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A COMPARISON OF THREE SETS OF HIGHER ORDER ADIABATIC PLASMA EQUATIONS

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ABSTRACT

A comparison is made among three methods of obtaining highly accurate approximate expressions for the average motion of plasmas in slowly varying electric and magnetic fields. It is shown that no inconsistencies are revealed by this comparison and that, in fact, two of the techniques are completely equivalent while most of the results of the third can be obtained from the equivalent pair. As a byproduct of the comparison, a function is obtained which can be shown to be conserved along the guiding center trajectory.

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A COMPARISON OF THREE SETS OF HIGHER ORDER ADIABATIC PLASMA EQUATIONS

I. INTRODUCTION

Three papers Frieman et al.[1] (I), Macmahon[2] (II), and Wilson[3] (III) have been presented describing separate approaches to the task of obtaining highly accurate approximate solutions to the Boltzmann and/or hydromagnetics equations for slowly varying fields of arbitrary geometry. Techniques (I) and (III) are very closely related as a closed system of self-consistent equations, correct to first order in $\epsilon = m/e$ the particle mass to charge ratio, is obtained by maintaining both microscopic and macroscopic equations. In (II), the ϵ expansion is introduced directly into the heirarchy of moments equations obtained from the Boltzmann equation. First-order portions of certain quantities can be determined explicitly in terms of higher moments but for this totally macroscopic description, closure may be obtained only through the imposition of additional approximations.[4]

It may easily be verified by inspection of the original papers that the assumptions upon which these works are based are completely equivalent in most plasma density and temperature realms of interest. Each set of conditions may be reduced to assuming that the expansion parameter ϵ may be as small as desired. Moreover, (I) and (II) are based on the Boltzmann equation with the Lorentz force expression while (III) is based on a separate solution of the Lorentz

equation combined with an expression of phase space conservation; an equivalent approach. Thus in as much as comparable expressions are calculated the results of the three must reasonably be the same. However, the veracity of this supposed equivalence is far from obvious, even though one of the results of (II) is verified in (III).[5] The equivalence of the microscopic equations in (I) and (III) is totally obscured by the nature of the special variables chosen in (III), and other results of (II) which should be obtainable from (III) have not been calculated. The purpose of this paper is to show that the expected equivalence does obtain; that the additional results of (II) can be derived from (III), and that (I) and (III) are as alike as to allow the calculation of the results of one from those of the other. A by-product of the inspection is the discovery of the form of a function which is conserved along the higher order guiding center trajectory.

With this in mind, in Section II the relevant results of (I) are rederived in the notation of (III), while Section III supplies the analogous results of (III). In Section IV, a detailed comparison is made between (I) and (III), the results of (II) are verified by use of (III), and the conservation of a proposed function along the guiding center path is demonstrated.

II. HIGHER ORDER CHEW-GOLDBERGER-LOW THEORY

(I) is an extension to higher order in ϵ of a procedure proposed by Chew, Goldberger and Low.[6] The method is based on an ordering scheme that requires, in addition to the assumption of slowly varying fields, plasma densities

and temperatures such that the plasma frequency is of order of magnitude ϵ^{-1} and the Debye length of order ϵ. Implied in this (as in all three theories discussed in this paper) is the assumption that as $\epsilon \to 0$ the component of the electric field parallel to **B** also vanishes.

One proceeds, then, to solve the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f + \frac{1}{\epsilon}\left(\mathbf{E}_\perp + \mathbf{E}_\parallel \hat{L} + \mathbf{v} \times \mathbf{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

in the limit as $\epsilon \to 0$. In (1), the subscripts $\parallel$ and $\perp$ refer to the directions parallel to and perpendicular to the magnetic field **B**; ∇_v is the divergence operator in velocity space; $\hat{L} = \mathbf{B}/|\mathbf{B}|$ with $\hat{M}$ in the direction of the principle radius of curvature of the lines of magnetic induction and $\hat{N} = \hat{L} \times \hat{M}$.

Referring to Figure 1, to all orders, **v** may be represented by

$$\mathbf{v} = \mathbf{U} + \eta\hat{L} + \sigma(\hat{M}\sin\nu + \hat{N}\cos\nu) \tag{2}$$

where

$$\text{(a)}\ \mathbf{U} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \qquad \text{(c)}\ \sigma = |\mathbf{v}_\perp - \mathbf{U}| \equiv |\mathbf{v}'|$$

$$\text{(b)}\ \eta = \hat{L} \cdot \mathbf{v} \qquad \text{(d)}\ \nu = \tan^{-1}\frac{\mathbf{v}' \cdot \hat{M}}{\mathbf{v}' \cdot \hat{N}}. \tag{3}$$

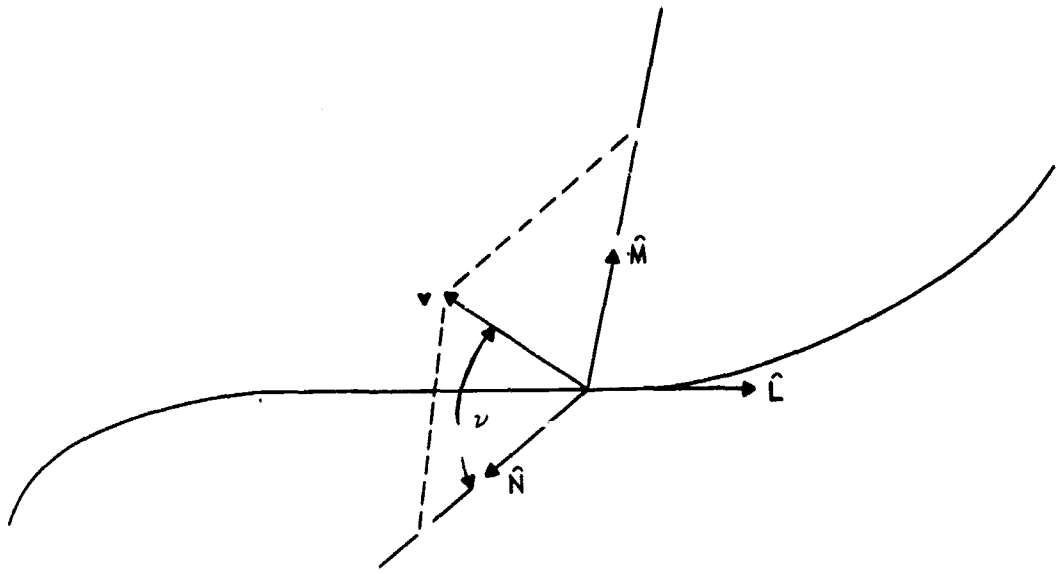

Figure 1. Cylindrical Coordinate System (Different From That of (1))

The distribution function f is assumed to be represented by a power series in ϵ and each term f_i is given in general by $f_i = \overline{f}_i + \tilde{f}_i$ where

$$\text{(a)} \quad \int_0^1 d\nu \, f_i = \overline{f}_i \, , \qquad\qquad \text{(b)} \quad \int_0^1 d\nu \, \tilde{f}_i = 0 \, . \tag{4}$$

Under the conditions stated, it follows from (1) that the zero order term of the distribution function is independent of ν. This knowledge plus the assumption that $\tilde{f}_i$ is periodic in ν makes it possible to arrive at an equation for $\overline{f}_0$ by averaging (1) over ν and neglecting all but the lowest order terms.

To next order, f is not independent of ν and the solution is algebraically more difficult to obtain. One may invert (1) to find $\partial \tilde{f}_i / \partial \nu$ in terms of the

previously defined $\bar{f}_0$. By integrating $\partial \tilde{f}_1/\partial \nu$ over phase from zero to ν and imposing 4b, one may obtain $\tilde{f}_1$. The result is:

$$\tilde{f}_1 = -\frac{\epsilon}{B}(\hat{L}\times \mathbf{v}')\cdot \nabla f_0 + \frac{\epsilon}{B}\left\{(\hat{L}\times \mathbf{v}')\cdot \frac{d\mathbf{V}}{dt} + \frac{\sigma^2}{2}\left[\mathbf{I}_\alpha \sin\nu\cos\nu + \mathbf{I}_\beta\left(\sin^2\nu - \frac{1}{2}\right)\right] : \nabla \mathbf{V}\right\}\frac{\partial f_0}{\partial\left(\frac{\sigma^2}{2}\right)}$$

$$- \frac{\epsilon}{B}\left\{(\hat{L}\times \mathbf{v}')\cdot \frac{d\hat{L}}{dt} + \sigma \mathbf{U}(\hat{L}\times \mathbf{v}') : \nabla\hat{L} + \frac{\sigma^2}{2}\left[\mathbf{I}_\alpha \sin\nu\cos\nu + \mathbf{I}_\beta\left(\sin^2\nu - \frac{1}{2}\right)\right] : \nabla\hat{L}\right\}\frac{\partial f_0}{\partial \eta} \qquad (5)^7$$

where:

$$\mathbf{I}_\alpha \equiv \hat{N}\hat{N} - \hat{M}\hat{M},$$

$$\mathbf{I}_\beta \equiv \hat{M}\hat{N} + \hat{N}\hat{M},$$

$$\mathbf{V} \equiv \mathbf{U} + \eta\hat{L}.$$

The notation d/dt refers to the convective derivative $\partial/\partial t + (\mathbf{U} + \eta\hat{L})\cdot\nabla$ and the double dot tensor product notation is exemplified by $\mathbf{ab} : \nabla\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\cdot\nabla)\,\mathbf{c}$.

By substituting (5) in (1) and integrating over ν one may obtain an equation of the following form for the first two terms of the power series

for $\overline{f}$:

$$\frac{\partial \overline{f}}{\partial t} + (\boldsymbol{\alpha} \cdot \nabla)\,\overline{f} + \beta \frac{\partial \overline{f}}{\partial \left(\frac{\sigma^2}{2}\right)} + \Gamma \frac{\partial \overline{f}}{\partial \eta} = 0 \tag{6a}$$

where

$$\boldsymbol{\alpha} = \mathbf{V} + \frac{\epsilon}{B}\hat{L} \times \frac{d\mathbf{V}}{dt} + \frac{\epsilon}{B}\frac{\sigma^2}{2B}(\hat{L} \times \nabla)\,B + \epsilon \frac{\sigma^2}{2B}\hat{L}\,\mathbf{I}_\gamma : \nabla\hat{L} \tag{6b}$$

$$\begin{aligned}\boldsymbol{\beta} = & -\frac{\sigma^2}{2}\mathbf{I}_\perp : \nabla\mathbf{V} \\ & + \epsilon \frac{\sigma^2}{2B}\left[B\nabla \cdot \left\{\hat{M}\frac{\hat{N}}{B} \cdot \frac{d\mathbf{V}}{dt} - \hat{N}\frac{\hat{M}}{B} \cdot \frac{d\mathbf{V}}{dt}\right\}\right. \\ & + \hat{M}\hat{L} : \nabla\hat{L}\hat{N} \cdot \frac{d\mathbf{V}}{dt} - \hat{N}\hat{L} : \nabla\hat{L}\hat{M} \cdot \frac{d\mathbf{V}}{dt} \\ & \left. + 2\left(\hat{N} \cdot \frac{d\hat{L}}{dt}\mathbf{U}\hat{M} : \nabla\hat{L} - \hat{M} \cdot \frac{d\hat{L}}{dt}\mathbf{U}\hat{N} : \nabla\hat{L}\right)\right]\end{aligned} \tag{6c}$$

$$
\begin{aligned}
\Gamma = {} & \frac{E_{\parallel}}{\epsilon} + \mathbf{U} \cdot \frac{d\hat{L}}{dt} + \frac{\sigma^2}{2} \nabla \cdot \hat{L} \\
& + \epsilon \frac{\sigma^2}{2B} \left[B\nabla \cdot \left\{ \frac{\hat{N}}{B} \left(\hat{M} \cdot \frac{d\hat{L}}{dt} + \mathbf{U}\hat{M} : \nabla\hat{L} \right) - \frac{\hat{M}}{B} \left(\hat{N} \cdot \frac{d\hat{L}}{dt} + \mathbf{U}\hat{N} : \nabla\hat{L} \right) \right\} \right. \\
& \left. + \frac{1}{2} \mathbf{I}_{\beta} : \nabla \mathbf{U} \mathbf{I}_{\alpha} : \nabla\hat{L} - \frac{1}{2} \mathbf{I}_{\alpha} : \nabla \mathbf{U} \mathbf{I}_{\beta} : \nabla\hat{L} \right] \\
& + \frac{\epsilon}{B} \left[\hat{M} \cdot \frac{d\mathbf{U}}{dt} \left(\hat{N} \cdot \frac{d\hat{L}}{dt} + \mathbf{U}\hat{N} : \nabla\hat{L} \right) - \hat{N} \cdot \frac{d\mathbf{U}}{dt} \left(\hat{M} \cdot \frac{d\hat{L}}{dt} + \mathbf{U}\hat{M} : \nabla\hat{L} \right) \right] \\
& + \frac{\epsilon\eta}{B} \left[\hat{M} \cdot \frac{d\hat{L}}{dt} \mathbf{U}\hat{N} : \nabla\hat{L} - \hat{N} \cdot \frac{d\hat{L}}{dt} \mathbf{U}\hat{M} : \nabla\hat{L} \right] \qquad \text{(6d)}
\end{aligned}
$$

where:

$$
\mathbf{I}_{\gamma} \equiv \hat{N}\hat{M} - \hat{M}\hat{N} ,
$$

$$
\mathbf{I}_{\perp} = \hat{M}\hat{M} + \hat{N}\hat{N} .
$$

To obtain a closed set of equations suitable for the study of plasma behavior, one must add to (6) equations which will determine the fields **E** and **B** self-consistently. This has been accomplished and reported in the original paper[8] and will not be discussed here. Equations (5) and (6) are the relevant equations to be studied in Section IV. It will be observed in Section III that an equation which serves the same purpose as (6) but which has a radically different appearance can be obtained. We will show that the two equations are consistent.

III. THE GUIDING CENTER PLASMA

The approach first outlined by Grad[9,10] and extended in (III) is quite different from that of the previous section. It is based on guiding center orbits obtained by solving the single particle equation of motion:

$$\frac{\delta \mathbf{v}}{\delta t} = \frac{1}{\epsilon}\left[\mathbf{E}_{\perp} + \mathbf{E}_{\parallel}\hat{\mathbf{L}} + \mathbf{v}\times\mathbf{B}\right] . \tag{7}$$

As outlined in (III),[11] the higher order terms of this solution can be calculated in terms of a "nice"[12] set of variables chosen to have the property that their equations of motion are free of any rapidly oscillating terms depending on phase. If the particle coordinates are represented in terms of "nice" variables as ($\mathbf{P}$, Σ, H, Φ) these variables are related to those of Section II by the following

$$\mathbf{P} = \mathbf{r} - \frac{\epsilon}{B}\hat{\mathbf{L}}\times\mathbf{v}' \tag{8a}$$

$$\frac{\Sigma^2}{2} = \frac{\sigma^2}{2} + \frac{\epsilon}{B}\left[\sigma\hat{\mathbf{M}} + \hat{\mathbf{L}}\times\mathbf{v}'\right]\cdot\frac{d\mathbf{V}}{dt} + \frac{\epsilon}{B}\frac{\sigma^2}{2}\left[\mathbf{I}_{\alpha}\sin\nu\cos\nu + \mathbf{I}_{\beta}\sin^2\nu\right] : \nabla\mathbf{V} \tag{8b}$$

$$H = \eta - \frac{\epsilon}{B}U\left[\sigma\hat{\mathbf{M}} + \hat{\mathbf{L}}\times\mathbf{v}'\right] : \nabla\hat{\mathbf{L}} - \frac{\epsilon}{B}\left[\sigma\hat{\mathbf{M}} + \hat{\mathbf{L}}\times\mathbf{v}'\right]\cdot\frac{d\hat{\mathbf{L}}}{dt} - \epsilon\frac{\sigma^2}{2B}\left[\mathbf{I}_{\alpha}\sin\nu\cos\nu + \mathbf{I}_{\beta}\sin^2\nu\right] : \nabla\hat{\mathbf{L}} \tag{8c}$$

$$\Phi = \nu + \theta(\epsilon) \tag{8d}$$

where the order $\epsilon(\theta(\epsilon))$ terms of (8d) are known but never used.

In the present case, the variables may reasonably be referred to as the guiding center variables since $\mathbf{P}$ is the position of the guiding center and the motion of the guiding center may be completely described in terms of $\mathbf{P}$, Σ, H and their derivatives. One may also describe the instantaneous motion of the particle itself in terms of these variables by using the inversion relations

$$\mathbf{v} = \mathbf{U}(\mathbf{r}) + \eta\hat{\mathbf{L}}(\mathbf{r}) + \sigma\left\{\hat{\mathbf{M}}(\mathbf{r})\sin\nu + \hat{\mathbf{N}}(\mathbf{r})\cos\nu\right\} \tag{9a}$$

$$\mathbf{r} = \mathbf{P} + \frac{\epsilon\sigma}{B(\mathbf{r})}\left(\hat{\mathbf{N}}(\mathbf{r})\sin\nu - \hat{\mathbf{M}}(\mathbf{r})\cos\nu\right) . \tag{9b}$$

In (9), the lengthy transformations (8) must be used to develop the expressions for $\mathbf{r}$ and $\mathbf{v}$ as power series expansions to the order of accuracy desired.[13]

The time derivatives of (8) are:

$$\frac{\delta\mathbf{P}}{\delta t} = \mathbf{U} + H\hat{\mathbf{L}} + \frac{\epsilon}{B}\hat{\mathbf{L}} \times \frac{d\mathbf{V}}{dt} + \frac{\epsilon}{B}\frac{\Sigma^2}{2B}(\hat{\mathbf{L}} \times \nabla)B$$

$$+ \frac{\epsilon}{B}\hat{\mathbf{L}}\left[\Sigma\left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U}\hat{\mathbf{M}} : \nabla\hat{\mathbf{L}}\right) + \frac{\Sigma^2}{2}\left(\mathbf{I}_\gamma + \frac{1}{2}\mathbf{I}_\beta\right) : \nabla\hat{\mathbf{L}}\right] \tag{10a}$$

$$
\begin{aligned}
\frac{\delta\left(\frac{\Sigma^2}{2}\right)}{\delta t} = & -\frac{\Sigma^2}{2}\,\mathbf{I}_\perp : \nabla(\mathbf{U}+\mathbf{H}\hat{\mathbf{L}}) \\
& + \epsilon\,\frac{\Sigma^2}{2B}\Bigg[B\nabla\cdot\left\{\frac{\hat{\mathbf{M}}}{B}\,\hat{\mathbf{N}}\cdot\frac{d\mathbf{V}}{dt} - \frac{\hat{\mathbf{N}}}{B}\,\hat{\mathbf{M}}\cdot\frac{d\mathbf{V}}{dt}\right\} \\
& + \hat{\mathbf{M}}\hat{\mathbf{L}} : \nabla\hat{\mathbf{L}}\,\hat{\mathbf{N}}\cdot\frac{d\mathbf{V}}{dt} - \hat{\mathbf{N}}\hat{\mathbf{L}} : \nabla\hat{\mathbf{L}}\,\hat{\mathbf{M}}\cdot\frac{d\mathbf{V}}{dt} \\
& + \frac{1}{2}\left(\mathbf{U}\cdot\frac{d\hat{\mathbf{L}}}{dt} + \frac{\mathbf{E}_\parallel}{\epsilon}\right)\mathbf{I}_\beta : \nabla\hat{\mathbf{L}} \\
& + 2\left(\hat{\mathbf{N}}\cdot\frac{d\hat{\mathbf{L}}}{dt}\,\mathbf{U}\hat{\mathbf{M}} : \nabla\hat{\mathbf{L}} - \hat{\mathbf{M}}\cdot\frac{d\hat{\mathbf{L}}}{dt}\,\mathbf{U}\hat{\mathbf{N}} : \nabla\hat{\mathbf{L}}\right) \\
& + \frac{1}{2}\,\mathbf{I}_\beta : \nabla\mathbf{V}\,\mathbf{I}_\perp : \nabla\mathbf{V} + \frac{1}{2}\frac{d}{dt}\,\mathbf{I}_\beta : \nabla\mathbf{V}\Bigg] \\
& + \frac{\epsilon\Sigma}{B}\left[2\left(\frac{\mathbf{E}_\parallel}{\epsilon} + \mathbf{U}\cdot\frac{d\hat{\mathbf{L}}}{dt}\right)\hat{\mathbf{M}}\cdot\frac{d\hat{\mathbf{L}}}{dt} + \frac{3}{2}\,\hat{\mathbf{M}}\cdot\frac{d\mathbf{V}}{dt}\,\mathbf{I}_\perp : \nabla\mathbf{V} + \frac{d}{dt}\,\hat{\mathbf{M}}\cdot\frac{d\mathbf{V}}{dt}\right] \\
& + \frac{\epsilon\Sigma^3}{2B}\left[\hat{\mathbf{M}}\hat{\mathbf{L}} : \nabla\mathbf{U} - \mathbf{U}\hat{\mathbf{M}} : \nabla\hat{\mathbf{L}}\right]\mathbf{I}_\perp : \nabla\hat{\mathbf{L}} + \epsilon\,\frac{\mathbf{H}\Sigma^3}{2B}\,\hat{\mathbf{M}}\hat{\mathbf{L}} : \nabla\hat{\mathbf{L}}\;\mathbf{I}_\perp : \nabla\hat{\mathbf{L}} \qquad \text{(10b)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta\mathbf{H}}{\delta t} = & \frac{\mathbf{E}_\parallel}{\epsilon} + \mathbf{U}\cdot\frac{d\hat{\mathbf{L}}}{dt} + \frac{\Sigma^2}{2}\,\nabla\cdot\hat{\mathbf{L}} \\
& + \frac{\epsilon\Sigma^2}{2B}\Bigg[B\nabla\cdot\left\{\frac{\hat{\mathbf{N}}}{B}\left(\hat{\mathbf{M}}\cdot\frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U}\hat{\mathbf{M}} : \nabla\hat{\mathbf{L}}\right) - \frac{\hat{\mathbf{M}}}{B}\left(\hat{\mathbf{N}}\cdot\frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U}\hat{\mathbf{N}} : \nabla\hat{\mathbf{L}}\right)\right\}
\end{aligned}
$$

$$+ \hat{\mathbf{M}}\hat{\mathbf{M}} : \nabla \mathbf{U} \mathbf{I}_\beta : \nabla \hat{\mathbf{L}} - \hat{\mathbf{M}}\hat{\mathbf{M}} : \nabla \hat{\mathbf{L}} \mathbf{I}_\beta : \nabla \mathbf{U}$$

$$- \frac{1}{2} \nabla \cdot \mathbf{V} \mathbf{I}_\beta : \nabla \hat{\mathbf{L}} - \frac{1}{2} \frac{d}{dt} \mathbf{I}_\beta : \nabla \hat{\mathbf{L}}$$

$$+ \frac{\epsilon \Sigma}{B} \Bigg[\mathbf{U} \hat{\mathbf{L}} : \nabla \hat{\mathbf{L}} \left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U} \hat{\mathbf{M}} : \nabla \hat{\mathbf{L}} \right)$$

$$- \frac{1}{2} \mathbf{I}_\perp : \nabla \mathbf{U} \left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U} \hat{\mathbf{M}} : \nabla \hat{\mathbf{L}} \right) - \hat{\mathbf{M}} \hat{\mathbf{L}} : \nabla \hat{\mathbf{L}} \left(\frac{\mathbf{E}_\parallel}{\epsilon} + \mathbf{U} \cdot \frac{d\hat{\mathbf{L}}}{\delta t} \right)$$

$$- \frac{d}{dt} \left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U} \hat{\mathbf{M}} : \nabla \hat{\mathbf{L}} \right) - \hat{\mathbf{M}} \cdot \frac{d\mathbf{U}}{dt} \nabla \cdot \hat{\mathbf{L}} \Bigg]$$

$$- \epsilon \frac{H\Sigma}{B} \left[\frac{3}{2} \hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \frac{1}{2} \mathbf{U} \hat{\mathbf{M}} : \nabla \hat{\mathbf{L}} \right] \nabla \cdot \hat{\mathbf{L}} - \frac{\epsilon}{B} \frac{\Sigma^3}{2} \hat{\mathbf{M}} \hat{\mathbf{L}} : \nabla \hat{\mathbf{L}} \nabla \cdot \hat{\mathbf{L}}$$

$$+ \frac{\epsilon}{B} \left[\hat{\mathbf{M}} \cdot \frac{d\mathbf{U}}{dt} \left(\hat{\mathbf{N}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U} \hat{\mathbf{N}} : \nabla \hat{\mathbf{L}} \right) - \hat{\mathbf{N}} \cdot \frac{d\mathbf{U}}{dt} \left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U} \hat{\mathbf{M}} : \nabla \hat{\mathbf{L}} \right) \right]$$

$$+ \frac{\epsilon H}{B} \left[\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} \mathbf{U} \hat{\mathbf{N}} : \nabla \hat{\mathbf{L}} - \hat{\mathbf{N}} \cdot \frac{d\hat{\mathbf{L}}}{dt} \mathbf{U} \hat{\mathbf{M}} : \nabla \hat{\mathbf{L}} \right] \qquad \text{(10c)}$$

$$2\pi \frac{\delta \Phi}{\delta t} = \frac{B}{\epsilon} + \mathbf{N} \cdot \frac{d\mathbf{M}}{dt} + \frac{1}{2} \mathbf{I}_\gamma : \nabla \mathbf{V} .$$

Here d/dt and $\mathbf{V}$ are defined as $\partial/\partial t + (\mathbf{U} + H\hat{\mathbf{L}}) \cdot \nabla$ and $\mathbf{U} + H\hat{\mathbf{L}}$ respectively. These definitions agree with those of the previous section to lowest order. In the next section the notation $\mathbf{V}$ and d/dt will be used only when higher order terms are to be neglected.

In (III), a further change of variables is constructed so that one velocity space component is characterized by the magnetic moment series correct to order ϵ. The resulting equation is of a simpler form and is more useful for the purposes of (III). This equation will be discussed and compared with an analogous equation in Section IV, but at this point we choose to present the consequences of the results thus far obtained. That is, one may determine the probability density F of guiding centers in the volume element $d^3 P \Sigma d\Sigma dH d\Phi$ through the conservation relation

$$\frac{\partial F}{\partial t} + \nabla \cdot \left(\frac{\delta \mathbf{P}}{\delta t} F\right) + \frac{\partial}{\partial \left(\frac{\Sigma^2}{2}\right)} \left(\frac{\delta \frac{\Sigma^2}{2}}{\delta t} F\right) + \frac{\partial}{\partial H} \left(\frac{\delta H}{\delta t} F\right) = 0 . \tag{11}$$

The theorem of phase independence of Reference 12[14] and Equation (10d) justify the absence of a term involving $\delta\Phi/\delta t$ in (11).

Equation (11) may be written in a form more suitable for comparison with (6) as follows

$$\frac{\partial F}{\partial t} + \left(\frac{\delta \mathbf{P}}{\delta t} \cdot \nabla\right) F + \frac{\delta\left(\frac{\Sigma^2}{2}\right)}{\delta t} \frac{\partial F}{\partial \left(\frac{\Sigma^2}{2}\right)} + \frac{\delta H}{\delta t} \frac{\partial F}{\partial H}$$

$$= -F \left[\nabla \cdot \left(\frac{\delta \mathbf{P}}{\delta t}\right) + \frac{\partial}{\partial \frac{\Sigma^2}{2}} \left(\frac{\delta\left(\frac{\Sigma^2}{2}\right)}{\delta t}\right) + \frac{\partial}{\partial H} \left(\frac{\delta H}{\delta t}\right)\right] . \tag{12}$$

Since the right-hand-side of (12) does not vanish, this equation expresses the fact that F is not conserved along a guiding center path. We will note presently that a function can be found that does satisfy such a criterion. Again, the discussion of the self-consistent closure of the system is to be omitted since (8), (10), and (12) are the results necessary for comparison with (6).

IV. COMPARISON

With the notation established and all necessary results stated, we can now proceed to demonstrate that the results of Sections II and III are consistent with each other. The connection with (II) will be dealt with separately since the microscopic equations of (I) and (III) are of primary interest.

The two distribution functions of Sections II and III

$$f\left(\mathbf{r}, \frac{\sigma^2}{2}, \eta, \nu\right)$$

and

$$F\left(\mathbf{P}, \frac{\Sigma^2}{2}, H, \Phi\right)$$

may be related by equating the number of particles in $d^3 r \sigma d\sigma d\eta d\nu$ to the number of guiding centers in $d^3 P \Sigma d\Sigma dH d\Phi$; an equation expected to be true on physical grounds. That is

$$f\, d^3 r \sigma d\sigma d\eta d\nu = F d^3 P \Sigma d\Sigma dH d\Phi \,. \tag{13}$$

The transformation Equations (8) make it possible to relate the particle and guiding center volume elements through the Jacobian of this transformation so that

$$d^3 r \, \sigma \, d\sigma \, d\eta \, d\nu = J \, d^3 P \, \Sigma \, d\Sigma \, dH \, d\Phi \, . \tag{14}$$

From (13) and (14) one may conclude that

$$F\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right) = J \, f\left(\mathbf{r}, \frac{\sigma^2}{2}, \eta, \nu\right) . \tag{15}$$

In (15) the argument Φ has been dropped in F because, as previously pointed out, F is known to be phase independent.

In an attempt to arrange (15) so that both sides of the equation are in terms of the same variables it is advantageous to expand f in Taylor series about the guiding center variables as follows

$$f\left(\mathbf{r}, \frac{\sigma^2}{2}, \eta, \nu\right) = f\left(\mathbf{P}, \frac{\Sigma^2}{2}, H, \Phi\right) + (\mathbf{r} - \mathbf{P}) \cdot \nabla f$$

$$+ \left(\frac{\sigma^2}{2} - \frac{\Sigma^2}{2}\right) \frac{\partial f}{\partial\left(\frac{\sigma^2}{2}\right)} + (\eta - H) \frac{\partial f}{\partial \eta} + (\nu - \Phi) \frac{\partial f}{\partial \nu} \cdot \tag{16}$$

Recalling from Section II that one may always write f as

$$f\left(\mathbf{P}, \frac{\Sigma^2}{2}, H, \Phi\right) = \bar{f}\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right) + \tilde{f}\left(\mathbf{P}, \frac{\Sigma^2}{2}, H, \Phi\right) . \tag{17}$$

we can substitute (5) into (17) and then (17) and (3) into (16) to obtain

$$f\left(\mathbf{r}, \frac{\sigma^2}{2}, \eta, \nu\right) = \bar{f}\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right)$$

$$- \frac{\epsilon}{B}\left\{\Sigma \hat{\mathbf{M}} \cdot \frac{d\mathbf{V}}{dt} + \frac{\Sigma^2}{4} \mathbf{I}_\beta : \nabla \mathbf{V}\right\} \frac{\partial \bar{f}}{\partial\left(\frac{\Sigma^2}{2}\right)}$$

$$+ \frac{\epsilon}{B}\left\{\Sigma\left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U}\hat{\mathbf{M}} : \nabla\hat{\mathbf{L}}\right) + \frac{\Sigma^2}{4} \mathbf{I}_\beta : \nabla \hat{\mathbf{L}}\right\} \frac{\partial \bar{f}}{\partial H}$$

$$= \bar{f}\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right) + \epsilon\gamma\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right) . \quad (18)$$

It is important to understand exactly what has been done here. Equation (17) represents no transformation of variables but only a renaming of arguments. The expression for $\tilde{f}$ is obtained by merely inserting into (5) the arguments indicated in (17). The function $\bar{f}(\mathbf{r}, \sigma^2/2, \eta)$ is determined by (6) with α, β, and Γ written as functions of $\mathbf{r}$, $\sigma^2/2$ and η. In like manner, the function $f(\mathbf{P}, \Sigma^2/2, H)$ that appears on the right-hand-side of (18) is determined by (6), provided the arguments of α, β and Γ are now replaced by $\mathbf{P}$, $\Sigma^2/2$ and H. From this point on then we assume (18) to have been enforced and all expressions of both Sections II and III are to be considered functions of $\mathbf{P}$, $\Sigma^2/2$ and H.

Taking note that the right hand side of (18) is independent of Φ, and recalling that F is also independent of Φ one may conclude that J must be phase independent for (15) to hold. From (8) it can be shown that this is true; J is

independent of Φ and of the form

$$J = 1 - \frac{\epsilon}{B\Sigma}\hat{M} \cdot \frac{d\mathbf{V}}{dt} + \frac{\epsilon\Sigma}{B}\hat{M}\hat{L} : \nabla\hat{L} - \frac{\epsilon}{2B}\mathbf{I}_{\beta} : \nabla\mathbf{V} \tag{19}$$

thus revealing no inconsistency.

Proceeding, one may invert (15) to obtain

$$\bar{f} = J^{-1}F - \epsilon\gamma . \tag{20}$$

Insertion of (20) and (6) yields

$$\frac{d'F}{dt} = \frac{F}{J}\frac{d'J}{dt} + \epsilon J\frac{d'\gamma}{dt} \tag{21}$$

where:

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \alpha\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right) \cdot \nabla + \beta\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right)\frac{\partial}{\partial\left(\frac{\Sigma^2}{2}\right)} + \Gamma\left(\mathbf{P}, \frac{\Sigma^2}{2}, H\right)\frac{\partial}{\partial H} .$$

Using some shorthand notation we rewrite (12) as

$$\frac{DF}{Dt} + F\Delta = 0 \tag{22}$$

where:

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \left(\frac{\delta\mathbf{r}}{\delta t} \cdot \nabla\right)F + \frac{\delta\left(\frac{\Sigma^2}{2}\right)}{\delta t}\frac{\partial F}{\partial\left(\frac{\Sigma^2}{2}\right)} + \frac{\delta H}{\delta t}\frac{\partial F}{\partial H}$$

and

$$\Delta = \nabla \cdot \left(\frac{\delta P}{\delta t}\right) + \frac{\partial}{\partial\left(\frac{\Sigma^2}{2}\right)} \frac{\delta\left(\frac{\Sigma^2}{2}\right)}{\delta t} + \frac{\partial}{\partial H} \frac{\delta H}{\delta t} .$$

Subtracting (21) from (22) one may obtain the following relation which must be satisfied by Equations (6) and (10)

$$\left(\frac{\delta P}{\delta t} - \alpha\right) \cdot \nabla F + \left(\frac{\delta\left(\frac{\Sigma^2}{2}\right)}{\delta t} - \beta\right) \frac{\partial F}{\partial\left(\frac{\Sigma^2}{2}\right)} + \left(\frac{\delta H}{\delta t} - \Gamma\right) \frac{\partial F}{\partial H}$$

$$= - \frac{F}{J} \left[\frac{d' J}{dt} + J \Delta\right] - \epsilon J \frac{d' \gamma}{dt} \cdot \quad (23)$$

Equations (10), (19), and (21) yield

$$\Delta = - \frac{1}{J} \frac{d' J}{dt} \quad (24)$$

which is to be expected since the left-hand-side of (23) contains no term proportional to F.

The comparison expressed by (23) now acquires additional significance. One may note that γ is obtained by an entirely separate calculation from (6) and (10), and that $d' \gamma/dt$ need be computed to lowest order only. Thus a simple calculation yields a check on the order ϵ terms of the left-hand-side of (23) which were originally obtained by means of tedious calculation.

The calculation of $d'\gamma/dt$ is straightforward. Second derivatives of F result but they can be eliminated by differentiating the lowest order form of (6) to find

$$\frac{d'}{dt}\frac{\partial F}{\partial\left(\frac{\Sigma^2}{2}\right)} = (\hat{M}\hat{M}+\hat{N}\hat{N}) : \nabla(U+H\hat{L})\frac{\partial F}{\partial\left(\frac{\Sigma^2}{2}\right)} - \nabla\cdot\hat{L}\frac{\partial F}{\partial H} \tag{24a}$$

$$\frac{d'}{dt}\frac{\partial F}{\partial H} = -(\hat{L}\cdot\nabla)F + \frac{\Sigma^2}{2}\nabla\cdot\hat{L}\frac{\partial F}{\partial\left(\frac{\Sigma^2}{2}\right)} - U\hat{L} : \nabla\hat{L}\frac{\partial F}{\partial H}\cdot \tag{24b}$$

The final result for $d'\gamma/dt$ is

$$\begin{aligned}
\epsilon\frac{d'\gamma}{dt} = \frac{\epsilon}{B}\Bigg\{&\frac{3}{2}\Sigma\hat{M}\cdot\frac{dV}{dt}I_\perp : \nabla V + \Sigma\frac{d}{dt}\hat{M}\cdot\frac{dV}{dt} \\
&+\left(\frac{E_\parallel}{\epsilon}+U\cdot\frac{d\hat{L}}{dt}\right)\left(2\Sigma\hat{M}\cdot\frac{d\hat{L}}{dt}+\frac{\Sigma^2}{4}I_\beta : \nabla\hat{L}\right) \\
&+\Sigma^3\hat{M}\cdot\frac{d\hat{L}}{dt}\nabla\cdot\hat{L}+\frac{\Sigma^2}{4}\frac{d}{dt}I_\beta : \nabla V \\
&+\frac{\Sigma^2}{4}I_\beta : \nabla V I_\perp : \nabla V - \frac{\Sigma^2}{2}\left(\hat{M}\cdot\frac{dL}{dt}+U\hat{M} : \nabla L\right)\Bigg\}\frac{\partial F}{\partial\left(\frac{\Sigma^2}{2}\right)} \\
-\frac{\epsilon}{B}\Bigg\{&\frac{\Sigma}{2}\left(\hat{M}\cdot\frac{d\hat{L}}{dt}+U\hat{M} : \nabla\hat{L}\right)I_\perp : \nabla V + \Sigma\frac{d}{dt}\left(\hat{M}\cdot\frac{d\hat{L}}{dt}+U\hat{M} : \nabla\hat{L}\right)
\end{aligned}$$

$$+ \Sigma \hat{\mathbf{M}}\hat{\mathbf{L}} : \nabla\hat{\mathbf{L}} \left(\frac{\mathbf{E}_{\parallel}}{\epsilon} + \mathbf{U} \cdot \frac{d\hat{\mathbf{L}}}{dt}\right) + \frac{\Sigma^3}{2} \hat{\mathbf{M}}\hat{\mathbf{L}} : \nabla\hat{\mathbf{L}} \nabla \cdot \hat{\mathbf{L}} + \frac{\Sigma^2}{4} \frac{d}{dt} \mathbf{I}_{\beta} : \nabla\hat{\mathbf{L}}$$

$$+ \Sigma \nabla \cdot \hat{\mathbf{L}}\hat{\mathbf{M}} \cdot \frac{d\mathbf{V}}{dt} + \frac{\Sigma^2}{2} \mathbf{I}_{\beta} : \nabla \mathbf{V} \nabla \cdot \hat{\mathbf{L}}$$

$$- \mathbf{U}\hat{\mathbf{L}} : \nabla\hat{\mathbf{L}} \left[\Sigma \left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U}\hat{\mathbf{M}} : \nabla\hat{\mathbf{L}}\right) + \frac{\Sigma^2}{4} \mathbf{I}_{\beta} : \nabla\hat{\mathbf{L}}\right]\Bigg\} \frac{\partial \mathbf{F}}{\partial \mathbf{H}}$$

$$+ \frac{\epsilon}{B} \left\{\Sigma \left(\hat{\mathbf{M}} \cdot \frac{d\hat{\mathbf{L}}}{dt} + \mathbf{U}\hat{\mathbf{M}} : \nabla\hat{\mathbf{L}}\right) + \frac{\Sigma^2}{4} \mathbf{I}_{\beta} : \nabla\hat{\mathbf{L}}\right\} (\hat{\mathbf{L}} \cdot \nabla) \mathbf{F} \quad . \quad (25)$$

Inserting the appropriate expressions (6) and (10) into the left-hand-side of (23) yields (25); which was to be shown.

Given the results of (III) one may derive (I) for given (8) and constraint (4b), J and $\tilde{f}_1$ can be obtained and thus (6). Some ambiguity is involved in the converse due to the arbitrariness involved in the choice of intermediate variables used to solve the single particle equation.[15] Nevertheless, a set of equations that is at least equivalent to those of (III) may be obtained by using (5) and (4b) to calculate transformation equations of the form of (8). These relations will yield J and γ (though possibly not the same J and γ) and (23) can be used to complete the derivation. The ambiguity is no shortcoming: those equations obtained could have been originally derived through the method of (III). We simply remark that in using (III) as a starting point the ambiguity is removed by specifying the allowed choice in variables.

Simplification in the form of each of these kinetic equations can be achieved by transforming velocity space variables so that the magnetic moment series makes up one component. The kinetic equation of (I) becomes[16]

$$\frac{\partial \bar{f}}{\partial t}(\mathbf{r}, \mu, \eta) + \boldsymbol{\alpha}(\mathbf{r}, \mu, \eta) \cdot \nabla \bar{f}(\mathbf{r}, \mu, \eta) + \Gamma(\mathbf{r}, \mu, \eta) \frac{\partial \bar{f}}{\partial \eta}(\mathbf{r}, \mu, \eta) = 0 \quad (26)$$

where

$$\mu = \frac{\sigma^2}{2B}\left[1 - \frac{\epsilon}{B}(\hat{N}\hat{M} - \hat{M}\hat{N}) : \nabla(\mathbf{U} + \eta\hat{L})\right]$$

$$\frac{d'\mu}{dt} = \theta\left(\epsilon^2\right).$$

Note that f remains as originally defined, the probability density of particles in the $\mathbf{r}$, $\sigma^2/2$, η, ν space, <u>not</u> in $\mathbf{r}$, μ, η, ν.

In (III), a new distribution function $\hat{f}$ is defined such that $\hat{f}$ represents the probability density of guiding centers in the space $\mathbf{P}$, Λ, $V_{\parallel}$, Φ where

$$\Lambda = \frac{\Sigma^2}{2B} - \frac{\epsilon\Sigma}{B^2}\hat{M} \cdot \frac{d}{dt}(\mathbf{U} + H\hat{L}) - \frac{\epsilon}{B}\frac{\Sigma^2}{2B}\left(\frac{3}{2}\hat{N}\hat{M} - \frac{1}{2}\hat{M}\hat{N}\right) : \nabla(\mathbf{U} + H\hat{L}) \quad (27)$$

and

$$V_{\parallel} = H + \frac{\epsilon\Sigma}{B}\left(\hat{M} \cdot \frac{d\hat{L}}{dt} + \mathbf{U}\hat{M} : \nabla\hat{L}\right) + \epsilon\frac{\Sigma^2}{2B}\left(\frac{3}{2}\hat{N}\hat{M} - \frac{1}{2}\hat{M}\hat{N}\right) : \nabla\hat{L}.$$

The equation expressing conservation of guiding centers in this space is

$$\frac{\partial \hat{f}}{\partial t} + \nabla \cdot \left(\frac{\delta' P}{\delta t} \hat{f}\right) + \frac{\partial}{\partial V_{\parallel}} \left(\frac{\delta' V_{\parallel}}{\delta t} \hat{f}\right) = 0 . \tag{28}$$

The expression for $\delta' P/\delta t$ can be obtained from (10a) by using (27) and

$$\begin{aligned}
\frac{\delta' V_{\parallel}}{\delta t} = {} & \frac{E_{\parallel}}{\epsilon} + U \cdot \frac{d\hat{L}}{dt} + B \Lambda \nabla \cdot \hat{L} \\
& + \epsilon \Lambda \Bigg[\frac{1}{2} I_{\alpha} : \nabla \hat{L} I_{\beta} : \nabla U - \frac{1}{2} I_{\alpha} : \nabla U I_{\beta} : \nabla \hat{L} + I_{\gamma} : \nabla U \nabla \cdot \hat{L} \\
& + \hat{L}\hat{L} : \nabla U I_{\gamma} : \nabla \hat{L} + \frac{d}{dt} I_{\gamma} : \nabla \hat{L} + B \nabla \cdot \left\{ \frac{\hat{L} \times (\hat{L} \cdot \nabla) U}{B} \right\} \\
& + \nabla \cdot \left\{ \hat{N} U \hat{M} : \nabla \hat{L} - \hat{M} U \hat{N} : \nabla \hat{L} \right\} \Bigg] \\
& + \epsilon V_{\parallel} \Lambda \left[I_{\gamma} : \nabla \hat{L} \nabla \cdot \hat{L} + B \nabla \cdot \left\{ \frac{\hat{L} \times (\hat{L} \cdot \nabla) L}{B} \right\} \right] \\
& + \frac{\epsilon}{B} \left(\hat{M} \cdot \frac{dU}{dt} \hat{N} \cdot \frac{d\hat{L}}{dt} - \hat{N} \cdot \frac{dU}{dt} \hat{M} \cdot \frac{d\hat{L}}{dt} \right) .
\end{aligned} \tag{29}$$ [17]

In exactly the same manner as outlined above, we form the equations

$$\hat{f}\, d^3P\, d\Lambda\, dV_{\parallel}\, d\Phi = f\, d^3 r\, d\left(\frac{\sigma^2}{2}\right) d\eta\, d\nu \tag{30a}$$

and

$$G\, d^3 P\, d\Lambda\, dV_{\parallel}\, d\Phi \;=\; d^3 r\, d\left(\frac{\sigma^2}{2}\right) d\eta\, d\nu \tag{30b}$$

and conclude that

$$\hat{f}(\mathbf{P}, \Lambda, V_{\parallel}) \;=\; Gf(\mathbf{r}, \mu, \eta, \nu) \;=\; G\Big[\bar{f}(\mathbf{P}, \Lambda, V_{\parallel})$$

$$- \epsilon\, \Lambda(\hat{N}\hat{M} - \hat{M}\hat{N}) : \nabla\hat{L}\, \frac{\partial f}{\partial V_{\parallel}}\Big] \;\equiv\; G[\bar{f} + \gamma']$$

where

$$G \;=\; B\left[1 + \frac{\epsilon}{B}\,(\hat{N}\hat{M} - \hat{M}\hat{N}) : \nabla(\mathbf{U} + V_{\parallel}\hat{L})\right].$$

Inserting (29) into (27) one may obtain

$$\left(\alpha - \frac{\delta' \mathbf{P}}{\delta t}\right) \cdot \nabla\bar{f} + \left(\Gamma - \frac{\delta' V_{\parallel}}{\delta t}\right) \frac{\partial\bar{f}}{\partial V_{\parallel}}$$

$$= -\left[\frac{\bar{f}}{G} + \frac{\epsilon\gamma'}{G}\right]\left[\frac{D' G}{Dt} + G\left(\nabla\cdot\left(\frac{\delta' \mathbf{P}}{\delta t}\right) + \frac{\partial}{\partial V_{\parallel}}\left(\frac{\delta' V_{\parallel}}{\delta t}\right)\right)\right]$$

$$- \epsilon\left\{\Lambda\, \frac{D'}{Dt}\,(\hat{N}\hat{M} - \hat{M}\hat{N}) : \nabla\hat{L} - \Lambda(\hat{N}\hat{M} - \hat{M}\hat{N}) : \nabla\hat{L}\,\mathbf{U}\hat{L} : \nabla\hat{L}\right\} \frac{\partial\bar{f}}{\partial V_{\parallel}}$$

$$+ \epsilon\, \Lambda(\hat{L}\cdot\nabla)\bar{f} \tag{31}$$

where

$$\frac{D'}{Dt} = \frac{\partial}{\partial t} + \left(\frac{\delta' P}{\delta t} \cdot \nabla\right) + \frac{\delta' V_{\parallel}}{\delta t} \frac{\partial}{\partial V_{\parallel}} .$$

Substituting the quantities specified into (31) and performing the operations indicated yields equality.

Since the sum of $\nabla \cdot (\delta' P/\delta t)$ and $\partial/\partial V_{\parallel} (\delta' V_{\parallel}/\delta t)$ does not vanish, Equation (28) expresses the fact that $D' \hat{f}/Dt \neq 0$ and, like F, $\hat{f}$ is not conserved along a guiding center trajectory. We postulate that a function that does satisfy such a requirement is $K(P, \Lambda, V_{\parallel}) = \hat{f}/G$. We then form the expression:

$$\frac{D' K}{Dt} = \frac{1}{G}\frac{D' \hat{f}}{Dt} - \frac{\hat{f}}{G^2}\frac{D' G}{Dt} = \frac{-\hat{f}}{G}\left[\nabla \cdot \left(\frac{\delta' P}{\delta t}\right) + \frac{\partial}{\partial V_{\parallel}}\left(\frac{\delta' V_{\parallel}}{\delta t}\right) + \frac{1}{G}\frac{D' G}{Dt}\right] = 0 .$$

Thus, correct to first order in ϵ, the function

$$\hat{f}/B \left\{1 + \frac{\varepsilon}{B} (\hat{N}\hat{M} - \hat{M}\hat{N}) : \nabla(U + V_{\parallel} \hat{L})\right\}$$

is conserved along a guiding center trajectory.[18]

The additional results of (II) that are to be verified here are the first order corrections to the heat flux tensor Q. To follow the procedure of (III) which was used to verify the pressure tensor of (II) (see footnote 5), we begin with the definition of the heat flux tensor:

$$Q = m \int f \, d^3 v \, (v - u)(v - u)(v - u) \tag{32}$$

where f is the solution of (1) and the definitions

$$\mathbf{u} = \frac{1}{n}\int f\,\mathbf{v}\,d^3 v \tag{33a}$$

and

$$n = \int f\,d^3 v \tag{33b}$$

will be used.

In order to calculate Q correct to order ϵ from the results of Section III, an equation of the general type (30a) must be enforced. Here we desire a different result from that obtained between f and $\hat{f}$. This is the relation:

$$f\,d^3 v = \hat{f}\,J'\,d\Lambda\,dV_{\parallel}\,d\Phi \tag{34a}$$

where

$$J' = 1 + \epsilon(2B\Lambda)^{1/2}\cos\Phi\nabla\cdot\left(\frac{\hat{M}}{B}\right) - \epsilon(2B\Lambda)^{1/2}\sin\Phi\nabla\cdot\left(\frac{\hat{N}}{B}\right)\cdot \tag{34b}$$

Now using (34) and definitions (33) in (32) with $\mathbf{v}$ given by (9a) and (8) we can arrive at the following portions of Q previously presented in (II)[19]

$$[Q:\hat{L}\hat{L}]_{\perp} = \frac{\epsilon}{B}\hat{L}\times\Big[\nabla R_2 - \frac{P_{\parallel}}{\rho}\nabla\cdot\mathbf{P} + \left(R_{\parallel} - 3R_2\right)(\hat{L}\cdot\nabla)\hat{L}$$

$$+ 2\left(Q_{\parallel} - Q\right)(\hat{L}\cdot\nabla)\,\mathbf{u}_{\perp} + 2Q\nabla u_{\parallel}$$

$$+ 2Q\left\{\hat{M}\hat{L}\hat{M} : \nabla\mathbf{u}_{\perp} + \hat{N}\hat{L}\hat{N} : \nabla\mathbf{u}_{\perp}\right\}\Bigg] \tag{35a}$$

$$\frac{1}{2}\left[Q : (\hat{M}\hat{M} + \hat{N}\hat{N})\right]_{\perp} = \frac{1}{2}\frac{\epsilon}{B}\hat{L}\times\Bigg[\nabla R_3$$

$$+ \left\{4\left(Q + R_2\right) - R_3\right\}(\hat{L}\cdot\nabla)\hat{L} - \frac{4P_{\perp}}{\rho}\nabla\cdot\mathbf{P}\Bigg] \tag{35b}$$

where, to lowest order

$$\rho = mn\,, \qquad u_{\parallel} = \frac{1}{n}\int \hat{f}\,V_{\parallel}\,d\Lambda\,dV_{\parallel}$$

$$P_{\parallel} = m\int \hat{f}\left(V_{\parallel} - u_{\parallel}\right)^2 d\Lambda\,dV_{\parallel}\,, \qquad P_{\perp} = m\int \hat{f}\,B\Lambda\,d\Lambda\,dV_{\parallel}$$

$$Q_{\parallel} = m\int \hat{f}\left(V_{\parallel} - u_{\parallel}\right)^3 d\Lambda\,dV_{\parallel}\,, \qquad Q = m\int \hat{f}\,B\Lambda\left(V_{\parallel} - u_{\parallel}\right) d\Lambda\,dV_{\parallel}$$

$$R_{\parallel} = m\int \hat{f}\left(V_{\parallel} - u_{\parallel}\right)^4 d\Lambda\,dV_{\parallel}\,, \qquad R_2 = m\int \hat{f}\,B\Lambda\left(V_{\parallel} - u_{\parallel}\right)^2 d\Lambda\,dV_{\parallel}$$

$$R_3 = m\int \hat{f}\,[B\Lambda]^2\,d\Lambda\,dv_{\parallel}\,, \qquad \mathbf{P} = P_{\parallel}\hat{L}\hat{L} + P_{\perp}\left(\hat{M}\hat{M} + \hat{N}\hat{N}\right)\,.$$

We hasten to note that one important result of (II) that cannot be obtained from (III) is that of the computation of u_1 correct to order ϵ^2. In principle, however, this can be accomplished by calculating the order ϵ^2 portions of relations (8) and using a higher order version of the procedure outlined above.

To sum up, one must conclude that the computations provided here lend strong credibility to the assertion that all three of the papers treated are correct and equivalent. To be sure, one would expect this to be true, but given the vast observable differences pointed out in the introduction, and given the possibility of algebraic as well as theoretical error, this demonstration is altogether reassuring.

ACKNOWLEDGMENTS

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FOOTNOTES

1. E. Frieman, R. Davidson, and B. Langdon, Phys. Fluids 9, 8, 1475 (1966).
2. A Macmahon, Phys. Fluids 8, 10, 1840 (1966).
3. G. Wilson, (to be published).
4. This necessity is verified in (I) where a closed system of hydromagnetics equations is obtained in the two dimensional case only.
5. See Ref. 3, eq. 38.
6. G. Chew, M. Goldberger, and F. Low, Proc. Roy. Soc. (London) A236, 112 (1956)
7. This equation and 6b are written in a slightly different form from that shown in (I). For the purposes of comparison it is convenient to adhere strictly to the notion that f represents probability density in the velocity space element $\sigma d\sigma d\eta d\nu$ and is, therefore, a function of $\sigma^2/2$, η and ν.
8. Ref. 1, pp. 1477-1479.
9. H. Grad, Proceedings of the Symposium on Electrodynamics and Fluid Dynamics of Gaseous Plasmas, (Polytechnic Press, Brooklyn, New York, 1961, pp. 37-64).
10. H. Grad, AEC Report NYO-1480-50, MF-48, New York University, 1966.
11. Ref. 1, Sec. II.
12. M. Kruskal, J. Math, Phys. 3, 806 (1962).
13. In principle, Relations (8) may be calculated to all orders in ϵ.
14. Ref. 11, p. 817, Sec. C.1.

15. See T. Northrop, M. Kruskal, and C. Liu, Phys. Fluids 9, 1503 (1966), pp. 1505-1506 for a discussion.

16. R. Davidson, Phys. Fluids 10, 669 (1967).

17. In (29) $d/dt\ \hat{L}$ is the convective derivative of $\hat{L}$, $\partial\hat{L}/\partial t + (\delta' P/\delta t \cdot \nabla)\hat{L}$ correct to order ϵ.

18. The fact that $\hat{f}/B$ is conserved along the lowest order guiding center trajectory has been pointed out by Y. Whang (to be published).

19. Ref. 2, eqs. 14-16. Note a difference in definition by a factor of 2 between 35a and that given in (II).

END

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