bar{R}_v} \] first derivative of \( \bar{R}_v \)

\[ \dot{R}_{vX}, \dot{R}_{vY}, \dot{R}_{vZ} \] first derivatives of \( R_{vX}, R_{vY}, \) and \( R_{vZ} \)

\[ \bar{r} = \bar{R}_v - \bar{R}_s \]

\( r_X, r_Y, r_Z \) elements of \( \bar{r} \)

\[ \dot{r}_X, \dot{r}_Y, \dot{r}_Z \] first derivatives of \( r_X, r_Y, \) and \( r_Z \)

\( s \) dummy integration variable

\[ sgn \rho \] signum function defined by

\[
\begin{cases}
1 & \text{if } \rho > 0 \\
-1 & \text{if } \rho < 0 \\
\text{Unspecified} & \text{if } \rho = 0
\end{cases}
\]

\( T \) magnitude of thrust vector

\( \bar{T} \) thrust vector

\( t \) time

\( t_0 \) initial time

\( t_f \) final time

\( [t_2, t_3] \) nonzero subinterval of \( [t_0, t_f] \)

\( t' \) isolated point at which \( M'\psi(t) = 0 \)

\( u_i \) elements of \( \hat{u} \) \( (i = 1, 2, 3) \)

\( u_4 \) magnitude of \( \bar{T} \)

\[ \ddot{u} = u_4 \hat{u} \]

\( \hat{u} \) unit vector in direction of \( \ddot{u} \)

\( u_4^* \) optimal form of \( u_4 \)

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\[ \hat{u}^* = u_4^* \hat{u}^* \]
\[ \hat{u}^* \text{ optimal form of } \hat{u} \]

\[ v_i, v_{k} \quad \text{variable } v_1, v_3, \text{ or } v_5 \]

\[ v_1 = x_1 + R_{sx} \]
\[ v_3 = x_3 + R_{sy} \]
\[ v_5 = x_5 + R_{sz} \]

\[ x, y, z \quad \text{rotating axis system defined by figure A-1} \]
\[ x', y', z' \quad \text{inertial axis system defined by figure A-2} \]

\[ \| \hat{x} \| = \sqrt{(x_1 + R_{sx})^2 + (x_3 + R_{sy})^2 + (x_5 + R_{sz})^2} \]

\[ x_i \quad \text{state variables } (i = 0, 1, \ldots, 8) \]

\[ \dot{x}_i \quad \text{first derivative of } x_i \]

\[ x_{i_0} \quad \text{initial values of } x_i \]

\[ \bar{x} = \text{col}(x_1, \ldots, x_6) \]

\[ \ddot{x} \quad \text{first derivative of } \bar{x} \]

\[ \ddot{x}_0 \quad \text{initial value of } \bar{x} \]

\[ \bar{Y}(\bar{x}, x_0) \quad \text{vector defined in equation (A9)} \]

\[ \alpha_i, \alpha_j \quad \text{unknown parameters} \]

\[ \overline{\alpha} \quad \text{seven-dimensional vector with elements } \alpha_i \]

\[ \beta \quad \text{bound on thrust magnitude} \]
\( \delta \alpha \) correction to \( \bar{\alpha} \)

\( \delta \alpha_7 \) variation in \( \alpha_7 \)

\( \theta_c, \varphi_c \) control angles (see fig. A-3)

\( \theta_v, \varphi_v \) initial vehicle angles (see fig. A-4)

\( \theta_o, \lambda, \phi_o \) angles determining target orbital plane

\( \lambda \) parameter governing step size of \( \delta \alpha \)

\( \mu \) universal gravitational constant multiplied by mass of attracting body

\( \rho \) switching function

\( \psi_e, \psi_1, \psi_j, \psi_k, \psi_m \) variables introduced by the maximum principle, with the subscripts equal to 0, 1, \ldots 8

\( \dot{\psi}_1 \) first derivative of \( \psi_1 \)

\( \overline{\psi} = \text{col}(\psi_1, \ldots, \psi_6) \)

\( \dot{\overline{\psi}} \) first derivative of \( \overline{\psi} \)

\( \sqrt{\overline{\psi}} = \sqrt{\psi_2^2 + \psi_4^2 + \psi_6^2} \)

\( \Omega \) constant angular velocity of the target in its orbital plane

\( \omega \) angular velocity of body about axis of rotation

Mathematical notation (with arbitrary symbols used as examples):

\( \dot{a}(t) \) first derivative of \( a(t) \) with respect to \( t \)

\( a(t) \) second derivative of \( a(t) \) with respect to \( t \)

\( \ddot{a} \cdot \ddot{b} = \sum_{i=1}^{n} a_i b_i \) where \( \ddot{a} = \text{col}(a_1, \ldots, a_n) \) and \( \ddot{b} = \text{col}(b_1, \ldots, b_n) \)
\[ \|\vec{a}\| = (\vec{a} \cdot \vec{a})^{1/2} \]

\[ [a,b] \quad \text{closed interval} \]

\[ [a,b) \quad \text{interval closed at } a \text{ and open at } b \]

\[ \frac{\partial b(\vec{a})}{\partial \vec{a}} \quad \text{Jacobian matrix with elements } c_{ij} = \frac{\partial b_i(\vec{a})}{\partial a_j} \]

\[ \frac{\partial b_i(\vec{a})}{\partial a_j} \quad \text{first partial derivative of } b_i(\vec{a}) \text{ with respect to } a_j \text{ evaluated at } \vec{a} = \text{col}(a_1, \ldots, a_n) \]

\[ \varepsilon \quad \text{belongs to a set} \]

\[ ' \quad \text{denotes matrix transpose} \]

\[ \vec{0} \quad \text{null vector} \]

\[ -1 \quad \text{as superscript to matrix, denotes inverse} \]

**DEVELOPMENT OF THE BOUNDARY-VALUE PROBLEM**

In appendix A the dynamic equations are developed for a space vehicle which is required to rendezvous with a target in a circular Keplerian orbit about a rotating body. The vehicle is a one-stage rocket, treated as a point mass, with bounded thrust magnitude. The controls are the magnitude and direction of the thrust vector.

From equation (A8), the dynamic equations when written as first-order differential equations in relative coordinates take the form

\[
\begin{align*}
\dot{x}_0 &= 1 \\
\dot{x} &= \frac{u_4 M_\mu}{x_7} + \overline{Y}(\bar{x}, \bar{x}_8) \quad \begin{cases} \bar{x}(t_0) = \bar{x}_0; & \bar{x}(t_f) = \vec{0} \\
\bar{x}_7(t_0) = m_0 & (x_7(t_0) = m_0) \\
\dot{x}_8 &= 1 \quad (x_8(t_0) = t_0) \end{cases}
\end{align*}
\]

(1)
The vector $\tilde{x}$ is a six-dimensional vector whose elements $x_1, x_2, \ldots, x_6$ are the relative position $(x_1, x_3, x_5)$ and velocity $(x_2, x_4, x_6)$ components of the vehicle and target. The variables $x_7$ and $x_8$ are the instantaneous vehicle mass and the time, respectively. The vector $\overline{Y}(\tilde{x}, x_8)$, given by equation (A9), includes the kinematic parts of the equations. The scalar $u_4$ and unit vector $\hat{u}$ are, respectively, the magnitude and direction of the thrust vector $\overline{T}$. The matrix $M$ is a constant matrix defined in appendix A.

Pontryagin's maximum principle (ref. 6) is now applied to find controls which minimize $\int_{t_0}^{t_f} dt$ while satisfying equation (1). In using the maximum principle, a new variable $x_0^*$ is introduced such that $\dot{x}_0 = 1(x_0(t_0) = 0)$ and the problem becomes one of minimizing $x_0(t_f)$. From reference 6 the following conditions must be satisfied:

(1) A function $u_4 \leq \beta$ and a function $\hat{u}$ (with $||\hat{u}|| = 1$) must be chosen to maximize

$$H(\psi_0, \ldots, \psi_8; x_1, \ldots, x_8; u_1, \ldots, u_4) = \psi_0 \dot{x}_0(t) + \overline{\psi}(t) \cdot \dot{x}(t) + \psi_7(t)x_7(t) + \psi_8(t)x_8(t)$$

for fixed $\psi_i$ and $x_i$ ($i = 0, 1, \ldots, 8$). The vector $\overline{\psi}$ is equal to $\text{col}(\psi_1, \ldots, \psi_6)$. The terms $\psi_i$ ($i = 0, 1, \ldots, 8$) are nine additional variables introduced by the Pontryagin maximum principle and defined by

$$\dot{\psi}_i = -\frac{\partial H}{\partial x_i} \quad (i = 0, 1, \ldots, 8)$$

(2) For any $t \in [t_0, t_f]$, $\psi_0(t) =$ Constant $\leq 0$ and the maximum value of

$$H(\psi_0, \ldots, \psi_8; x_1, \ldots, x_8; u_1, \ldots, u_4)$$

with respect to $u_1, \ldots, u_4$, given by

$$M(\psi_0, \ldots, \psi_8; x_1, \ldots, x_8),$$

must be identically zero over $[t_0, t_f]$.

(3) The transversality condition must be satisfied.

Since $\tilde{x}(t_f)$ is specified and $x_7(t_f)$ and $x_8(t_f)$ are unspecified, the transversality condition discussed in reference 6 yields

$$\psi_7(t_f) = \psi_8(t_f) = 0$$

From a consideration of condition (1),

$$H = \psi_0 + \overline{\psi} \cdot \left[ \frac{u_4 M \hat{u}}{x_7} + \overline{Y}(\tilde{x}, x_8) \right] - \frac{\psi_7 u_4}{c} + \psi_8$$

whereby, from equation (3),
\[
\begin{align*}
\dot{\psi}_0 &= 0 & (\psi_0 &= 0) \\
\dot{\psi} &= -\frac{\partial Y(x, x_8)}{\partial x} \bar{\psi} & (\bar{\psi}(t_o) &= \text{undetermined}) \\
\dot{\psi}_7 &= -\frac{u_4 \bar{\psi} \cdot M \bar{u}}{x_7^2} & (\psi_7(t_f) &= 0) \\
\dot{\psi}_8 &= -\frac{\partial Y(x, x_8)}{\partial x} \cdot \bar{\psi} & (\psi_8(t_f) &= 0)
\end{align*}
\]

If \( M'\bar{\psi} \neq 0 \), the \( \hat{u} \) which maximizes \( H \) and satisfies \( \|\hat{u}\| = 1 \) is

\[
\hat{u}^* = \frac{M'\psi}{\|M'\psi\|}
\]

since \( \bar{\psi} \cdot M \hat{u} \) can be written as \( M'\bar{\psi} \cdot \hat{u} \). Then \( H \) becomes

\[
H = \psi_0 + u_4 \left( \frac{\|M'\bar{\psi}\|}{x_7} - \frac{\psi_7}{c} \right) + \bar{\psi} \cdot \nabla(x, x_8) + \psi_8
\]

and the function \( u_4 \leq \beta \) which maximizes \( H \), if \( \frac{\|M'\bar{\psi}\|}{x_7} - \frac{\psi_7}{c} \) does not vanish identically over a nonzero interval in \([t_0, t_f]\), is

\[
u_4^* = \frac{\beta}{2}(1 + \text{sgn} \rho) = \begin{cases} 
\beta & \text{if } \rho(t) > 0 \\
0 & \text{if } \rho(t) < 0
\end{cases}
\]

with

\[
\rho = \frac{\|M'\bar{\psi}\|}{x_7} - \frac{\psi_7}{c}
\]

The complete control now takes the form

\[
\tilde{u}^* = u_4 \hat{u}^* = \frac{\beta}{2}(1 + \text{sgn} \rho) \frac{M'\bar{\psi}}{\|M'\bar{\psi}\|}
\]

The function \( \rho(t) \) is the switching function for the system. The function \( \rho(t) \) has no zeros on \([t_0, t_f]\) except possibly at \( t_f \), which simply means the thrust is always at its maximum value. Since \( \psi_7(t) \) is such that

\[
\dot{\psi}_7 = \frac{\beta}{2}(1 + \text{sgn} \rho) \frac{\|M'\bar{\psi}\|}{x_7^2} \leq 0
\]
and \( \psi_7(t_f) = 0 \), it follows that \( \psi_7(t) \leq 0 \) for all \( t \in [t_0, t_f] \). If \( \psi_7(t') = 0 \) at some \( t' \in [t_0, t_f] \), the condition \( \rho(t) < 0 \) must be satisfied for \( t > t' \) in order to meet the condition \( \psi_7(t_f) = 0 \). The implication is that from \( t' \) until \( t_f \), the vehicle coasts; that is, \( T = 0 \) from \( t' \) to \( t_f \). However, to rendezvous at \( t_f \) requires that the vehicle and target have the same position and velocity. Therefore, \( \psi_7(t) \) can vanish only at \( t_f \), since it is not possible for the vehicle to coast into the same position and velocity as the target; that is, \( \psi_7(t) < 0 \) and \( \rho(t) > 0 \) for \( t \in [t_0, t_f] \). Equation (7) then takes the form

\[
\tilde{u}^* = \beta \frac{M'\tilde{\psi}}{\|M'\tilde{\psi}\|}
\]

and \( M \) becomes

\[
M = \psi_0 + \beta \rho + \tilde{\psi} \cdot \tilde{\nabla}(\tilde{x}, x_8) + \psi_8
\]

In an optimal system, \( M'\tilde{\psi} = (\psi_2, \psi_4, \psi_6)' = 0 \) over a nonzero interval (for example, over \([t_2, t_3]\) of \([t_0, t_f]\)) cannot be allowed. For, if it is, the differential equations for \( \psi_2, \psi_4, \text{ and } \psi_6 \) imply \( (\psi_1, \psi_3, \psi_5) = 0 \) over \([t_2, t_3]\). In fact, \( \tilde{\psi}(t) = 0 \) would be the solution of the \( \tilde{\psi} \) system over \([t_2, t_f]\). Since \( \psi_7(t_f) = 0 \) and \( \psi_8(t_f) = 0 \) over \([t_2, t_f]\). Also \( M = 0 \) implies that \( \psi_0 = 0 \) over \([t_2, t_f]\), giving 

\[
(\psi_0, \tilde{\psi}(t), \psi_7(t), \psi_8(t)) = 0 \quad \text{over } [t_2, t_f],
\]

which is a contradiction to the maximum principle. The maximum principle states that for each \( t \in [t_0, t_f] \), the vector \( (\psi_0(t), \psi_1(t), \ldots, \psi_8(t)) \) is nonzero. Isolated points at which \( M'\tilde{\psi} = 0 \) have no effect on the solution, since \( \tilde{u}(t) \) is bounded. For definiteness,

\[
\tilde{u}(t') = \lim_{t \to t'} \frac{M'\tilde{\psi}(t)}{\|M'\tilde{\psi}(t)\|}
\]

at any isolated points \( t' \) where \( M'\tilde{\psi}(t') = 0 \).

Equations (1) and (4) now take the form

\[
\begin{align*}
\dot{x}_0 &= 1 \\
\dot{x} &= \frac{\beta M'M' \tilde{\psi}}{\|M'M'\| x_7} - \frac{\Omega^2 R_s^3}{\|M'M'\| x_7} \left[ N \tilde{x} + M R_s(x_8) \right] + \Omega^2 M R_s(x_8) \\
&+ \left( N' + 2\omega K + \omega^2 L \right) \tilde{x} \\
\dot{x}_7 &= -\frac{\beta}{c} \\
\dot{x}_8 &= 1
\end{align*}
\]

(9a)
and
\[ \dot{\psi} = 0 \quad (\psi_0 \equiv 0) \]
\[ \dot{\psi} = \frac{\Omega^2 R_s^3 N' \overline{\psi}}{\| \dot{A\bar{x}} + \bar{R}_s(x_8) \|^3} - \frac{3\Omega^2 R_s^3}{\| \dot{A\bar{x}} + \bar{R}_s(x_8) \|^5} \left( [\dot{N\bar{x}} + M\bar{R}_s(x_8)] \cdot \overline{\psi} \right) A' [\dot{A\bar{x}} + \bar{R}_s(x_8)] \]
\[ \quad - (N + 2\omega K' + \omega^2 L') \overline{\psi} \quad (\overline{\psi}(t_0) \text{ undetermined}) \]
\[ \dot{\psi}_7 = \frac{\beta M' \overline{\psi}}{x_7^2} \quad (\psi_7(t_f) = 0) \]
\[ \dot{\psi}_8 = \frac{\Omega^2 R_s^3 \overline{\psi} \cdot M\bar{R}_s(x_8)}{\| \dot{A\bar{x}} + \bar{R}_s(x_8) \|^3} - \Omega^2 \overline{\psi} \cdot M\bar{R}_s(x_8) \]
\[ \quad - \frac{3\Omega^2 R_s^3}{\| \dot{A\bar{x}} + \bar{R}_s(x_8) \|^5} \left( R_s(x_8) \cdot [\dot{A\bar{x}} + \bar{R}_s(x_8)] \right) \left( \overline{\psi} \cdot [\dot{N\bar{x}} + M\bar{R}_s(x_8)] \right) \quad (\psi_8(t_f) = 0) \]

By comparing equations (9a) and (9b), a two-point boundary-value problem is recognized; \( \psi_0, \overline{\psi}(t_0), \psi_7(t_0), \psi_8(t_0), \) and \( t_f \) need to be found such that \( \bar{x}, \bar{M}, \psi_7, \) and \( \psi_8 \) are zero at \( t_f \). This boundary-value problem can be placed in the simpler form
\[ M(\psi_0, \ldots, \psi_8; x_1, \ldots, x_8) = 0 \]
which yields
\[ \psi_8(t_0) = -\left[ \psi_0 + \beta \rho(t_0) + \overline{\psi}(t_0) \cdot \overline{x}(t_0, t_0) \right] \quad (10) \]

Since \( M = 0 \) over \( [t_0, t_f] \), since \( \psi_8(t_f) = 0 \), since \( \bar{x}(t_f) = \bar{0} \), and since \( \overline{y}(\bar{0}, t_f) = \bar{0} \), it follows that
\[ M = \psi_0 + \beta \rho(t_f) = 0 \]
or
\[ \psi_0 = -\beta \rho(t_f) \quad (11) \]

The boundary-value problem then reduces to satisfying equations (9a), (9b), (10), and (11) and finding parameters \( \psi_1(t_0), \ldots, \psi_7(t_0), \) and \( t_f \) such that \( \bar{x}(t_f) \) and \( \psi_7(t_f) \) are zero. There appear to be eight parameters and seven boundary conditions. However, because of the homogeneous form of their differential equations, one of the parameters \( \psi_1(t_0) \) (i = 1, 2, ..., 7) can be removed by normalization – that is, by fixing its value.
Also, by observing the differential equation for \( \psi_7 \), it can be noted that the parameter \( \psi_7(t_0) \) could be determined after the other parameters are found by setting

\[
\psi_7(t_0) = -\beta \int_{t_0}^{t_f} \frac{\|M^T \psi(s)\|}{x_7^2(s)} \, ds
\]

The minimal combination is then six parameters (\( \psi(t_0) \) and \( t_f \), with one of the elements of \( \psi(t_0) \) normalized) and six boundary conditions (\( x(t_f) = 0 \)). However, since the correct algebraic sign of an element of \( \psi(t_0) \) may not be known a priori, the problem is best solved by finding seven parameters (\( \psi(t_0) \) and \( t_f \), with \( \psi_7(t_0) \) normalized) such that the seven boundary conditions (\( x(t_f) = 0 \) and \( \psi_7(t_f) = 0 \)) are met.

**SOLUTION OF THE BOUNDARY-VALUE PROBLEM**

**Development of Iterative Logic**

The approach taken to obtain a solution of the foregoing boundary-value problem is that discussed in reference 5. In reference 5, given a similar boundary-value problem, a vector \( \tilde{e}(\alpha) \) is defined such that when \( \tilde{e}(\alpha) = 0 \) the boundary conditions are satisfied and the unknown parameters for the problem are \( \alpha \).

The vectors \( \alpha \) and \( \tilde{e}(\alpha) \) correspond to \( \text{col}(\alpha_i) \) and \( \text{col}(e_i(\alpha)) \) \((i = 1, 2, \ldots, 7)\), respectively, with elements

\[
\begin{align*}
\alpha_1 &= \psi_1(t_0) \\
\alpha_2 &= \psi_2(t_0) \\
& \quad \vdots \\
\alpha_6 &= \psi_6(t_0) \\
\alpha_7 &= t_f
\end{align*}
\]

\[
\begin{align*}
e_1(\alpha) &= x_1(\alpha) \\
e_2(\alpha) &= x_2(\alpha) \\
& \quad \vdots \\
e_6(\alpha) &= x_6(\alpha) \\
e_7(\alpha) &= \psi_7(\alpha)
\end{align*}
\]  \( (12) \)

The quantities \( x_i \) \((i = 1, 2, \ldots, 6)\) and \( \psi_7 \) are written \( x_i(\alpha) \) and \( \psi_7(\alpha) \) to indicate their implicit dependence on \( \alpha \).

The magnitude of \( \tilde{e}(\alpha) \) is measured by a scalar quantity

\[
\mathbb{E}[\tilde{e}(\alpha)] = \frac{\tilde{e}(\alpha) \cdot B \tilde{e}(\alpha)}{2}
\]

where \( B \) is a positive definite diagonal matrix of weighting elements. Here

\[
\mathbb{E}[\tilde{e}(\alpha)] = \frac{1}{2} \left[ b_1 x_1^2(\alpha) + b_2 x_2^2(\alpha) + \ldots + b_6 x_6^2(\alpha) + b_7 \psi_7^2(\alpha) \right]
\]  \( (13) \)

where \( b_i > 0 \) \((i = 1, 2, \ldots, 7)\).
A value of $\bar{\alpha}$ is assumed, the differential equations are integrated forward in time until $t = t_f = \alpha_7$, and $E[\delta(\bar{\alpha})]$ is evaluated. If $\delta(\bar{\alpha})$ vanishes or, for practical purposes, is sufficiently small, the boundary-value problem is considered solved. Otherwise, the assumed $\bar{\alpha}$ is corrected by

$$\delta\bar{\alpha} = -\left[\frac{\partial \delta(\bar{\alpha})}{\partial \bar{\alpha}} B \frac{\partial \delta(\bar{\alpha})}{\partial \bar{\alpha}} + \lambda I\right]^{-1} \frac{\partial \delta(\bar{\alpha})}{\partial \bar{\alpha}} B \delta(\bar{\alpha})$$

(14)

Where $\lambda > 0$ is adjusted such that

$$E[\delta(\bar{\alpha} + \delta\bar{\alpha})] < E[\delta(\bar{\alpha})]$$

(15)

For this problem $\frac{\partial \delta(\bar{\alpha})}{\partial \bar{\alpha}}$ is given as the partitioned matrix

$$\frac{\partial \delta(\bar{\alpha})}{\partial \bar{\alpha}} = \begin{bmatrix} \frac{\partial \delta X(\bar{\alpha})}{\partial \bar{\alpha}} & \tilde{\delta X(\bar{\alpha})} \\ \frac{\partial \delta \psi_1(\bar{\alpha})}{\partial \bar{\alpha}} & \tilde{\delta \psi_1(\bar{\alpha})} \\ \frac{\partial \delta \psi_7(\bar{\alpha})}{\partial \bar{\alpha}} & \tilde{\delta \psi_7(\bar{\alpha})} \end{bmatrix}$$

(16)

where $\frac{\partial \delta X(\bar{\alpha})}{\partial \bar{\alpha}}$ is a Jacobian matrix with elements $\frac{\partial \delta x_i(\bar{\alpha})}{\partial \alpha_j}$ ($i = 1, 2, \ldots, 6$; $j = 1, 2, \ldots, 6$) and $\frac{\partial \delta \psi_7(\bar{\alpha})}{\partial \bar{\alpha}}$ is a row vector with elements $\frac{\partial \delta \psi_7(\bar{\alpha})}{\partial \alpha_j}$ ($j = 1, 2, \ldots, 6$).

The first six elements of $\delta\bar{\alpha}$ are the corrections on $\psi_1(t_0)$ ($i = 1, 2, \ldots, 6$). The seventh element $\delta \alpha_7$ is the correction on the last value of $t_f$. The next final time is given by $\alpha_7 + \delta \alpha_7$.

The procedure is designed to be applied iteratively and generate a monotone decreasing sequence of $E[\delta(\bar{\alpha})]$ converging to the smallest $E[\delta(\bar{\alpha})]$ available relative to the initial choice of $\bar{\alpha}$. Success of the method is dependent on the user's ability to find starting values of $\bar{\alpha}$ and at each stage find values of $\lambda$ such that equation (15) is satisfied. At each stage a one-dimensional search may be performed to find the values of $\lambda$. However, for this problem it was found that a value of $\lambda$ of 10 or 1 would suffice throughout. In general, the method does not guarantee a solution for an arbitrary boundary-value problem. It has, however, been highly successful in yielding solutions to the boundary-value problem under consideration herein and to others (see ref. 5).
Expanded versions of equations (9a) and (9b), with \( x_8 \) replaced by \( t \), are

\[
\begin{align*}
\dot{x}_1 &= x_{i+1} && (i = 1, 3, 5; \quad x_1(t_0) = x_{i0}; \quad x_1(t_f) = 0) \\
\dot{x}_2 &= \frac{\beta \psi_2}{x_7 \psi} - \Omega^2 R_s^3 \frac{x_1 + R_s x}{(\sqrt{x})^3} + \Omega^2 R_s x + \omega^2 x_1 + 2\omega x_4 && (x_2(t_0) = x_{20}; \quad x_2(t_f) = 0) \\
\dot{x}_4 &= \frac{\beta \psi_4}{\sqrt{x}} - \Omega^2 R_s^3 \frac{x_3 + R_s y}{(\sqrt{x})^3} + \Omega^2 R_s y - 2\omega x_2 + \omega^2 x_3 && (x_4(t_0) = x_{40}; \quad x_4(t_f) = 0) \\
\dot{x}_6 &= \frac{\beta \psi_6}{x_7 \psi} - \Omega^2 R_s^3 \frac{x_5 + R_s z}{(\sqrt{x})^3} + \Omega^2 R_s z \\
\dot{x}_7 &= -\frac{\beta}{c} \\
&= (x_7(t_0) = m_0)
\end{align*}
\]  

and

\[
\begin{align*}
\dot{\psi}_1 &= \frac{\Omega^2 R_s^3 \psi_2}{(\sqrt{x})^3} - 3\Omega^2 R_s^3 d \frac{x_1 + R_s x}{(\sqrt{x})^5} - \omega^2 \psi_2 && (\psi_1(t_0) = \alpha_1) \\
\dot{\psi}_2 &= -\psi_1 + 2\omega \psi_4 \\
\dot{\psi}_3 &= \frac{\Omega^2 R_s^3 \psi_4}{(\sqrt{x})^3} - 3\Omega^2 R_s^3 d \frac{x_3 + R_s y}{(\sqrt{x})^5} - \omega^2 \psi_4 && (\psi_3(t_0) = \alpha_3) \\
\dot{\psi}_4 &= -2\omega \psi_2 - \psi_3 \\
\dot{\psi}_5 &= \frac{\Omega^2 R_s^3 \psi_6}{(\sqrt{x})^3} - 3\Omega^2 R_s^3 d \frac{x_5 + R_s z}{(\sqrt{x})^5} && (\psi_5(t_0) = \alpha_5) \\
\dot{\psi}_6 &= -\psi_5 \\
\dot{\psi}_7(t) &= \frac{\beta \sqrt{\psi}}{x_7^2} && (\psi_7(t_0) \text{ normalized}; \quad \psi_7(t_f) = 0)
\end{align*}
\]
where

\[ \sqrt{\psi} = \sqrt{\psi_2^2 + \psi_4^2 + \psi_6^2} \]

\[ \sqrt{\kappa} = \left( x_1 + R_s x \right)^2 + \left( x_3 + R_s y \right)^2 + \left( x_5 + R_s z \right)^2 \]

and

\[ d = \psi_2(x_1 + R_s x) + \psi_4(x_3 + R_s y) + \psi_6(x_5 + R_s z) \]

The derivatives needed to form equation (15) can be obtained (ref. 7) by solving the following system in conjunction with equations (17a) and (17b) from \( t_0 \) to \( t_f \), with \( j = 1, 2, \ldots, 6 \):

\[ \frac{d}{dt} \left( \frac{\partial x_i}{\partial \alpha_j} \right) = \frac{\partial x_{i+1}}{\partial \alpha_j} - \left( \frac{\partial x_i(t_0)}{\partial \alpha_j} = 0; \quad i = 1, 3, 5 \right) \]

\[ \frac{d}{dt} \left( \frac{\partial x_2}{\partial \alpha_j} \right) = A_2 \frac{\partial \psi_2}{\partial \alpha_j} + B_{24} \frac{\partial \psi_4}{\partial \alpha_j} + B_{26} \frac{\partial \psi_6}{\partial \alpha_j} + \left[ C_{11} - \frac{\Omega^2 R_s^3}{(\sqrt{\kappa})^3} + \omega^2 \right] \frac{\partial x_1}{\partial \alpha_j} + C_{13} \frac{\partial x_3}{\partial \alpha_j} + 2\omega \frac{\partial x_4}{\partial \alpha_j} + C_{15} \frac{\partial x_5}{\partial \alpha_j} \]

\[ \frac{d}{dt} \left( \frac{\partial x_4}{\partial \alpha_j} \right) = B_{24} \frac{\partial \psi_2}{\partial \alpha_j} + A_4 \frac{\partial \psi_4}{\partial \alpha_j} + B_{46} \frac{\partial \psi_6}{\partial \alpha_j} + C_{13} \frac{\partial x_1}{\partial \alpha_j} - 2\omega \frac{\partial x_2}{\partial \alpha_j} + \left[ C_{33} - \frac{\Omega^2 R_s^3}{(\sqrt{\kappa})^3} + \omega^2 \right] \frac{\partial x_3}{\partial \alpha_j} + C_{35} \frac{\partial x_5}{\partial \alpha_j} \]

\[ \frac{d}{dt} \left( \frac{\partial x_6}{\partial \alpha_j} \right) = B_{26} \frac{\partial \psi_2}{\partial \alpha_j} + B_{46} \frac{\partial \psi_4}{\partial \alpha_j} + A_6 \frac{\partial \psi_6}{\partial \alpha_j} + C_{15} \frac{\partial x_1}{\partial \alpha_j} + C_{35} \frac{\partial x_3}{\partial \alpha_j} + \left[ C_{55} - \frac{\Omega^2 R_s^3}{(\sqrt{\kappa})^3} \right] \frac{\partial x_5}{\partial \alpha_j} \]
\frac{d}{dt}\left(\frac{\partial \psi_1}{\partial \alpha_j}\right) = \left[\frac{\Omega^2 R_s^3}{(\pi k)^3} - C_{11} - \omega^2\right] \frac{\partial \psi_2}{\partial \alpha_j} - C_{13} \frac{\partial \psi_4}{\partial \alpha_j} - C_{15} \frac{\partial \psi_6}{\partial \alpha_j} + (D_{21} + E_{11}) \frac{\partial x_1}{\partial \alpha_j}
+ (F_{24} + E_{13}) \frac{\partial x_3}{\partial \alpha_j} + (F_{26} + E_{15}) \frac{\partial x_5}{\partial \alpha_j}
+ \left(\frac{\partial \psi_1(t_0)}{\partial \alpha_j}\right) = \begin{cases} 1 & (j = 1) \\ 0 & (j \neq 1) \end{cases}

\frac{d}{dt}\left(\frac{\partial \psi_2}{\partial \alpha_j}\right) = 2\omega \frac{\partial \psi_4}{\partial \alpha_j} - \frac{\partial \psi_1}{\partial \alpha_j}
+ \left(\frac{\partial \psi_2(t_0)}{\partial \alpha_j}\right) = \begin{cases} 1 & (j = 2) \\ 0 & (j \neq 2) \end{cases}

\frac{d}{dt}\left(\frac{\partial \psi_3}{\partial \alpha_j}\right) = -C_{13} \frac{\partial \psi_2}{\partial \alpha_j} + \left[\frac{\Omega^2 R_s^3}{(\pi k)^3} - C_{33} - \omega^2\right] \frac{\partial \psi_4}{\partial \alpha_j} - C_{35} \frac{\partial \psi_6}{\partial \alpha_j} + (F_{42} + E_{13}) \frac{\partial x_1}{\partial \alpha_j}
+ (D_{43} + E_{33}) \frac{\partial x_3}{\partial \alpha_j} + (F_{46} + E_{35}) \frac{\partial x_5}{\partial \alpha_j}
+ \left(\frac{\partial \psi_3(t_0)}{\partial \alpha_j}\right) = \begin{cases} 1 & (j = 3) \\ 0 & (j \neq 3) \end{cases}

\frac{d}{dt}\left(\frac{\partial \psi_4}{\partial \alpha_j}\right) = -2\omega \frac{\partial \psi_2}{\partial \alpha_j} - \frac{\partial \psi_3}{\partial \alpha_j}
+ \left(\frac{\partial \psi_4(t_0)}{\partial \alpha_j}\right) = \begin{cases} 1 & (j = 4) \\ 0 & (j \neq 4) \end{cases}

\frac{d}{dt}\left(\frac{\partial \psi_5}{\partial \alpha_j}\right) = -C_{15} \frac{\partial \psi_2}{\partial \alpha_j} - C_{35} \frac{\partial \psi_4}{\partial \alpha_j} + \left[\frac{\Omega^2 R_s^3}{(\pi k)^3} - C_{55}\right] \frac{\partial \psi_6}{\partial \alpha_j} + (F_{62} + E_{15}) \frac{\partial x_1}{\partial \alpha_j}
+ (F_{64} + E_{35}) \frac{\partial x_3}{\partial \alpha_j} + (D_{65} + E_{55}) \frac{\partial x_5}{\partial \alpha_j}
+ \left(\frac{\partial \psi_5(t_0)}{\partial \alpha_j}\right) = \begin{cases} 1 & (j = 5) \\ 0 & (j \neq 5) \end{cases}

\frac{d}{dt}\left(\frac{\partial \psi_6}{\partial \alpha_j}\right) = -\frac{\partial \psi_5}{\partial \alpha_j}
+ \left(\frac{\partial \psi_6(t_0)}{\partial \alpha_j}\right) = \begin{cases} 1 & (j = 6) \\ 0 & (j \neq 6) \end{cases}

\frac{d}{dt}\left(\frac{\partial \psi_7}{\partial \alpha_j}\right) = \frac{\beta \left(\psi_2 \frac{\partial \psi_2}{\partial \alpha_j} + \psi_4 \frac{\partial \psi_4}{\partial \alpha_j} + \psi_6 \frac{\partial \psi_6}{\partial \alpha_j}\right)}{x_7^2 \sqrt{\psi}}
+ \left(\frac{\partial \psi_7(t_0)}{\partial \alpha_j}\right) = 0

In this system

\[ A_1 = \beta \left[\frac{1}{x_7 \sqrt{\psi}} - \frac{\psi_1^2}{x_7 (\sqrt{\psi})^3}\right] \]
\[ B_{ik} = -\frac{\beta \psi_i \psi_k}{\sqrt{x}} (\sqrt{\psi})^3 \]
\[ C_{ik} = \frac{3\Omega^2 R_s}{(\sqrt{x})^5} v_i v_k \]
\[ D_{ik} = -\frac{3\Omega^2 R_s}{(\sqrt{x})^5} (2\psi_i v_k + d) \]
\[ E_{ik} = \frac{15\Omega^2 R_s}{(\sqrt{x})^7} d v_i v_k \]

and
\[ F_{ik} = -\frac{3\Omega^2 R_s}{(\sqrt{x})^5} (\psi e v_i + \psi_m v_k) \]

with
\[ v_1 = x_1 + R_s x \]
\[ v_3 = x_3 + R_s y \]

and
\[ v_5 = x_5 + R_s z \]

**Iteration Sequence**

In summary, the procedure used to solve the boundary-value problem presented in the preceding section is as follows:

1. Assume a value of \( \alpha \) given by equation (12).
2. Solve equations (17a) and (17b) to \( t_f = \alpha T \) and evaluate \( \bar{e}(\alpha) \) given by equation (12).
3. If \( \bar{e}(\alpha) \) is sufficiently small, then the problem is solved; \( \bar{\alpha} \) gives the unknown initial conditions and final time, and the corresponding solution of equation (13) gives the optimal trajectory.
4. Otherwise, with the use of the solution given by equation (13), for the assumed \( \bar{\alpha} \), solve equations (18a) and (18b) and evaluate \( \bar{e}(\alpha) \) given by equation (16).
5. If previous value is not satisfactory, find \( \lambda \) such that equation (15) follows.
6. Replace \( \bar{\alpha} \) by \( \bar{\alpha} + \delta \bar{\alpha} \) and return to step (2).

A digital computer program for this procedure is discussed in the next section.
DESCRIPTION OF PROGRAM

General

The program was written in FORTRAN IV language for the Control Data series 6000 computer systems at the Langley Research Center. A complete listing of the program is found in appendix B.

Upon acceptance of an assumed value of $\bar{\alpha}$ and appropriate system constants characterizing the particular rendezvous problem to be solved, the program proceeds according to the steps listed in the preceding section. The program does not contain a search routine for the determination of $\lambda$ in step (5). It is expected that in each instance a value of $\lambda$ which will work for the complete iteration process can be found.

The mathematical symbols used in the theory and their FORTRAN equivalents are given in table 1.

<table>
<thead>
<tr>
<th>Mathematical symbol</th>
<th>FORTRAN equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>$B(I)$</td>
</tr>
<tr>
<td>$c$</td>
<td>$C$</td>
</tr>
<tr>
<td>$e_i$</td>
<td>$E(I)$</td>
</tr>
<tr>
<td>$E_2(M)$</td>
<td>$EDP$</td>
</tr>
<tr>
<td>$t_0$</td>
<td>$IO$</td>
</tr>
<tr>
<td>$r_s$</td>
<td>$RS$</td>
</tr>
<tr>
<td>$r_{sy}, r_{sz}$</td>
<td>$RSV(I)$</td>
</tr>
<tr>
<td>$T_0$</td>
<td>$DRV(I)$</td>
</tr>
<tr>
<td>$T_1$</td>
<td>$RV$</td>
</tr>
<tr>
<td>$T_2$</td>
<td>$RVX, RVY, RVZ$</td>
</tr>
<tr>
<td>$u_1, u_2, u_3$</td>
<td>$DRVX, DRVY, DRVZ$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$VAR(I)$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$TF$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$XT1(L,J)$</td>
</tr>
<tr>
<td>$x_{12}$</td>
<td>$XT2(L,J)$</td>
</tr>
<tr>
<td>$x_{13}$</td>
<td>$UX, UY, UZ$</td>
</tr>
<tr>
<td>$x_{23}$</td>
<td>$VAR(I)$</td>
</tr>
<tr>
<td>$\frac{\partial x_1}{\partial \alpha}$</td>
<td>$PEMAT(L,J)$</td>
</tr>
<tr>
<td>$\frac{\partial x_2}{\partial \alpha}$</td>
<td>$PX(L,J)$</td>
</tr>
<tr>
<td>$\frac{\partial x_3}{\partial \alpha}$</td>
<td>$PEVEC(L,J)$</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>$PEVEC(I,L)$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$BETA$</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>$THETA$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$THETVO$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$LAMBDA$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$MU$</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>$PHIO$</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>$PSI(I,J)$</td>
</tr>
<tr>
<td>$\frac{\partial \psi_1}{\partial \alpha}$</td>
<td>$PFSI(L,J)$</td>
</tr>
<tr>
<td>$\frac{\partial \psi_2}{\partial \alpha}$</td>
<td>$PSI(I,J)$</td>
</tr>
</tbody>
</table>

18
Subroutines

Seven subroutines are used in addition to the main program. The purpose of each is given in table 2:

<table>
<thead>
<tr>
<th>Subroutine</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>DERSUB</td>
<td>Evaluates all differential equations to be solved</td>
</tr>
<tr>
<td>CHSUB</td>
<td>Tests for the end of a trajectory</td>
</tr>
<tr>
<td>COMP</td>
<td>Evaluates the position and rate of ( \overline{R}_b(t) ), the vector from the origin to the target</td>
</tr>
<tr>
<td>ITERAT</td>
<td>Computes and applies the correction ( \delta \overline{a} ) to the initial ( \overline{a} )</td>
</tr>
<tr>
<td>BLOCK DATA</td>
<td>Initializes ( \frac{\partial x_i}{\partial a_j} ), ( \frac{\partial y_i}{\partial a_j} ), and ( \frac{\partial z_i}{\partial a_j} ) (( i = 1, 2, \ldots 6; j = 1, 2, \ldots 6 ))</td>
</tr>
<tr>
<td>INT2</td>
<td>Numerically integrates the differential equations with a fixed-step size method by employing a fourth-order Adams-Bashforth predictor formula and a fourth-order Adams-Moulton corrector formula</td>
</tr>
<tr>
<td>MATINV</td>
<td>Obtains the inverse of the matrix ( \frac{\partial \delta}{\partial \alpha} B \frac{\partial \delta}{\partial \alpha} + \lambda I )</td>
</tr>
</tbody>
</table>

Input

Input is of the form shown in table 3:

<table>
<thead>
<tr>
<th>Card number</th>
<th>FORTRAN variable name</th>
<th>Description</th>
<th>FORTRAN format</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>NO</td>
<td>Case number</td>
<td>120</td>
</tr>
<tr>
<td>2</td>
<td>SQMEG, BETA, C, TF</td>
<td>( \omega, \beta, c, t_f = \alpha_7 )</td>
<td>4E20.8</td>
</tr>
<tr>
<td>3</td>
<td>PHIO, THETVO, RV</td>
<td>( \varphi_0^0, \theta_0^0, R_V(t_0) )</td>
<td>3E20.8</td>
</tr>
<tr>
<td>4</td>
<td>DRVXO, DRVYO, DRVZO</td>
<td>( \hat{R}<em>{Vx}(t_0), \hat{R}</em>{Vy}(t_0), \hat{R}_{Vz}(t_0) )</td>
<td>3E20.8</td>
</tr>
<tr>
<td>5</td>
<td>PHIO, THETAO, IO, RS</td>
<td>( \varphi_o, \theta_o, \psi_o, R_S )</td>
<td>4E20.8</td>
</tr>
<tr>
<td>6</td>
<td>VAR(1), VAR(8), MU</td>
<td>( t_o, m_o, \mu )</td>
<td>3E20.8</td>
</tr>
<tr>
<td>7</td>
<td>VAR(9) to VAR(12)</td>
<td>( \alpha_1 ) to ( \alpha_4 )</td>
<td>4E20.8</td>
</tr>
<tr>
<td>8</td>
<td>VAR(13) to VAR(15)</td>
<td>( \alpha_5, \alpha_6, \psi_7(t_0) )</td>
<td>3E20.8</td>
</tr>
<tr>
<td>9</td>
<td>CI, SPEC</td>
<td>Computing interval, printing frequency (see &quot;Output&quot; section)</td>
<td>2E20.8</td>
</tr>
<tr>
<td>10</td>
<td>IPRINT, IERØR, IMAT</td>
<td>(See &quot;Output&quot; section)</td>
<td>3120</td>
</tr>
<tr>
<td>11</td>
<td>LAMBDA, CRIT, MAXIT</td>
<td>( \lambda, ) stopping criterion, maximum number of iterations (see &quot;Output&quot; section)</td>
<td>2E20.8, 120</td>
</tr>
<tr>
<td>12</td>
<td>B(1) to B(4)</td>
<td>( b_1 ) to ( b_4 )</td>
<td>4E20.8</td>
</tr>
<tr>
<td>13</td>
<td>B(5) to B(7)</td>
<td>( b_5 ) to ( b_7 )</td>
<td>3E20.8</td>
</tr>
</tbody>
</table>
All the input variables are dimensionalized and angles are in radians; $a_i$ $(i = 1, 2, \ldots 6)$, $b_i$ $(i = 1, 2, \ldots 7)$, and $\psi_i$ $(i = 0, 1, \ldots 8)$ are considered dimensionless.

Output

The program offers several options for output. Regardless of the options, the input data are always printed initially. Afterwards, output is presented at each iteration according to the following input variables: SPEC, IPRINT, IEROR, IMAT.

SPEC specifies how often results are to be printed. If SPEC = $10^{10}$, output will be printed only at $t = t_0$ and $t = t_f$. If SPEC = nCI, where CI is the integration computing interval and n is a positive integer, results will be printed every n integration step. At $t = t_0$, the variables that are printed are $t_0$, $\psi_i(t_0)$ $(i = 1, 2, \ldots 7)$, and $u_i$ $(i = 1, 2, 3)$. At any other time $t$, determined by SPEC, the variables that are printed are $t$, $\psi_i(t)$ $(i = 1, 2, \ldots 7)$, $R_S(t)$, $\dot{R}_S(t)$, $\ddot{R}_S(t)$, $R_V(t)$, $\dot{R}_V(t)$, $\ddot{R}_V(t)$, $x_i$ $(i = 1, 2, \ldots 7)$, $u_i$ $(i = 1, 2, 3)$, $\dot{x}_2(t)$, $\dot{x}_4(t)$, $\dot{x}_6(t)$, $\beta$, $\|R_V(t)\|$, and the relative distances and velocities $\sqrt{x_1^2 + x_3^2 + x_5^2}$ and $\frac{d}{dt}\sqrt{x_1^2 + x_3^2 + x_5^2}$. At $t = t_f$, $E[\bar{e}(\bar{\alpha})]$ and $\delta\bar{\alpha}$ are also printed.

IPRINT provides the option for printing the partial derivatives $\frac{\partial \psi_i(t)}{\partial \alpha_j}$, $\frac{\partial x_i(t)}{\partial \alpha_j}$, and $\frac{\partial \psi_i(t)}{\partial \alpha_j}$ $(i = 1, 2, \ldots 6; j = 1, 2, \ldots 6)$. If IPRINT = 0, the partial derivatives are not printed; if IPRINT = 1, the partial derivatives are printed.

IEROR provides the option for printing the truncation errors for the differential equations. If IEROR = 0, truncation errors for the differential equations are not printed; if IEROR = 1, the truncation errors are printed.

IMAT provides the option for printing the matrix $\left[\frac{\partial \bar{e}(\bar{\alpha})}{\partial \alpha} B \frac{\partial \bar{e}(\bar{\alpha})}{\partial \alpha} + \lambda I\right]$, its inverse, and the product of the two. If IMAT = 0, the matrices are not printed; if IMAT = 1, the matrices are printed.

The program terminates when convergence is reached ($E[\bar{e}(\bar{\alpha})] \leq CRIT$) or when the maximum number of iterations (MAXIT) has been reached.

EXAMPLE COMPUTATIONS

The use of the foregoing procedures is demonstrated by the problem of a space vehicle launched from the surface of the moon to rendezvous, in a minimum of flight time, with a target in a circular orbit. The vehicle has a bounded thrust magnitude which, along
with the thrust direction, acts as a control variable. There is no terminal mass constraint. The system constants used in this study are given in Table 4.

**TABLE 4.- SYSTEM CONSTANTS**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial time, $t_0$, sec</td>
<td>$0$</td>
</tr>
<tr>
<td>Upper bound on thrust magnitude, $\beta$, lbf (kg)</td>
<td>$3504$ (1589.4)</td>
</tr>
<tr>
<td>Initial mass, $m_0$, slugs (kg)</td>
<td>$285.5$ (4166.3)</td>
</tr>
<tr>
<td>Effective exhaust velocity, $c$, ft/sec (m/sec)</td>
<td>$9853.2$ (279.86)</td>
</tr>
<tr>
<td>$R_v(t_0)$ set equal to radius of moon, ft (km)</td>
<td>$5.707 \times 10^6$ (1739.4)</td>
</tr>
<tr>
<td>$R_s$ set equal to radius of 80-nautical-mile satellite circular orbit, ft (km)</td>
<td>$6.1934 \times 10^6$ (1887.7)</td>
</tr>
<tr>
<td>Universal gravitational constant multiplied by the moon mass, $\mu$, ft$^3$/sec$^2$ (m$^3$/sec$^2$)</td>
<td>$1.727 \times 10^{14}$ (48.90 $\times$ 10$^{11}$)</td>
</tr>
<tr>
<td>Angular velocity of moon about axis of rotation, $\omega$, rad/sec</td>
<td>$2.66 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

The satellite orbital plane was placed in the xy-plane of the rotating system (Fig. A-1). Studies were made with the vehicle launched from rest from the surface of the moon with the launch site both in and out of this plane (planar and nonplanar case, respectively).

It was found that a workable set of $b_i$ ($i = 1, 2, \ldots, 7$) and $\lambda$ for convergence was $b_1 = b_3 = b_5 = b_7 = 1$, $b_2 = b_4 = b_6 = 10$, and $\lambda = 10$ or $\lambda = 1$. These values were used throughout. It was observed that increasing the value of $\lambda$ yielded a slower converging process, while a decrease was apt to produce divergence. It was also observed that by increasing a particular $b_i$, greater influence could be applied to the correction of the error $e_i$; that is, this error would be corrected more quickly than before, but at the expense of the other errors.

It was found that in the planar case, with the vehicle launched such that the satellite lead angle $\phi_0 - \phi_v^0$ was $90^\circ$ (with $\phi_0 = 890^\circ$), the set of values

$$\alpha_1 = -100\tilde{R}_{s_x}(t_0) = 7.8565$$
$$\alpha_2 = \tilde{R}_{s_x}(t_0) = 5.296 \times 10^3$$
$$\alpha_3 = \tilde{R}_{s_y}(t_0) = -4.5016$$
$$\alpha_4 = \tilde{R}_{s_y}(t_0) = -92.44$$
$$\alpha_5 = \tilde{R}_{s_z}(t_0) = 0$$
$$\alpha_6 = \tilde{R}_{s_z}(t_0) = 0$$
$$\alpha_7 = 500 \text{ seconds}$$
having been normalized at $-1.184 \times 10^5$ leads to convergence with the values of $b_1$ previously mentioned and $\lambda = 10$ in 12 iterations. This procedure gave a solution to be used as a nominal, or guessed, solution for neighboring lead angles. Table 5 shows the planar results obtained. For each lead angle the iteration was stopped when
\[
\tilde{x}(t_f) \cdot \tilde{x}(t_f) + \psi_7^2(t_f) \leq 1
\]
with $\psi_7(t_0)$ normalized at $-1.184 \times 10^5$. Since the results are planar, $\psi_5(t) \equiv \psi_6(t) \equiv 0$.

It can be seen from Table 5 that near the lead angle 13.7°, the smallest value of $t_f$ and hence the largest terminal mass occur. Figure 1 shows a graph of these results.

<table>
<thead>
<tr>
<th>Lead angle, $\phi_o - \phi_v^0$, deg</th>
<th>Unknown parameters</th>
<th>Percent initial mass at $t_f$</th>
</tr>
</thead>
</table>
|                                     | $\psi_1(t_0)$ | $\psi_2(t_0)$ | $\psi_3(t_0)$ | $\psi_4(t_0)$ | $t_f$, sec |%
| 8                                   | 11.662         | 5577.3          | 5.6883         | 2051.8         | 547.9          | 31.7 |
| 9                                   | 12.929         | 5948.3          | 7.4703         | 2638.8         | 524.8          | 34.6 |
| 10                                  | 14.120         | 6239.4          | 10.026         | 3422.5         | 499.5          | 37.8 |
| 12                                  | 12.530         | 5464.5          | 17.992         | 5427.7         | 453.0          | 43.6 |
| 13                                  | 6.9429         | 3774.3          | 20.577         | 5758.3         | 443.3          | 44.8 |
| 13.7                                | 2.4694         | 2500.4          | 20.507         | 5501.0         | 442.3          | 44.9 |
| 14                                  | 73674          | 2012.3          | 20.108         | 5315.8         | 443.0          | 44.8 |
| 16                                  | -6.2567        | -42.810         | 15.897         | 4005.4         | 454.8          | 43.3 |
| 18                                  | -8.4150        | -882.65         | 12.256         | 3110.3         | 471.7          | 41.2 |
| 22                                  | -8.5773        | -1472.0         | 7.8560         | 2155.4         | 507.3          | 36.8 |

TABLE 5.- TIME-OPTIMAL PLANAR RESULTS

---

Figure 1.- Percentage of initial mass at rendezvous as a function of lead angle $\phi_o - \phi_v^0$. Planar case; $\theta_v^0 = 0°$. 22
Schematic views of the flight paths for planar solutions with lead angles 80°, 13.70°, and 220° are shown in figure 2. From this figure an idea can be gained as to the different maneuvers required for different lead angles. Arrows placed along the trajectories indicate the true direction of the thrust vector at 50-second intervals. The spatial coordinates used in this plot are to different scales. The xy-plane is viewed as being inertial since the total rotation of the moon was less than 0.1° for the longest flight time.

Examples of out-of-plane results were obtained by holding Φ0 and Φv0 fixed at 93.7° and 80°, respectively, and allowing nonzero values of θv0. For θv0 = 2°, the planar solution for Φ0 - Φv0 = 13.7° was used as a nominal. The results for this non-planar case exemplify a typical sequence of iterations, and this sequence is tabulated in table 6. For θv0 = 2°, b1 = b3 = b5 = b7 = 1, b2 = b4 = b6 = 10, and λ = 1. Table 7 shows other out-of-plane results for the fixed lead angle of 13.7°. A graph of the percentage of initial mass at rendezvous as a function of out-of-plane angle θv0 is shown in figure 3.

![Figure 2. Comparison of planar time-optical trajectories. Φv0 = 80°; Φ0 = 88°, 93.7°, and 102°; θv0 = 0°. (0.3048 meter = 1 foot)](image-url)
### TABLE 6. - TYPICAL TIME-OPTIMAL ITERATION SEQUENCE

\[\varphi_0 = 93.7^\circ, \quad \varphi_\nu^0 = 80^\circ; \quad \theta_\nu^0 = 2^\circ \text{ (except for nominal results)}\]

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Unknown parameters</th>
<th></th>
<th>Percent initial mass at (t_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\psi_1(t_0))</td>
<td>(\psi_2(t_0))</td>
<td>(\psi_3(t_0))</td>
</tr>
<tr>
<td>Nominal(^a)</td>
<td>2.4694</td>
<td>2500.4</td>
<td>20.507</td>
</tr>
<tr>
<td>1</td>
<td>2.6048</td>
<td>2526.5</td>
<td>20.415</td>
</tr>
<tr>
<td>2</td>
<td>3.5498</td>
<td>2718.7</td>
<td>19.445</td>
</tr>
<tr>
<td>3</td>
<td>3.6100</td>
<td>2732.8</td>
<td>19.461</td>
</tr>
<tr>
<td>4</td>
<td>3.6263</td>
<td>2736.9</td>
<td>19.469</td>
</tr>
<tr>
<td>5</td>
<td>3.6325</td>
<td>2738.6</td>
<td>19.474</td>
</tr>
<tr>
<td>(b6)</td>
<td>3.6351</td>
<td>2739.2</td>
<td>19.476</td>
</tr>
</tbody>
</table>

\(^a\theta_\nu^0 = 0; \quad x_1(t_f) = -582.7, \quad x_2(t_f) = -0.03386, \quad x_3(t_f) = -3275.5, \quad x_4(t_f) = 0.15997, \quad x_5(t_f) = 1.8728 \times 10^5, \quad x_6(t_f) = -27.581, \quad \text{and} \quad \psi_\gamma(t_f) = 9.9489.\)

\(^b\)\(x_1(t_f) = -0.0017, \quad x_2(t_f) = 0.230, \quad x_3(t_f) = -0.006, \quad x_4(t_f) = -0.026, \quad x_5(t_f) = 0.001, \quad x_6(t_f) = 0.015, \quad \text{and} \quad \psi_\gamma(t_f) = -0.042 \quad \text{for a computing time of 23 seconds.}\)

### TABLE 7. - TIME-OPTIMAL OUT-OF-PLANE RESULTS

\[\varphi_0 = 93.7^\circ, \quad \varphi_\nu^0 = 80^\circ\]

<table>
<thead>
<tr>
<th>(\theta_\nu^0), deg</th>
<th>Unknown parameters</th>
<th>Percent initial mass at (t_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\psi_1(t_0))</td>
<td>(\psi_2(t_0))</td>
</tr>
<tr>
<td>0</td>
<td>2.4694</td>
<td>2500.4</td>
</tr>
<tr>
<td>2</td>
<td>3.6351</td>
<td>2739.2</td>
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<td>3233.6</td>
</tr>
<tr>
<td>8</td>
<td>6.5482</td>
<td>3215.7</td>
</tr>
<tr>
<td>10</td>
<td>5.9679</td>
<td>2970.9</td>
</tr>
</tbody>
</table>
CONCLUDING REMARKS

A technique has been presented for obtaining three-dimensional rendezvous trajectories between a one-stage rocket vehicle and a target in a circular Keplerian orbit. The trajectories satisfy the necessary conditions of the Pontryagin maximum principle for time-optimal rendezvous in which no terminal mass constraint is placed on the rocket.

The use of Pontryagin's theory leads to a two-point boundary-value problem. Certain initial conditions on a set of differential equations introduced by the maximum principle had to be found such that certain boundary conditions were met. A digital program was given for the solution of this problem based on an iteration method. Given assumed values of the initial conditions, which do not yield rendezvous, the program attempts to correct these values in such a way that rendezvous is more closely attained. Iterative use of this procedure gives a sequence of trajectories converging to one yielding rendezvous.

The computational time for obtaining both planar and nonplanar trajectories was less than 1 minute.
The program was successfully applied to a problem in which a space vehicle was launched from the surface of the moon and required to rendezvous with a target in an 80-nautical-mile circular orbit. Both planar and nonplanar trajectories were obtained with equal ease in less than 1 minute of computational time on the Control Data series 6000 computer systems.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., November 8, 1968,
125-17-05-10-23.
APPENDIX A

FORMULATION OF DYNAMIC EQUATIONS

The dynamic equations for a space vehicle which is required to rendezvous with a target in a circular Keplerian orbit about a rotating body are derived in reference 5. The formulation of these equations is summarized in this appendix. The vehicle is a one-stage rocket, treated as a point mass, with bounded thrust magnitude. The controls are the magnitude and direction of the thrust vector.

Let $x$, $y$, and $z$ be Cartesian coordinates of a rotating axis system located in the center of the body with the $z$-axis through the axis of rotation of the body. The geometry is represented in figure A-1.

![Figure A-1: The rotating axis system.](image)

The vector $\mathbf{R}_s(t)$ is from the origin to the target, $\mathbf{R}_v(t)$ is from the origin to the vehicle, and $\omega$ is the angular velocity of the body about its axis of rotation. The relative distance between the target and vehicle is given by

$$\mathbf{r}(t) = \mathbf{R}_v(t) - \mathbf{R}_s(t)$$

(A1)

Since the target is assumed to be in a circular orbit, it moves in its orbital plane at a constant distance $R_s$ from the center of the body with a constant angular velocity $\Omega = (\mu/R_s^3)^{1/2}$, where $\mu$ is the universal gravitational constant multiplied by the body mass and where the magnitude of $R_s$ is $[\mathbf{R}_s(t) \cdot \mathbf{R}_s(t)]^{1/2}$. Consider an inertial $XYZ$-axis system fixed in the center of the body such that, at the initial time $t_0$, it is alined with the rotating $xyz$-axis system. In this framework the target can be pictured as in figure A-2.
APPENDIX A

Figure A-2. Target viewed in the inertial axis system.

The angles $\vartheta_0$ and $\vartheta_0$ define the normal and line of nodes, respectively, of the target orbital plane relative to the inertial system. The $x'$ and $y'$ axes therefore define the orbital plane of the target. If at $t_0$ the target is in the position $(x',y') = (R_s \cos \varphi_0, R_s \sin \varphi_0)$ and moves toward the line of nodes, then

$$\overline{R_s}[x'(t), y'(t), z'(t)] = R_s \begin{bmatrix} \cos[\varphi_0 - \Omega(t - t_0)] \\
\sin[\varphi_0 - \Omega(t - t_0)] \\
0 \end{bmatrix}$$

(A2)

and

$$\overline{R_s}(X,Y,Z) = T_1 \overline{R_s}(x',y',z')$$

(A3)

where

$$T_1 = \begin{bmatrix} \cos \vartheta_0 & -\cos \vartheta_0 \sin \vartheta_0 & \sin \vartheta_0 \sin \vartheta_0 \\
\sin \vartheta_0 & \cos \vartheta_0 \cos \vartheta_0 & -\sin \vartheta_0 \cos \vartheta_0 \\
0 & \sin \vartheta_0 & \cos \vartheta_0 \end{bmatrix}$$

(A4)

Since the xyz-axis system rotates about the Z-axis with a constant angular velocity $\omega$,

$$\overline{R_s}(x,y,z) = \overline{R_s}(t) = T_2(t) \overline{R_s}(X,Y,Z)$$

(A5)
APPENDIX A

where

\[ T_2(t) = \begin{bmatrix} \cos \omega(t - t_0) & \sin \omega(t - t_0) & 0 \\ -\sin \omega(t - t_0) & \cos \omega(t - t_0) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Also

\[ \frac{d\bar{R}_S(t)}{dt} = \hat{T}_2(t)\bar{R}_S(X,Y,Z) + T_2(t)\dot{\bar{R}}_S(X,Y,Z) \]  \hspace{1cm} (A6)

where

\[ \hat{T}_2(t) = \omega \begin{bmatrix} -\sin \omega(t - t_0) & \cos \omega(t - t_0) & 0 \\ -\cos \omega(t - t_0) & -\sin \omega(t - t_0) & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

and

\[ \dot{\bar{R}}_S(X,Y,Z) = R_S \Omega T_1 \begin{bmatrix} \sin[\phi_0 - \Omega(t - t_0)] \\ -\cos[\phi_0 - \Omega(t - t_0)] \\ 0 \end{bmatrix} \]

Thus the position and rate of $\bar{R}_S(t)$ can be obtained by specifying $R_S$, $\Omega$, $\theta_0$, and $\phi_0$ at $t_0$ and by using equations (A5) and (A6).

The thrust control vector $\bar{T}$ is related to the rotating axis system by

\[ \bar{T} = T \begin{bmatrix} \cos \theta_c & \cos \phi_c \\ \cos \theta_c & \sin \phi_c \\ \sin \theta_c \end{bmatrix} \] \hspace{1cm} (A7)

as shown in figure A-3.

The vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the direction of the $x$, $y$, and $z$ axes, respectively, and $T$ is the magnitude of the thrust vector. Let

\[ \hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos \theta_c & \cos \phi_c \\ \cos \theta_c & \sin \phi_c \\ \sin \theta_c \end{bmatrix} \]
APPENDIX A

\[ u_4 = T \]

\[
\vec{r} = \begin{bmatrix}
  r_x \\
  r_y \\
  r_z
\end{bmatrix} = \begin{bmatrix}
  R_{vX} - R_{sX} \\
  R_{vY} - R_{sY} \\
  R_{vZ} - R_{sZ}
\end{bmatrix}
\]

\[
\dot{\vec{r}} = \begin{bmatrix}
  \dot{r}_x \\
  \dot{r}_y \\
  \dot{r}_z
\end{bmatrix}
\]

and

\[ x_1 = r_x \] (relative x distance)
\[ x_2 = \dot{r}_x \] (relative x velocity)
\[ x_3 = r_y \] (relative y distance)
\[ x_4 = \dot{r}_y \] (relative y velocity)
\[ x_5 = r_z \] (relative z distance)
\[ x_6 = \dot{r}_z \] (relative z velocity)
\[ x_7 = m(t) \] (instantaneous vehicle mass)
\[ x_8 = t \] (time)

In this framework the dynamic equations can be written as (ref. 5):

\[
\dot{x} = \frac{u_4 M \dot{\mu}}{x_7} + Y(x, x_8) \quad \text{where} \quad \begin{cases} 
\dot{x}(t_0) = x_0; \quad \ddot{x}(t_f) = 0 \\
\text{and (A8)} \quad x_7(t_0) = m(t_0) = m_0 \\
\dot{x}_8 = 1 \quad (x_8(t_0) = t_0)
\end{cases}
\]

where

\[
Y(x, x_8) = -\frac{\Omega R_s^3}{||A\ddot{x} + R_s||^3} \left[ N\dddot{x} + MR_s(x_8) \right] + \Omega^2 MR_s(x_8) + \left( N' + 2\omega K + \omega^2 L \right)\ddot{x}
\]

with

\[ \dddot{x} = \text{col}(x_1, \ldots, x_8) \]
APPENDIX A

\[ M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

and

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
APPENDIX A

In order to compute the initial value of $\tilde{x}(t_0)$, the initial value of $\vec{R}_V(t_0)$ can be specified by

$$\vec{R}_V(t_0) = \begin{bmatrix} \cos \theta_v^0 & \cos \varphi_v^0 \\ \cos \theta_v^0 & \sin \varphi_v^0 \\ \sin \theta_v^0 \end{bmatrix}$$

where $\theta_v^0$ and $\varphi_v^0$ are as shown in figure A-4. Also

$$\vec{R}_V(t_0) = \begin{bmatrix} \dot{R}_{Vx}(t_0) \\ \dot{R}_{Vy}(t_0) \\ \dot{R}_{Vz}(t_0) \end{bmatrix}$$

and $\vec{R}_S(t_0)$ and $\vec{\dot{R}}_S(t_0)$ are computed from equations (A2) to (A6).

The act of rendezvous requires that the vehicle and target have the same position and velocity at $t_f$; hence, the condition $\tilde{x}(t_f) = 0$. In addition, $u_4 \leq \beta$, the largest value obtainable for the thrust magnitude.
APPENDIX B

PROGRAM LISTING

The program presented on the following pages is written in FORTRAN IV language for the Control Data series 6000 computer systems at the Langley Research Center.
APPENDIX B

PROGRAM E1257(INPUT=INPUT,OUTPUT=OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)

TIME OPTIMAL RENDEZVOUS STUDY

DEFINITIONS

CASE NO. CASE NO.
ANG VEL OF CENTRAL BODY ABOUT ITS OWN AXIS
MAX. THRUST
EFFECTIVE EXHAUST SPEED
FINAL TIME - GUESS
INITIAL POSITION OF VEHICLE
MAG. OF RAD. VECTOR TO VEHICLE
INITIAL POSITION OF ORBITING VEHICLE
MAG. OF RAD. VECTOR TO ORBITING VEHICLE
INITIAL TIME
INITIAL MASS
GRAVITATIONAL CONSTANT OF THE CENTRAL BODY
I.C. ON PSI1 - PSI7
COMPUTING INTERVAL
DO NOT PRINT AT SUB-INTERVALS
USE AS PRINT INTERVAL
0 = DO NOT PRINT TRUNCATION ERRORS - 1 = PRINT
PARAMETER USED IN ITERATION SCHEME
MAX NO. OF ITERATIONS DESIRED
DIAGONAL MATRIX OF WEIGHTING FACTORS

INPUT AS FOLLOWS

<table>
<thead>
<tr>
<th>INPUT</th>
<th>CARD NO.</th>
<th>QUANTITY</th>
<th>FORMAT</th>
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<td>I</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>SOMEGBETA*CTF</td>
<td>E</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>PHIVOTHETVO*RV</td>
<td>E</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>DRVXODRVYODRVZO</td>
<td>E</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>PHIO<em>THETAO</em>IO</td>
<td>E</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>VAR(1)VAR(8)*MU</td>
<td>E</td>
<td></td>
</tr>
<tr>
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<td>VAR(9) - VAR(12)</td>
<td>E</td>
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<td>8</td>
<td>VAR(13) - VAR(15)</td>
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<tr>
<td>9</td>
<td>CI*SPEC</td>
<td>E</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX B

COMMON /SPACE/
1 VAR*CUVAR*SAVE*PE*PEMAT*PEVEC*ERRVAL*SOMEGR
2 E*RS*LEMDA*DP1*TF*E1
3 DER*F1*DP2*CI*ELE2*DRS
4 RSV*E1*II*SOMEGR*COMEGR*IPRINT
5 S2*E2*N*COMEGR*EN2*KOUNT
6 Q1*E3*TEMPCL*SI0*PHIO*Q2
7 E4*SPEC*ETO*LAMBDAMB*TEMPT*Q3
8 T2*TEMPSP*CIO*CRIT*EN1*IKOUNT
9 RVX0*RVYO*RVZ0*DRXO*DRY0*DRZ0
10 T4*MU*CIO*CT0*B*XTI

REAL LAMBDA,MU
REAL IO
LOGICAL FIRST

DIMENSION VAR(93),*CUVAR(93),*DER(93)
1 ELE1(92),*ELE2(92),*ERRVAL(92)
3 RSV(3),*DRSV(3),*PX(6*6)
4 PPSI(6*6),*PPSI7(6),*CUPX(6*6)
5 CUPSI(6*6),*CUPSI7(6),*DRPX(6*6)
6 DRPSI(6*6),*DRPSI7(6)*ERX(6*6)
7 ERPSI(6*6),*ERPSI7(6),*E(7)
8 PE(7*7),*PEMAT(7*7),*PEVEC(7*1)
9 B(7),*SAVE(93)*XTI(3*3)

EQUIVALENCE
1 (VAR(16),*PX(1*1)) *(VAR(52),*PPSI(1*1))
2 (VAR(88),*PPSI7(1)) *(CUVAR(16),*CUPX(1*1))
3 (CUVAR(52),*CUPSI(1*1)) *(CUVAR(88),*CUPSI7(1))
4 (DER(16),*DRPX(1*1)) *(DER(52),*DRPSI(1*1))
5 (DER(88),*DRPSI7(1)) *(ERRVAL(15),*ERX(1*1))
6 (ERRVAL(51),*ERPSI(1*1)) *(ERRVAL(87),*ERPSI7(1))

EXTERNAL DERSUB,CHSUB

35
EQUIVALENCE( VAR(1), T1)

FORMAT STATEMENTS

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500 FORMAT(30H TIME OPTIMAL RENDEZVOUS STUDY / 10X 8HCASE NO. 13//)
16H INPUT / 5H BETA 2X E16.8 / 10X 2HR8 8X E16.8 / 10X 7HOMEGA M 3X
2E16.8 / 10X 6HLAMBD A2X E16.8 / 2H C 5X E16.8 / 10X 2HRV 8X E16.8 / 10X
37HOMEGA T 3X E16.8 / 10X 2HMU 6X E16.8//)
501 FORMAT(19H WEIGHTING ELEMENTS / 7E16.7//)
502 FORMAT(19H INITIAL CONDITIONS / 3H TO 4X E16.8 / 10X 6HPHI V0 4X,
1E16.8 / 10X 8HRVX0 DOT 2X E16.8 / 10X 4HPHI0 4X E16.8 / 5H MASS 2X E16.8
2 / 10X 8HTHETA V0 2X E16.8 / 10X 8HRVY0 DOT 2X E16.8 / 10X 6HTHETA0 2X
3E16.8 / 5H MDOT2XE16.8 / 46X 8HRVZ0 DOT 2X E16.8 / 10X 2H10 6XE16.8//)
503 FORMAT(19H ASSUMED CONDITIONS / 3H TF 4X E16.8 / 10X 4PSI11 6X E16.8
110X 4HPSI2 6X E16.8 / 5H PSI7 2X E16.8 / 10X 4HPSI3 6X E16.8 / 10X
24HPSI4 6X E16.8 / 33X 4HPSI15 6X E16.8 / 10X 4HPSI6 6X E16.8//)
504 FORMAT(28H COMPUTED INITIAL CONDITIONS / 3H XR 4X E16.8 / 10X 6HRX DO
1T 4X E16.8 / 3H YR 4X E16.8 / 10X 6HYR DOT 4X E16.8 / 3H ZR 4X E16.8
210X 6HXR DOT 4X E16.8 / / 4H RSX 3X E16.8 / 10X 7HRSX DOT 3X E16.8 / 10X
110X 4HPSI2 6X E16.8 / 5H PSI7 2X E16.8 / 10X 4HPSI3 6X E16.8 / 10X
24HPSI4 6X E16.8 / 33X 4HPSI15 6X E16.8 / 10X 4HPSI6 6X E16.8//)
505 FORMAT(9H PARTIALS / 8H X/ALPHA/ 6(6E18.8/) //
110H PSI/ALPHA/ 6(6E18.8/) / 11H PSI/ALPHA/ 6(6E18.8/) //)
800 FORMAT(5H TIME 2X E16.8 / 10X 4HRLEL 5X E16.8 / 10X 6HHRTHUST 3X E16.8
15H MASS 2X E16.8 / 10X 4HRELV 5X E16.8 / 10X 2HRV 7X E16.8//)
804 FORMAT(15H PSI1 2X E16.8 / 10X 4HPSI2 5X E16.8 / 10X 4HPSI15 5X E16.8
15H PS13 2X E16.8 / 10X 4HPSI4 5X E16.8 / 5H PS15 2X E16.8 / 10X 4HPSI6
25X E16.8//)
805 FORMAT(9H PARTIALS / 8H X/ALPHA/ 6(6E18.8/) / 110H PSI/ALPHA/ 6(6E18.8/) / 11H PSI/ALPHA/ 6(6E18.8/) /)
866 FORMAT(3H XR 4X E16.8 / 10X 6HXR DOT 3X E16.8 / 10X 7HDXR DOT 2X
1E16.8 / 10X 2HUX 8X E16.8 / 3H YR 4X E16.8 / 10X 6HYR DOT 3X E16.8 / 10X
27HDYR DOT 2X E16.8 / 10X 2HU3 8X E16.8 / 3H ZR 4X E16.8 / 10X 6HUXR DOT
33X E16.8 / 10X 7HDXR DOT 2X E16.8 / 10X 2HU3 8X E16.8//)
808 FORMAT(4H RSX 3X E16.8 / 10X 7HRSX DOT 2X E16.8 / 10X 3HRVX 6X E16.8
110X 7HRVX DOT 3X E16.8 / 4H RSY 3X E16.8 / 10X 7HRSY DOT 3X E16.8 / 10X
23HRVY 6X E16.8 / 10X 7HRVY DOT 3X E16.8 / 4H RSZ 3X E16.8 / 10X
37HRVZ DOT 2X E16.8 / 10X 7HRVZ DOT 3X E16.8//)
810 FORMAT(5H TIME 2X E16.8//)
811 FORMAT(5H PSI1 2X E16.8 / 10X 4HPSI2 5X E16.8 / 10X 4HPSI15 5X E16.8
110X 2HU3 8X E16.8 / 5H PS13 2X E16.8 / 10X 4HPSI4 5X E16.8 / 45X
22HU3 8X E16.8 / 5H PS15 2X E16.8 / 10X 4HPSI6 5X E16.8 / 45X 2HU3 8X
3E16.8//)
APPENDIX B

900 FORMAT(24H LOCAL TRUNCATION ERRORS//2H X/6E18.8//3H X7/E18.8//
14H PSI/6E18.8//5H PSI7/E18.8//11H P(X/ALPHA)/6(6E18.8//)
213H P(PSI/ALPHA)/6(6E18.8//)/14H P(PSI7/ALPHA)/
36E18.8//)

C
4900 FORMAT(/)
901 FORMAT(40H MAX NO. OF ITERATIONS HAS BEEN REACHED)
C
903 FORMAT(9H E(ALPHA) 2X E16.8//)
C
START
C
READ INPUT
C
1 READ(5,100) NO,SOMEGBETA,CTF,PHIVO,THETVO,RV,DRVX0,DRVY0,DRVZ0,
1PHIO,THETA0,10,RS,VAR(1),VAR(8),MU,(VAR(1),I=1,9,15)
2 READ(5,200)CI,SPEC,IPRINT,IEROR,IMAT,LAMBDA,CRT,MAXIT,
1B(I),I=1,7
C
COMPUTE CONSTANT TERMS AND INITIAL CONDITIONS
C
COMEGB=MU/RS**3
COMEGB=SQRT(COMEGB)
SOMEGB=SOMEGB**2
SIO=SIN(IO)
STO=SIN(THETAO)
CIO=COS(IO)
CTO=COS(THETAO)
C
T MATRIX
C
XT1(1,1)=CTO
XT1(1,2)=-CIO*STO
XT1(1,3)=SIO*STO
XT1(2,1)=STO
XT1(2,2)=CIO*CTO
XT1(2,3)=-SIO*CTO
XT1(3,1)=0.0
XT1(3,2)=SIO
XT1(3,3)=CIO
C
INITIALIZATION
C
RVX=RV*COS(THETVO)*COS(PHIVO)
RVY=RV*COS(THETVO)*SIN(PHIVO)
RVZ=RV*SIN(THETVO)
APPENDIX B

RVX0=RVX  
RVY0=RVY  
RVZ0=RVZ  
CALL COMP(0,0)  
VAR(2)=RVX-RSV(1)  
VAR(3)=DRVX0-DRSV(1)  
VAR(4)=RVY-RSV(2)  
VAR(5)=DRVY0-DRSV(2)  
VAR(6)=RVZ-RSV(3)  
VAR(7)=DRVZ0-DRSV(3)  
DER(8)=-BETA/C  
WRITE(6,500) NO,BETA,RS,SEMEG,LAMBDA,C,RV,COME,MU  
WRITE(6,501) (B(I),I=1,7)  
WRITE(6,502) VAR(1),PHIV0,DRVX0,PHI0,VAR(8),THETV0,DRVY0,THETA0,  
1DER(8),DRVZO*10  
WRITE(6,503) TF,VAR(9),VAR(10),VAR(15),VAR(11),VAR(12),VAR(13),  
1VAR(14)  
WRITE(6,504) (VAR(I),I=2:7),RSV(1),DRSV(1),RVX*RSV(2),DRSV(2),RVY,  
2RSV(3),DRSV(3),RVZ,CI  

C INITIALIZATION FOR INTEGRATION ROUTINE  

DO 17 I=1,15  
17 SAVE(I)=VAR(I)  
DO 14 I=16,93  
14 VAR(I)=SAVE(I)  

C TEMPCI=CI  
TEMPSP=SPEC  
IKOUNT=0  
10 CALL INT2(I,N,NT,CI,SPEC,CIMAX,IERR,VAR,CUVAR,DER,ELE1,ELE2,ELT,  
1ERRVAL,DERSUB,CHSUB,ITEXT)  

C RETURN FROM INTEGRATION ROUTINE  
11 IF (((ABS(T1-TF) .LE. 1.0E-06) .OR. FIRST .OR.  
1((SPEC .LT. 1.0E10) .AND. (SPEC .NE. 0.0)) ) ) GO TO 16  
GO TO 20  

C
APPENDIX B

C WRITE OUTPUT

C 16 RELD=SQRT(VAR(2)**2+VAR(4)**2+VAR(6)**2)
     RELV=SQRT(VAR(3)**2+VAR(5)**2+VAR(7)**2)
     RVX=VAR(2)+RSV(1)
     RVY=VAR(4)+RSV(2)
     RVZ=VAR(6)+RSV(3)
     RVMAG=SQRT(RVX**2+RVY**2+RVZ**2)
     DRVX=VAR(3)+DRSV(1)
     DRVY=VAR(5)+DRSV(2)
     DRVZ=VAR(7)+DRSV(3)

C U4=BETA
     UX=VAR(10)/EN1
     UY=VAR(12)/EN1
     UZ=VAR(14)/EN1
     IF (FIRST) GO TO 60
     WRITE(6,800) VAR(1)*RELD+U4*VAR(8)*RELV*RVMAG
     WRITE(6,804) VAR(9)*VAR(10)+VAR(15)*VAR(11)+VAR(12)*
     1VAR(13)+VAR(14)
     WRITE(6,806) VAR(2)*VAR(3)*DER(3)+UX*VAR(4)+VAR(5)+DER(5)+UY,
     1VAR(6)*VAR(7)+DER(7)+UZ
     WRITE(6,808) RSV(1)+DRSV(1)+RVX+DRVX+RSV(2)+DRSV(2)+RVY+DRVY,
     1RSV(3)+DRSV(3)+RVZ+DRVZ
     GO TO 61

60 WRITE(6,810) VAR(1)
     WRITE(6,811) VAR(9)+VAR(10)+VAR(15)+UX+VAR(11)+VAR(12)+UY,
     1VAR(13)+VAR(14)+UZ

61 IF (PRINT *NE. 1) GO TO 50
     WRITE(6,805) ((PX(I+J),J=1,6),I=1,6),
     1((PPS(I+J),J=1,6),I=1,6)*(PPS7(I+J),I=1,6)

C 50 IF(IEROR *EQ. 1)GO TO 9
     GO TO 49

9 WRITE(6,900) (ERRVAL(I),I=1,14)*((ERX(I+J),J=1,6),I=1,6),
     1(ERPS(I+J),J=1,6),I=1,6)*(ERPS7(I),I=1,6)
49 WRITE(6,4900)
20 IF (FIRST) FIRST=.FALSE.
     IF (ABS(TI-TF) .LE. 1.0E-06) GO TO 13
     IF ((T1+CI) .LE. TF) GO TO 10
     CI=TF-T1
     II=0
     SPEC=0*0
     GO TO 10
APPENDIX B

13 IKOUNT=IKOUNT+1
  IF(IKOUNT .GT. MAXIT) GO TO 15
  COMPUTE (E+B**2)/2

  EDP=0.0
  DO 19 I=2,7
  19 EDP=EDP+B(I-1)*VAR(I)**2
     EDP=.5*(EDP+B(7)*VAR(15)**2)
     WRITE(6,903) EDP
     IF (EDP .LE. CRIT) GO TO 1
     CALL ITERAT
     DO 18 I=1,93
  18 VAR(I)=SAVE(I)
  GO TO 1010

15 WRITE(6,901)
  GO TO 1

ENC

SUBROUTINE DERSUB

COMMON /SPACE/
  VAR (93), CUVAR (93), SAVE (93), C (93), MAXIT (93), IMAT (93)

  E, PE (92), PEMAT (92), PEVEC (92), ERRVAL (92), SOME (92)

  DER (93), ELE2 (92), DPI (92), TF (93), ELE1 (93)

  RSV (93), DP2 (93), CI (93), ELE2 (92), DRSV (93)

  F2 (93), E1 (93), II (93), SOME (92), COMEG (92)

  F3 (93), E2 (93), N (93), COMEG (92), EN2 (93)

  Q1 (93), E3 (93), TEMPCI (93), SI0 (93), PHI10 (93), Q2 (93)

  E4 (93), SPEC (93), STO (93), LAMBD (93), TEMPT (93), Q3 (93)

  T2 (93), TEMSP (93), CI0 (93), CRIT (93), EN1 (93), IKOUNT (93)

  RVX0 (93), RVY0 (93), RVZ0 (93), DRVX0 (93), DRVY0 (93), DRVZ0 (93)

  T4 (93), MU (93), CT0 (93), B (93), XT1 (93)

REAL LAMBDA, MU

DIMENSION VAR(93), CUVAR(93), DER(93)
APPENDIX B

EQUIVALENCE
1  (VAR(16) * PX(1,1))  (VAR(52) * PPSI(1,1))
2  (VAR(88) * PPSI(1,1))  (CUVAR(16) * CUPX(1,1))
3  (CUVAR(52) * CUPSI(1,1))  (CUVAR(88) * CUPSI(1,1))
4  (DER(16) * DRPX(1,1))  (DER(52) * DRPSI(1,1))
5  (DER(88) * DRPSI(1,1))  (ERRVAL(15) * ERX(1,1))
6  (ERRVAL(51) * ERPSI(1,1))  (ERRVAL(87) * ERPSI(1,1))

EQUIVALENCE (CUVAR(1), T1)

TCOMP= TJ-SAVE(1)
CALL COMP(TCOMP)
Q1= CUVAR(2) + RSV(1)
Q2= CUVAR(4) + RSV(2)
Q3= CUVAR(6) + RSV(3)
EN1= SQRT(CUVAR(10)**2 + CUVAR(12)**2 + CUVAR(14)**2)
EN2= SQRT(Q1**2 + Q2**2 + Q3**2)
DP1= CUVAR(2)* CUVAR(10) + CUVAR(4)* CUVAR(12) + CUVAR(6)* CUVAR(14)
DP2= RSV(1)* CUVAR(10) + RSV(2)* CUVAR(12) + RSV(3)* CUVAR(14)
F2= (COMEGS* RS**3)/EN2**3
F3= (3.0* F2/EN2**2)* (DP1 + DP2)
E1= EN1/ (CUVAR(8) * EN1)
E2= E1 / CUVAR(8)
E3= E1 / EN1**2
E4= EN1 / CUVAR(8)**2
T2= (3.0* F2)/ EN2**2
TA= 5.0* F3/ EN2**2

STATE VARIABLES

DER(9)= F2* CUVAR(10) - F3* Q1 - COMEGS* CUVAR(10)
DER(10)= - CUVAR(9) - 2.0* COMEGS* CUVAR(12)
DER(11)= F2* CUVAR(12) - F3* Q2 - COMEGS* CUVAR(12)
DER(12)= - 2.0* COMEGS* CUVAR(10) - CUVAR(11)
DER(13)= F2* CUVAR(14) - F3* Q3
DER(14)= - CUVAR(13)
F1= BETA / (CUVAR(8) * EN1)
S1 DER(2)= CUVAR(3)
DER(3)= F1* CUVAR(10) - F2* Q1 + COMEGS* RSV(1) + COMEGS* CUVAR(2) + 2.0* COMEGS* CUVAR(5)
DER(4)= CUVAR(5)
DER(5)= F1* CUVAR(12) - F2* Q2 + COMEGS* RSV(2) - 2.0* COMEGS* CUVAR(3) + COMEGS*
APPENDIX B

\[ \begin{align*}
&1\text{CUVAR}(4) \\
&\text{DER}(6) = \text{CUVAR}(7) \\
&\text{DER}(7) = F1*\text{CUVAR}(14) - F2*Q3 + \text{SOMEWSRSV}(3) \\
&\text{DER}(8) = -BETA/C \\
&\text{DER}(15) = (BETA*E1)/\text{CUVAR}(8)**2 \\
\end{align*} \]

**PARTIALS**

\[ \begin{align*}
&\text{DPX}(1,1) = \text{CUPX}(2,1) \\
&\text{DPX}(2,1) = BETA* (E1 - E3*\text{CUVAR}(10)**2) * \text{CUPSI}(2,1) - E3*\text{CUVAR}(10) \\
&\text{CUPSI}(12)*\text{CUPSI}(4,1) - E3*\text{CUVAR}(10)*\text{CUVAR}(14)*\text{CUPSI}(6,1)) + T2*Q1**2 - 2F2 + \text{SOMEWSRSV})*\text{CUPX}(1,1) + T2*Q1*Q2*\text{CUPX}(3,1) + 2*0*\text{SOMEWSRSV})*\text{CUPX}(4,1) + T2*Q1*3Q3*\text{CUPX}(5,1) \\
&\text{DPX}(3,1) = \text{CUPX}(4,1) \\
&\text{DPX}(4,1) = BETA* (-E3*\text{CUVAR}(10)*\text{CUVAR}(12)*\text{CUPSI}(2,1) + (E1 - E3* \text{CUPSI}(12)**2)*\text{CUPSI}(4,1) - E3*\text{CUVAR}(12)*\text{CUVAR}(14)*\text{CUPSI}(6,1)) + T2*Q1*2Q2*\text{CUPX}(1,1) - 2*0*\text{SOMEWSRSV})*\text{CUPX}(2,1) + (T2*Q2**2 - F2 + \text{SOMEWSRSV})*\text{CUPX}(3,1) + 3T2*Q2*Q3*\text{CUPX}(5,1) \\
&\text{DPX}(5,1) = \text{CUPX}(6,1) \\
&\text{DPX}(6,1) = BETA* (-E3*\text{CUVAR}(10)*\text{CUVAR}(14)*\text{CUPSI}(2,1) - E3* \text{CUPSI}(12)*\text{CUVAR}(14)*\text{CUPSI}(4,1) + (E1 - E3*\text{CUVAR}(14)**2)*\text{CUPSI}(6,1)) + 2T2*Q1*Q3*\text{CUPX}(1,1) + T2*Q2*Q3*\text{CUPX}(3,1) + (T2*Q3**2 - F2)*\text{CUPX}(5,1) \\
&\text{DPX}(7,1) = (F2 - T2*Q1**2 - \text{SOMEWSRSV})*\text{CUPSI}(2,1) - T2*Q1*Q2*\text{CUPSI}(4,1) - T2*103*Q1*\text{CUPSI}(6,1) + (L2*0*\text{CUVAR}(10)*\text{Q1} - F3 + T4*Q1**2)*\text{CUPX}(1,1) + (-T2 \\
&2*\text{CUVAR}(10)*\text{Q2} + \text{CUVAR}(12)*Q1 + T4*Q1*Q2)*\text{CUPX}(3,1) + (-T2*(\text{CUVAR}(10)*3Q3 + \text{CUVAR}(14)*Q1) + T4*Q1*Q3)*\text{CUPX}(5,1) \\
&\text{DPX}(10,1) = -\text{CUPSI}(1,1) + 2*0*\text{SOMEWSRSV})*\text{CUPSI}(4,1) \\
&\text{DPX}(9,1) = -T2*Q1*Q2*\text{CUPSI}(2,1) + (F2 - T2*Q2**2 - \text{SOMEWSRSV})*\text{CUPSI}(4,1) - \\
&1T2*Q2*Q3*\text{CUPSI}(6,1) + (-T2*(\text{CUVAR}(12)*Q1 + \text{CUVAR}(10)*Q2) + T4*Q1*Q2)* \\
&2\text{CUPX}(1,1) + (-2*0*\text{T2*CUVAR}(12)*Q2 - F3 + T4*Q2**2)*\text{CUPX}(3,1) + (-T2* \\
&3 \text{CUVAR}(12)*Q3 + \text{CUVAR}(14)*Q2) + T4*Q2*Q3)*\text{CUPX}(5,1) \\
&\text{DPX}(4,1) = -2*0*\text{SOMEWSRSV})*\text{CUPSI}(2,1) - \text{CUPSI}(3,1) \\
&\text{DPX}(5,1) = -T2*Q1*Q3*\text{CUPSI}(2,1) - T2*Q2*Q3*\text{CUPSI}(4,1) + (F2 - T2*Q3**2)*
APPENDIX B

1CUPSI(6,I)+(-T2*(CWAR(14)*Q1+CUVAR(10)*Q3)+T4*Q1*Q3)*CUPX(1,I)+
2(-T2*(CWAR(14)*Q2+CUVAR(12)*Q3)+T4*Q2*Q3)*CUPX(3,I)+(-2*0*T2*
3CWAR(14)*Q3-F3+14*Q3**2)*CUPX(5,I)

C
DRPSI(6,I)=-CUPSI(5,I)
C
DRPSI7(I)=BETA*E2*(CUVAR(10)*CUPSI(2,I)+CUVAR(12)*CUPSI(4,I)
1+CUVAR(14)*CUPSI(6,I))
C
60 CONTINUE
RETURN
END
SUBROUTINE CHSUB
COMMON /SPACE/
VAR *CWAR
SAVE *C_MAXIT *IMAT *
E PEMAT PEVEC ERRVAL SOMEG *
DER *BETA DP1 TF ELE1 *
RSV *F1 DP2 CI ELE2 DRSV *
F2 I1 SOMEGS COMEGS IPRINT *
F3 E2 N EN2 KOUNT *
Q1 E3 TEMP CI SIO PH10 Q2 *
E4 SPEC STO LAMBDA TEMPT Q3 *
T2 TEMPS CI0 CRIT EN1 IKOUNT *
RVX0 RVY0 RVZ0 DRVXO DRVY0 DRVZ0 *
T4 MU CT0 B XT1 *

C
REAL LAMBDA,MU
C
DIMENSION
VAR(93) *CUVAR(93) *DER(93) *
ELE1(92) *ELE2(92) *ERRVAL(92) *
RSV(3) *DRSV(3) *PX(6,6) *
PSS1(6,6) *PSS1(6) *CUPX(6,6) *
CUPSI(6,6) *CUPSI1(6) *DRPX(6,6) *
DRPSI(6,6) *DRPSI1(6) *ERX(6,6) *
ERPSI(6,6) *ERPSI1(6) *E(7) *
PE(7,7) *PEMAT(7,7) *PEVEC(7,1) *
B(7) *SAVE(93) *XT1(3,3) *

C
EQUIVALENCE
VAR(16,1)*PX(1,1)) *VAR(52,1)*Psis(1,1)) *
(VAR(88,1)*PSS1(1,1)) *CUVAR(16,1)*CUPX(1,1)) *
(CUVAR(52,1)*CUPSI(1,1)) *(CUVAR(88,1)*CUPSI1(1,1))
APPENDIX B:

4 (DER(16)*DRPX(1,1)) (DER(52)*DRPSI(1,1))
5 (DER(88)*DRPSI7(1)) (ERRVAL(15)*ERX(1,1))
6 (ERRVAL(51)*ERPSI(1,1)) (ERRVAL(87)*ERPSI7(1))

C EQUIVALENCE (CUVAR(1)*TI)
C IF ((TI+CI) *LE* TF) GO TO 78
II=3
78 RETURN
END
SUBROUTINE COMP(DT)
COMMON /SPACE/
1 VAR *CUVAR *SAVE *C *MAXIT *IMAT *
2 E *PE *PEMAT *PEVEC *ERRVAL *SOMEQ *
3 DER *RS *BETA *DP1 *TF *ELE1 *
4 RSV *F1 *DP2 *CI *ELE2 *DRSV *
5 F2 *E1 *II *SOMEQS *COMEGS *IPRINT *
6 F3 *E2 *N *COMEG *EN2 *KOUNT *
7 Q1 *E3 *TEMPC1 *S10 *PHI0 *Q2 *
8 E4 *SPEC *ST0 *LAMBDA *TEMPT *Q3 *
9 T2 *TEMPSP *C10 *CRIT *EN1 *KOUNT *
1 VAR *CUVAR *SAVE *C *MAXIT *IMAT *
2 E *PE *PEMAT *PEVEC *ERRVAL *SOMEQ *
3 DER *RS *BETA *DP1 *TF *ELE1 *
4 RSV *F1 *DP2 *CI *ELE2 *DRSV *
5 F2 *E1 *II *SOMEQS *COMEGS *IPRINT *
6 F3 *E2 *N *COMEG *EN2 *KOUNT *
7 Q1 *E3 *TEMPC1 *S10 *PHI0 *Q2 *
8 E4 *SPEC *ST0 *LAMBDA *TEMPT *Q3 *
9 T2 *TEMPSP *C10 *CRIT *EN1 *KOUNT *
1 VAR *CUVAR *SAVE *C *MAXIT *IMAT *
2 E *PE *PEMAT *PEVEC *ERRVAL *SOMEQ *
3 DER *RS *BETA *DP1 *TF *ELE1 *
4 RSV *F1 *DP2 *CI *ELE2 *DRSV *
5 F2 *E1 *II *SOMEQS *COMEGS *IPRINT *
6 F3 *E2 *N *COMEG *EN2 *KOUNT *
7 Q1 *E3 *TEMPC1 *S10 *PHI0 *Q2 *
8 E4 *SPEC *ST0 *LAMBDA *TEMPT *Q3 *
9 T2 *TEMPSP *C10 *CRIT *EN1 *KOUNT *
1 REAL LAMBDA,MU
C C DIMENSION
1 VAR(93) *CUVAR(93) *DER(93) *
2 ELE1(92) *ELE2(92) *ERRVAL(92) *
3 RSV(3) *DRSV(3) *PX(6,6) *
4 PPSI(6,6) *PPSI7(6) *CUPX(6,6) *
5 CUPSI(6,6) *CUPSI7(6) *DRPX(6,6) *
6 DRPSI(6,6) *DRPSI7(6) *ERX(6,6) *
7 ERPSI(6,6) *ERPSI7(6) *E(7) *
8 PE(7,7) *PEMAT(7,7) *PEVEC(7,1) *
9 B(7) *SAVE(93) *XT1(3,3) *
C C EQUIVALENCE
1 (VAR(16)*PX(1,1)) (VAR(52)*PPSI(1,1)) *
2 (VAR(88)*PPSI7(1)) (CUVAR(16)*CUPX(1,1)) *
3 (CUVAR(52)*CUPSI(1,1)) (CUVAR(88)*CUPSI7(1)) *
4 (DER(16)*DRPX(1,1)) (DER(52)*DRPSI(1,1)) *
5 (DER(88)*DRPSI7(1)) (ERRVAL(15)*ERX(1,1)) *
APPENDIX B

6 (ERRVAL(SL),ERPSI(1+1)),(ERRVAL(87),ERPSI7(1))
DIMENSION XTO(3),XT2(3,3),XT3(3),XTD2(3,3),RSXYZ(3),TDRS(3),
1RSXYZD(3),TRSDOT(3)
PHIOMT=PHIO-COME*DT

T0 MATRIX

XTO(1)=RS*COS(PHIOMT)
XTO(2)=RS*SIN(PHIOMT)
XTO(3)=0*0

T2 MATRIX

SOT=SIN(SOME*DT)
COT=COS(SOME*DT)
XT2(1,1)=COT
XT2(1,2)=SOT
XT2(1,3)=0*0
XT2(2,1)=-SOT
XT2(2,2)=COT
XT2(2,3)=0*0
XT2(3,1)=0*0
XT2(3,2)=0*0
XT2(3,3)=1*0

T2 DOT MATRIX

XTD2(1,1)=-SOME*SOT
XTD2(1,2)=-SOME*COT
XTD2(1,3)=0*0
XTD2(2,1)=-SOME*COT
XTD2(2,2)=-SOME*SOT
XTD2(2,3)=0*0
XTD2(3,1)=0*0
XTD2(3,2)=0*0
XTD2(3,3)=0*0

T3 MATRIX

XT3(1)=COME*XTO(2)
XT3(2)=-COME*XTO(1)
XT3(3)=0*0

T1T0 MATRIX
APPENDIX B

DO 100 I=1,3
   RSXYZ(I)=0.0
DO 100 J=1,3
100 RSXYZ(I)=RSXYZ(I)+XT1(I,J)*XT0(J)

C C RSV MATRIX
C
DO 101 I=1,3
   RSV(I)=0.0
DO 101 J=1,3
101 RSV(I)=RSV(I)+XT2(I,J)*RSXYZ(J)
DO 102 I=1,3
   TDRS(I)=0.0
DO 102 J=1,3
102 TDRS(I)=TDRS(I)+XTD2(I,J)*RSXYZ(J)
DO 103 I=1,3
   RSXYZD(I)=0.0
DO 103 J=1,3
103 RSXYZD(I)=RSXYZD(I)+XT1(I,J)*XT3(J)
DO 104 I=1,3
   TRSDOT(I)=0.0
DO 104 J=1,3
104 TRSDOT(I)=TRSDOT(I)+XT2(I,J)*RSXYZD(J)

C C RSV DOT MATRIX
C
DO 105 I=1,3
105 DRSV(I)=TDRS(I)+TRSDOT(I)
RETURN
END

SUBROUTINE ITERAT
COMMON /SPACE/
   1 VAR CUVAR SAVE C MAXIT IMAT *
   2 E PE PEMAT PEVEC ERRVAL SOMEQ *
   3 DER RS BETA DP1 TF ELE1 *
   4 RSV F1 DP2 CI ELE2 DRSV *
   5 F2 E1 II SOMEQS COMEGS IPRINT *
   6 F3 E2 N COMEG EN2 KOUNT *
   7 G1 E3 TEMPCI SIO PH10 Q2 *
   8 E4 SPEC STO LAMBD A TEMPT Q3 *
   9 T2 TEMPS P C10 CRIT EN1 KOUNT *
  1 RVXO RY0 RVZO DRVXO DRVY0 DRVZO *
  2 T4 MU CTO B XT1 *
C
   REAL LAMBDA,MU
APPENDIX B

DIMENSION
1 VAR(93) CUVAR(93) DER(93)
2 ELE1(92) ELE2(92) ERRVAL(92)
3 RSV(3) DRSV(3) PX(6,6)
4 PPSI(6,6) PPSI7(6) CUPX(6,6)
5 CUPSI(6,6) CUPSI7(6) DRPX(6,6)
6 DRPSI(6,6) DRPSI7(6) ERX(6,6)
7 ERPSI(6,6) ERPSI7(6) E(7)
8 PE(7,7) PEMAT(7,7) PEVEC(7,1)
9 B(7) SAVE(93) XTI(3,3)

EQUIVALENCE
1 (VAR(16)*PX(1,1)) (VAR(52)*PPSI(1,1))
2 (VAR(88)*PPSI7(1)) (CUVAR(16)*CUPX(1,1))
3 (CUVAR(52)*CUPSI(1,1)) (CUVAR(88)*CUPSI7(1))
4 (DER(16)*DRPX(1,1)) (DER(52)*DRPSI(1,1))
5 (DER(88)*DRPSI7(1)) (ERRVAL(15)*ERX(1,1))
6 (ERRVAL(51)*ERPSI(1,1)) (ERRVAL(87)*ERPSI7(1))

DIMENSION IPIVOT(7), INDEX(7,2)

DIMENSION SAVMAT(7,7), UNIT(7,7)

FORM MATRIX OF PARTIALS

DO 25 I=1,6
DO 25 J=1,6
PE(I,J)=PX(I,J)
25 CONTINUE

J=1
DO 26 I=1,5,2
J=J+2
PE(I,7)=VAR(J)
PE(I+1,7)=DER(J)
26 CONTINUE
DO 27 I=1,6
27 PE(7,I)=PPSI7(I)

PE(7,7)=DER(15)
FORM E VECTOR

DO 38 I=1,6
38 E(I)=VAR(I+1)
E(7)=VAR(15)

T

FORM PE *B*PE+LAMBDA*I MATRIX

DO 40 I=1,7
DO 40 J=1,7
PEMAT(I,J)=0.0
DO 40 K=1,7
PEMAT(I,J)=PEMAT(I,J)+PE(K,I)*B(K)*PE(K,J)
40 CONTINUE

DO 41 I=1,7
41 PEMAT(I,I)=PEMAT(I,I)+LAMBDA

SOLVE THE MATRIX EQ (PE *B*PE+LAMBDA*I)DELTA=PE *B*E

DO 50 I=1,7
PEVEC(I)=0.0
DO 50 J=1,7
50 PEVEC(I)=PEVEC(I)-PE(J)*B(J)*E(J)
IF(IMAT .NE. 1) GO TO 13
DO 14 I=1,7
DO 14 J=1,7
14 SAVMAT(I,J)=PEMAT(I,J)
IF(IMAT .EQ. 1) WRITE(6,11) ((PEMAT(I,J),J=1,7),I=1,7)
11 FORMAT(6H PEMAT/7(7E15.7/)//)
13 CALL MATINV(PEMAT,7,PEVEC,1,DETERM,IPIVOT,INDEX,7,ISCALE)

IF(IMAT .EQ. 1) WRITE(6,12) ((PEMAT(I,J),J=1,7),I=1,7)
12 FORMAT(8H INVERSE/7(7E16.7/)//)
IF(IMAT .NE. 1) GO TO 15
DO 16 I=1,7
DO 16 J=1,7
UNIT(I,J)=0.0
DO 16 K=1,7
16 UNIT(I,J)=UNIT(I,J)+SAVMAT(I,K)*PEMAT(K,J)
WRITE(6,17) ((UNIT(I,J),J=1,7),I=1,7)
17 FORMAT(9H IOENTITY/7(7E16.7/)//)
APPENDIX B

15 WRITE(6,10) IKOUNT,PEVEC(7*1),(PEVEC(I*1),I=1,6)
10 FORMAT(34H CORRECTIONS ON INITIAL CONDITIONS,10X,13H ITERATION NO.,
10X,13//9H DELTA TF,4X,E16.8,10X,10H DELTA P
2SI1*2X,E16.8,10X,10H DELTA PS12,2X,E16.8/39X,10H DELTA PS13,2X*
3E16.8*10X,10H DELTA PS14,2X,E16.8/39X,10H DELTA PS15,2X,E16.8*10X,
410H DELTA PS16,2X,E16.8//)

C    TF=TF+PEVEC(7*1)
DO51 I=1,6
   SAVE(I+8)=SAVE(I+8)+PEVEC(I*1)
51 CONTINUE
C
RETURN
END

BLOCK DATA
COMMON /SPACE /

1 VAR  CUVAR  SAVE  C  MAXIT  IMAT *
2 E    PE     PEMAT  PEVEC  ERRVAL  S OMEG *
3 DER  RS     BETA  DP1   TF     ELE1 *
4 RSV  F1     DP2   CI     ELE2   DRSV *
5 F2   E1     II    SOMEGS COMEGS  IPRINT *
6 F3   E2     N     COMEG  EN2    KOUNT *
7 Q1   E3     TEMPCI  S10   PH10   02 *
8 E4   SPEC   STO   LAMBDA TEMPT   Q3 *
9 T2   TEMPS  CI0   CRIT   EN1    IKOUNT *
1 RVX0  RYVO  RVZO  DRVX0  DRYVO  DRVZ0 *
2 T4   MU     CT0   B     XT1 *
C
REAL LAMBDA,MU
C
C
DIMENSION VAR(93), CUVAR(93), DER(93),
       ELE1(92), ELE2(92), ERRVAL(92),
       RSV(3), DRSV(3), PX(6*6),
       PPSI(6*6), CUPS17(6), CUPX(6*6),
       DRPSI(6*6), DRPS17(6), DRPX(6*6),
       ERPSI(6*6), E(7),
       PE(7,7), PEMAT(7,7), PEVEC(7*1),
       B(7), SAVE(93), XT1(3,3)
C
EQUIVALENCE (VAR(16)*PX(1,1))
       (VAR(52)*PPSI(1,1))
APPENDIX B

```
2 (VAR(88) * PPS17(1)) *(CUVAR(16) * CUPX(1*1)) *
3 (CUVAR(52) * CPUS17(1*1)) *(CUVAR(88) * CPUS17(1)) *
4 (DER(16) * DREVX(1*1)) *(DER(52) * DREVX(1*1)) *
5 (DER(88) * DREVX(1*1)) *(DERVAL(15) * ERX(1*1)) *
6 (ERRVAL(51) * ERPS17(1*1)) *(ERRVAL(87) * ERPS17(1)) *

C DATA N/92/
DATA (SAVE(I), I=16*93)/78*0.0/
DATA SAVE(52)/1*0/, SAVE(59)/1*0/, SAVE(66)/1*0/, SAVE(73)/1*0/
ISAVE(80)/1*0/, SAVE(87)/1*0/
END
SUBROUTINE INT(NINT, CI, SPEC, AX, IERR, VAR, CUVAR, DER, ELE, ELE2, ELE3, ERRVAL, DERSUB, CHKSUB, ITEXT)
DIMENSION VAR(93), CUVAR(93)
DIMENSION DER(92), ELE1(92), ELE2(92), ELE3(92), ERRVAL(92)
DIMENSION TEMP, DER1(92), DER2(92), DER3(92)
DIMENSION S1VAR(93)
IF(IIF)1, 1, 2
C INITIALIZATION SECTION
1 IF(C1) 3, 4, 3
1000 FORMAT(11HOCI=O STOP)
CALL EXIT
C SAVE CI
3 H=CI
18 IERR=1
TO=SPEC+VAR(1)
MODE=1
II= I
N1=N+1
DO 5 J=1, N1
CUVAR(J)=VAR(J)
5 CONTINUE
C EVALUATION SECTION HERE
8 CALL DERSUB
   IF(MODE.EQ.1) GO TO 6
   IF(II-3)36, 36, 7
36 CALL CHKSUB
   IF(II.EQ.2) GO TO 1
37 DO 38 J=1, N1
38 VAR(J)=CUVAR(J)
   IF(II-3)6, 7, 7
7 RETURN
6 IF(SPEC) 9, 7, 9
9 DEL=VAR(I)-TO
```
APPENDIX B

```
DELP=DEL*(1+1*0E-6)
IF(ABS(DELP)-ABS(SPEC)) 2*10*10
10 TO=VAR(1)
GO TO 7
2 II=1
IF(MODE=4) II=12
RUNGE-KUTTA
C
11 DO 20 J=2*N1
DER3(J-1)=DER2(J-1)
DER2(J-1)=DER1(J-1)
DER1(J-1)=DER(J)
ELE1(J-1)=DER(J)
CUVAR(J)=0*0D+00
DELT=0*04*ELE1(J-1)*H
S1VAR(J)=VAR(J)
CUVAR(J)=S1VAR(J)+DELT
20 CONTINUE
S1VAR(1)=VAR(1)
CUVAR(1)=S1VAR(1)+0*4*H
CALL DERSUB
IF(II-3)23,23,7
23 CUVAR(1)=S1VAR(1)+0*45573725*H
DO 24 J=2*N1
ELE2(J-1)=DER(J)
DELT=(0*29697761*ELE1(J-1)+0*15875964*ELE2(J-1))*H
CUVAR(J)=S1VAR(J)+DELT
24 CONTINUE
CALL DERSUB
IF(II-3)25,25,7
25 CUVAR(1)=S1VAR(1)+H
DO 26 J=2*N1
TEMP(J-1)=DER(J)
DELT=(0*21810040*ELE1(J-1)-3*05096516*ELE2(J-1)
1+3*83286476*TEMP(J-1))*H
CUVAR(J)=S1VAR(J)+DELT
26 CONTINUE
CALL DERSUB
IF(II-3)27,27,7
27 DH=H
CUVAR(1)=VAR(1)+DH
DO 28 J=2*N1
DOUB=0*17476028*ELE1(J-1)-0*55148066*ELE2(J-1)
1+1*20953560*TEMP(J-1)+0*17118478*DER(J)
CUVAR(J)=VAR(J)+DH*D0UB
28 CONTINUE
```
APPENDIX B

MODE=MODE+1
GO TO 8
C ADAMS-MOULTON
C ADAMS-BASHFORTH PREDICTOR
12 CUVAR(1)=VAR(1)+H
DH=H/24*0
DO 13 J=2,N1
DOUB=55*0*DER(J)-59*0*DER1(J-1)+37*0*DER2(J-1)-9*0*DER3(J-1)
CUVAR(J)=VAR(J)+DH*DOUB
13 CONTINUE
DO 14 J=1,N
DER3(J)=DER2(J)
DER2(J)=DER1(J)
14 DER1(J)=DER(J+1)
CALL DERSUB
IF(II-3)15*15*7
C ADAMS-MOULTON CORRECTOR
15 DO 16 J=2,N1
TEMP=CUVAR(J)
DOUB=9*0*DER(J)+19*0*DER1(J-1)-5*0*DER2(J-1)+DER3(J-1)
CUVAR(J)=VAR(J)+DH*DOUB
16 ERRVAL(J-1)=(TEMP-CUVAR(J))/14*210526
19 GO TO 8
END
SUBROUTINE MATINV(A,N,B,M,DETERM,IPIVOT,INDEX,NMAX,ISCALE)
**********
REVISED 08/01/68
C MATRIX INVERSION WITH ACCOMPANYING SOLUTION OF LINEAR EQUATIONS
C DIMENSION IPIVOT(N),A(NMAX,N),B(NMAX,M),INDEX(NMAX*2)
EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)
C INITIALIZE
C
5 ISCALE=0
6 R1=10*0**100
7 R2=1.0/R1
10 DETERM=1.0
15 DO 20 J=1,N
20 IPIVOT(J)=0
30 DO 550 I=1,N
C SEARCH FOR PIVOT ELEMENT
C
40 AMAX=0.0

52
APPENDIX B

45 DO 105 J=1+N
50 IF (IPIVOT(J)-1) 60, 105, 60
60 DO 100 K=1+N
70 IF (IPIVOT(K)-1) 80, 100, 740
80 IF (ABS(AMAX)-ABS(A(J*K))) 85, 100, 100
85 IROW=J
90 ICOLUMN=K
95 AMAX=A(J*K)
100 CONTINUE
105 CONTINUE
110 IF (AMAX) 110, 106, 110
106 DETERM=0*0
ISCALE=0
GO TO 740
110 IPIVOT(ICOLUMN)=IPIVOT(ICOLUMN)+1

C
C INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
130 IF (IROW-ICOLUMN) 140, 260, 140
140 DETERM=-DETERM
150 DO 200 L=1+N
160 SWAP=A(IROW*L)
170 A(IROW*L)=A(ICOLUMN*L)
200 A(ICOLUMN*L)=SWAP
205 IF(M) 260, 260, 210
210 DO 250 L=1+M
220 SWAP=B(IROW*L)
230 B(IROW*L)=B(ICOLUMN*L)
250 B(ICOLUMN*L)=SWAP
260 INDEX(I*1)=IROW
270 INDEX(I*2)=ICOLUMN
310 PIVOT=A(ICOLUMN,ICOLUMN)
1 IF (PIVOT) 1000, 106, 1000

C
C SCALE THE DETERMINANT
C
1000 PIVOT=PIVOT
1005 IF (ABS(DETERM)-R1) 1030, 1010, 1010
1010 DETERM=DETERM/R1
ISCALE=ISCALE+1
1 IF (ABS(DETERM)-R1) 1060, 1020, 1020
1020 DETERM=DETERM/R1
ISCALE=ISCALE+1
GO TO 1060
1030 IF (ABS(DETERM)-R2) 1040, 1040, 1060
APPENDIX B

1040 DETERM = DETERM * R1  
   ISCALE = ISCALE - 1  
   IF (ABS (DETERM) - R2) 1050, 1050, 1060
1050 DETERM = DETERM * R1  
   ISCALE = ISCALE - 1  
1060 IF (ABS (PIVOT) - R1) 1090, 1070, 1070
1070 PIVOT = PIVOT / R1  
   ISCALE = ISCALE + 1  
   IF (ABS (PIVOT) - R1) 320, 1080, 1080
1080 PIVOT = PIVOT / R1  
   ISCALE = ISCALE + 1  
   GO TO 320
1090 IF (ABS (PIVOT) - R2) 2000, 2000, 320
2000 PIVOT = PIVOT * R1  
   ISCALE = ISCALE - 1  
   IF (ABS (PIVOT) - R2) 320, 2010, 2010
2010 PIVOT = PIVOT * R1  
   ISCALE = ISCALE - 1  
320 DETERM = DETERM * PIVOT

   DIVIDE PIVOT ROW BY PIVOT ELEMENT

330 A (ICOLUM, ICOLUM) = 1.0  
340 DO 350 L = 1, N  
350 A (ICOLUM + L) = A (ICOLUM + L) / PIVOT  
355 IF (M) 380, 380, 360
360 DO 370 L = 1, M  
370 B (ICOLUM + L) = B (ICOLUM + L) / PIVOT

   REDUCE NON-PIVOT ROWS

380 DO 550 L = 1, N  
390 IF (L + ICOLUM) 400, 550, 400
400 T = A (L + ICOLUM)  
420 A (L + ICOLUM) = 0.0  
430 DO 450 L = 1, N  
450 A (L + L) = A (L + L) - A (ICOLUM + L) * T  
455 IF (M) 550, 550, 460
460 DO 500 L = 1, M  
500 B (L + L) = B (L + L) - B (ICOLUM + L) * T  
550 CONTINUE

   INTERCHANGE COLUMNS

600 DO 710 I = 1, N
APPENDIX B

610 L=N+1-I
620 IF (INDEX(L*1)-INDEX(L*2)) 630 710 630
630 JROW=INDEX(L*1)
640 JCOLUMN=INDEX(L*2)
650 DO 705 K=1*N
660 SWAP=A(K+JROW)
670 A(K+JROW)=A(K+JCOLUMN)
700 A(K+JCOLUMN)=SWAP
705 CONTINUE
710 CONTINUE
740 RETURN
END
REFERENCES


"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

— NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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