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THE CALCULATION OF ANTENNA RADIATION PATTERNS BY A VECTOR THEORY USING DIGITAL COMPUTERS

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Richard F. Schmidt
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ABSTRACT

This document continues with the application of the field equations to the solution of antenna problems begun in X-525-68-201. The electric and magnetic fields at the reflector boundary after reflection are related to the incident fields via the boundary conditions. It is shown that the evaluation of the scattered radiation pattern can be accomplished in an efficient manner by converting electric field terms to their magnetic equivalents and utilizing only a simple magnetic polarization vector. The history of the wave emanating from the prime-feed source is next written into the formulation explicitly, but without introducing a specific illumination function. A generally-applicable technique for accommodating feed translation and rotation with six degrees of freedom is then developed for numerical integration of the field equations. An appendix containing the subroutines for surface normal and differential area of the set of conic surfaces (paraboloid, hyperboloid, ellipsoid, sphere, and cone) is included so that this document provides the means for evaluating a large class of antenna configurations.

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GLOSSARY OF NOTATION

Symbol	Meaning
∇	vector operator del
r, R	radial distances, subscripted
$\psi(r)$	scalar wave without time dependence
$\bar{n}$	normal to surface or line, a vector
$\underset{\sim}{S}, s$	surface, subscripted
C	contour
ω	angular frequency
$x, y, z; x', y', z'; x'', y'', z''$	Cartesian coordinates for reference frames
k	wave number
$\bar{E}, \bar{H}$	electric and magnetic field vectors, subscripted
σ_i	electrical conductivity, subscripted
μ	magnetic permeability, subscripted
ϵ	inductive capacity, subscripted
$\bar{D}, \bar{B}$	dielectric displacement and magnetic flux density
η	surface charge density
$\bar{K}$	sheet current, subscripted
$d\bar{l}$	differential length, a vector
j	a constant equal to $\sqrt{-1}$ in complex variable theory
$\hat{l}_r, \hat{l}_\theta, \hat{l}_\phi$	unit vectors in the r, θ, ϕ directions, spherical coordinate system

GLOSSARY OF NOTATION (Continued)

Symbol	Meaning
α, β, γ	Euler angles of the rigid body transformation
$\overline{\rho}_\epsilon$	feed translation vector
$G(\Theta, \Phi)$	feed gain function
$\overline{e}_i, \overline{h}_i$	electric and magnetic polarization vectors
σ, ζ	radial and angular variables of the cylindrical coordinate system
F	focal length, and certain defined functions
a, c	parameters of hyperboloids, ellipsoids, spheres, and cones
k_1	curvature of a curve

THE CALCULATION OF ANTENNA RADIATION PATTERNS BY A VECTOR THEORY USING DIGITAL COMPUTERS

INTRODUCTION

The field equations

$$\bar{E}(x', y', z') = -\frac{1}{j\omega\epsilon_0}\frac{1}{4\pi}\oint_c \nabla\psi\, \bar{H}_1 \cdot d\bar{l} - \frac{1}{4\pi}\int_{S_1}\left[j\omega\mu_0(\bar{n} \times \bar{H}_1)\psi + (\bar{n}\cdot\bar{E}_1)\nabla\psi\right]dS$$

and

$$\bar{H}(x', y', z') = -\frac{1}{4\pi}\int_{S_1}(\bar{n} \times \bar{H}_1) \times \nabla\psi\, dS$$

are applicable to various geometrical and physical situations. In specific cases the $\bar{E}_1$ and $\bar{H}_1$ fields must be determined, usually from some radiation source or prime feed removed from the reflector surface. Expressions for the surface normal $\bar{n}$ and the differential area dS can ordinarily be found by standard mathematical techniques, but moderate amounts of analysis are sometimes required. The specification of the boundary edge $d\bar{l}$ is inherent in the definition of the surface. Only physical constants and other terms which are independent of the reflector geometry remain and integration can, in principle, be carried out by means of high-speed digital computers.

Certain general observations can be made prior to the introduction of a specific reflector geometry, and the approach to the overall problem can be organized somewhat to facilitate computation. Electromagnetic boundary conditions provide the transition from the $\bar{E}_1$ and $\bar{H}_1$ fields to the incident fields $\bar{E}_i$ and $\bar{H}_i$. An inspection of the field equations shows that the electric field appears in only one of the four integrals. If the reflecting surface is in the spherical wave zone of the prime feed, which is the usual case, then it is possible to relate $\bar{E}_i$ directly to $\bar{H}_i$ by a simple expression and cast the entire formulation in terms of the incident magnetic field $\bar{H}_i$. A consequence of this organization is the elimination of the unit electric polarization vector in the computations. It will be shown that the unit magnetic polarization vector is also simpler to express and utilize than the corresponding electric vector. The selection of the magnetic vector is particularly significant since the field integrals will be treated

as summations of a large number[1] of weighted differential areas for every pattern field point.

APPLICATION OF THE FIELD EQUATIONS

The transition from the fields $\bar{E}_1$ and $\bar{H}_1$ of the nonconducting medium to the incident fields $\bar{E}_i$ and $\bar{H}_i$ is made by regarding the antenna surface as locally plane so that the usual boundary conditions apply.

$$\bar{n}\cdot(\bar{B}_2-\bar{B}_1) = 0 \qquad \bar{n}\times(\bar{E}_2-\bar{E}_1) = 0$$

$$\bar{n}\cdot(\bar{D}_2-\bar{D}_1) = \eta \qquad \bar{n}\times(\bar{H}_2-\bar{H}_1) = \bar{K}_2 \qquad \Big|\begin{matrix}\sigma_1\to 0\\ \sigma_2\to\infty\end{matrix}$$

The boundary conditions state that the normal components of $\bar{B}$ and the tangential components of $\bar{E}$ are continuous across the specified boundary. The normal components of $\bar{D}$ and the tangential components of $\bar{H}$ are discontinuous, however, for that boundary. Since all fields vanish within the perfect conductor, it follows that the incident and reflected fields combine to satisfy the boundary conditions when $\bar{B}_2 = \bar{H}_2 = \bar{D}_2 = \bar{E}_2 = 0$. Then

$$\bar{H}_1\cdot d\bar{l} = 2\bar{H}_i\cdot d\bar{l},$$

$$\bar{n}\times\bar{H}_1 = 2\bar{n}\times\bar{H}_i,$$

and

$$\bar{n}\cdot\bar{E}_1 = 2\bar{n}\cdot\bar{E}_i$$

in the field equations.

Electric and magnetic fields from the prime feed are related by the expressions[2]

---

[1]The magnetic vector is calculated approximately 10^4 times in computing the field intensity at one point of the radiation pattern of a large S-band antenna.

[2]Reference 1, page 284

$$\bar{\rho}_1 \times \bar{E}_i = \sqrt{\mu_0/\epsilon_0}\,\bar{H}_i .$$

and

$$\bar{E}_i = \sqrt{\mu_0/\epsilon_0}\,\bar{H}_i \times \bar{\rho}_1 ,$$

where $\bar{\rho}_1$ is the direction of propagation at the prime feed. Here $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance of free space, and $\frac{|\bar{E}|}{|\bar{H}|} = Z_0$. It is noted that the spherical-zone fields in the equations for $\bar{E}(x', y', z')$ and $\bar{H}(x', y', z')$ are orthogonal in space, synchronous in time, and also yield the intrinsic impedance of free space.

The field equations can now be rewritten as

$$\bar{E}(x', y', z') = -\frac{1}{j\omega\epsilon_0}\frac{1}{2\pi}\oint_C \nabla\psi \bar{H}_i \cdot d\bar{l} - \frac{1}{2\pi}\int_{S_1}\Big[j\omega\mu_0 (\bar{n} \times \bar{H}_i)\psi - \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} \bar{n} \cdot (\bar{\rho}_1 \times \bar{H}_i)\nabla\psi \Big] dS \tag{1}$$

and

$$\bar{H}(x', y', z') = -\frac{1}{2\pi}\int_{S_1} (\bar{n} \times \bar{H}_i) \times \nabla\psi\, dS, \tag{2}$$

taking $\mu_r = \epsilon_r = 1$ in the space in front of the reflector. At this point in the development the unit magnetic polarization vector is needed, and the history of the wave from the prime feed can be introduced. In addition, the history of the waves radiated from the reflector should be considered before proceeding to a specific reflector geometry.

Assuming that the reflector is in the spherical wave zone of the source, the unit electric and magnetic polarization vectors $\bar{e}_i$ and $\bar{h}_i$ must be orthogonal to $\bar{\rho}_1$ and to each other. The unit electric vector will be assumed[1] to be coplanar

[1] Ref. 2, page 3; Ref. 3, page 3

with an electric source polarization vector $\bar{v}$ as shown (Figure 1). Then

$$\bar{e}_i = \frac{\bar{\rho}_1 \times (\bar{v} \times \bar{\rho}_1)}{|\bar{\rho}_1 \times (\bar{v} \times \bar{\rho}_1)|}$$

since $\bar{e}_i \perp \bar{\rho}_1$ and $\bar{e}_i \perp (\bar{v} \times \bar{\rho}_1)$.

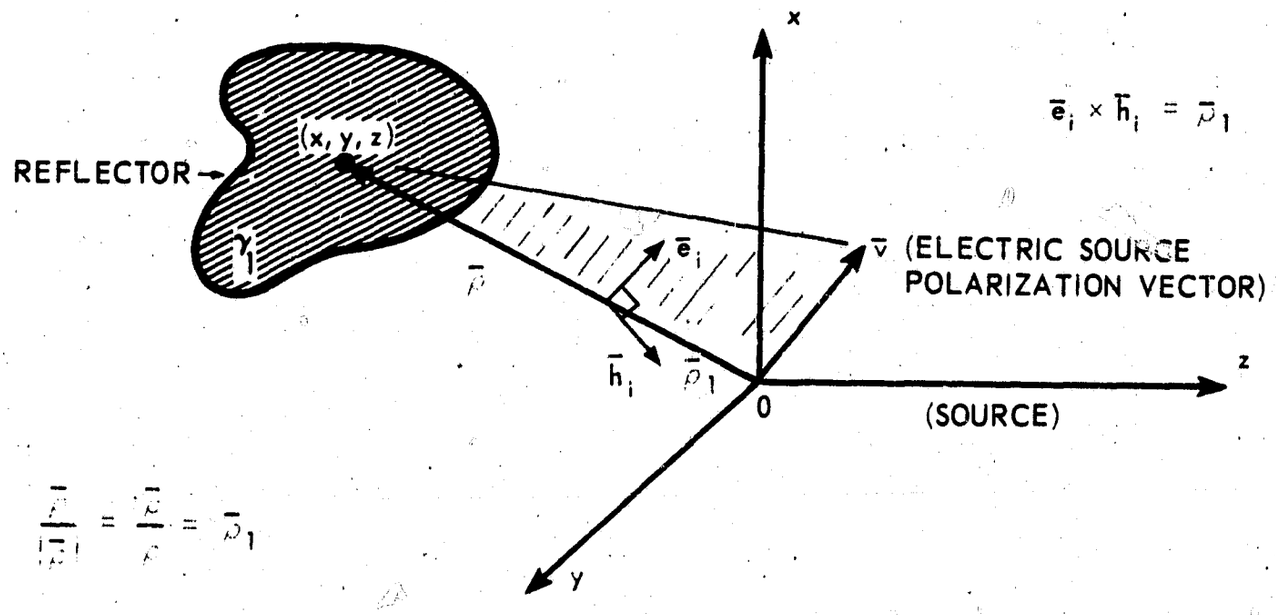

Figure 1. Polarization Vectors

For cases where magnetic fields are encountered, as in $(n \times \bar{H}_i)$ of the field equations, it is necessary to write[1]

$$\bar{n} \times \bar{H}_i = {}^{*}M_{const}\left[\bar{n} \times (\bar{\rho}_1 \times \bar{e}_i)\right]$$

at a given dS on γ_1.

[1] Ref. 4, page 150

Effectively,

$$\bar{h}_i = \frac{\bar{\rho}_1 x \left[\bar{\rho}_1 x (\bar{v} x \bar{\rho}_1) \right]}{\left| \bar{\rho}_1 x \left[\bar{\rho}_1 x (\bar{v} x \bar{\rho}_1) \right] \right|}$$

via $\bar{e}_i$.

But

$$\bar{h}_i = \frac{\bar{\rho}_1 x \bar{v}}{|\bar{\rho}_1 x \bar{v}|}$$

directly by inspection of the physical picture or the reduction

$$\bar{A} \times [\bar{A} \times (\bar{B} \times \bar{A})] = - \bar{A} \times [\bar{A} \times (\bar{A} \times \bar{B})] = \bar{A} \times \bar{B}$$

when $\bar{A}$ is a unit vector and $\bar{B}$ is any vector. The elimination of two of the three vector cross-products is achieved through the use of $\bar{h}_i$ instead of $\bar{e}_i$ for the three field integrals which contain $\bar{H}_i$. The remaining field integral contains $\bar{E}_i$ (or $\bar{\rho}_1 x \bar{H}_i$) and two vector cross-products are necessary, using either $\bar{e}_i$ or $\bar{h}_i$, to account for the polarization from the prime feed. The orientation of the electric source polarization vector $\bar{v}$ is arbitrary and it will be treated as a free vector[1], not necessarily bound to origin 0, in subsequent developments.

The history of the wave from the prime feed is now written into the problem explicitly. It is convenient to begin with the expression[2]

$$\bar{E}_i (\rho, \Theta, \Phi) = \frac{1}{\rho} \left[2 \left(\frac{\mu_0}{\epsilon_0} \right)^{1/2} \frac{P}{4\pi} G_f (\Theta, \Phi) \right]^{1/2} \bar{e}_i (\Theta, \Phi) \, e^{-jk\rho}$$

[1]Ref. 5, page 33. In a sense $\bar{v}$ is "bound" to the feed at point $\bar{\rho}_e$ in later developments.

[2]Ref. 4, page 150

where ρ is the distance from feed to reflector, P is the total radiated power and G_f is the gain function of the feed. Capital letters Θ, Φ are used for the spherical coordinate angles associated with the feed, and θ, ϕ are reserved for the field points on a surface (usually a sphere) of observation. Then

$$\bar{H}_i\ (\rho,\Theta,\Phi) = \frac{1}{\rho}\left[2\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}\frac{P}{4\pi}\,G_f\ (\Theta,\Phi)\right]^{1/2}\bar{h}_i\ (\Theta,\Phi)\ e^{-jk\rho}.$$

The $1/\rho$ space divergence (attenuation) from feed to reflector surface and the accompanying phase change $e^{-jk\rho}$ are now a part of the history of the waves designated $\bar{E}_i$ and $\bar{H}_i$ in the field equations.

The history of waves from the surface of an antenna is contained in the scalar ψ and the vector $\nabla\psi$ of the field equations. Here

$$\psi = \frac{e^{-jkr}}{r}$$

and

$$\nabla\psi = -\left(jk + \frac{1}{r}\right)\psi\,\hat{1}_r.$$

and r is the distance from the reflector surface to the point of the observer. These factors are under the integral sign for the general case. Distance r is variable for the various differential areas of surface γ_1, and vector $\hat{1}_r$ is a free vector whose Cartesian components vary across γ_1 as shown in Figure 2a.

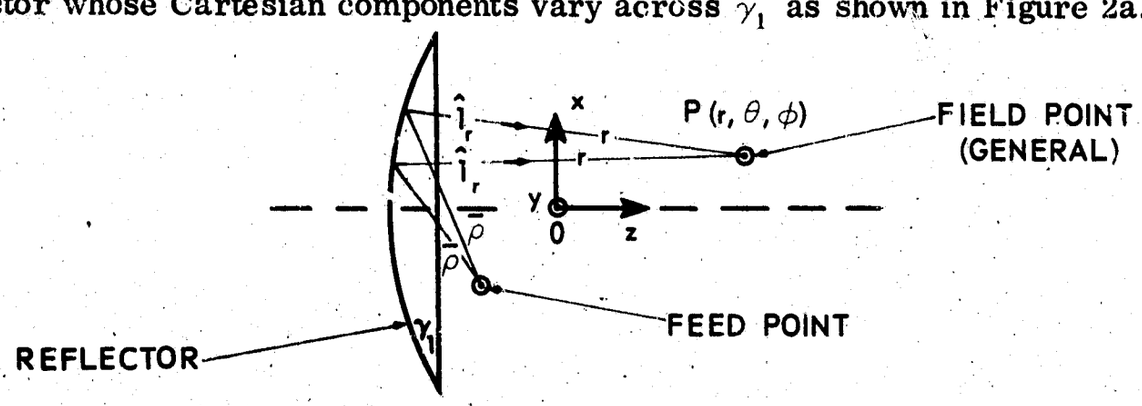

Figure 2a. Near-Field Geometry

If the observer is sufficiently far from the reflector γ_1, a spherical wave having only transverse field components results. This field decays as $1/r$ and the phase delay from the surface γ_1 to the field point can be written as $e^{+jk\bar{\rho}\cdot\hat{1}_r}$ to obtain the diffraction pattern. (See Figure 2b.) In this case the $1/r$ divergence term can be taken outside of the integral sign, and the transverse components of

$$\bar{E}(x', y', z') = -\frac{1}{4\pi}\int_{S_1} j\omega\mu_0(\bar{n}\times\bar{H}_1)\psi dS$$

(Equation 1, page 3)

are obtained by forming

$$E_\theta = \bar{E}(x', y', z') \cdot \hat{1}_\theta$$

and

$$\bar{E}_\phi = E(x', y', z') \cdot \hat{1}_\phi .$$

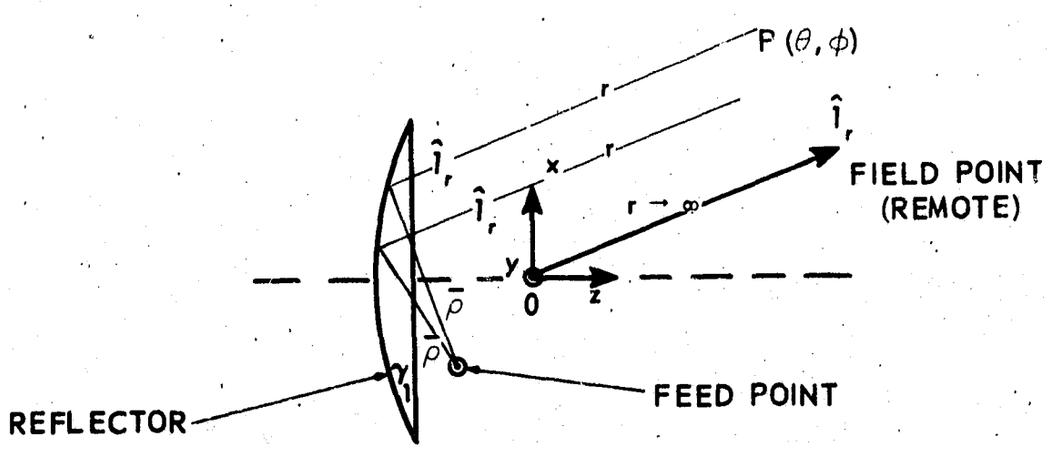

Figure 2b. Far-Field Geometry

FEED TRANSLATION AND ROTATION

The combined translation and rotation of a prime feed provides six additional degrees of freedom for controlling the illumination at a reflector surface. It is assumed that this surface has a parametric representation so that a translation

vector ($\overline{\rho}_\epsilon$) and an orthogonal rotation matrix (A) can be introduced to obtain the correct magnitude, phase, and polarization of the complex vectors $\overline{E}_i$ and $\overline{H}_i$ at each dS of γ_1. Initially the source pattern function is taken to be $F(\Theta,\Phi) = G_1^{1/2}(\Theta,\Phi)$. See Figure 3.

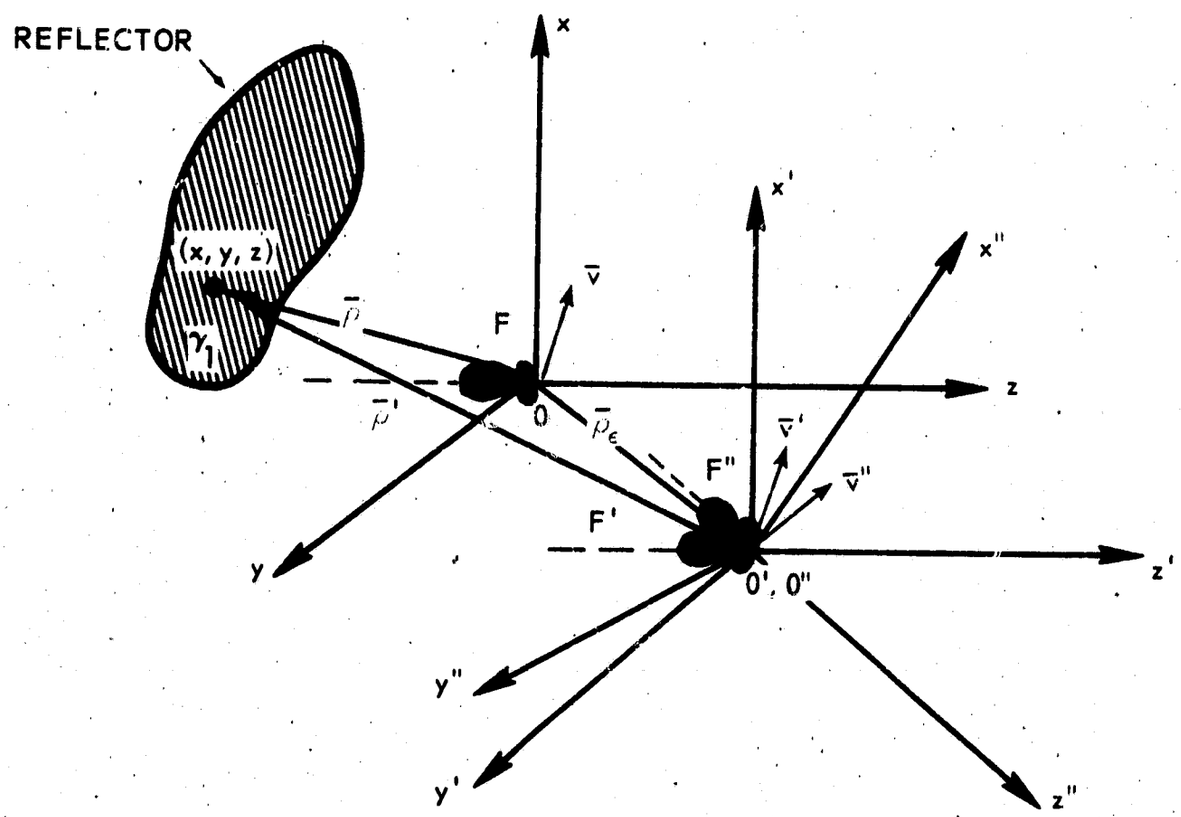

Figure 3. Coordinate Reference Frames

If the prime feed undergoes simple translation, the distance from the feed to a point on the reflector is given by

$$\overline{\rho}' = \overline{\rho}-\overline{\rho}_\epsilon = (x-x_\epsilon)\,\hat{i} + (y-y_\epsilon)\,\hat{j} + (z-z_\epsilon)\,\hat{k} = x'\,\hat{i} + y'\,\hat{j} + z'\,\hat{k}.$$

Then

$$\frac{1}{\rho} \rightarrow \frac{1}{\rho'}, \quad e^{-jk\rho} \rightarrow e^{-jk\rho'}$$

and

$$\bar{h}_i = \frac{\bar{\rho}_1 \times \bar{v}}{|\bar{\rho}_1 \times \bar{v}|} \rightarrow \frac{\bar{\rho}_1' \times \bar{v}}{|\bar{\rho}_1' \times \bar{v}|} .$$

The components of vectors in the coordinate frames with origins 0 and 0' are indistinguishable and the indicated vector cross-product is well-defined. The intensity directed toward a point on surface γ_1 can be found in terms of the displacement $\bar{\rho}_\epsilon$ since F $(\Theta,\Phi) \rightarrow$ F (Θ',Φ'). See Figure 3. Now $|\bar{\rho}'| = (x'^2 + y'^2 + z'^2)^{1/2}$ can be calculated, and the two unknowns Θ', Φ' can be found from a system of three equations.

$$x' = \rho' \sin\Theta' \cos\Phi'$$

$$y' = \rho' \sin\Theta' \sin\Phi'$$

$$z' = \rho' \cos\Theta'$$

If the translation is degenerate, $x_\epsilon = y_\epsilon = z_\epsilon = 0$, and the angles Θ' and Φ' become Θ and Φ, respectively. Then F (Θ, Φ) is evaluated with the ordinary spherical coordinate angles, as before.

If the feed is displaced and disoriented, or simply disoriented, the vector $\bar{\rho}_\epsilon$ and a 3 x 3 rotation matrix[1] A with Euler angles α, β, γ can be used to determine the illumination at the reflector surface. Ordinarily the prime feed function F and the source polarization vector $\bar{v}$ are known in <u>body</u> coordinates, but the reflector surface and the field points are designated in <u>space</u> coordinates. The inverse matrix transformation,

$$\underset{\text{space}}{\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}} = \begin{bmatrix} \\ A^{-1} \\ \\ \end{bmatrix} \begin{bmatrix} v_x'' \\ v_y'' \\ v_z'' \end{bmatrix} = \begin{bmatrix} (\cos\gamma\cos\alpha - \cos\beta\sin\alpha\sin\gamma) & (-\sin\gamma\cos\alpha - \cos\beta\sin\alpha\cos\gamma) & (\sin\beta\sin\alpha) \\ (\cos\gamma\sin\alpha + \cos\beta\cos\alpha\sin\gamma) & (-\sin\gamma\sin\alpha + \cos\beta\cos\alpha\cos\gamma) & (-\sin\beta\cos\alpha) \\ (\sin\beta\sin\gamma) & (\sin\beta\cos\gamma) & (\cos\beta) \end{bmatrix} \underset{\text{body}}{\begin{bmatrix} v_x'' \\ v_y'' \\ v_z'' \end{bmatrix}}$$

[1] Ref. 6, page 107

provides the components of the source polarization vector in space coordinates so that the magnetic polarization vector becomes

$$\bar{h}_i = \frac{\bar{\rho}_1' \times \bar{v}''}{|\bar{\rho}_1' \times \bar{v}''|}$$

The vector cross-product is well-defined when the components of $\bar{\rho}_1'$ and $\bar{v}''$ are all relative to space coordinates.

The intensity of the feed pattern F which is directed toward the reflector at a point (x,y,z) in space coordinates can be determined by the direct rather than the inverse transformation. Vector $\bar{\rho}'$ has components which are identical in both 0 and 0' coordinate systems. The components of $\bar{\rho}'$ relative to body coordinates, frame 0", can be found via the matrix transformation

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

where

$$A^{-1} = A^T$$

since the rotation matrix is an orthogonal matrix.

But

$$x'' = \rho'' \sin\Theta'' \cos\Phi''$$

$$y'' = \rho'' \sin\Theta'' \sin\Phi''$$

$$z'' = \rho'' \cos\Theta''$$

again provides a system of three equations in two unknowns, Θ'', and Φ''. If the rotation is degenerate ($\alpha = \beta = \gamma = 0$) so that $A = I$, the identity matrix, the equations for the case of simple feed translation reappear.

The preceding methods for introducing feed translation and rotation into the problem are adapted to numerical computation rather than analysis. The reflector area is presumed to be subdivided suitably into a number of differential areas. If the subdivision or sampling[1] is adequate, then the field intensity due to the illumination can be used to "weight" each differential area prior to summation over the reflector. The precise manner[2] in which the area is "weighted" is a topic sufficiently broad to deserve detailed treatment, and will not be considered in this document. It remains to provide the general methods for obtaining the local normals ($\bar{n}$) and the differential areas (dS) on reflector surfaces. The rotationally symmetric surfaces—paraboloid, hyperboloid, ellipsoid, sphere and cone—are presented in Appendix A as they are representative cases which find frequent application in reflector antenna systems.

SUMMARY

This report provides mathematical subroutines which are required to transport the illumination of a prime feed to a reflector surface. The feed translation vector and the rotation matrix afford an unusual amount of flexibility to the program. Utilization of the magnetic quantities throughout greatly simplifies the formulation and external control. Although the appendix contains only subroutines for the normal and differential area of rotationally symmetric surfaces derived from the conic sections, the field equations are not restricted to these particular surfaces. Other smooth surfaces, symmetric or non-symmetric, can be introduced providing that their sizes and minimum radii of curvature exceed one-wavelength.

Dual or multiple-reflector systems can be analyzed by the techniques outlined herein. Complex prime-feed arrangements can also be accommodated by superimposing field solutions or prime feed-functions. In conclusion, a library of feed and reflector-surface subroutines can be formed and utilized for many practical configurations.

ACKNOWLEDGMENTS

The author acknowledges the help received through many discussions with personnel of the Antenna Systems Branch. An attempt has been made to give credit to those authors whose ideas have been incorporated in this document by numerous references.

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[1] Ref. 7, page 323

[2] Various integration techniques may be applied: direct method, trapezoidal rule, Simpson's rule, Filon's method, Ralston's method, etc.

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APPENDIX A

Normals and Differential Areas from Parametric Equations in Cylindrical Variables (σ, ζ)

$x = \sigma \sin \zeta$ $\qquad$ $y = -\sigma \cos \zeta$

paraboloids $\qquad z = \dfrac{\sigma^2}{4F} + z_1$

hyperboloids $\qquad z = c(1 + \sigma^2/a^2)^{1/2} + z_1$

ellipsoids $\qquad z = c(1 - \sigma^2/a^2)^{1/2} + z_1$

spheres $\qquad z = (c^2 - \sigma^2)^{1/2} + z_1$

cones $\qquad z = c\sigma + z_1$

Tangents to Surface: $\bar{\rho}_\zeta = \dfrac{\partial \bar{\rho}}{\partial \zeta}$, $\bar{\rho}_\sigma = \dfrac{\partial \bar{\rho}}{\partial \sigma}$, $\bar{\rho}(x,y,z) = \hat{i}\, x + \hat{j}\, y + \hat{k}\, z = (x,y,z)$

$\bar{\rho}_\zeta = (\sigma \cos \zeta, \sigma \sin \zeta, 0)$

paraboloids $\qquad \bar{\rho}_\sigma = (\sin \zeta, -\cos \zeta, \sigma/2F)$

hyperboloids $\qquad \bar{\rho}_\sigma = \left[\sin \zeta, -\cos \zeta, \sigma c/a(a^2 + \sigma^2)^{1/2}\right]$

ellipsoids $\qquad \bar{\rho}_\sigma = \left[\sin \zeta, -\cos \zeta, -\sigma c/a(a^2 - \sigma^2)^{1/2}\right]$

spheres $\qquad \bar{\rho}_\sigma = \left[\sin \zeta, -\cos \zeta, -\sigma/(c^2 - \sigma^2)^{1/2}\right]$

cones $\qquad \bar{\rho}_\sigma = (\sin \zeta, -\cos \zeta, c)$

Normal to Surface: $\bar{n} = (\bar{\rho}_\sigma \times \bar{\rho}_\zeta)/|\bar{\rho}_\sigma \times \bar{\rho}_\zeta|$

Differential Area:

$dS = (EG - F)^{1/2}\, d\sigma\, d\zeta$

$E = x_\sigma^2 + y_\sigma^2 + z_\sigma^2$, $F = x_\sigma x_\zeta + y_\sigma y_\zeta + z_\sigma z_\zeta$, $G = x_\zeta^2 + y_\zeta^2 + z_\zeta^2$

paraboloids $\qquad dS = \sigma(1 + \sigma^2/4F^2)^{1/2}\, d\sigma\, d\zeta$

hyperboloids	$dS = \sigma \left[1 + \sigma^2 c^2 / a^2 (a^2 + \sigma^2)\right]^{1/2} d\sigma \, d\zeta$
ellipsoids	$dS = \sigma \left[1 + \sigma^2 c^2 / a^2 (a^2 - \sigma^2)\right]^{1/2} d\sigma \, d\zeta$
spheres	$dS = \sigma c / (c^2 - \sigma^2)^{1/2} \, d\sigma \, d\zeta$
cones	$dS = \sigma (c^2 + 1)^{1/2} \, d\sigma \, d\zeta$

See also:

Ref. 8, page 3
Ref. 5, page 206
Ref. 9, page 3.5-10
Ref. 10, page 106

APPENDIX B

Minimum Radius of Curvature from Parametric Equations in Cylindrical Variables (σ, ζ)

Curvature $k_1 = 1/R_c$, where R_c = radius of curvature.[1]

$x = x(\sigma)$, $y = y(\sigma)$, $z = z(\sigma)$

$$k_1^2 = \frac{\begin{vmatrix} x'' & y'' \\ x' & y' \end{vmatrix}^2 + \begin{vmatrix} y'' & z'' \\ y' & z' \end{vmatrix}^2 + \begin{vmatrix} z'' & x'' \\ z' & x' \end{vmatrix}^2}{(x'^2 + y'^2 + z'^2)^3} = \frac{z''^2}{(1 + z'^2)^3}$$

since

$$x' = \sin\zeta,\ x'' = 0,\ y' = -\cos\zeta,\ y'' = 0,\ z' = \hat{k}\cdot\overline{\rho}_\sigma .$$

k_1 = maximum at $\sigma = 0$.

paraboloids $\quad z'' = 1/2F, \qquad \min R_c = 2F$

hyperboloids $\quad z'' = \frac{c}{a}\left[\frac{-\sigma^2}{(a^2+\sigma^2)^{3/2}} + \frac{1}{(a^2+\sigma^2)^{1/2}}\right] = \frac{c}{a^2}, \qquad \min R_c = \frac{a^2}{c}$

ellipsoids $\quad z'' = -\frac{c}{a}\left[\frac{+\sigma^2}{(a^2-\sigma^2)^{3/2}} + \frac{1}{(a^2-\sigma^2)^{1/2}}\right] = -\frac{c}{a^2}, \quad \min R_c = \frac{a^2}{c}$

spheres $\quad z'' = \left[\frac{-\sigma^2}{(c^2-\sigma^2)^{3/2}} - \frac{1}{(c^2-\sigma^2)^{1/2}}\right] = -\frac{1}{c}, \qquad \min R_c = c$

cones $\quad z'' = 0 \qquad \min R_c = \infty$

[1]Ref. 10, page 51