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COLLAPSE LOADS OF PARTIALLY LOADED
SHALLOW SPHERICAL CAPS

By

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ACKNOWLEDGMENT

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All notation will be defined where it is first introduced.

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$r, R_1, R_2$ . . . Cap radii of curvature
$r_1, r_2$ . . . Dimensionless cap radii
$S$ . . . Dimensionless shear force resultant
$t$ . . . Thickness of uniform shell
$V$ . . . Cap displacement in the direction of the tangent to the meridian
$W$ . . . Cap displacement in the direction of the normal to the middle surface
$v, w$ . . . Dimensionless velocities
$\varepsilon_\theta, \varepsilon_\phi$ . . . Dimensionless strain-rates
$\gamma_\theta, \gamma_\phi$ .
$\theta$ . . . Cap angle in circumferential direction
$\phi$ . . . Cap angle in meridional direction
$\phi_L$ . . . Cap loading angle
$\phi_0$ . . . Maximum cap angle in meridional direction
$\lambda$ . . . Arbitrary positive factor
$\sigma_o$ . . . Tensile and compressive yield stress for uniform shell
$\sigma'_o$ . . . Yield stress for thin sheet of sandwich shell
$(\cdot)^+$ . . . Refers to upper
$(\cdot)^-$ . . . Refers to lower
$(\cdot)_l$ . . . Refers to internal
$(\cdot)_e$ . . . Refers to external
COMPUTER PROGRAM SYMBOLS

\( \alpha \rightarrow \text{ALP} \quad \rightarrow \text{V} \)

\( \Delta \phi \rightarrow \text{DPHI} \quad \rightarrow \text{W} \)

\( \psi \rightarrow \text{PHI} \quad \lambda_1 \rightarrow \text{XLO} \)

\( \phi \rightarrow \text{PHIH} \quad \lambda_2 \rightarrow \text{XLT} \)

\( \phi_{\text{H}} \rightarrow \text{PHIM} \quad m_\rho \rightarrow \text{XM} \)

\( \phi_{\text{Hmax}} \rightarrow \text{PHIMM} \quad m_\rho' \rightarrow \text{XMP} \)

\( \phi_{\text{L}} \rightarrow \text{PHIL} \quad m_\phi \rightarrow \text{XMT} \)

\( \phi_1 \rightarrow \text{PHIO} \quad n_\phi \rightarrow \text{XMTP} \)

\( \phi_2 \rightarrow \text{PHIT} \quad n_\phi' \rightarrow \text{XNP} \)

\( \phi_0 \rightarrow \text{PHIZ} \quad n_\phi'' \rightarrow \text{XNP} \)

\( P_{\text{Min}} \rightarrow \text{PMIN} \quad n_0 \rightarrow \text{XNT} \)

\( \phi_{1_{\text{Min}}} \rightarrow \text{PHMIN} \quad n_0' \rightarrow \text{ZNTP} \)
ABSTRACT

One of the problems of current interest to the National Aeronautics and Space Administration is the prediction of damage to spacecraft during water-landing impact. Technically speaking, this problem is one of the damage to a shallow spherical cap impacting a relatively incompressible liquid. For many purposes, the problem can be further reduced to one of the quasi-static collapse load on partially loaded, rotationally symmetric, shallow spherical caps with simply-supported and clamped edges. The objective of the research reported here was to develop appropriate techniques for obtaining bounds and exact solutions for the collapse loads of such partially loaded, shallow spherical caps.

In this investigation the theory of limit analysis is used to investigate collapse loads of partially loaded, shallow spherical caps. The treatment is restricted to rotationally symmetric loads and geometry, with both simply supported and clamped boundary conditions. The cap material is considered to be rigid-perfectly plastic and to obey the Tresca yield condition and flow law. The structure is approximated by a sandwich shell cross section; thus, the yield condition becomes piecewise linear.

Upper and lower bounds are obtained for the simply supported cap. Lower bounds are determined through the use of a statically admissible stress field based on fields previously used in determination of lower bounds for uniformly loaded caps. Upper bounds for the simply supported cap are determined through the use of appropriate kinematically admissible velocity fields. For very shallow caps, a field based upon the results for the uniformly loaded spherical cap is used. For deeper caps, a different field, involving discontinuities, was required. The resulting
bounds were very close for the very shallow caps but diverged appreciably
for caps approaching the limit of shallow cap classification. Results
presented include curves of collapse pressure vs. cap angle for several
cases of loading and cap thickness.

For the partially loaded clamped spherical cap it was possible to ob-
tain an exact solution by determining fields which were shown to be both
statically and kinematically admissible. Results include presentation of
distribution of stress and velocity distributions as well as collapse pres-
sure as a function of cap angle.
CHAPTER I

INTRODUCTION

In recent years there has been considerable interest in determining the load carrying capacity of shell structures loaded beyond the elastic limit of the material. Even though portions of the structure material may have reached yield, more load usually may be carried by the structure. If the structure is made of a rigid-plastic material, then the theory of limit analysis may be used to determine the load carrying capacity of the structure. In this theory, it is assumed that a structure made of such a material is subjected to gradually increasing loads which are fully prescribed to within a magnitude parameter, $p$. For sufficiently small loads the structure will remain rigid. For larger values of the load, part of the structure will become plastic; however, the structure as a whole will remain in equilibrium due to the surrounding material which is rigid. If the loads are increased further, an increasing portion of the structure will become plastic, and, for a particular value of the load intensity $p$, the surrounding material will no longer be able to maintain the structure in equilibrium. With a load of such intensity, the structure will undergo plastic deformations and will no longer be serviceable. The load intensity at which this takes place is called the "collapse load." In this treatment the assumption has been made
that the deformations are sufficiently small that changes in geometry may be ignored. Also, it has been assumed that inertia forces are negligible and therefore the deformations take place under quasistatic conditions.

A review of the literature shows that much work has been done in determining the load carrying capacity of shell-type structures. Limit analysis of uniformly and partially loaded circular plates \([1,2]^{*}\) has led to exact predictions of collapse loads. Rotationally symmetric shells, in particular the spherical cap which is loaded over the entire surface with uniform intensity, have been investigated extensively; however, in contrast, the spherical cap which is loaded only over a portion of the cap surface has received little attention. The purpose of this investigation is to examine certain partially loaded spherical caps and to determine exactly or to bound the intensity of the collapse load, using the theory of limit analysis.

The distribution of the loading over the cap surface is to be circular, concentric with the cap apex axis, and have uniform intensity. Complete rotational symmetry is assumed; therefore, no loads or deformations in the circumferential direction of the cap are allowed. The cap is assumed to be made of a rigid-plastic material and obey the Tresca yield condition and the associated flow rule. The edge boundary conditions of the caps to be investigated are those of simple support and clamping. For the simply supported case, there will be no normal or tangential displacements and no restraint against rotation along

*Numbers in brackets refer to the List of References.*
the outer edge of the cap. The clamped case allows no displacement or rotation along the outer edge.

One of the first investigations of a uniformly loaded plastic spherical cap was performed by Onat and Prager [3] in 1954. The load carrying capacity of shells of revolution made of a rigid-plastic material that obeys the Tresca yield condition and the associated flow rule was investigated. The work considered axially symmetric conditions of loading and support. The yield criterion was specified in terms of the stress resultants of the shell, and the associated flow rule was given in terms of the rates of extension and curvature of the middle surface of the uniform thick shell. Through the use of the yield condition and flow rule, the load intensity at which a rigid-plastic shell would first begin to deform could be obtained. A uniformly loaded spherical cap, clamped along the boundary, was used as an example to demonstrate the theory. The bounds obtained for the collapse load were very crude, especially for shallow caps, where the solution was based on the assumption of a membrane-type stress field.

The complexity of the mathematical problem associated with a given physical problem will depend partially on the complexity of the yield surface. For this reason, the actual yield surface is often approximated by a simple shape before specific solutions are attempted. However, even if the exact yield surface is determined, it is not always obvious how to construct a linear approximation. In 1958, Hodge [4] replaced a homogeneous section of the uniform shell by a section consisting of a sandwich shell so that the overall properties
of the structure were unchanged. For this nonhomogeneous structure
the actual yield condition was found to be piecewise linear. Therefore,
it was found that for complex structures, it is often simpler to deter-
mine the piecewise linear yield surface for the nonhomogeneous struc-
ture than it is to determine exact surface for the uniform structure.

In 1959, Hodge [5,6] applied this technique of the approximate
structure to uniformly loaded spherical caps with simply supported
and with clamped boundaries. Upper and lower bounds on the load inten-
sity were obtained. There was good agreement for the simply supported
caps; the worst bounds differed by about 22% of the lower bound. For
the clamped cap, the agreement was less satisfactory, the discrepancy
being as high as 50% for a considerable range of cap angles. It was
pointed out that improvement in the bounds using this technique
appeared unfeasible in view of the complexity of the formulas; however,
the results were still superior to any results previously presented
in the literature.

In 1960, Hodge [7,8] suggested a new approximation which
combined the advantages of mathematical simplicity and reasonable accu-
curacy. It was argued that in most shell problems the moments and
direct forces will not be of simultaneous importance so that coupling
of moment and force in the yield relations are of limited importance.
However, either moment or direct force may dominate so that all
resultants must be included in the yield condition. Thus, all inter-
action between force and force or between moment and moment is main-
tained, but all interaction between force and moment is neglected.
The result was the two-moment limited-interaction yield condition, a linear yield surface in four-dimensional space defined by twelve planes. The exact collapse load for uniformly loaded spherical caps with both clamped and simply supported edges were found, using this approximate yield condition.

One case has been treated involving the collapse load of a simply supported partially loaded spherical cap. This investigation was conducted by Sankaranarayanan [9], and the concentrated collapse load at the apex of the cap was determined. It was pointed out by Sankaranarayanan that it is possible to obtain an exact solution to a rotationally symmetric shell problem by using the piecewise linear yield conditions previously discussed. However, in practice, the equations become quite complex and it is doubtful if the resulting labor is worthwhile. Therefore, a much simpler yield surface was proposed, a square yield condition. Based on the proposed yield condition, considerable mathematical simplicity was found to exist in the analysis; however, since the proposed yield surface circumscribed the Tresca yield condition, the collapse load is an upper bound to that obtained on the basis of the Tresca yield condition.

More recently, numerical techniques and high speed computers have been applied to find bounds or exact solutions of the uniformly loaded spherical cap. Lance and Rickert [10], in 1966, found lower bounds for the clamped spherical caps by using a linear programming technique. The cap was assumed to be of sandwich construction and obey the Tresca yield condition and associated flow rule. In 1968,
Lee and Onat [11] obtained exact collapse loads for uniformly loaded, clamped spherical shells—both shallow and deep. The shell was assumed to be of sandwich construction and obey the Tresca yield condition and associated flow rule. Also, the entire shell was assumed to undergo plastic deformations at the yield point state. With these assumptions, a search for appropriate plastic regimes on the yield surface was made. When the proper regimes were found, it was shown that the solution was both statically and kinematically admissible and thus the exact solution.

Olszak, Mróz, and Perzyna [12] present a comprehensive survey of the literature of the field aimed at determining trends in the development of the theory of plasticity. They conclude that no results of investigations of other types of shells or loading conditions other than rotationally symmetric shells under axisymmetric loading have been reported. Principal results reported in this area include the work of Hodge [7] dealing with bounds for the uniformly loaded spherical cap, the work of Feinberg [13] and Hazalia [14] who considered uniformly loaded shallow spherical caps and the work on approximate solutions for spherical shells by Rozhdestvensky [15,16]. A survey of the material concerning the loading of spherical caps was reviewed by Vann [17] in 1968. Additional work on uniformly loaded spherical caps was reported by Chuh-Sheng Nien [18] and by Shablii [19] to the present time; however, only one case of a partially loaded spherical cap has been found—that of Sankaranarayanan [9] previously mentioned.
As seen from the review of the literature, there has been little work reported on determining the collapse load intensity of partially loaded spherical caps. However, the work on uniformly loaded caps, that has been reported, may be used as guidance in seeking a complete solution or bounds on the load intensity for the partially loaded structure. Most of the previous studies sought solutions by making appropriate assumptions about the shell behavior—stress and/or velocity assumptions which were sufficiently close to the actual shell performance. The closer the assumption is to the actual behavior, the closer are the bounds. The present problem of the partially loaded cap is sufficiently complex so that guessing or assuming a part of the exact solution is very likely impossible; however, it should be possible to make sufficiently accurate assumptions that reasonable load bounds for limited ranges may be obtained.

The theorems of limit analysis were first formulated by Drucker, Greenberg, and Prager [21,22] and by Hill [23,24]. The theory of limit analysis is concerned only with the infinitesimal motions that take place at the instant of plastic collapse. Therefore, we may speak of either deformations or deformation-rates. Since the motions are infinitesimal, it is assumed that the behavior of the shell is adequately approximated by a theory in which normals to the middle surface of the shell remain straight and normal to the deformed middle surface. The loads and displacements of the shell are assumed to be rotationally symmetric, that is, there are no loads or displacements in the circumferential direction of the shell.
There are two fundamental theorems of limit analysis, the Lower-Bound and the Upper-Bound Theorems. The Lower-Bound Theorem [1] is stated in terms of a statically admissible stress field. A statically admissible stress field is, by definition, a field of generalized stresses, \( n_0, n_\rho, m_0, m_\rho, \) shear \( s^- \), and load \( p^- \), such that all equilibrium requirements (satisfaction of equilibrium equations, stress boundary conditions, and continuity of \( n_0, s^-, \) and \( m_\rho \)) are satisfied and the yield condition is nowhere violated. Therefore, if there exists any statically admissible stress field for the load \( p^- \), then \( p^- \) is a lower bound on the yield-point load.

The Upper-Bound Theorem [1] is concerned with any velocity field which satisfies the velocity boundary conditions. From the velocity field a strain-rate field is computed and the strain-rate vector determines an associated stress field by means of the yield condition and flow rule. The external energy is defined as the energy due to a load \( p^+ \) on the above velocity field. Equating the internal and external energy expressions, a kinematically admissible field is defined provided the external energy is positive. Now it can be stated that if there exists any kinematically admissible field for the load \( p^+ \), then \( p^+ \) is an upper bound on the yield-point load. Proofs of the two fundamental theorems of limit analysis may be found in the work of Hodge [1].

In obtaining lower bounds for the partially loaded, simply supported cap, the available solutions for the uniformly loaded
spherical cap will be used as guidance in determining which plastic regimes of the yield surface to use. For the regimes of the yield surface assumed, the solution will be shown to satisfy the stress equilibrium and stress boundary conditions. Also, it will be shown that the solution does not violate the yield condition. Such a solution is statically admissible and thus a valid lower bound.

A velocity field may be determined by requiring compatibility with the assumption that the stresses are on certain faces of the yield surface. If these faces correspond to those which were utilized in finding a close lower bound, then the velocity field may be expected to yield a solution which is reasonably close to the lower bound. From this velocity field, the strain-rates may be found and the cap internal rate of dissipation of energy may be obtained. Equating the internal and external energy expressions, an upper bound on the collapse load is found.

Usually in problems of this type, where upper and lower bounds are determined, certain cap parameters must be restricted in range in order that solutions remain valid. For the load bounds found in this investigation, the range of validity of the solutions has been determined.

In considering the partially loaded, clamped spherical caps, an approach similar to that of Lee and Onat [11] is utilized. The same regions of the yield surface which define the stress and velocity fields will be used. Beginning at a corner of the yield surface, the stress state of the shell moves along certain faces of the yield
surface satisfying the equations of equilibrium simultaneously. The motion of the stress point will continue until a predetermined stopping point on the yield surface is reached. At this point a statically admissible field is obtained. Then, moving backward along the same path of the yield surface to the originating point, a kinematically admissible field will be obtained if the velocity requirements associated with the yield surface regimes are not violated. Thus, statically and kinematically admissible fields are constructed and a complete solution is determined.

Up to the present time a complete solution has been obtained for the uniformly loaded, simply supported spherical cap \([1,7,8]\) which obeys the simple two-moment limited-interaction yield condition. This yield condition neglects all interaction between force and moment. However, a complete solution has not been found for the uniformly loaded simply supported cap obeying the Tresca yield condition for the sandwich shell. In view of the fact that an exact solution has not been found for this latter problem, it may be suspected that an exact solution for the partially loaded, simply supported spherical cap is highly improbable. However, a complete solution has been found for the uniformly loaded clamped cap \([11]\), and, therefore, a cap which is loaded over a major portion of the surface may possibly be expected to yield an exact solution using similar techniques.
CHAPTER II

GOVERNING EQUATIONS

2.1 Equilibrium Requirements

In considering the geometry of the shell, the angle between a normal to the shell and the positive Z-direction is chosen as the independent variable. The shell geometry is then described by its two radii of curvature: \( R_1 \) in the \( \varphi \)-direction and \( R_2 \) in the \( \theta \)-direction, as shown in Figure 2.1, and

\[
R_0 = R_2 \sin \varphi
\]  
(2.1)

The stress state of the shell is defined by stress resultants per unit length, which include two direct forces \( N_\varphi \) and \( N_\theta \), two moments \( M_\varphi \) and \( M_\theta \), and a shear force \( S \). These resultants and sign conventions are illustrated in Figure 2.1.

For a typical element of the shell in a state of equilibrium under the five stress resultants applied to its edges and the loads \( P_n \) and \( P_\varphi \) applied to its surface, the governing equations of equilibrium are

\[
(R_0 N_\varphi)' - R_1 N_\theta \cos \varphi - R_0 S + R_1 P_\varphi = 0
\]

\[
(R_0 S)' + R_1 N_\theta \sin \varphi + R_0 N_\varphi + R_1 P_n = 0
\]

\[
(R_0 M_\varphi)' - R_1 M_\theta \cos \varphi - R_0 R_1 S = 0
\]

(2.2)
Figure 2.1. Shell element.
where primes indicate differentiation with respect to \( \theta \). These equations are derived and discussed in detail in [1,5,20]. Let \( L \) be a typical length of the shell and define dimensionless quantities by

\[
\begin{align*}
\sigma &= \frac{S}{N_o}, \\
P_n &= \frac{P_n L}{2N_o}, \\
P_\theta &= \frac{P_\theta L}{2N_o}, \\
n_\theta &= \frac{N_\theta}{N_o}, \\
n_\phi &= \frac{N_\phi}{N_o}, \\
m_\theta &= \frac{M_\theta}{M_o}, \\
m_\phi &= \frac{M_\phi}{M_o}
\end{align*}
\]

(2.3)

\[
\begin{align*}
R_o &= \frac{R_o}{L}, \\
R_1 &= \frac{R_1}{L}, \\
R_2 &= \frac{R_2}{L}, \\
k &= \frac{M_o}{N_o L^2} = \frac{t}{4L}
\end{align*}
\]

(2.4)

(2.5)

where \( N_o \) and \( M_o \) are the maximum uniaxial direct force and the maximum bending moment, respectively, that the shell can withstand, and \( t \) is the thickness of the shell. Using the dimensionless quantities (2.3) through (2.5), and letting \( R = R_1 = R_2 = L \), the case of the spherical cap, the equations of equilibrium (2.2) may be written as

\[
\begin{align*}
(n_\phi \sin \theta)' - n_\theta \cos \theta &= s \sin \theta \\
(n_\phi + n_\theta + 2p) \sin \theta &= - (s \sin \theta)' \\
k[n_\phi \sin \theta)' - m_\phi \cos \theta] &= s \sin \theta
\end{align*}
\]

(2.6)

where the \( P_\phi \) term has been omitted, since rotational symmetry has been assumed and \( P_n = P \), since only one normal external load is applied (see Hodge [1,5]). If the shell does not have the externally applied load \( P \), then the equations of equilibrium become
Now the equations of equilibrium are to be written in a form which does not contain $s$, since the shear strains have been neglected because of the assumption that straight normals to the middle surface remain straight and normal. The shear force $s$ does no work and is treated only as a reaction and does not enter into the yield criteria. Thus the equations of equilibrium may be written as

\[
(n_0 \sin \phi)' - n_0 \cos \phi = s \sin \phi
\]

\[
(n_0 + n_0) \sin \phi = - (s \sin \phi)'
\]

\[
k[(m_0 \sin \phi) - m_0 \cos \phi] = s \sin \phi
\]  

(2.7)

\[
n_0' = [n_0 - n_0 - (n_0 + p) \tan^2 \phi] \cot \phi
\]

\[
m_0' = [m_0 - m_0 - \frac{1}{k}(n_0 + p) \tan^2 \phi] \cot \phi
\]  

(2.8)

for the region under the loaded portion of the shell and

\[
n_0' = [n_0 - n_0 - n_0 \tan^2 \phi - p \frac{\sin^2 \phi}{\cos^2 \phi}] \cot \phi
\]

\[
m_0' = [m_0 - m_0 - \frac{1}{k} n_0 \tan^2 \phi - \frac{1}{k} p \frac{\sin^2 \phi}{\cos^2 \phi}] \cot \phi
\]  

(2.9)

for the region outside of the loaded region of the cap where $p = 0$ (see Appendix A). Now either of two forms of the equations of equilibrium may be used, (2.6) and (2.7) or (2.8) and (2.9). It will be seen that the form of this latter set, (2.8) and (2.9), is more convenient to use whenever the computer is required to obtain numerical results.
2.2 Strain-Rate-Velocity Relations

Since deformations or deformation-rates may be used, the strain-rate-velocity relations will be the same as the strain-displacement relations. Since only symmetrical displacements are being considered, a small displacement of a point on the middle surface of the shell can be resolved into two components: $V$ in the direction of the tangent to the meridian and $W$ in the direction of the normal to the middle surface. The extensions and curvatures of the middle surface of the shell are

\[
\varepsilon_\theta = \frac{V \cot \theta - W}{R_2}, \quad \varepsilon_\phi = \frac{V' - W}{R_1}, \quad K_\theta = -\frac{k \cot \theta}{R_2} \left(\frac{V + W'}{R_1}\right), \quad K_\phi = -\frac{k}{R_1} \left(\frac{V + W'}{R_1}\right)
\]

These equations are derived and discussed in detail in [1,5,20].

Next we define the dimensionless strain-rates

\[
\varepsilon_\theta = \dot{\varepsilon}_\theta, \quad \varepsilon_\phi = \dot{\varepsilon}_\phi, \quad \kappa_\theta = \frac{M_0}{N_0} \dot{\kappa}_\theta, \quad \kappa_\phi = \frac{M_0}{N_0} \dot{\kappa}_\phi
\]

and dimensionless velocities

\[
v = \frac{\dot{V}}{L}, \quad w = \frac{\dot{W}}{L}
\]

Then the middle surface strain-rates, using (2.10), (2.11) and (2.12) may be written [1,5] as
\[ \epsilon_\theta = \frac{v \cot \theta - w}{r_2}, \quad \epsilon_\phi = \frac{v' - w}{r_1} \]  
\[ \kappa_\theta = - \frac{k \cot \theta}{r_2} \left( \frac{v + w'}{r_1} \right), \quad \kappa_\phi = - \frac{k}{r_1} \left( \frac{v + w'}{r_1} \right) \]

For the case of the spherical cap, \( r_1 = r_2 = 1 \), (2.13) becomes

\[ \epsilon_\theta = v \cot \theta - w \quad , \quad \epsilon_\phi = v' - w \]
\[ \kappa_\theta = - k \cot \theta \left( v + w' \right), \quad \kappa_\phi = - k \left( v + w' \right) \]

If \( v \) and \( w \) are eliminated from (2.14), then the equations of compatibility for the generalized strain-rates are [11]

\[ \epsilon_\theta' = (\epsilon_\phi - \epsilon_\theta + \frac{1}{k} \kappa_\theta \tan^2 \theta) \cot \theta \]
\[ \kappa_\theta' = (\kappa_\phi - \kappa_\theta \sec^2 \theta) \cot \theta \]

The principle of virtual work [1] states that if for any system of loads and stresses in internal and external equilibrium, and any system of strains and displacements which satisfy the strain-displacement equations, the internal work done by the stress field on the strains must equal the external work done by the loads on the displacements. For the shell boundary value problem, the internal work is that done by the generalized stresses on the generalized strains

\[ W_1 = 2\pi \int (N_\theta \epsilon_\theta + N_\phi \epsilon_\phi + M_\theta \kappa_\theta + M_\phi \kappa_\phi) R_o r_1 d\theta \]
The external work is that done by the loads on the displacements

$$W_e = 2\pi \int (PW) R_o R_1 \, d\theta$$

(2.17)

where $P$ is the only load and is applied normal to the cap surface.

For a perfectly plastic material the role of work is taken by the rate of energy dissipation, where velocities $\dot{V}$ and $\dot{W}$ and generalized strain-rates $\dot{\varepsilon}_\theta$, $\dot{\varepsilon}_\phi$, $\dot{\varepsilon}_r$, $\dot{\varepsilon}_\rho$, $\dot{\varepsilon}_\theta$, $\dot{\varepsilon}_\phi$, $\dot{\varepsilon}_r$, $\dot{\varepsilon}_\rho$ are used rather than strains and deformations. In this way equations (2.16) and (2.17) may be written as

$$D_i = 2\pi \int (N_\theta \dot{\varepsilon}_\theta + N_\phi \dot{\varepsilon}_\phi + M_\theta \dot{\varepsilon}_\theta + M_\phi \dot{\varepsilon}_\phi) R_o R_1 \, d\theta$$

(2.18)

and

$$D_e = 2\pi \int (\dot{P}\dot{\varepsilon}_\rho \sin \theta) R_o R_1 \, d\theta$$

(2.19)

With the help of the dimensionless quantities defined in (2.3), (2.4), (2.5), and (2.11), the rates of energy of dissipation (2.18) and (2.19) may be written, for the case of the spherical cap, as

$$d_i = \int (n_\theta \dot{\varepsilon}_\theta + n_\phi \dot{\varepsilon}_\phi + m_\theta \dot{\varepsilon}_\theta + m_\phi \dot{\varepsilon}_\phi) \sin \theta \, d\theta$$

(2.20)

and

$$d_e = 2\int p \dot{w} \sin \theta \, d\theta$$

(2.21)

In determining lower bounds, the stress equilibrium requirements must be satisfied along with the yield condition requirements. Thus, equations (2.6) and (2.7), or (2.8) and (2.9) give the stress
equilibrium equations. The yield condition requirements will be presented in the following chapter. Equations (2.12) give the strain-rate-velocity relations which must be used in order to obtain upper bounds, and (2.20) and (2.21) provide expressions for determining the internal and external energy.
The structure material used throughout this investigation is assumed to be rigid-perfectly plastic. It is also assumed to obey the Tresca yield condition and the associated flow rule. The homogeneous structure will be approximated by a nonhomogeneous structure; that is, the uniform shell will be replaced by the idealized sandwich shell and the exact yield condition for this structure approximation will be used as the yield criteria.

3.1 Tresca Condition for Sandwich Shell

First the uniform shell of thickness $t$ and tensile and compressive yield stress $\sigma_0$ is replaced by the idealized sandwich shell with the same resistance to pure tension or pure bending. The sandwich shell is composed of upper and lower thin sheets of thickness $\ell$ separated by a core of thickness $2H'$ (see Figure 3.1(a)). Each sheet is assumed to obey the Tresca yield condition, has a yield stress of $\sigma_0'$ and is so thin that the stress variation across it can be neglected. The core has sufficient stiffness to maintain the separation of the sheets but its strength is negligible compared with the face sheets. With these definitions, the maximum uniaxial direct stress resultant per unit width is
Figure 3.1. Sandwich shell approximation.  
(a) Thin sheets and core.  
(b) Stress distribution.
for the sandwich shell and

$$N_0 = 2\sigma_0 J$$

(3.1)

for the sandwich shell and

$$N_0 = \sigma_0 t$$

(3.2)

for the uniform shell. The maximum bending moment per unit width is

$$M_0 = 2\sigma_0' H'J$$

(3.3)

for the sandwich shell and

$$M_0 = \frac{1}{4} \sigma_0 t^2$$

(3.4)

for the uniform shell. The two shells will be equivalent if

$$\sigma_0' J = \frac{1}{2} \sigma_0 t, \quad H' = \frac{1}{4} t$$

(3.5)

The stress distribution of a typical point of the idealized sandwich shell is shown in Figure 3.1(b). The stress resultants are

$$N_\theta = J(\sigma^+_\theta + \sigma^-_\theta), \quad N_\varphi = J(\sigma^+_\varphi + \sigma^-_\varphi)$$

(3.6)

$$M_\theta = H'J(\sigma^-_\theta - \sigma^+_\theta), \quad M_\varphi = H'J(\sigma^-_\varphi - \sigma^+_\varphi)$$

Solving (3.6) for the stresses,

$$\sigma^+_\theta = \frac{H'N_\theta - M_\theta}{2H'J}, \quad \sigma^+_\varphi = \frac{H'N_\varphi - M_\varphi}{2H'J}$$

(3.7)

$$\sigma^-_\theta = \frac{H'N_\theta + M_\theta}{2H'J}, \quad \sigma^-_\varphi = \frac{H'N_\varphi + M_\varphi}{2H'J}$$
Now from (3.1) and (3.3), it is seen that

\[
2H'J = \frac{M}{\sigma_0}, \quad H' = \frac{M}{N_0}
\]  

(3.8)

Therefore, using (3.8), the stresses in terms of the dimensionless stress resultants defined in (2.4) are

\[
\begin{align*}
\sigma_0^+ &= \sigma'_0(n_0 - m_0), & \sigma_0^- &= \sigma'_0(n_0 - m_0) \\
\sigma_y^+ &= \sigma'_0(n_y - m_y), & \sigma_y^- &= \sigma'_0(n_y + m_y)
\end{align*}
\]

(3.9)

Tresca's yield condition states that the maximum shearing stress is less than half the tensile yield stress [1]; hence,

\[
\max (|\sigma_0|, |\sigma_y|, |\sigma_0 - \sigma_y|) \leq \sigma'_0
\]

(3.10)

The six inequalities implied by (3.10) must be satisfied by the stresses (3.9) in both the top and bottom sheets of the sandwich shell. Therefore, the Tresca yield condition for the idealized sandwich shell consists of twelve linear expressions as listed in Table 3.1.

The direction of the strain-rate vector is the same for all stress points on a given face of the yield surface, since each face is represented by a linear expression, and is given by the gradient of the equation of the face according to the associated flow law. The direction of the strain-rate vector is indicated in the last column of Table 3.1. At the intersection of two or more faces, the
<table>
<thead>
<tr>
<th>Face</th>
<th>Stress Equation</th>
<th>Strain-Rate Vectors ${\varepsilon_\theta, \varepsilon_\phi, \kappa_\theta, \kappa_\phi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$\sigma_{\theta} - \sigma_{\phi} = 1$</td>
<td>$\lambda(1, 0, -1, 0)$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\phi} - \sigma_{\phi} = 1$</td>
<td>$\lambda(0, 1, 0, -1)$</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_{\theta} + \sigma_{\phi} + \sigma_{\phi} - \sigma_{\phi} = 1$</td>
<td>$\lambda(-1, 1, 1, -1)$</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_{\theta} + \sigma_{\phi} = 1$</td>
<td>$\lambda(-1, 0, 1, 0)$</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_{\phi} + \sigma_{\phi} = 1$</td>
<td>$\lambda(0, -1, 0, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\phi} - \sigma_{\phi} - \sigma_{\phi} + \sigma_{\phi} = 1$</td>
<td>$\lambda(1, -1, -1, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\phi} - \sigma_{\phi} = 1$</td>
<td>$\lambda(1, 0, -1, 0)$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\phi} + \sigma_{\phi} = 1$</td>
<td>$\lambda(0, 1, 0, 1)$</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_{\theta} + \sigma_{\phi} - \sigma_{\phi} + \sigma_{\phi} = 1$</td>
<td>$\lambda(-1, 1, -1, 1)$</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_{\theta} - \sigma_{\phi} = 1$</td>
<td>$\lambda(-1, 0, -1, 0)$</td>
</tr>
<tr>
<td></td>
<td>$-\sigma_{\phi} - \sigma_{\phi} = 1$</td>
<td>$\lambda(0, -1, 0, -1)$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\phi} - \sigma_{\phi} + \sigma_{\phi} - \sigma_{\phi} = 1$</td>
<td>$\lambda(1, -1, 1, -1)$</td>
</tr>
</tbody>
</table>
resulting strain-rate vector may be any linear combination, with positive coefficients, of the strain-rate vectors for the intersecting sides.

### 3.2 Geometrical Representation

It is convenient to give the yield surface a geometrical representation. Since the yield surface is described in terms of four dimensionless stress resultants, \( n_\theta, n_\varphi, m_\theta, \) and \( m_\varphi, \) and consequently in four-dimensional space, a direct representation of the surface is not possible. However, it is possible to visualize the yield surface partially by considering two, two-dimensional descriptions of the surface simultaneously.

Consider two Cartesian coordinate systems of \( (n_\varphi-m_\varphi, n_\theta-m_\theta) \) and \( (n_\varphi+m_\varphi, n_\theta+m_\theta) \). The stress equations in Table 3.1 may be represented as shown in Figure 3.2. The faces labeled in the figure correspond with those given in the table.

It has been shown [5,11] (see Appendix B) that plastic flow can take place only if the stress state of the shell is simultaneously on one or more of the faces of each profile in Figure 3.2. Thus the associated flow rule is

\[
\varepsilon_\theta = \lambda_1 \frac{\partial f_{io}}{\partial n_\theta} + \lambda_2 \frac{\partial f_{oi}}{\partial n_\theta}, \quad \kappa_\theta = \lambda_1 \frac{\partial f_{io}}{\partial m_\theta} + \lambda_2 \frac{\partial f_{oi}}{\partial m_\theta}
\]

\[
\varepsilon_\varphi = \lambda_1 \frac{\partial f_{io}}{\partial n_\varphi} + \lambda_2 \frac{\partial f_{oi}}{\partial n_\varphi}, \quad \kappa_\varphi = \lambda_1 \frac{\partial f_{io}}{\partial m_\varphi} + \lambda_2 \frac{\partial f_{oi}}{\partial m_\varphi}
\]

If the stress state is on two of these faces simultaneously, it may be termed a fully plastic side. A plane representation of the fully
Figure 3.2. Geometrical representation of the Tresca condition for sandwich shell.
plastic sides of the yield surface has been previously presented [5] and is given in Figure 3.3. Each square represents a fully plastic side.

Comparing Figures 3.2 and 3.3, it is seen that a fully plastic side corresponds to being on one face of each profile at the same time. A line between two fully plastic sides in Figure 3.3 corresponds to a corner of one profile and a face of the other simultaneously. Likewise, the intersection of four fully plastic sides in Figure 3.3 corresponds to being at a corner of each profile of Figure 3.2 simultaneously.

The equations of equilibrium, the strain-rate-velocity relations and the yield condition have now been presented. Thus, the necessary equations to obtain upper and lower bounds on the collapse load of partially loaded shallow spherical caps have been obtained.
Figure 3.3. Outline of the fully plastic sides of the yield surface [5].
In order to obtain a lower bound, it is necessary everywhere to satisfy the stress equilibrium requirements and nowhere violate the yield condition. If the stress profile of the cap is assumed to lie on (or within) certain plastic regimes of the yield surface and it is shown that the equilibrium requirements are everywhere satisfied, then a statically admissible stress field will have been constructed and a valid lower bound obtained.

The difficulty in this problem arises in determining the appropriate plastic regimes for the cap. It may be expected that the closer the choice of the stress profile is to the actual stress state of the cap, the better will be the lower bound. Since very little previous limit analysis work has been reported for partially loaded spherical caps, the solution for the uniformly loaded spherical cap is used as guidance in choosing appropriate plastic regimes.

4.1 Stress Profile

At the center of the cap, $\vartheta = 0$, the stresses must remain finite and because of symmetry

$$n_\vartheta = n_0, \quad m_\vartheta = m_0 \quad \text{at } \vartheta = 0 \quad (4.1)$$

must hold. Of all the possible stress states producing plastic flow for the considered yield surface, the only ones which can satisfy (4.1) are
\[ n_\theta = n_\phi = 0, \quad m_\theta = m_\phi = \pm 1 \quad (4.2) \]

or

\[ n_\theta = n_\phi = \pm 1, \quad m_\theta = m_\phi = 0 \quad (4.3) \]

The states of stress (4.2) and (4.3) correspond to faces 4578, 120A, 1278 and 450A, respectively, of the stress profiles in Figure 3.2. This has been previously pointed out by Lee and Onat [11] in studying the uniformly loaded spherical cap.

Since the applied pressure is external and inward membrane forces are expected to be negative at \( \phi = 0 \), then the negative sign must be employed in (4.3). Similarly, the positive sign is expected to hold in (4.2). For a shallow cap, it is expected that the solution would be primarily bending; therefore, it is assumed that the stress profile begins at faces 4578 (see Figure 4.1). This starting point also corresponds to that of the partially loaded circular plate and the uniformly loaded spherical cap mentioned in Hopkins and Prager [2] and Lee and Onat [11], respectively. Using the uniformly loaded simply supported spherical cap stress profile as guidance, it is postulated that the entire stress profile for the partially loaded spherical cap on faces 47 of the stress profiles (see Figure 4.1).

4.2 Lower-Bound Collapse Load

In determining the collapse load of the partially loaded spherical cap, there are two regions of the cap to consider. There is the region \( 0 \leq \phi \leq \phi_L \) in which the cap has the distributed load with uniform intensity \( p \), and the region \( \phi_L < \phi \leq \phi_o \) which has no external load, i.e., \( p = 0 \) (see Figure 4.2).
Figure 4.1. Stress profile for lower-bound collapse load.
Figure 4.2. Partially loaded shallow spherical cap.
For the stress profile considered in the previous section (see Figure 4.1)

\[ n_\theta - m_\theta = -1, \quad n_\phi + m_\phi = 1 \]

which implies that

\[ n_\phi = 0, \quad m_\phi = 1 \]

throughout the cap, \( 0 \leq \phi \leq \phi_c \). Now if (4.4) is substituted into the three equations of equilibrium, then only three unknown stress resultants remain, \( s, n_\phi, \) and \( m_\phi \). The three equations can be solved to obtain the three unknowns. Once all stress resultants are found and the stress boundary conditions satisfied, then it will be shown that the yield condition is not violated in order to obtain a statistically admissible lower bound.

Consider the region of the cap \( 0 \leq \phi \leq \phi_L \). Substituting (4.4) into the equations of equilibrium (2.6) yields

\[ (n_\phi \sin \phi)'' = s \sin \phi \quad (4.5) \]
\[ (n_\phi + 2p) \sin \phi = -(s \sin \phi)'' \quad (4.6) \]
\[ (m_\phi \sin \phi)'' - \cos \phi = \frac{1}{k} s \sin \phi \quad (4.7) \]

Combining (4.5) and (4.6) and solving for \( n_\phi \), then

\[ n_\phi = C_1 \cot \phi + C_2 - p(1 - \phi \cot \phi) \quad (4.8) \]

Using (4.8) in (4.5) leads to

\[ s = C_2 \cot \phi - p \phi \quad (4.9) \]
From the condition (4.1) and \( s = 0 \) at \( \theta = 0 \), \( C_1 \) and \( C_2 \) can be evaluated from (4.8) and (4.9)

\[
C_1 = 0, \quad C_2 = 0 \tag{4.10}
\]

Therefore, from (4.8), (4.9) and (4.10)

\[
n_\theta = - p(1 - \theta \cot \theta) \quad \text{for} \quad (0 \leq \theta \leq \theta_L) \tag{4.11}
\]

and

\[
s = - p \theta \quad \text{for} \quad (0 \leq \theta \leq \theta_L) \tag{4.12}
\]

Now substituting (4.12) into (4.7) and solving for \( m_\theta \), then

\[
m_\theta = 1 - \frac{1}{k} p(1 - \theta \cot \theta) + \frac{C_3}{\sin \theta} \tag{4.13}
\]

Again from condition (4.1), it is seen from (4.13) that

\[
C_3 = 0 \tag{4.14}
\]

Therefore,

\[
m_\theta = 1 - \frac{1}{k} p(1 - \theta \cot \theta) \quad \text{for} \quad (0 \leq \theta \leq \theta_L) \tag{4.15}
\]

Consider the region of the cap \( \theta_L < \theta \leq \theta_0 \). Substituting (4.4) into the equations of equilibrium (2.7) yields

\[
(\eta_\theta \sin \theta)' = s \sin \theta \tag{4.16}
\]

\[
\eta_\theta \sin \theta = -(s \sin \theta)' \tag{4.17}
\]

\[
(m_\theta \sin \theta)' - \cos \theta = \frac{1}{k} s \sin \theta \tag{4.18}
\]

Combining (4.16 and (4.17) then

\[
\eta_\theta = C_4 \cot \theta + C_5 \quad \text{for} \quad (\theta_L < \theta \leq \theta_0) \tag{4.19}
\]
Substituting (4.19) into (4.15) then
\[ s = -C_4 + C_5 \cot \theta \quad \text{for} \quad (\theta_L < \theta \leq \theta_o) \quad (4.20) \]

Since continuity in \( s \) and \( n_\theta \) must be maintained at \( \theta = \theta_L \), and
using (4.11) and (4.19), and (4.12) and (4.20), then \( C_4 \) and \( C_5 \) are
\[ C_4 = p(\theta_L - \sin \theta_L \cos \theta_L) \quad (4.21) \]
\[ C_5 = -p \sin^2 \theta_L \quad (4.22) \]

Now substituting (4.20) into (4.18) and solving for \( m_\theta \) then
\[ m_\theta = 1 + \frac{1}{k}(C_4 \cot \theta + C_5) + \frac{C_6}{\sin \theta} \quad (4.23) \]

Again, since continuity in \( m_\theta \) must be maintained at \( \theta = \theta_L \), then
from (4.15) and (4.23)
\[ C_6 = 0 \quad (4.24) \]

Therefore,
\[ m_\theta = 1 + \frac{1}{k}(C_4 \cot \theta + C_5) \quad \text{for} \quad (\theta_L < \theta \leq \theta_o) \quad (4.25) \]

From (4.11) it is seen that
\[ n_\theta' = -p \csc^2 \theta (\theta - \sin \theta \cos \theta) \quad (4.26) \]

and
\[ n_\theta'' = -2p \csc^2 \theta (1 - \theta \cot \theta) \quad (4.27) \]

Since \( p \), \( (\theta - \sin \theta \cos \theta) \), and \( (1 - \theta \cot \theta) \) are positive, then
\( n_\theta' \) and \( n_\theta'' \) are both negative. Therefore, in the region \( 0 \leq \theta \leq \theta_L \),
\( n_\theta \) is zero at \( \theta = 0 \) and is concave downward with negative slope.
From (4.19) it is seen that

$$n_{\varphi}' = -C_4 \csc^2 \varphi$$  \hspace{1cm} (4.28)

and

$$n_{\varphi}'' = 2C_4 \csc^2 \varphi \cot \varphi$$  \hspace{1cm} (4.29)

From (4.21), $C_4$ is positive, therefore, in the region $\varphi_L < \varphi \leq \varphi_o$, $n_{\varphi}'$ is negative and $n_{\varphi}''$ is positive. Thus $n_{\varphi}$ is concave upward with negative slope and is at a maximum negative value at $\varphi = \varphi_o$.

In both regions

$$n_{\varphi} = 1 + \frac{1}{k} n_{\varphi}'$$  \hspace{1cm} (.30)

Therefore, $n_{\varphi} = 1$ at $\varphi = 0$ and decreases with negative slope in the same manner as $n_{\varphi}'$, since $1/k$ is positive. Now if (4.25) is to satisfy the boundary condition $m_{\varphi}' = 0$ at $\varphi = \varphi_o$, the collapse load must be given by

$$p = \frac{k}{[\sin^2 \varphi_L - \cot \varphi_o (\varphi_L - \sin \varphi_L \cos \varphi_L)]}$$  \hspace{1cm} (4.31)

From (4.19) at $\varphi = \varphi_o$

$$n_{\varphi} = -p[\sin^2 \varphi_L - \cot \varphi_o (\varphi_L - \sin \varphi_L \cos \varphi_L)]$$  \hspace{1cm} (4.32)

where (4.21) and (4.22) have been employed. Now using (4.31) in (4.32)

$$n_{\varphi} = -k$$  \hspace{1cm} (4.33)
In order to remain on faces 47 of the yield surface, then from Figure 4.1 it is seen that

\[ 0 \leq n_y + m_y \leq 1 \quad (4.34) \]

\[ -1 \leq n_y - m_y \leq 0 \]

must hold. Using (4.30), then (4.34) can be written as

\[ 0 \leq [1 + \left( \frac{1+k}{k} \right) n_y] \leq 1 \quad (4.35) \]

\[ -1 \leq [1 + \left( \frac{1-k}{k} \right) n_y] \leq 0 \]

or

\[ -\frac{k}{1+k} \leq n_y \leq 0 \quad (4.36) \]

\[ -\frac{k}{1-k} \leq n_y \leq 0 \]

Comparing with (4.33), it is seen that the first of (4.36) is violated. However, if the pressure \( p \) and the stresses are each divided by the factor \( 1+k \), the resulting solution will still be in equilibrium and will also satisfy the yield condition. Therefore, from (4.31)

\[ p^- = \frac{(\frac{k}{1+k})}{[\sin^2 \varphi_L - \cot \varphi_o (\varphi_L - \sin \varphi_L \cos \varphi_L)\]} \quad (4.37) \]

is a lower bound on the collapse load.
Thus (4.37) is a lower bound for the simply supported, partially loaded spherical cap and is valid for all values of $\theta$ within the range here treated as shallow caps. The results of this equation are presented in Figures 5.2 and 5.3 in Section 5.7, where they are compared with upper bounds obtained in Chapter V.
CHAPTER V

UPPER-BOUND LOAD - SIMPLY SUPPORTED CAP

In determining an upper bound, it is necessary to find a kinematically admissible velocity field. In order to obtain such a field, the assumed velocity field must satisfy any velocity constraints on the structure and the total external rate of work done by the actual loads on the assumed velocities must be positive. Using the strain-rates derived from the velocity field, the internal rate of dissipation of energy may be calculated. Likewise, the external rate may be determined from the applied load and velocity field. Then according to the upper-bound theorem of limit analysis, an upper bound on the collapse load may be obtained by equating the internal and external energy expressions.

In the following sections, the cap angle $\phi_0$ will always be considered within the range of a shallow cap. Even though the terms small, intermediate, and large cap angle will be used, it is understood that the largest cap angle possible here is still a shallow cap.

5.1 Velocity Field (Small Cap Angles)

For caps with small $\phi_0$ angles, a solution is sought based on the uniformly loaded, simply supported spherical cap solutions. Note that in the previous chapter concerning lower bounds, the stress profile was on faces 47 of the yield surface. If an upper bound can be
found, using the same two faces of the yield surface, then a reasonably
good upper bound may be expected. Therefore, if the entire stress
profile for the cap is on faces 47, then from Table 3.1 and equations
(2.14) and (3.11), it follows that the strain-rate components are

\[
\begin{align*}
\varepsilon_\theta &= v \cot \phi - w = -\lambda_1 + \lambda_2 \\
\varepsilon_\phi &= v' - w = 0 \\
\kappa_\theta &= -k \cot \phi (v + w') = \lambda_1 + \lambda_2 \\
\kappa_\phi &= -k (v + w')' = 0
\end{align*}
\]

The velocities \( v \) and \( w \) can be determined from (5.1b) and (5.1d) to
within three constants of integration. Two of these constants may be
determined by the velocity boundary conditions \( v = w = 0 \) at \( \phi = \phi_0 \).
Therefore, the velocity field for faces 47 is

\[
v = -C \left[ 1 - \cos (\phi_0 - \phi) \right]
\]

\[
w = C \sin (\phi_0 - \phi)
\]

where \( C \) is a positive constant.

With \( v \) and \( w \) known, the factors \( \lambda_1 \) and \( \lambda_2 \) are determined from
(5.1a) and (5.1c) as

\[
\lambda_1 = \frac{C}{2 \sin \phi} \left[ (1 + k) \cos \phi - \cos \phi_0 \right]
\]

\[
\lambda_2 = \frac{C}{2 \sin \phi} \left[ \cos \phi_0 - (1 - k) \cos \phi \right]
\]
If the velocity field is to be kinematically admissible, both of these factors must be non-negative. This requirement is satisfied by $\lambda_1$, but $\lambda_2$ will be positive only if

$$\cos \phi_0 \geq (1-k)$$

Therefore, (5.2) are kinematically admissible only if the parameters $\phi_0$ and $k$ satisfy (5.4).

5.2 Upper-Bound Collapse Load (Small Cap Angles)

For the stress profile being considered, faces 47, it follows from (2.20) that the internal rate of dissipation of energy is

$$d_i = \int \phi_0 \sin \theta \, d\theta$$

since from (5.1) it is seen that $e_\theta = n_\theta = 0$, and from (4.4) $n_\theta = 0$ and $m_\theta = 1$. Now, using (5.5), (5.1c) and (5.2), the internal rate of dissipation of energy is

$$d_i = \int \phi_0 \cot \theta [-C(1 - \cos (\phi_0 - \phi) - C \cos (\phi_0 - \phi)) \sin \phi \, d\phi$$

or

$$d_i = k \, C \sin \phi_0$$

The external energy rate, from (2.21) and (5.2), is

$$d_e = 2 \int p \, C \sin (\phi_0 - \phi) \sin \phi \, d\phi$$
\[ d_e = p C \sin \Theta_0 \left( \sin^2 \Theta_L - \cot \Theta_0 (\Theta_L \sin \Theta_L \cos \Theta_L) \right) \] (5.7)

Now according to the upper-bound theorem of limit analysis, an upper bound on the yield-point load may be found by equating the internal and external rates of energy. Therefore, from (5.6) and (5.7)

\[ \rho^* = \frac{k}{[\sin^2 \Theta_L - \cot \Theta_0 (\Theta_L \sin \Theta_L \cos \Theta_L)]} \] (5.8)

and is a valid upper bound provided the parameters \( \Theta_0 \) and \( k \) satisfy (5.4).

The upper-bound solution (5.8) is compared graphically with the lower-bound solution (4.37) in Section 5.7. Upon comparison of the upper bound (5.8) with the lower bound (4.37), it is seen that relatively good agreement has been obtained within the region specified by (5.4), since \( k \) is small. Next it is desired to seek an upper-bound solution for values of \( k \) and \( \Theta_0 \) outside of the region specified by (5.4).

5.3 Velocity Field (Intermediate Cap Angles)

In the work of Hodge [1], an upper-bound solution was sought for larger cap angles of the uniformly loaded spherical cap by replacing face 7 of the yield surface by its parallel face 0. If a similar assumption is made here, then the cap stress point is on faces 40 for \( 0 \leq \Theta \leq \Theta \) and on faces 47 for \( \Theta < \Theta \leq \Theta_0 \). If the stress profile for the cap is on faces 40, then for Table 3.1, (3.11) and equation (2.14), it follows that the strain-rate components are
The velocities \( v \) and \( w \) determined from (5.9) are the same as (5.2). However, the factors \( \lambda_1 \) and \( \lambda_2 \) are determined by substituting (5.2) into (5.9a) and (5.9c) and are found to be

\[
\lambda_1 = \frac{C}{2 \sin \theta} \left[ (1 + k) \cos \theta - \cos \theta_o \right]
\]

\[
\lambda_2 = \frac{C}{2 \sin \theta} \left[ (1 - k) \cos \theta - \cos \theta_o \right]
\]

If the velocity field is to be kinematically admissible, both factors must be non-negative. This requirement is satisfied by \( \lambda_1 \) but \( \lambda_2 \) will be positive only if

\[
\cos \theta_o \leq (1 - k) \cos \theta
\]

If \( \eta \) is defined by

\[
\cos \eta = \frac{\cos \theta_o}{1 - k}
\]

then the stress point associated with (5.2) is on faces 40 for \( 0 \leq \theta \leq \eta \) on faces 47 for \( \eta < \theta \leq \theta_o \).

5.4 Upper-Bound Collapse Load (Intermediate Cap Angles)

If the stress profile is on faces 40, then

\[
\eta_0 - m_0 = -1, \quad \eta_0 + m_0 = 1
\]
must hold which implies that
\[ n_\theta = -1, \quad m_\theta = 0 \] (5.14)

Since from (5.9) it is seen that \( \epsilon_\phi = \kappa_\phi = 0 \), and in view of (5.14), the expression for the internal rate of dissipation of energy (2.20), for \( 0 \leq \phi \leq \eta \) becomes
\[ \dot{d}_{i40} = \int_0^\eta \epsilon_\theta \sin \phi \, d\phi \] (5.15)

Using (5.15), (5.9a), and (5.2), the internal rate of dissipation of energy for \( 0 \leq \phi \leq \eta \) becomes
\[ \dot{d}_{i40} = \int_0^\eta \{ -C \cot \phi \sin (\phi - \phi_0) - C \sin (\phi_0 - \phi) \cos (\phi - \phi_0) \} \, d\phi \] (5.16)

or
\[ \dot{d}_{i40} = C(\sin \eta - \eta \cot \phi_0) \] (5.17)

If the stress profile is on faces 47, then as previously determined \( \epsilon_\phi = \kappa_\phi = 0 \) and \( n_\theta = 0, m_\theta = 1 \) and
\[ \dot{d}_{i47} = \int_\phi^\phi_0 \kappa_\phi \sin \phi \, d\phi \] \[ \dot{d}_{i47} = \int_\phi^\phi_0 \{ -k \cot \phi \sin (\phi - \phi_0) - C \cos (\phi_0 - \phi) \} \sin \phi \, d\phi \] or
\[ \dot{d}_{i47} = kC (\sin \phi_0 - \sin \eta) \] (5.18)
Thus, the total rate of dissipation of energy is the sum of (5.17) and (5.18), or

\[ d_1 = C \left\{ \left(1-k\right) \sin \eta + k \sin \phi_o - \eta \cos \phi_o \right\} \]  

(5.19)

Since the velocity field is the same as before, (5.2), the external rate of dissipation of energy is given by (5.7). Equating the internal and external rates of energy, an upper bound on the yield-point load is

\[ p^+ = \frac{k + \cot \phi_o (\tan \eta - \eta)}{[\sin^2 \phi_L - \cot \phi_o (\phi_L - \sin \phi_L \cos \phi_L)]} \]  

(5.20)

provided \( \phi_o \), \( k \), and \( \eta \) satisfy (5.11) and (5.12).

The upper-bound solution (5.20) is compared graphically with the lower-bound solution (4.37) in Figures 5.2 and 5.3. It is found that the difference between the two bounds begins to increase rapidly as cap angle increases; therefore, another approach must be used to obtain better bounds for the larger cap angles.

5.5 Velocity Field (Large Cap Angles)

For values of \( \phi_o \) exceeding the restriction (5.4), it is seen that the upper and lower bounds begin to diverge rather rapidly. Thus another solution is pursued in an effort to decrease the difference between the upper and lower bounds. In the work of Hodge \( ^6 \), a collapse mode was considered in which a central portion of the cap was plastic, while an outer annular section of the cap remained rigid. A hinge circle was formed around the cap where the rigid and plastic portions joined together.
Intuitively it seems plausible that a similar situation would occur here. Thus it is assumed that a similar collapse mode will occur for the case of the partially loaded spherical cap. The hinge circle is regarded as a circle on the shell across which the slope rate $w'$ and/or extension rate $v$ are discontinuous.

First, a collapse mode is assumed such that a hinge circle forms on the shell surface at some value $\theta = \theta_H$, where $\theta_L \leq \theta_H \leq \theta_0$. The region of the shell outside of the hinge circle, $\theta_H < \theta < \theta_0$, is considered rigid, since the shell material is rigid-perfectly plastic. For the stress profile shown in Figure 5.1, it is seen that the velocity field defined by

$$\begin{align*}
w &= C \cos \theta [\sec \theta_H + \ln (\tan \frac{\theta_H}{2} \cot \frac{\theta}{2}) - \tan \theta \csc \theta] \quad (5.21a) \\
v &= w \tan \theta - (B - C \sec \theta) \csc \theta_1 \quad (0 \leq \theta \leq \theta_1) \quad (5.21b) \\
w &= C \cos \theta [\sec \theta_H + \ln (\tan \frac{\theta_H}{2} \cot \frac{\theta}{2}) - \sec \theta] \quad (\theta_1 < \theta \leq \theta_H) \quad (5.21c) \\
v &= w \tan \theta - (B - C \sec \theta) \csc \theta \quad (5.21d)
\end{align*}$$

provides continuous $v$, $w$, and $w'$ at $\theta = \theta_1$, and furnishes $w = 0$ at $\theta = \theta_H$.

For the stress profile shown in Figure 5.1, it follows from Table 3.1 and (3.11) that the strain-rate components are

$$\begin{align*}
\varepsilon_\theta &= v \cot \theta - w = - \lambda_1 + \lambda_2 \quad (5.22a) \\
\varepsilon_\theta' &= v' - w = 0 \quad (5.22b) \\
\varepsilon_\theta &= - k \cot \theta (v + w') = \lambda_1 + \lambda_2 \quad (5.22c) \\
\varepsilon_\theta' &= - k (v + w')' = 0 \quad (5.22d)
\end{align*}$$
Figure 5.1. Stress profile for upper-bound collapse load (large cap angles) [6].
for faces 47, and

\[ e^\theta = v \cot \theta - w = -\lambda_1 + \lambda_2 \]  \hspace{1cm} (5.23a)

\[ e^\phi = v' - w = \lambda_1 - \lambda_2 \]  \hspace{1cm} (5.23b)

\[ \kappa^\theta = -k \cot \theta (v + w') = \lambda_1 + \lambda_2 \]  \hspace{1cm} (5.23c)

\[ \kappa^\phi = -k (v + w')' = -\lambda_1 - \lambda_2 \]  \hspace{1cm} (5.23d)

for faces 3B. If the velocity field (5.21) is to be kinematically admissible, the factors \( \lambda_1 \) and \( \lambda_2 \) must be non-negative and (5.22b) and (5.22d) must be satisfied. From (5.21a) and (5.21b) it is seen that

\[ w' = -w \tan \theta - C \sec \theta \csc \theta_1 \]

\[ v' = w' \tan \theta + w \sec^2 \theta + C \sec \theta \tan \theta \csc \theta_1 \]

\[ = w (\sec^2 \theta - \tan^2 \theta) = w \]

Now

\[ e^\phi = v' - w = 0 \]

and

\[ \kappa^\phi = -k (v + w')' = -k (-B \csc \theta_1)' = 0 \]

Substituting (5.21) into (5.22) and (5.23) leads to

\[ \lambda_1 = \frac{1}{2} \csc \theta \csc \theta_1 [(1+k) B \cos \theta - C] \]  \hspace{1cm} (5.24)  \hspace{1cm} (for Faces 47)

\[ \lambda_2 = \frac{1}{2} \csc \theta \csc \theta_1 [C - (1-k) B \cos \theta] \]  \hspace{1cm} (5.25)

and

\[ \lambda_1 = \frac{1}{2} \csc^2 \theta [(1+k) B \cos \theta - C] \]  \hspace{1cm} (5.26)  \hspace{1cm} (for Faces 3B)

\[ \lambda_2 = \frac{1}{2} \csc^2 \theta [C - (1-k) B \cos \theta] \]  \hspace{1cm} (5.27)
Thus, the condition that the factors all be positive leads to the four inequalities

\[ - C + (1 + k) B \cos \theta \geq 0, \quad C \geq 0 \quad (5.28) \]
\[ C - (1 - k) B \geq 0, \quad B \geq 0 \quad (5.29) \]

From Table 3.1 and (3.11), the strain-rate components for faces 2A are

\[ \epsilon_\theta = v \cot \theta - w = 0 \quad (5.30a) \]
\[ \epsilon_\varphi = v' - w = \lambda_1 - \lambda_2 \quad (5.30b) \]
\[ \kappa_\theta = - k \cot \theta (v + w') = 0 \quad (5.30c) \]
\[ \kappa_\varphi = - k (v + w')' = - \lambda_1 - \lambda_2 \quad (5.30d) \]

If the velocity field is to remain kinematically admissible, the factors \( \lambda_1 \) and \( \lambda_2 \) must be non-negative.

Substituting (5.21) into (5.30), it is found that

\[ \lambda_1 = \frac{1}{2 \sin^2 \theta} [ (1 + k) B \cos \theta - C ] \quad (5.31) \]
\[ \lambda_2 = \frac{1}{2 \sin^2 \theta} [ C - (1 - k) B \cos \theta ] \quad (5.32) \]

Therefore, as long as the inequalities in (5.28) and (5.29) hold, \( \lambda_1 \) and \( \lambda_2 \) will be positive.
5.6 Upper-Bound Collapse Load (Large Cap Angles)

In calculating the internal rate of dissipation of energy, the two plastic regions where (5.21) is valid and the hinge circle at $\phi = \phi_H$ must be considered. On faces 47, it was previously found that

$$n_\theta = 0, \quad m_\theta = 1$$  \hspace{1cm} (5.33)

and on faces 3B, it is found from Table 3.1 that

$$-n_\theta + n_\phi + m_\theta - m_\phi = 1$$

$$n_\theta - n_\phi + m_\theta - m_\phi = 1$$  \hspace{1cm} (5.34)

must hold. Therefore, from (5.34)

$$n_\theta = n_\phi, \quad m_\theta - m_\phi = 1$$  \hspace{1cm} (5.35)

From (5.22), it is seen that $\epsilon_\phi = \kappa_\phi = 0$ and since $n_\theta = 0$ the only term contributing to $d_1$ in the plastic region on faces 47 is $m_\theta \kappa_\theta$. Similarly, for faces 3B, it is seen from (5.23) that $\epsilon_\theta = -\epsilon_\phi$, and since $n_\theta = n_\phi$, then only the $m_\theta \kappa_\theta$ and $m_\phi \kappa_\phi$ terms will contribute to $d_1$. From (5.23), $\kappa_\phi = -\kappa_\theta$ and using (5.35) then

$$m_\theta \kappa_\theta + m_\phi \kappa_\phi = (m_\theta - m_\phi) \kappa_\theta = \kappa_\theta$$  \hspace{1cm} (5.36)

depth, the internal rate of dissipation of energy in the plastic region is

$$d_{1p} = \int_0^{\phi_1} \kappa_\theta \sin \phi \, d\phi + \int_{\phi_1}^{\phi_H} \kappa_\theta \sin \phi \, d\phi$$
or using (5.22) and (5.23)

\[ d_i^p = \int_0^{\pi/2} -k \cot \theta (v+w') \sin \theta \, d\theta + \int_0^{\pi/2} -k \cot \theta (v+w') \sin \theta \, d\theta \]

Substituting (5.21) into (5.37)

\[ d_i^p = \int_0^{\pi/2} k B \cot \theta \csc \theta_1 \sin \theta \, d\theta + \int_0^{\pi/2} k B \cot \theta \d\theta \]

or

\[ d_i^p = k B \left[ 1 + \ln \left( \frac{\sin \theta_H}{\sin \theta_1} \right) \right] \]

(5.38)

Now considering the hinge circle, it is found that on the faces 2A,

\[ n_\theta = 0 , \quad m_\theta = -1 \]

(5.39)

Using (5.30) and (5.39) along with (2.20), the internal rate of dissipation of energy in the hinge circle is

\[ d_i = \sin \theta \left[ k (v + w') \right] \]

(5.40)

where the symbol \( \) stands for the jump in the quantity preceding.

Substituting (5.21) into (5.40), then

\[ d_i = \sin \theta \left[ k B \csc \theta \right] = kB \]

(5.41)

The total internal rate of dissipation of energy, combining (5.38) and (5.41), is

\[ d_i = kB \left[ 2 + \ln \left( \frac{\sin \theta_H}{\sin \theta_1} \right) \right] \]

(5.42)
In determining the lower-bound load in the previous chapter, the load angle $\varphi_L$ was less than or equal to the angle $\varphi_1$. Therefore, again considering the situation of $\varphi_L \leq \varphi_1$, the external rate of dissipation of energy is

$$d_e = 2p \int_0^\varphi L \cos \varphi \left[ \sec \varphi_H + \ln \left( \tan \frac{\varphi_H}{2} \cot \frac{\varphi_1}{2} \right) \right] - \tan \varphi \csc \varphi_1 \sin \varphi \, d\varphi$$

where (5.21a) has been substituted into (2.21). Thus

$$d_e = pC \left[ \sec \varphi_H + \ln \left( \tan \frac{\varphi_H}{2} \cot \frac{\varphi_1}{2} \right) \sin^2 \varphi_L \right] - \csc \varphi_1 (\varphi_L - \sin \varphi_L \cos \varphi_L)$$

Now equating the internal and external rates of dissipation of energy, we find, using (5.42) and (5.43), that

$$p^+ = \frac{\sin \varphi_1}{k(B/C)[2 + \ln(\frac{\varphi_H}{\sin \varphi_1})]}$$

is an upper bound on the yield-point load, provided the inequalities (5.28) and (5.29) are satisfied.

From (5.44) it is seen that the upper bound is proportional to $(B/C)$. The best (lowest) upper bound will occur when $(B/C)$ is a minimum subject to (5.28) and (5.29). Therefore, from (5.28) and (5.29)
\[ B(1+k) \cos \varphi_H \geq C \geq (1-k)B \]

or

\[ C = B(1+k) \cos \varphi_H \quad (5.45) \]

will give minimum \((B/C)\) and thus the inequalities \((5.28)\) and \((5.29)\) reduce to

\[ (1+k) \cos \varphi_H \geq (1-k), \quad C \geq 0 \quad (5.46) \]

Substituting \((5.45)\) into \((5.44)\), the upper bound is

\[ p^+ = \frac{\sin \varphi_H}{(\frac{k}{1+k})[2 + \ln(\frac{\sin \varphi_H}{\sin \varphi_L})]} \]

\[ \frac{1}{(1+\cos \varphi_H \ln(\tan \frac{\varphi_H}{2} \cot \frac{\varphi_L}{2})) \sin^2 \varphi_L \cos \varphi_H - \frac{\cos \varphi_H \sin \varphi_L - \sin \varphi_H \cos \varphi_L}{\sin \varphi_L}} \]

subject to the inequalities \((5.46)\).

Now it is desired to further minimize \((5.47)\). For given values of \(k, \varphi_L, \) and \(\varphi_o\), the upper bound may be minimized with respect to \(\varphi_1\) and \(\varphi_H\). Since \((5.47)\) is quite complex, this minimization has been done numerically on the computer. For specific values of \(k, \varphi_L, \) and \(\varphi_o\), small incremental values of \(\varphi_H\) are stepped off beginning at the loading angle \(\varphi_L\) and stopping at either the cap angle \(\varphi_o\) or the maximum value of \(\varphi_H\), since \(\varphi_H\) is limited by the parameter \(k\) as given by \((5.46)\). After each step of \(\varphi_H\), small incremental values of \(\varphi_1\) are stepped off beginning at the loading angle \(\varphi_L\) and stopping at the hinge circle angle \(\varphi_H\). For each combination of \(\varphi_1\) and \(\varphi_H\), an upper bound according to \((5.47)\) is calculated and from all the combinations the lowest value is obtained.

The computer program minimizing \((5.47)\) is given in Appendix C.

The above work was for \(\varphi_L \leq \varphi_1\). The case of \(\varphi_L > \varphi_1\) was considered and found to yield upper bounds which were greater than \((5.47)\).
5.7 Results for Simply Supported Cap

The results of Chapters IV and V are presented in Figures 5.2 and 5.3. The collapse load $p$ is plotted against the cap angle $\varphi_o$ for two values of $k$ and various ratios of cap angle to loading angle $\varphi_o/\varphi_L$.

A plot of the stresses versus cap angle for one value of $k$ and $\varphi_o$ and two values of $\varphi_o/\varphi_L$ are given in Appendix D.

The lower bound cut-off value of $p = 1$ for the uniformly loaded cap, i.e., $\varphi_o/\varphi_L = 1.0$ in Figures 5.2 and 5.3, is the solution obtained by Hodge [1,5].
Figure 5.2. Load carrying capacity for simply supported shells with $k = 1/50$. 
Figure 5.3. Load carrying capacity for simply supported shells with \( k = 1/200 \).
CHAPTER VI

EXACT SOLUTION-CLAMPED CAP

In the investigation of the simply supported cap it was pointed out that an exact solution for the partially loaded cap was highly improbable, since an exact solution for the uniformly loaded, simply supported cap has not been found. However, in the case of the clamped cap, an exact solution for the partially loaded clamped cap, which is loaded over most of its surface, seemed possible, since an exact solution has been found for the uniformly loaded cap [11].

In this section of the investigation the problem is considered beginning with the lower-bound theorem of limit analysis. If a stress field can be defined for a given pressure $p$ which is everywhere in equilibrium and nowhere violates the yield condition, then a statically admissible field is defined and $p$ is a lower bound on the collapse pressure. The lower-bound theorem states that $p^*$, the exact collapse pressure, is the largest admissible pressure.

Now the complete solution which is sought involves the determination of the critical pressure $p^*$ and the associated fields of stresses and velocity. A complete solution must contain the following:

The stress field in the presence of $p^*$ must be in equilibrium throughout the interval $0 \leq \theta \leq \theta_0$ and each section of the shell must nowhere violate the yield condition.
The state of stress at any point of a plastic deforming element of the shell must be on a fully plastic side of the yield surface.

The velocity components $v(\phi)$ and $w(\theta)$ must satisfy the flow rule associated with the plastic regimes considered for the stress field.

The generalized strain-rates must be zero in the rigid portions of the shell.

The stress and velocity fields must satisfy continuity and boundary conditions.

In the work of Lee and Onat [11] complete solutions were found for the uniformly loaded cap. It seems reasonable, therefore, that a complete solution for the partially loaded cap which is loaded over a majority of its surface may be found, using the identical plastic regimes of the yield surface. If a $p^*$ is assumed, then the cap angle $\phi_0$ and the associated fields of stress and velocity for which $p^*$ is the critical pressure is desired.

The analysis begins by choosing a $p^*$ and determining the stresses starting from the center of the shell. The starting point on the yield surface is the intersection of the faces 45 and 78. The analysis continues along the yield surface until the built-in edge of the cap is reached. The built-in edge of the cap was determined by Lee and Onat [11] to be on face 3 and in the corner AB of the yield surface. At this point a lower bound has been determined.

In order to show that a complete solution has been obtained, $\lambda_1$ and $\lambda_2$ and the velocities are determined starting from the built-in edge. If the $\lambda$'s are found to be non-negative, then a complete solution has been obtained.
In the following discussion the same plastic regimes of the yield surface will be used as were used in the work of Lee and Onat [11] for the uniformly loaded shallow cap. These plastic regions are shown in Figure 6.1, and in order to remain on the yield surface for the three regions defined, i.e., faces 47, 4B, 3B, the following must hold:

\[
\begin{align*}
\eta - \mu &= -1, \\
\eta + \mu &= 1
\end{align*}
\]

or

\[
\begin{align*}
\eta &= 0, \\
\mu &= 1 \\
(\text{Faces 47})
\end{align*}
\]

\[
\begin{align*}
\eta - \mu &= -1, \\
(n_\theta + \mu_\theta) - (n_\theta + \mu_\theta) &= -1 \\
(\text{Faces 4B})
\end{align*}
\]

and

\[
\begin{align*}
(n_\theta - \mu_\theta) - (n_\theta - \mu_\theta) &= 1, \\
(n_\theta + \mu_\theta) - (n_\theta + \mu_\theta) &= -1
\end{align*}
\]

or

\[
\begin{align*}
\eta &= n_\theta, \\
\mu &= 1 + \mu_\theta \\
(\text{Faces 3B})
\end{align*}
\]

Now if a value of \( p^* \) is chosen, then two equations from the yield condition and two equilibrium equations are available with which to solve for the four stress resultants, \( n_\theta, n_\theta, n_\theta \) and \( m_\theta \). Several different cases will have to be considered, depending on which face of the yield surface the stress point is on, faces 47, 4B, or 3B, and which region of the shell the stress point is in, i.e., under the loaded portion \( 0 < \phi < \phi_L \) or within the unloaded portion of the shell \( \phi_L < \phi < \phi_0 \).

6.1 Boundary Conditions

It is assumed that the entire shell will undergo plastic deformation at the yield-point state. At the clamped boundary of the shell
Figure 6.1. Yield surface for clamped cap.
\( \phi = \phi_0 \), the kinematical boundary conditions are

\[
v = w = 0 \quad \text{and} \quad w' = 0
\]  \hspace{1cm} (6.4)

Now \( w \) is continuous over the shell interval \( 0 \leq \phi \leq \phi_0 \), since the transverse shear strains have been neglected. Considering the boundary conditions (6.4) and the generalized strain-rates (2.14), then at \( \phi = \phi_0 \)

\[
\varepsilon_\phi = \kappa_\phi = 0
\]

\[
\varepsilon_\psi = v', \quad \kappa_\psi = -k(v + w')'
\]  \hspace{1cm} (6.5)

where \( \varepsilon_\psi \) and \( \kappa_\psi \) will, in general, be different from zero. A study of the flow rule associated with the plastic regimes listed in Table 3.1 shows that only four of the faces can give rise to strain-rates of the type given in (6.5). They are faces 2, 5, 8 and A, or

\[
\pm (n_\phi \pm m_\phi) = 1
\]  \hspace{1cm} (6.6)

Since the pressure is applied externally and inward, it is expected that \( n_\phi \leq 0 \) and \( m_\phi \leq 0 \) at the built-in edge so that the regime of interest at \( \phi = \phi_0 \) will be face A, or

\[
- n_\psi - m_\psi = 1
\]  \hspace{1cm} (6.7)

6.2 Stress Field

From Figure 6.1, it is seen that the stress point begins at the corner of the yield surface where faces 45 and faces 78 intersect and moves along faces 47 until the corner 7B is reached, i.e., at

\[
n_\psi + m_\psi = 0
\]  \hspace{1cm} (6.8)
Next the stress point moves along faces 4B until the corner 34 is reached, i.e., at

\[ n_\varphi - m_\varphi = 0 \]  
(6.9)

The stress point continues to move along faces 3B until the corner AB is reached, i.e., at

\[ - n_\varphi - m_\varphi = 1 \]  
(6.10)

At this point computations stop, since the boundary condition (6.7) is fulfilled and a possible location for the built-in edge of the cap has been reached.

Now the analysis begins with the stress point in the corners of the yield surface at the intersection of faces 45 and 78. As the stress point begins to move along the yield surface, it advances along the faces 47. If equilibrium and the yield condition are to be satisfied on faces 47, then (6.1) and (2.8) must be satisfied simultaneously. Substituting (6.1) into (2.8), a first-order system of differential equations for \( n_\varphi \) and \( m_\varphi \) is obtained

\[ n'_\varphi = [- n_\varphi - (n_\varphi + p^*) \tan^2 \varphi] \cot \varphi \]  
(6.11)

\[ m'_\varphi = [1 - m_\varphi - \frac{1}{k(n_\varphi + p^*)} \tan^2 \varphi] \cot \varphi \]  
(6.12)

Solving (6.11) and (6.12) with the initial conditions

\[ n_\varphi = 0, \quad m_\varphi = 1 \text{ at } \varphi = 0 \]  
(6.13)

the following is obtained

\[ n_\varphi = -p^*(1 - \varphi \cot \varphi) \]  
(6.14)

\[ m_\varphi = 1 - \frac{1}{k} p^*(1 - \varphi \cot \varphi) \]  
(6.15)
Now (6.1), (6.14) and (6.15) determine the trajectory of the stress point in the region $0 \leq \phi \leq \phi_1$.

The motion of the stress point on faces 47 with increasing $\phi$ will be interrupted when the corner 7B is reached. At this point

$$n_\phi + m_\phi = 0 \quad (6.16)$$

or using (6.14) and (6.15)

$$1 - (\frac{1+k}{k})p^*(1 - \phi \cot \phi) = 0 \quad (6.17)$$

Let the smallest positive root of this equation be denoted by $\phi_1$. If plastic deformations are to occur everywhere, then further motion of the stress point on the yield surface is possible along faces 4B. This motion is controlled by the yield conditions (6.2) and by the equations of equilibrium (2.8). From (6.2) it is seen that

$$m_\phi = 1 + n_\phi \quad (6.18)$$

and

$$n_\phi = \frac{1}{2} (n_\phi + m_\phi) \quad (6.19)$$

Substituting (6.19) into the first of (2.8), then

$$n_\phi' = \left[ \frac{1}{2} m_\phi - \frac{1}{2} n_\phi - (n_\phi p^*) \tan^2 \phi \right] \cot \phi \quad (6.20)$$

Substituting (6.15) and (6.19) into the second equation of (2.8), then

$$m_\phi' = \left[ 1 - \frac{1}{2} m_\phi + \frac{1}{2} n_\phi - \frac{1}{k} (n_\phi p^*) \tan^2 \phi \right] \cot \phi \quad (6.21)$$

Solving (6.20) for $m_\phi'$, then

$$m_\phi' = 2n_\phi' \tan \phi + n_\phi + 2(n_\phi p^*) \tan^2 \phi \quad (6.22)$$
and from this

\[ m' = 2n'' \tan \theta + 2n'(\sec^2 \theta + \tan^2 \theta) + n' + 4(n' + p^*) \tan \theta \sec^2 \theta \]  \hspace{1cm} (6.23)

Now substituting (6.22) into (6.21)

\[ m' = \left[ 1 - n' \tan \theta - \left( \frac{1+k}{k} \right)(n' + p^*) \tan^2 \theta \right] \cot \theta \]  \hspace{1cm} (6.24)

Equating (6.23) and (6.24), a second-order differential equation in \( n' \) is obtained which is

\[ n'' + n' \left[ \frac{2}{\sin \theta \cos \theta} \right] + n' \left[ 2 \sec^2 \theta + \frac{1}{2} \frac{(1+k)}{k} \right] = \frac{1}{2} \cot^2 \theta - p^* \left[ 2 \sec^2 \theta + \frac{1}{2} \frac{(1+k)}{k} \right] \]  \hspace{1cm} (6.25)

No closed form solution for this equation could be found, therefore, recourse was made to numerical integration with a computer. The values of \( n' \) and \( m' \) at \( \theta_1 \), obtained from the previous steps of the analysis, provide initial conditions for this step. Appendix E provides the equations used to solve (6.25). Once \( n' \) has been found, then \( m' \), \( n'^* \) and \( m'^* \) are obtained from (6.22), (6.18) and (6.19), respectively.

Now (6.2), (6.18), (6.19), (6.22) and (6.25) determine trajectory of the stress point in the region \( \theta_1 \leq \theta \leq \theta_2 \).

The motion of the stress point on faces 4B with increasing \( \theta \) will be interrupted when the corner 34 is reached. At this point, \( \theta_2 \),

\[ n' - m' = 0 \]  \hspace{1cm} (6.26)

Further motion of the stress point on the yield surface is possible along faces 3B. This motion is controlled by the yield conditions (6.3)
and by the equilibrium equations (2.8). Substituting the first equa-
tion of (6.3) into the first equation of (2.8), then

\[ n'_\varphi = [- (n_\varphi + p^*) \tan^2 \varphi] \cot \varphi \]  
(6.27)

Substituting the second equation of (6.3) into the second equation of 
(2.8), then

\[ m'_\varphi = [1 - \frac{1}{k} (n_\varphi + p^*) \tan^2 \varphi] \cot \varphi \]  
(6.28)

Solving (6.27) for \( n_\varphi \), then

\[ n_\varphi = C_1 \cos \varphi - p^* \]  
(6.29)

Now substituting (6.29) into (6.28) and solving for \( m_\varphi \), then

\[ m_\varphi = \ln (\sin \varphi) + \frac{1}{k} C_1 \cos \varphi + C_2 \]  
(6.30)

Now motion of the stress point on faces 3B is controlled by (6.29) and 
(6.30) where constants \( C_1 \) and \( C_2 \) are to be determined for the values 
of \( n_\varphi \) and \( m_\varphi \) at \( \varphi = \varphi_2 \) obtained in the previous step of the analysis. 
The motion of the stress point continues until \( \varphi = \varphi_L \) is reached. 
Since this is the unloaded region of the cap, further motion along 
faces 3B is controlled by (6.3) and (2.9). Substituting (6.3) into 
(2.9), a system of first-order differential equations is obtained

\[ n'_\varphi = [- n_\varphi \tan^2 \varphi - p^* \frac{\sin^2 \varphi_L}{\cos^2 \varphi}] \cot \varphi \]  
(6.31)

\[ m'_\varphi = [1 - \frac{1}{k} n_\varphi \tan^2 \varphi - \frac{1}{k} p^* \frac{\sin^2 \varphi_L}{\cot^2 \varphi}] \]  
(6.32)
The solutions to these equations are

\[ n_\theta = C_3 \cos \theta - p^* \sin^2 \theta_L \left[ 1 + \cos \theta \ln \left( \tan \frac{\theta}{2} \right) \right] \]  

(6.33)

\[ m_\theta = \ln \left( \sin \theta \right) + \frac{1}{k} C_3 \cos \theta - \frac{1}{k} p^* \sin^2 \theta_L \cos \theta \ln \left( \tan \frac{\theta}{2} \right) + C_4 \]  

(6.34)

where the constants \( C_3 \) and \( C_4 \) are to be determined from the values of \( n_\theta \) and \( m_\theta \) at \( \theta = \theta_L \) in the previous step of the analysis.

The motion of the stress point on faces 3B continues until it reaches corner AB where

\[ -n_\theta - m_\theta = 1 \]  

(6.35)

The angle \( \theta = \theta_o \), where the above condition is fulfilled, is, in view of (6.7), a possible location for the built-in edge of the cap and therefore the calculations for the stresses are terminated at this point. Now a stress field, which is statically admissible, has been obtained and the chosen \( p^* \) is equal to or less than the critical pressure for a shell with half opening angle \( \theta = \theta_o \).

6.3 Velocity Field

The plastic regimes of the yield surface are considered to be the same as those which governed the stresses. The velocities \( v \) and \( w \) and the multipliers \( \lambda_1 \) and \( \lambda_2 \) are related through the flow rule (3.11) and the generalized strain-rates (2.14), by
\[ \varepsilon_\theta = v \cot \gamma - w = \lambda_1 \frac{\partial f_{i0}}{\partial \varepsilon} + \lambda_2 \frac{\partial f_{i1}}{\partial \varepsilon} \]
\[ \varepsilon_\gamma = v' - w = \lambda_1 \frac{\partial f_{i0}}{\partial \gamma} + \lambda_2 \frac{\partial f_{i1}}{\partial \gamma} \]
\[ \kappa_\theta = -k \cot \beta (v + w') = \lambda_1 \frac{\partial f_{i0}}{\partial \beta} + \lambda_2 \frac{\partial f_{i1}}{\partial \beta} \]
\[ \kappa_\gamma = -k(v + w')' = \lambda_1 \frac{\partial f_{i0}}{\partial \gamma} + \lambda_2 \frac{\partial f_{i1}}{\partial \gamma} \]

(6.36)

where \( \lambda_1 \) and \( \lambda_2 \) are non-negative.

Since there is a sign requirement on the \( \lambda \)'s, it would be easier to work with equations which involve only \( \lambda_1 \) and \( \lambda_2 \). Now, using Table 3.1 and (6.36), the generalized strain-rates for the faces 3B are found to be

\[ \varepsilon_\theta = -\lambda_1 + \lambda_2 , \quad \varepsilon_\gamma = \lambda_1 - \lambda_2 \]
\[ \kappa_\theta = \lambda_1 + \lambda_2 , \quad \kappa_\gamma = -\lambda_1 - \lambda_2 \]

(6.37)

Similarly for the faces 4B it is found that

\[ \varepsilon_\theta = -\lambda_1 + \lambda_2 , \quad \varepsilon_\gamma = -\lambda_2 \]
\[ \kappa_\theta = \lambda_1 + \lambda_2 , \quad \kappa_\gamma = -\lambda_2 \]

(6.38)

and for the faces 47

\[ \varepsilon_\theta = -\lambda_1 + \lambda_2 , \quad \varepsilon_\gamma = 0 \]
\[ \kappa_\theta = \lambda_1 + \lambda_2 , \quad \kappa_\gamma = 0 \]

(6.39)
Substituting the right-hand side of the above system into the equations of compatibility (2.15), then

\[
\frac{d\lambda_1}{d\theta} = \frac{1}{2} \left[ -4 \cot \phi - (1+\alpha) \tan \theta \right] \lambda_1 - \left[ (1+\alpha) \tan \theta \right] \lambda_2
\]

(6.40)

\[
\frac{d\lambda_2}{d\theta} = \frac{1}{2} \left[ - \left( 1 - \alpha \right) \tan \theta \right] \lambda_1 - \left[ 2 \cot \phi + (1-\alpha) \tan \theta \right] \lambda_2
\]

(6.41)

where \( \alpha = \frac{1}{k} \). Similarly for faces 4B

\[
\frac{d\lambda_1}{d\theta} = \frac{1}{2} \left[ -2 \cot \phi + (1+\alpha) \tan \theta \right] \lambda_1 - \left[ (1+\alpha) \tan \theta \right] \lambda_2
\]

(6.42)

\[
\frac{d\lambda_2}{d\theta} = \frac{1}{2} \left[ - \left( 1 - \alpha \right) \tan \theta \right] \lambda_1 - \left[ 2 \cot \phi + (1-\alpha) \tan \theta \right] \lambda_2
\]

If the above system of equations is to be integrated, the boundary conditions on the \( \lambda \)'s must be known. Also, it must be known how the \( \lambda \)'s change from one plastic regime to another. In the work of Lee and Onat [11], it was shown that the boundary conditions on the \( \lambda \)'s at the built-in edge of the cap were

\[
\lambda_1 = 0, \quad \lambda_2 = \cot \phi_0
\]

(6.43)

and that for the plastic regimes considered, i.e., faces 47, 4B and 3B, the \( \lambda \)'s were continuous in passing from one regime to another. Since the problem considered here, i.e., the partially loaded cap, is different
from [1:1] only in relation to the loading, then the same velocity field may be considered. If the λ's satisfy the above requirements and are found to be non-negative, then a complete solution is obtained.

The equations (6.42) governing the λ's in the region of the shell near the center may be solved to obtain

\[
\begin{align*}
\lambda_1 &= \frac{1}{2} [(1+\alpha) A_1 \cot \theta - A_2 \csc \theta] \\
\lambda_2 &= \frac{1}{2} [(1-\alpha) A_1 \cot \theta + A_2 \csc \theta]
\end{align*}
\]  
(6.44)

Substituting these equations into the first two equations of (6.36), the velocities are found to be

\[
\begin{align*}
v &= A_3 \sin \theta + A_2 \cos \theta - A_1 \alpha \\
w &= A_3 \cos \theta - A_2 \sin \theta
\end{align*}
\]  
(6.45)

where \(A_1\), \(A_2\) and \(A_3\) are constants to be determined by the previous step of integration.

From (6.44) it is seen that \(\lambda_1\) and \(\lambda_2\) become infinite at the center of the cap. If the λ's are to remain non-negative there, then

\[(1+\alpha)A_1 - A_2 \geq 0, \quad (1-\alpha)A_1 + A_2 \geq 0 \]  
(6.46)

must hold. Since \(\alpha \gg 1\), then (6.46) implies that \(A_1 \geq 0\) and \(A_2 \geq 0\). Therefore, the λ's will remain non-negative if the inequalities (6.46) are satisfied.
6.4 Results for the Clamped Cap

The results obtained for the clamped cap are presented in Figure 6.2 where the pressure $p$ versus cap angle $\theta_0$ has been plotted. The curve of $p$ versus $\theta_0$ for the uniformly loaded cap was determined first (see Figure 6.2). The computer program for this curve is presented in Appendix F. Once this was satisfactory, that is, when the results were very close to the results of the work of Lee and Onat [11], then the later part of the program was altered to take into account the discontinuity in the load. The computer program for the partially loaded cap is given in Appendix G.

For very small angles, the pressure $p$ tends to infinity; therefore, the results may be expressed in terms of the dimensionless pressure

$$\bar{p} = \frac{p \sin^2 \theta_0}{6k} \quad (6.47)$$

A table consisting of values of $\bar{p}$ for small angles of $\theta_0$ is presented in Appendix H. Also, curves of stress and velocity distribution for the uniformly and partially loaded caps are presented in Appendix I.
Figure 6.2. Load carrying capacity for clamped shells with $k = 1/50$. 

The graph shows the load carrying capacity as a function of $\theta_s$ (rad.) with various values of $\phi_L$: $0.19$ rad., $0.20$ rad., and $0.22$ rad.
CHAPTER VII

DISCUSSION

7.1 Simply Supported Cap

The lower- and upper-bound results are presented in Figures 5.2 and 5.3. The collapse load $p$ is plotted against the cap angle for two values of $k$ and various ratios of cap angle to loading angle $\varphi_o/\varphi_L$. It is seen that as $k$ increases, the bounds are close for a larger range of cap angle $\varphi_o$. Also, as the ratio of cap angle to loading angle decreases, the bounds become progressively better.

If the lower- and upper-bound equations (4.37), (5.8) and (5.20) are taken to the limit by letting the loading angle equal the cap angle, i.e., $\varphi_o/\varphi_L = 1.0$, then (4.37) becomes

$$p^- = \frac{(1-k)}{(1-\varphi_o \cot \varphi_o)}$$  \hspace{1cm} (7.1)

(5.8) becomes

$$p^+ = \frac{k}{(1-\varphi_o \cot \varphi_o)}$$  \hspace{1cm} (7.2)

and (5.20) becomes

$$p^+ = \frac{k + \cot \varphi_o (\tan \eta - \eta)}{(1 - \varphi_o \cot \varphi_o)}$$  \hspace{1cm} (7.3)

These bounds correspond exactly to the bounds given previously by Hodge [1] for a uniformly loaded, simply supported cap. It is
fortunate that a relative simple membrane solution can be constructed for the uniformly loaded cap and provide a cut-off lower-bound value. However, no such simple solution could be found, if one exists, for the partially loaded cap.

Certainly the difference between the bounds is appreciable for large cap angles. However, it must be noted that the exact solution for the case of the simply supported, uniformly loaded spherical cap has not yet been found and thus far no other results for the partially loaded cap have been reported. An exact collapse load has been obtained only for the special case of the simply supported, uniformly loaded spherical cap, using a yield condition which neglects all interaction between forces and moments \([1,7,8]\).

### 7.2 Clamped Cap

It was felt that the case of the clamped cap loaded over most of its surface should produce a pressure versus cap angle curve close to that of the uniformly loaded cap. Since the cap is loaded over a major portion of its surface, then \(\mathcal{P}_L\) is considered to be on faces 3B of the yield surface. A slight alteration of the loading should not cause a large variation in the results. This is found to be true as seen in the pressure versus cap angle curve in Figure 6.2. Also, the stress and velocity distributions are not altered significantly from the uniformly loaded cap results, as seen in Appendix I.

Several curves for the partially loaded cap were computed. The case for \(\mathcal{P}_L = 0.19 \text{ rad.}, 0.20 \text{ rad.}, \text{ and } 0.22 \text{ rad.}\) were considered (see Figure 6.2). Values of \(\mathcal{P}_L\) less than 0.19 rad. resulted in \(\mathcal{P}_L\)
not being on faces 3B. It is seen that as the loading over the cap increases toward that of being uniformly loaded, the pressure intensity decreases as would be expected. It was found in [11] that the shallow shell range continued out to approximately 0.28 rad. before the $\lambda$'s became negative. It was found here that by slightly altering the load, this cap range is not changed.

In the computer programs the fourth-order Runge-Kutta method [25] was used for the numerical integration. In order to determine what step size should be used in the numerical integration technique, several values were computed with different step sizes. Whenever the step size caused little change in the results, then that was the step size used. For the uniformly loaded cap it was found that the maximum cap angle which could be used before the $\lambda$'s became negative was $\varphi_o = 0.28093$ rad. and the load was $p^* = 1.42$. These values are slightly different from those found in [11], $\varphi_o = 0.2817$ rad., $p^* = 1.39$. The maximum cap angle which could be used before the $\lambda$'s became negative in the case of the partially loaded cap, $\varphi_L = 0.20$ rad., was $\varphi_o = 0.28071$ rad. and the loading was $p^* = 1.715$.

7.3 Summary

In this work, collapse load bounds have been obtained for certain simply supported, partially loaded, shallow spherical caps and an exact solution has been found for the clamped, partially loaded shallow spherical cap.

The next appropriate steps are thought to be an extension, again following Lee and Onat [11], to clamped, partially loaded deep shells and
efforts to obtain an exact solution to the case of the uniformly loaded simply supported cap. Since almost no work has been reported on partially loaded spherical caps, it is hoped that these results will be of assistance in guiding further efforts in the area.
APPENDIX A

EQUATIONS OF EQUILIBRIUM

In considering the equilibrium of the shell element in Figure 2.1, the equations of equilibrium in dimensionless form are

\[
\begin{align*}
(n_0 \sin \phi)' - n_0 \cos \phi &= s \sin \phi \\
(n_0 + n_0 + 2p) \sin \phi &= -(s \sin \phi)' \\
k[(m_0 \sin \phi)' - m_0 \cos \phi] &= s \sin \phi
\end{align*}
\] (A.1)

Now it is desired to rewrite the equations of equilibrium into a form which does not contain \( s \), since the assumption has been made that straight normals remain straight and normal and thus the shear strains have been neglected. Thus the shear force \( s \) does no work and is treated only as a reaction.

Multiply (A.1a) by \( \sin \phi \)

\[
(n_0 \sin \phi)' \sin \phi - n_0 \sin \phi \cos \phi = s \sin^2 \phi
\] (A.2)

and (A.1b) by \( \cos \phi \)

\[
(n_0 \sin \phi \cos \phi + n_0 \sin \phi \cos \phi + 2p \sin \phi \cos \phi = - (s \sin \phi)' \cos \phi
\] (A.3)

then add (A.2) and (A.3) to obtain

\[
\begin{align*}
n_0' \sin^2 \phi + 2n_0 \sin \phi \cos \phi + 2p \sin \phi \cos \phi + s' \sin \phi \cos \phi + s' \cos^2 \phi - s \sin^2 \phi &= 0
\end{align*}
\] (A.4)
Now
\[
(n_\phi \sin^2 \phi)' = n_\phi \sin^2 \phi + 2n_\phi \sin \phi \cos \phi \tag{A.5}
\]
and
\[
(s \sin \phi \cos \phi)' = s' \sin \phi \cos \phi + s \cos^2 \phi - s \sin^2 \phi \tag{A.6}
\]
Therefore, using (A.5) and (A.6) then write (A.4) as
\[
(n_\phi \sin^2 \phi)' + (s \sin \phi \cos \phi)' + 2p \sin \phi \cos \phi = 0
\]
Now integrating
\[
n_\phi \sin^2 \phi + s \sin \phi \cos \phi + p \sin^2 \phi + C_1 = 0 \tag{A.7}
\]
Using the condition \( s = 0, \) at \( \phi = 0, \) then from (A.7)
\[
C_1 = 0
\]
Thus
\[
s = -(n_\phi + p) \tan \phi \tag{A.8}
\]
Substituting (A.8) into (A.1a) then
\[
(n_\phi \sin \phi)' - n_\phi \cos \phi = -(n + p) \sin \phi \tan \phi
\]
\[
n_\phi' \sin \phi + n_\phi \cos \phi - n_\phi \cos \phi = -(n_\phi + p) \sin \phi \tan \phi
\]
\[
n_\phi' \sin \phi = n_\phi \cos \phi - n_\phi \cos \phi - (n_\phi + p) \sin \phi \tan \phi
\]
\[
n_\phi' = [n_\phi - n_\phi - (n_\phi + p) \tan^2 \phi] \cot \phi
\]
and substituting (A.8) into (A.1c),
\[
k[(m_\phi \sin \phi)' - m_\phi \cos \phi] = -(n_\phi + p) \sin \phi \tan \phi
\]
\[
m_\phi' \sin \phi + m_\phi \cos \phi - m_\phi \cos \phi = -\frac{1}{k} (n_\phi + p) \sin \phi \tan \phi
\]
\[ n'_y \sin \theta = m_\theta \cos \phi - m_\phi \cos \theta - \frac{1}{k} (n_\phi + p) \sin \phi \tan \phi \]
\[ m'_y = [m_\theta - m_\phi - \frac{1}{k} (n_\phi + p) \tan^2 \phi] \cot \phi \] (A.10)

Equations (A.9) and (A.10) are valid only where loading occurs from \( \phi = 0 \) to \( \phi = \phi_L \).

It is seen from (A.8) that a discontinuity in \( p \) will cause a discontinuity in \( s \), which is not allowed. Going back to (A.7) rewrite as

\[ n_\phi \sin^2 \phi + s \sin \phi \cos \phi + \bar{p} \sin^2 \phi + C_2 = 0 \] (A.11)

where \( \bar{p} \) is the intensity of the load in the region \( \phi > \phi_L \). Now at \( \phi = \phi_L \), \( C_2 \) can be evaluated using (A.11)

\[ n_\phi \sin^2 \phi_L + s_\phi \sin \phi_L \cos \phi_L + \bar{p} \sin^2 \phi_L + C_2 = 0 \] (A.12)

\[ C_2 = - n_\phi \sin^2 \phi_L - s_\phi \sin \phi_L \cos \phi_L - \bar{p} \sin^2 \phi_L \]

Substituting (A.8) into (A.12) at \( \phi = \phi_L \), since \( s \) must be continuous, then

\[ C_2 = - n_\phi \sin^2 \phi_L + (n_\phi + p) \tan \phi_L \sin \phi_L \cos \phi_L - \bar{p} \sin^2 \phi_L \]

or

\[ C_2 = (p - \bar{p}) \sin^2 \phi_L \] (A.13)

Substituting (A.13) into (A.11),

\[ n_\phi \sin^2 \phi + s \sin \phi \cos \phi + \bar{p} \sin^2 \phi + (p - \bar{p}) \sin^2 \phi_L = 0 \]

\[ s \sin \phi \cos \phi = - n_\phi \sin^2 \phi - \bar{p} \sin^2 \phi - (p - \bar{p}) \sin^2 \phi_L \] (A.14)

\[ s = - (n_\phi + \bar{p}) \tan \phi - (p - \bar{p}) \frac{\sin^2 \phi_L}{\sin \phi \cos \phi} \]
Now consider the case where \( \vec{p} = p \), or the cap is uniformly loaded, then from (A.14),

\[
s = -(n_\theta + p) \tan \theta
\]

which is exactly the same as (A.8).

Now if \( \vec{p} = 0 \), then (A.14) becomes

\[
s = -n_\theta \tan \theta - p \frac{\sin^2 \theta_L}{\sin \theta \cos \theta}
\]  
(A.15)

Substituting (A.14) into (A.1a),

\[
(n_\theta \sin \theta)' - n_\theta \cos \theta = -(n_\theta + \vec{p}) \sin \theta \tan \theta - (p - \vec{p}) \frac{\sin^2 \theta_L}{\cos \theta}
\]

\[
(m_\theta \sin \theta)' - m_\theta \cos \theta = \frac{1}{k} (n_\theta + \vec{p}) \sin \theta \tan \theta - \frac{1}{k} (p - \vec{p}) \frac{\sin^2 \theta_L}{\cos \theta}
\]

Substituting (A.14) into (A.1c),

\[
(m_\theta \sin \theta)' - m_\theta \cos \theta = \frac{1}{k} (n_\theta + \vec{p}) \sin \theta \tan \theta - \frac{1}{k} (p - \vec{p}) \frac{\sin^2 \theta_L}{\cos \theta}
\]

Equations (A.16) and (A.17) are valid in the region \( \theta_L < \theta < \theta_0 \).
Now consider the case where \( \bar{p} = p \), or the cap is uniformly loaded, then from (A.16),

\[
n'_{\theta} = [n_{\theta} - n_{\theta} - (n_{\theta} + p) \tan^{2} \theta] \cot \theta
\]

and from (A.17),

\[
m'_{\theta} = [m_{\theta} - m_{\theta} - \frac{1}{k}(n_{\theta} + p) \tan^{2} \theta] \cot \theta
\]

which corresponds to (A.9) and (A.10), respectively.

Now if \( \bar{p} = 0 \), then (A.16) becomes

\[
n'_{\theta} = [n_{\theta} - n_{\theta} - n_{\theta} \tan^{2} \theta - p \frac{\sin^{2} \theta_{L}}{\cos^{2} \theta}] \cot \theta \quad (A.18)
\]

\( (\theta_{L} < \theta < \theta_{0}) \)

and (A.17) becomes

\[
m'_{\theta} = [m_{\theta} - m_{\theta} - \frac{1}{k} n_{\theta} \tan^{2} \theta - \frac{1}{k} p \frac{\sin^{2} \theta_{L}}{\cos^{2} \theta}] \cot \theta \quad (A.19)
\]

\( (\theta_{L} < \theta < \theta_{0}) \)

Now (A.18) and (A.19) are the equilibrium equations to be used when the stress point is in the unloaded region of the cap, \( \theta_{L} < \theta < \theta_{0} \).
APPENDIX B

PLASTIC FLOW

It is desired to show that plastic flow cannot take place if the stress point is on only one face of the yield surface.

Face 1

From Table 4.1 it is found that

\[ \varepsilon_\theta = \lambda, \quad \varepsilon_\varphi = 0, \quad \kappa_\theta = -\lambda, \quad \kappa_\varphi = 0 \]  \hspace{1cm} (B.1)

Combining (B.1) with the compatibility equations (2.15) then

\[ \lambda' = (\cot \varphi + \frac{1}{k} \tan \varphi) \lambda \]  \hspace{1cm} (B.2)

\[ \lambda' = \lambda \sec^2 \varphi \cot \varphi \]

Thus it is seen that the only solution of (B.2) is \( \lambda = 0 \).

Face 2

From Table 4.1 it is found that

\[ \varepsilon_\theta = 0, \quad \varepsilon_\varphi = \lambda, \quad \kappa_\theta = 0, \quad \kappa_\varphi = -\lambda \]  \hspace{1cm} (B.3)

Combining (B.3) with the compatibility equations (2.15) then

\[ 0 = \lambda \cot \varphi \]  \hspace{1cm} (B.4)

\[ 0 = -\lambda \cot \varphi \]

Thus it is seen that the only solution of (B.4) is \( \lambda = 0 \).
From Table 4.1 it is found that
\[ \varepsilon_\theta = -\lambda, \quad \varepsilon_\phi = \lambda, \quad \kappa_\theta = \lambda, \quad \kappa_\phi = -\lambda \] (B.5)

Combining (B.5) with the compatibility equations (2.15) then
\[ \lambda' = (2 \cot \phi + \frac{1}{k} \tan \phi)\lambda \]
\[ \lambda' = - (\cot \phi + \sec \phi \csc \phi)\lambda \] (B.6)
Thus it is seen that the only solution of (B.6) is \( \lambda = 0 \).

From Table 4.1 it is found that
\[ \varepsilon_\theta = -\lambda, \quad \varepsilon_\phi = 0, \quad \kappa_\theta = \lambda, \quad \kappa_\phi = 0 \] (B.7)

Combining (B.7) with the compatibility equations (2.15) then
\[ \lambda' = - (\cot \phi + \frac{1}{k} \tan \phi)\lambda \]
\[ \lambda' = - \lambda \sec \phi \csc \phi \] (B.8)
Thus it is seen that the only solution of (B.8) is \( \lambda = 0 \).

From Table 4.1 it is found that
\[ \varepsilon_\theta = 0, \quad \varepsilon_\phi = -\lambda, \quad \kappa_\theta = 0, \quad \kappa_\phi = \lambda \] (B.9)

Combining (B.9) with the compatibility equations (2.15) then
\[ 0 = (- \cot \phi + \frac{1}{k} \tan \phi)\lambda \] (B.10)
\[ 0 = \lambda \cot \phi \]
Thus it is seen that the only solution of (B.10) is \( \lambda = 0 \).
From Table 4.1 it is found that
\[ \epsilon_\theta = \lambda, \quad \epsilon_\phi = -\lambda, \quad \kappa_\theta = -\lambda, \quad \kappa_\phi = \lambda \quad \text{(B.11)} \]
Combining (B.11) with the compatibility equations (2.15) then
\[ \lambda' = - (2 \cot \beta + \frac{1}{k} \tan \phi)\lambda \]
\[ \lambda' = (\cot \beta + \sec \phi \csc \phi)\lambda \quad \text{(B.12)} \]
Thus it is seen that the only solution of (B.12) is \( \lambda = 0 \).

From Table 4.1 it is found that
\[ \epsilon_\theta = \lambda, \quad \epsilon_\phi = 0, \quad \kappa_\theta = \lambda, \quad \kappa_\phi = 0 \quad \text{(B.13)} \]
Combining (B.13) with the compatibility equations (2.15) then
\[ \lambda' = (- \cot \beta + \frac{1}{k} \tan \phi)\lambda \]
\[ \lambda' = - \lambda \sec \phi \csc \phi \quad \text{(B.14)} \]
Thus it is seen that the only solution of (B.14) is \( \lambda = 0 \).

From Table 4.1 it is found that
\[ \epsilon_\theta = 0, \quad \epsilon_\phi = \lambda, \quad \kappa_\theta = 0, \quad \kappa_\phi = \lambda \quad \text{(B.15)} \]
Combining (B.15) with the compatibility equations (2.15) then
\[ 0 = \lambda \cot \phi \quad \text{(B.16)} \]
Thus it is seen that the only solution of (B.16) is \( \lambda = 0 \).
From Table 4.1 it is found that
\[ \epsilon_\theta = -\lambda, \quad \epsilon_\phi = \lambda, \quad \kappa_\theta = -\lambda, \quad \kappa_\phi = \lambda \]  
(B.17)

Combining (B.17) with the compatibility equations (2.15) then
\[ -\lambda' = (2 \cot \phi - \frac{1}{k} \tan \phi)\lambda \]  
\[ -\lambda' = (\cot \phi + \sec \phi \csc \phi)\lambda \]  
(B.18)

Thus it is seen that the only solution of (B.18) is \( \lambda = 0 \).

From Table 4.1 it is found that
\[ \epsilon_\theta = 0, \quad \epsilon_\phi = -\lambda, \quad \kappa_\theta = 0, \quad \kappa_\phi = -\lambda \]  
(B.19)

Combining (B.19) with the compatibility equations (2.15) then
\[ -\lambda' = (\cot \phi - \frac{1}{k} \tan \phi)\lambda \]  
\[ -\lambda' = \lambda \sec \phi \csc \phi \]  
(B.20)

Thus it is seen that the only solution of (B.20) is \( \lambda = 0 \).

From Table 4.1 it is found that
\[ \epsilon_\theta = 0, \quad \epsilon_\phi = -\lambda, \quad \kappa_\theta = 0, \quad \kappa_\phi = -\lambda \]  
(B.21)

Combining (B.21) with the compatibility equations (2.15) then
\[ 0 = -\lambda \cot \phi \]  
\[ 0 = -\lambda \cot \phi \]  
(B.22)

Thus it is seen that the only solution of (B.22) is \( \lambda = 0 \).
From Table 4.1 it is found that
\[ T = \lambda, \quad e = -\lambda, \quad K = \lambda, \quad K = -\lambda \]  
(B.23)

Combining (B.23) with the compatibility equations (2.15) then
\[ \lambda' = (-2 \cot \varphi + \frac{1}{k} \tan \varphi)\lambda \]  
(B.24)
\[ \lambda' = -(\cot \varphi + \sec \varphi \csc \varphi)\lambda \]  

Thus it is seen that the only solution of (B.24) is \( \lambda = 0. \)
APPENDIX C

COMPUTER PROGRAM FOR UPPER BOUND WITH HINGE CIRCLE
FOR SIMPLY SUPPORTED CAP

The following is a computer program, Fortran IV language, which
determines upper bounds on the collapse load for a simply supported,
shallow spherical cap. The cap is partially loaded by a ring load,
concentric with the cap apex, and has uniform intensity.

The only input data required are the values of k. The output
gives $\theta / \theta_L$ of 10, 5, 2.5 and in increments of 0.05 rad. from
$\theta = 0.10$ rad. to $\theta = 0.35$ rad.
FORTRAN IV LANGUAGE

UPPER BOUND FOR SIMPLY SUPPORTED CAP WITH HINGE CIRCLE
RATIO OF PHI-ZERO/PHI-LOADING OF 10,5,2.5,
PHI-ZERO=0.10 TO 0.35 RAD.

READ (5,1) SK
1 FORMAT (E10.3)
ALP=1.0/SK
AB=(1.0-SK)/(1.0+SK)
PHIZ=0.10
DO 900 J=1,6
PHIHM=ARCS(AB)
IF (PHIZ .LE. PHIHM) PHIHM=PHIZ
PHIL=0.10*PHIZ
DO 800 J=1,3
RATIO=PHIZ PHIL
SINLS=SIN(PHIL)*SIN(PHIL)
BE=PHIL-(SIN(PHIL)**COS(PHIL))
PHIH=PHIL
DO 700 K=J,100
IND=1
IF (PHIH .GE. PHIHM) GO TO 20
CONTINUE
GO TO 50
20 PHIH=PHIHM
IND=2
50 PHIO=PHIL
PMIN=1.0E+06
DO 600 L=J,100
IF (PHIO .GE. PHIH) GO TO 80
CONTINUE
GO TO 90
80 PHIO=PHIH
90 BB=SK/(1.0+SK)
BC=2.0+ALOG(SIN(PHIIH)/SIN(PHIO))
XNUM=BB*BC
BD=1.0+(COS(PHIH)*ALOG(TAN(PHIH/2.0)/TAN(PHIO/2.0)))
DENOM=(BD*SINLS)-(BE*COS(PHIH)/SIN(PHIO))
P=XNUM/DENOM
IF (P .LT. PMIN) GO TO 100
CONTINUE
GO TO 500
100 CONTINUE
PMIN=P
PHDMIN=PHIO
500 IF (PHIO .GE. PHIH) GO TO 650
IF (J .EQ. 1) GO TO 550
CONTINUE
GO TO 600
550 IF ((I .EQ. 2).OR.(I .EQ. 4).OR.(I .EQ. 6)) GO TO 575
CONTINUE
GO TO 600
575 IF (L .EQ. 1) PHIO=PHIO+0.005
600 PHIO=PHIO+0.01
650 WRITE (6,1000) ALP, RATIO, PHIL
1000 FORMAT (2X,'ALPHA=',F4.0,7X,'PHI-ZERO/PHI-LOAD=',F8.3,
17X,'PHI-LOAD=',F8.5)
WRITE (6,1001) PHIH, PHIZ, PHOMIN, PMIN
1001 FORMAT (28X,'PHI-HINGE=',F8.5,'PHI-ZERO=',F8.5,'P-MIN=',F10.5)
IF (J .EQ. 2) GO TO 750
IF (J .EQ. 1) GO TO 675
CONTINUE
GO TO 700
675 IF ((J .EQ. 2).OR.(J .EQ. 4).OR.(J .EQ. 6)) GO TO 680
CONTINUE
GO TO 700
680 IF (K .EQ. 1) PHIH=PHIH+0.005
700 PHIH=PHIH+0.01
750 CONTINUE
800 PHIL=2.0*PHIL
900 PHIZ=PHIZ+0.05
STOP
END
APPENDIX D

STRESS DISTRIBUTIONS FOR SIMPLY SUPPORTED CAP

Presented here are stress distributions for two particular cases of loading. These distributions are not for the exact collapse load, but are for a valid lower-bound collapse load.
Figure D.1. Stress distribution for $\phi_o/\phi_L = 1.0$, $k = 1/50$, $\phi_o = 0.25$ rad.
Figure D.2. Stress distributions for $\psi_o/\psi_L = 5.0$, $k = 1/50$, $\psi_o = 0.25$ rad.
APPENDIX E

NUMERICAL INTEGRATION EQUATIONS

The fourth-order Runge-Kutta method is used to solve the second-order differential equation

\[ n'' + f(\theta) \, n' + g(\theta) \, n = r(\theta) \quad (C.1) \]

Now if

\[ \frac{dn}{d\theta} = v \quad (C.2) \]

Then (C.1) can be written

\[ \frac{dv}{d\theta} = - f(\theta) \, v - g(\theta) \, n + r(\theta) = H(\theta, v, n) \quad (C.3) \]

Now write (C.2) and (C.3) as 2 fourth-order Runge-Kutta formulas

\[ v_{i+1} = v_i + \frac{1}{6} (B_1 + 2B_2 + 2B_3 + B_4) \quad (C.4) \]

where

\[ B_1 = (\Delta \theta) \, H(\theta_i, n_{1i}, v_i) \]

\[ B_2 = (\Delta \theta) \, H(\theta_i + \frac{\Delta \theta}{2}, n_{1i} + \frac{D_1}{2}, v_i + \frac{B_1}{2}) \quad (C.5) \]

\[ B_3 = (\Delta \theta) \, H(\theta_i + \frac{\Delta \theta}{2}, n_{1i} + \frac{D_2}{2}, v_i + \frac{B_2}{2}) \]

\[ B_4 = (\Delta \theta) \, H(\theta_i + \Delta \theta, n_{1i} + D_3, v_i + B_3) \]
and

\[ n_{g_{i+1}} = n_{g_i} + \frac{1}{6} (D_1 + 2D_2 + 2D_3 + D_4) \]  \hspace{1cm} (C.6) \]

where

\[ D_1 = (\Delta \phi) (v_i) \]

\[ D_2 = (\Delta \phi) (v_i + \frac{1}{2}) \]  \hspace{1cm} (C.7) \]

\[ D_3 = (\Delta \phi) (v_i + \frac{2}{2}) \]

\[ D_4 = (\Delta \phi) (v_i + B_3) \]

If (C.7) is substituted into (C.5) and (C.6) then

\[ B_1 = (\Delta \phi) H(\gamma^2, n_{g_i}, v_i) \]

\[ B_2 = (\Delta \phi) H[\gamma^2 + \frac{\Delta \phi}{2}, n_{g_i} + \frac{\Delta \phi}{2}, v_i + \frac{1}{2}] \]  \hspace{1cm} (C.8) \]

\[ B_3 = (\Delta \phi) H[\gamma^2 + \frac{\Delta \phi}{2}, n_{g_i} + \frac{\Delta \phi}{2}, v_i + \frac{\Delta \phi}{4}, B_1, v_i + \frac{1}{2}] \]

\[ B_4 = (\Delta \phi) H[\gamma^2 + \Delta \phi, n_{g_i} + (\Delta \phi) v_i + \frac{\Delta \phi}{2}, B_2, v_i + B_3] \]

and

\[ n_{g_{i+1}} = n_{g_i} + \gamma_i (\Delta \phi) + \frac{1}{6} (\Delta \phi)(B_1 + B_2 + B_3) \]  \hspace{1cm} (C.9) \]

Now \( v_{i+1} \) and \( n_{g_{i+1}} \) may be found from (C.4) and (C.9), respectively, with the help of (C.8).
APPENDIX F

COMPUTER PROGRAM FOR UNIFORMLY LOADED CLAMPED CAP

The following is a computer program, Fortran IV language, which determines the exact collapse load for a clamped, shallow spherical cap. The cap is loaded over the entire surface and has uniform intensity.

The input data required are the values of $P$, the value of $k$, the loading angle $\varphi_L$, and the increments of $\varphi$ desired for the different regions of the yield surface. $\Delta\varphi_1$, $\Delta\varphi_2$ and $\Delta\varphi_4$ are increments of $\varphi$ used in stepping of algebraic equations. $\Delta\varphi_3$ is the increment of $\varphi$ used in the numerical integration technique.

Any number of data cards may be used; however, the last card should contain some number other than zero in column 71.
FORTRAN IV LANGUAGE

UNIFORM LOAD FOR CLAMPED CAP

COMMON AX, PHI, DPHI, XY, XNP, XM, XMP, SNM, DNM, XNT, XMT, P, CX, BX, 1XL0, XLT
2 READ (5,1) P, SK, PHIL, DPHI1, DPHI2, DPHI3, DPHI4, ID
1 FORMAT (7E10.0, I2)
AX=(1.0+SK)/SK
BX=(1.0-SK)/SK
PHI=0.0
WRITE (6,1000) SK, P
1000 FORMAT ('I', 'F5.3,3X,'P='F10.5//)
WRITE (6,1001)
DO 100 I=1, 1500
IF (PHI .EQ. 0.0) GO TO 3
CONTINUE
GO TO 5
3 XN=0.0
XM=1.0
GO TO 30
5 COT=COS(PHI)/SIN(PHI)
XN=-P*(1.0-(PHI*COT))
XM=1.0+(XN/SK)
30 XNT=0.0
XMT=1.0
SNM=XN+XM
DNM=XN-XM
SNMT=XNT+XMT
DNMT=XNT-XMT
IF (SNM .LT. 0.0) GO TO 155
WRITE(6,1002) PHI, XN, XM, XNT, XMT, DNM, DNMT, SNM, SNMT
1002 FORMAT (5X,'F7.5,5X,'F9.5,5X,'F9.5,5X,'F9.5,5X,'F9.5,5X,'F9.5,5X,'F9.5,5X,'F9.5)
CONTINUE
100 PHI=PHI+DPHI1
155 IF ((SNM .GE. -1.0E-05) .AND. (SNM .LE. 1.0E-05)) GO TO 160
DO 201 IA=1, 20
SINPS=SIN(PHI)*SIN(PHI)
AB=PHI-(SIN(PHI)*COS(PHI))
SNMP=-(AX*P*AB)/SINPS
PHI=PHI-(SNM/SNMP)
COT=COS(PHI)/SIN(PHI)
XN=-P*(1.0-(PHI*COT))
XM=1.0+(XN/SK)
SNM=XN+XM
DNM=XN-XM
IF ((SNM .GE. -1.0E-05) .AND. (SNM .LE. 1.0E-05)) GO TO 160
201 CONTINUE
202 WRITE (6,1003)
1003 FORMAT (2X,'ERROR')
160 CONTINUE
PHI0=PHI
WRITE (6,1004) PHI0,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
DO 300 J=1,500
DPHI=DPHI3
CALL RKINT
SNM=XN+XM
DNM=XN-XM
SNMT=XNT+XMT
DNMT=XNT-XMT
IF (DNM .GT. 0.0) , GO TO 170
WRITE (6,1005) PHI1,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
CONTINUE

300 PHI=PHI+DPHI3
170 IF ((DNM .LE. 1.0E-05) .AND. (DNM .GE. -1.0E-05)) GO TO 180
DO 400 JA=1,20
TAN=SIN(PHI)/COS(PHI)
TANS=TAN*TAN
COT=1.0/TAN
DNMP=COT*(-1.0+XM-XN+(BX*(XM+P)*TANS))
DPHI=-(DNM/DNMP)
CALL RKINT
SNM=XN+XM
DNM=XN-XM
SNMT=XNT+XMT
DNMT=XNT-XMT
IF ((DNM .LE. 1.0E-05) .AND. (DNM .GE. -1.0E-05)) GO TO 180
400 CONTINUE
402 WRITE (6,1003)
180 CONTINUE
PHI1=PHI
WRITE (6,1005) PHI1,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
COST=COS(PHI1)
C3=(XM+P)/COST
C4=XM-ALOG(SIN(PHI1))-((C3*COST)/SK)
PHI1=PHI1+DPHI4
DO 500 K=1,500
CONTINUE
403 XN=(C3*COS(PHI))—P
XM=ALOG(SIN(PHI))+(C3*COS(PHI)/SK)+C4
SNM=XN+XM
DNM=XN-XM
XNT=XN
XMT=1.0+XM
SNMT = XNT + XMT  
DNMT = XNT - XMT  
IF (SNM .LT. -1.0) GO TO 750  
WRITE (6,1002) PHI,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT  
500 PHI = PHI + DPHI4  
750 ADD = SNM + 1.0  
IF ((ADD .GE. -1.0E-05) .AND. (ADD .LE. 1.0E-05)) GO TO 900  
DO 800 LA = 1,30  
TAN = SIN(PHI)/COS(PHI)  
COT = 1.0/TAN  
SNMP = COT - (AX *(XN + P) * TAN)  
PHI = PHI - (ADD/SNMP)  
XN = ALUG(SIN(PHI)) + {C3*COS(PHI)/SK} + C4  
XNT = XN  
XMT = 1.0 + XM  
SNM = XN + XM  
DNM = XN - XM  
SNMT = XNT + XMT  
DNMT = XNT - XMT  
ADD = SNM + 1.0  
IF ((ADD .GE. -1.0E-05) .AND. (ADD .LE. 1.0E-05)) GO TO 900  
800 CONTINUE  
802 WRITE(6,1003)  
900 CONTINUE  
WRITE (6,1013) PHIZ,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT  
1013 FORMAT (2X,'PHIZ=',F9.5,5X,'XM=',F9.5,5X,'XMT=',F9.5,5X,'XNT=',F9.5,5X,'DNM=',F9.5,5X,'SNM=',F9.5,5X,'SNMT=',F9.5,5X)  
15X,F9.5,5X,F9.5,5X,F9.5//)  
SINZS = SIN(PHI)*SIN(PHI)  
P8 = (P*SINZS)/(6.0*SK)  
WRITE (6,2000) P8  
2000 FORMAT (2X,'PHIZ=',F9.5,3X,'PHIT=',F9.5,3X,'PHIO=',F9.5,3X)  
C CHECK LAMBDA'S FOR KINEMATIC FIELD  
C  
WRITE (6,1009) SK,PHIZ,PHIT,PHIO  
1009 FORMAT (15X,'PHIZ=',F9.5,3X,'PHIT=',F9.5,3X,'PHIO=',F9.5,3X)  
1'PHIZ= F9.5,5X,'PHIT= F9.5,5X,'PHIO= F9.5//)  
ALP = 1.0/SK  
PHI = PHIZ  
XLO = 0.0  
XLT = COS(PHI)/SIN(PHI)  
V = 1.0  
W = 0.0  
WRITE (6,1010) PHI, XLO, XLT, V, W  
1010 FORMAT (2X,'PHIZ=',F9.5,3X,'PHIT=',F9.5,3X,'PHIO=',F9.5,3X)  
C REDEFINING BX. DIFFERENT FROM BX IN LINE 5.  
BX = 1.0 - ALP
125 \text{DEL} = \text{PHI} - \text{PHIT} \\
\text{IF} \ (\text{DEL} \ \cdot \ \text{GE.} \ \text{DPHI}3) \ \text{GO TO} \ 110 \\
\text{DPHI} = -\text{DEL} \\
\text{IND} = 2 \\
\text{GO TO} \ 115 \\
110 \text{IND} = 1 \\
\text{DPHI} = -\text{DPHI}3 \\
115 \ V = (\text{DPHI} \cdot (\text{XLO} - \text{XLT} + \text{W})) + \text{V} \\
\text{CALL RK3B} \\
\text{COT} = \text{COS} (\text{PHI}) / \text{SIN} (\text{PHI}) \\
\text{W} = (\text{V} \cdot \text{COT}) + \text{XLO} - \text{XLT} \\
\text{IF} \ (\text{IND} \ \cdot \ \text{EQ.} \ 2) \ \text{GO TO} \ 120 \\
\text{WRITE} \ (6,1011) \ \text{PHI}, \text{XLO}, \text{XLT}, \text{V}, \text{W} \\
1011 \text{FORMAT} \ (2X, \ 'PHI' = 'F9.5, 3X, 'LAMBD\-A-1' = 'F10.5, 3X, \\
1 'LAMBD\-A-2' = 'F10.5, 3X, 'V' = 'F9.5, 3X, 'W' = 'F9.5) \\
\text{GO TO} \ 105 \\
120 \text{CONTINUE} \\
\text{PHIT} = \text{PHI} \\
\text{WRITE} \ (6,1006) \ \text{PHIT}, \text{XLO}, \text{XLT}, \text{V}, \text{W} \\
1006 \text{FORMAT} \ (2X, \ 'PHIT' = 'F9.5, 3X, 'LAMBD\-A-1' = 'F10.5, 3X, \\
1 'LAMBD\-A-2' = 'F10.5, 3X, 'V' = 'F9.5, 3X, 'W' = 'F9.5) \\
125 \text{DEL} = \text{PHI} - \text{PHIO} \\
\text{IF} \ (\text{DEL} \ \cdot \ \text{GE.} \ \text{DPHI}3) \ \text{GO TO} \ 130 \\
\text{DPHI} = -\text{DEL} \\
\text{IND} = 4 \\
\text{GO TO} \ 140 \\
130 \text{IND} = 3 \\
\text{DPHI} = -\text{DPHI}3 \\
140 \ V = (\text{DPHI} \cdot (\text{XLO} - \text{XLT} + \text{W})) + \text{V} \\
\text{CALL RK43} \\
\text{COT} = \text{COS} (\text{PHI}) / \text{SIN} (\text{PHI}) \\
\text{W} = (\text{V} \cdot \text{COT}) + \text{XLO} - \text{XLT} \\
\text{IF} \ (\text{IND} \ \cdot \ \text{EQ.} \ 4) \ \text{GO TO} \ 150 \\
\text{WRITE} \ (6,1011) \ \text{PHI}, \text{XLO}, \text{XLT}, \text{V}, \text{W} \\
\text{GO TO} \ 125 \\
150 \text{CONTINUE} \\
\text{PHIO} = \text{PHI} \\
\text{WRITE} \ (6,1007) \ \text{PHIO}, \text{XLO}, \text{XLT}, \text{V}, \text{W} \\
1007 \text{FORMAT} \ (2X, \ 'PHIO' = 'F9.5, 3X, 'LAMBD\-A-1' = 'F10.5, 3X, \\
1 'LAMBD\-A-2' = 'F10.5, 3X, 'V' = 'F9.5, 3X, 'W' = 'F9.5) \\
\text{TAN} = \text{SIN} (\text{PHIO}) / \text{COS} (\text{PHIO}) \\
\text{COT} = 1.0 / \text{TAN} \\
\text{SECO} = 1.0 / \text{COS} (\text{PHIO}) \\
\text{A1} = \text{TAN} \cdot (\text{XLO} + \text{XLT}) \\
\text{A2} = \text{SIN} (\text{PHIO}) \cdot ((\text{AX} \cdot \text{COT} \cdot \text{A1}) - (2.0 \cdot \text{XLO})) \\
\text{A3} = (\text{W} \cdot \text{SECO}) + (\text{A2} \cdot \text{TAN}) \\
\text{PHI} = \text{PHI} - \text{DPHI}1 \\
\text{DO} \ 200 \ M = 1, 500 \\
101 \ \text{CSC} = 1.0 / \text{SIN} (\text{PHI}) \\
\text{COT} = \text{COS} (\text{PHI}) / \text{SIN} (\text{PHI}) \\
\text{XLO} = 0.5 \cdot ((\text{AX} \cdot \text{A1} \cdot \text{COT}) - (\text{A2} \cdot \text{CSC}))
XLT = 0.5 * ((BX * A1 * COT) + (A2 * CSC))
V = (A3 * SIN(PHI)) + (A2 * COS(PHI)) - (A1 * ALP)
W = (A3 * COS(PHI)) - (A2 * SIN(PHI))
IF (IND .EQ. 5) GO TO 250
IF (PHII .LE. 0.001) GO TO 175
WRITE (6, 1011) PHI, XLO, XLT, V, W
GO TO 200
175 PHI = 0.001
IND = 5
GO TO 101
200 PHI = PHI - DPHII
250 WRITE (6, 1011) PHI, XLO, XLT, V, W
XINEQ1 = (AX * A1) - A2
XINEQ2 = (BX * A1) + A2
WRITE (6, 1003) XINEQ1, XINEQ2, A1, A2
1003 FORMAT (2X, '(1+ALP)A1-A2=', F9.5, 3X, '(1-ALP)A1+A2=', F9.5*)
1010 FORMAT (2X, 'A1=', F9.5, 3X, 'A2=', F9.5)
IF (IND .EQ. 0) GO TO 2
950 CONTINUE
STOP
END
SUBROUTINE RKINT
COMMON AX, PHI, DPHII, XN, XNP, XM, XMP, SNM, DNM, XNT, XMT, P, CX, BX,
1 XLO, XLT
COT = COS(PHI) / SIN(PHI)
SEC = 1.0 / COS(PHI)
TAN = 1.0 / COS(PHI)
F = 2.0 * SEC * CSC
G = (2.0 * SEC * SEC) + (0.5 * AX)
R = (0.5 * COT * COT) - (F * G)
XNP = COT ** (1.0 * XM) - (0.5 * XN) - (XN + P) * TAN * TAN)
H = -(F * XNP) - (G * XN) + R
B1 = DPHII * H
PHII = PHI + (0.5 * DPHII)
XN2 = XN + (0.5 * DPHII * XNP)
XNP2 = XNP + (0.5 * B1)
COT = COS(PHI) / SIN(PHI)
SEC = 1.0 / COS(PHI)
TAN = 1.0 / COS(PHI)
F = 2.0 * SEC * CSC
G = (2.0 * SEC * SEC) + (0.5 * AX)
R = (0.5 * COT * COT) - (F * G)
H = -(F * XNP2) - (G * XN2) + R
B2 = DPHII * H
XN3 = XN + (0.5 * DPHII * XNP) + (0.25 * DPHII * B1)
XNP3 = XNP + (0.5 * B2)
H = -(F * XNP3) - (G * XN3) + R
B3 = DPHII * H
PHII = PHI + DPHII
\begin{align*}
XN4 &= XN*(DPHI*XNP)+(0.5*DPHI*B2) \\
XNP4 &= XNP+B3 \\
CUT &= \cos(PHI1)/\sin(PHI2) \\
SEC &= 1.0/\cos(PHI2) \\
TAN &= 1.0/COT \\
CSC &= 1.0/\sin(PHI2) \\
G &= (2.0*SEC*SEC)+(0.5*AX) \\
R &= (0.5*COT*COT)-(P*G) \\
H &= -(F*XNP4)-(G*XN4)+R \\
B4 &= DPHI*H \\
XN &= XN+(DPHI*XNP)+(DPHI*(B1+B2+B3))/6.0) \\
XNP &= XNP+((B1+(2.0*B2)+(2.0*B3)*B4)/6.0) \\
PHI &= PHI1 \\
XM &= (2.0*XNP*TAN)+XN+(2.0*(XN+P)*TAN*TAN) \\
SNM &= XN*XN \\
XNT &= 0.5*SNM \\
XMT &= 1.0*XNT \\
RETURN \\
END \\
SUBROUTINE RK3B \\
COMMON AX, PHI, DPHI, XN, XNP, XM, XMP, SNM, DNM, XNT, XMT, P, CX, BX, \\
XL0, XLT \\
TAN &= \sin(PHI)/\cos(PHI) \\
COT &= 1.0/TAN \\
F1 &= -(2.0*COT)-(0.5*AX*TAN) \\
F2 &= -(0.5*AX*TAN) \\
F3 &= -(0.5*AX*TAN) \\
F4 &= -(2.0*COT)-(0.5*AX*TAN) \\
E1 &= (F1*XL0)+(F2*XLT) \\
E2 &= (F3*XL0)+(F4*XLT) \\
AK1 &= DPHI*E1 \\
Q1 &= DPHI*E2 \\
PHI1 &= PHI+(0.5*DPHI) \\
XL02 &= XL0+(0.5*AK1) \\
XLT2 &= XLT+(0.5*Q1) \\
TAN &= \sin(PHI)/\cos(PHI) \\
COT &= 1.0/TAN \\
F1 &= -(2.0*CUT)-(0.5*AX*TAN) \\
F2 &= -(0.5*AX*TAN) \\
F3 &= -(0.5*AX*TAN) \\
F4 &= -(2.0*CUT)-(0.5*AX*TAN) \\
E1 &= (F1*XL02)+(F2*XLT2) \\
E2 &= (F3*XL02)+(F4*XLT2) \\
AK2 &= DPHI*E1 \\
Q2 &= DPHI*E2 \\
XL03 &= XL0+(0.5*AK2) \\
XLT3 &= XLT+(0.5*Q2) \\
E1 &= (F1*XL03)+(F2*XLT3) \\
E2 &= (F3*XL03)+(F4*XLT3) \\
AK3 &= DPHI*E1 \\
\end{align*}
Q3 = DPHI * E2
PHII = PHI + DPHI
XL04 = XL0 + AK3
XLT4 = XLT + Q3
TAN = SIN(PHII) / COS(PHII)
COT = 1.0 / TAN
F1 = -(2.0 * COT) - (0.5 * AX * TAN)
F2 = -(0.5 * AX * TAN)
F3 = -(0.5 * BX * TAN)
F4 = -(2.0 * COT) - (0.5 * BX * TAN)
E1 = (F1 * XL04) + (F2 * XLT4)
E2 = (F3 * XL04) + (F4 * XLT4)
AK4 = DPHI * E1
Q4 = DPHI * E2
XL0 = XL0 + ((AK1 + (2.0 * AK2) + (2.0 * AK3) + AK4) / 6.0)
XLT = XLT + ((Q1 + (2.0 * Q2) + (2.0 * Q3) + Q4) / 6.0)
PHI = PHII
RETURN
END

SUBROUTINE RK4B
COMMON AX, PHI, DPHI, XN, XNP, XM, XMP, SNM, DNM, XNT, XMT, P, CX, BX,
XL0, XLT
TAN = SIN(PHI) / COS(PHI)
COT = 1.0 / TAN
F5 = -COT - (0.5 * AX * TAN)
F6 = -(0.5 * AX * TAN)
F7 = -(0.5 * BX * TAN)
F8 = -COT - (0.5 * BX * TAN)
E3 = (F5 * XL02) + (F6 * XLT2)
E4 = (F7 * XL02) + (F8 * XLT2)
AK1 = DPHI * E3
Q1 = DPHI * E4
PHII = PHI + (0.5 * DPHI)
XL02 = XL0 + (0.5 * AK1)
XLT2 = XLT + (0.5 * Q1)
TAN = SIN(PHII) / COS(PHII)
COT = 1.0 / TAN
F5 = -COT - (0.5 * AX * TAN)
F6 = -(0.5 * AX * TAN)
F7 = -(0.5 * BX * TAN)
F8 = -COT - (0.5 * BX * TAN)
E3 = (F5 * XL02) + (F6 * XLT2)
E4 = (F7 * XL02) + (F8 * XLT2)
AK2 = DPHI * E3
Q2 = DPHI * E4
XL03 = XL0 + (0.5 * AK2)
XLT3 = XLT + (0.5 * Q2)
E3 = (F5 * XL03) + (F6 * XLT3)
E4 = (F7 * XL03) + (F8 * XLT3)
AK3 = DPHI * E3
Q3 = DPHI * E4
\begin{verbatim}
PHI1 = PHI + DPHI
XLU4 = XLU + AK3
XLT4 = XL1 + Q3
TAN = SIN(PHI1) / COS(PHI1)
COT = 1.0 / TAN
F5 = -CUT - (0.5 * AX * TAN)
F6 = -(0.5 * AX * TAN)
F7 = -(0.5 * BX * TAN)
F8 = -COT - (0.5 * BX * TAN)
E3 = (F5 * XLO4) + (F6 * XLT4)
E4 = (F7 * XLO4) + (F8 * XLT4)
AK4 = DPHI * E3
Q4 = DPHI * E4
XLU = XLG + (AK1 + (2.0 * AK2) + (2.0 * AK3) + AK4) / 6.0
XLT = XLT + (Q1 + (2.0 * Q2) + (2.0 * Q3) + Q4) / 6.0
PHI = PHI
RETURN
END
\end{verbatim}
APPENDIX G

COMPUTER PROGRAM FOR PARTIALLY LOADED CLAMPED CAP

The following is a computer program, Fortran IV language, which determines the exact collapse load for a clamped, shallow spherical cap. The cap is partially loaded by a ring load, concentric with the cap apex, and has uniform intensity.

The input data required are the value of P, the value of k, the loading angle $\theta$, and the increments of $\theta$ desired for the regions of the yield surface. DPHI1, DPHI2 and DPHI4 are increments of $\theta$ used in stepping of algebraic equations. DPHI3 is the increment of $\theta$ used in the numerical integration technique.

Any number of data cards may be used; however, the last card should contain some number other than zero in column 71.
FORTRAN IV LANGUAGE

LARGE LOADING ANGLE FOR CLAMPED CAP PHIL ON FACES 3B

COMMON AX,PHI,DPHI,XN,XNP,XM,XMP,SNM,DNM,XNT,XMT,P,CX,BX,
1XL0,XLT
2 READ (5,1) P,SK,PHIL,DPHI1,DPHI2,DPHI3,DPHI4,1D
1 FORMAT (/E10.0,12)
   AX=(1.0+SK)/SK
   BX=(1.0-SK)/SK
   PHI=0.0
   WRITE (6,1000) PHI,SK,P
1000 FORMAT ('PHI='F8.5,'K='F5.3,'P='F10.5//)
   WRITE (6,1001)
1001 FORMAT ('PHI','N','NT','M','M+NT','NT-M','N-M','N+M','N+M+NT'/)
   DO 100 I=1,1500
   IF (PHI.EQ.0.0) GO TO 3
   CONTINUE
   GO TO 5
3 XN=0.0
   XM=1.0
   GO TO 30
5 COT=COS(PHI)/SIN(PHI)
   XN=-P*(1.0-(PHI*COT))
   XM=1.0+(XN/SK)
30 XNT=0.0
   XMT=1.0
   SNM=XN+XM
   DNM=XN-XM
   SNMT=XNT+XMT
   DNM=SNMT-XMT
   IF (SNM.LT.0.0) GO TO 155
   WRITE (6,1002) PHI,XN,XM,XNT,XMT,DNM,SNMT
1002 FORMAT ('PHI','N','NT','M','M+NT','NT-M','N-M','N+M','N+M+NT'/)
   CONTINUE
100 PHI=PHI+DPHI1
155 IF (((SNM.GE.-1.0E-05).AND.((SNM.LE.1.0E-05))) GO TO 160
   DO 201 IA=1,20
      SINPS=SIN(PHI)*SIN(PHI)
      AB=PHI-(SIN(PHI)*COS(PHI))
      SNMP=-{(4*P*AB)/SINPS}
      PHI=PHI-(SNMP/SNMP)
      COT=COS(PHI)/SIN(PHI)
   XN=-P*(1.0-(PHI*COT))
   XM=1.0+(XN/SK)
   SNM=XN+XM
   DNM=XN-XM
   CONTINUE
IF ((SNM .GE. -1.0E-05) .AND. (SNM .LE. 1.0E-05)) GO TO 160
201 CONTINUE
202 WRITE (6,1003)
1603 FORMAT (2X,'ERROR')
160 CONTINUE
PHIU=PHI
WRITE (6,1004) PHIU,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
1004 FORMAT (2X,'PHI, XN, XM, XNT, XMT, DNM, DNMT, SNM, SNMT
DO 300 J=1,500
DPHI=DPHI3
CALL RKINT
SNM=XN+XM
DNM=XN-XM
SNMT=XNT+XMT
DNMT=XNT-XMT
IF (DNM .GT. 0.0) GO TO 170
WRITE (6,1002) PHI,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
CONTINUE
300 PHI=PHI+DPHI3
170 IF ((DNM .LE. 1.0E-05) .AND. (DNM .GE. -1.0E-05)) GO TO 180
DO 400 J=1,20
TAN=SIN(PHI)/COS(PHI)
TANS=TAN*TAN
COT=1.0/TAN
DNMP=COT*(-1.0+XM-XN+B*X(N+P)*TANS)
DPHI=-(DNM/DNMP)
CALL RKINT
SNM=XN+XM
DNM=XN-XM
SNMT=XNT+XMT
DNMT=XNT-XMT
IF (DNM .LE. 1.0E-05) .AND. (DNM .GE. -1.0E-05)) GO TO 180
400 CONTINUE
402 WRITE (6,1003)
180 CONTINUE
PHIT=PHI
WRITE (6,1005) PHIT,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
1005 FORMAT (2X,'PHI, XN, XM, XNT, XMT, DNM, DNMT, SNM, SNMT
COST=COS(PHIT)
C3=(XN+P)/COST
C4=XM-ALOG(SIN(PHIT))-((C3*COST)/SK)
PHI=PHIT+DPHI4
DO 500 K=1,500
IF (PHI .LT. PHIT) GO TO 403
PHI=PHIT
CONTINUE
(403 XN=(C3*COS(PHI))-P
XM=ALOG(SIN(PHI))+((C3*COS(PHI)/SK)+C4
SNM=XN+XM
DNM=XN-XM
XNT=XN
XMT=1.0+XM
SNMT=XNT+XMT
DNMT=XNT-XMT
IF (PHI .EQ. PHIL) GO TO 650
WRITE (6,1002) PHI,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
500 PHI=PHI+DPHI4
650 WRITE (6,1012) PHIL,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
1012 FORMAT (2X, 'PL 'F7.5,5X,F9.5,5X,F9.5,5X,F9.5,5X,F9.5,
15X,F9.5,5X,F9.5,5X,F9.5,5X,F9.5)
TANHP=SIN(0.5*PHIL)/COS(0.5*PHIL)
SINPS=SIN(PHI)*SIN(PHI)
CX=P*SINPS
DX=(COS(PHI)*ALOG(TANHP))
EX=1.0+DX
C1=(XN*(CX*EX))/COS(PHI)
C2=XN-ALOG(SIN(PHI))-(C1*COS(PHI)/SK)+(CX*DX/SK)
PHI=PHI+DPHI4
DO 700 L=1,500
TANHP=SIN(0.5*PHI)/COS(0.5*PHI)
SINPS=SIN(PHI)*SIN(PHI)
FX=COS(PHI)*ALOG(TANHP)
GX=1.0+FX
XN=(C1*COS(PHI))-(CX*GX)
XM=ALOG(SIN(PHI))+(C1*COS(PHI)/SK)-(CX*FX/SK)+C2
XNT=XN
XMT=1.0+XM
SNM=XN+XM
DNM=XN-XM
SNMT=XNT+XMT
DNMT=XNT-XMT
IF (SNM .LT. -1.0) GO TO 750
WRITE (6,1002) PHI,XN,XM,XNT,XMT,DNM,DNMT,SNM,SNMT
700 PHI=PHI+DPHI4
750 ADD=SNM+1.0
IF ((ADD .GE. -1.0E-05) .AND. (ADD .LE. 1.0E-05)) GO TO 900
DO 800 LA=1,20
TAN=SIN(PHI)/COS(PHI)
COT=1.0/TAN
SEC=1.0/COS(PHI)
SNMP=C01*(1.0-(AX*XN*TAN*TAN)-(AX*CX*SEC*SEC))
PHI=PHI-(ADD/SNMP)
TANHP=SIN(0.5*PHI)/COS(0.5*PHI)
SINPS=SIN(PHI)*SIN(PHI)
FX=COS(PHI)*ALOG(TANHP)
GX=1.0+FX
XN=(C1*COS(PHI))-(CX*GX)
XM=ALOG(SIN(PHI))+(C1*COS(PHI)/SK)-(CX*FX/SK)+C2
XNT=XN
XMT=1.0+XM
SNM=XN+XM
DNM=XN-XM
5NMT=XNT+XMT
UNMT=XNT-XMT
ADD=SNM+1.0
IF (ADD .GE. -1.0E-05) .AND. (ADD .LE. 1.0E-05)) GO TO 900
600 CONTINUE
b02 WRITE(6,1003)
900 CONTINUE
PHIZ=PHI
WRITE (6,1013) PHIZ,XN,XM,XNT,XMT,DNM,DNMT,SN,M,SNMT
SINZS=SIN(PHI)*SIN(PHI)
PZ=(P*SINZS)/(6.0*SK)
WRITE (6,2000) PB
2000 FORMAT (12X,'P-BAR=',F10.5)
C CHECK LAMBDA'S FOR KINEMATIC FIELD
C
WRITE (6,1009) SK,PHIZ,PHIT,PHIO
1009 FORMAT (12X,'PHIZ=',F9.5,3X,'PHIT=',F9.5,3X,'PHIO=',F9.5,3X,'PHIZ=PHI',F9.5,3X,'ALP=',1.0/SK)
XLO=0.0
XLT=TRIGO(PHIZ)*SIGN(1013)
V=1.0
W=0.0
WRITE (6,1010) PHIZ,XLO,XLT,V,W
1010 FORMAT (12X,'PHIZ=',F9.5,3X,'LAMBDAT=',F10.5,3X,'LAMBDAT=',F10.5,3X,'ALP=1.0/SK')
BX=1.0-ALP
C REDEFINING BX. DIFFERENT FROM BX IN LINE 5.
105 DEL=PHIT-PHI
IF (DEL .GE. DPHI3) GO TO 110
DPHI=-DEL
IND=2
GO TO 115
110 IND=1
DPHI=DPHI3
115 V=(DPHI*(XLO-XLT+W)) + V
CALL RK3B
COT=TRISO(PHI)*SIGN(PHI)
W=(V*COT)+XLO-XLT
IF (IND .EQ. 2) GO TO 120
WRITE (6,1011) PHIZ,XLO,XLT,V,W
1011 FORMAT (12X,'PHIZ=',F9.5,3X,'LAMBDAT=',F10.5,3X,'ALP=1.0/SK')
V=1.0-ALP
C
105 GO TO 105
120 CONTINUE
PHIT=PHI
write (6,1006) PHIT,XLO,XLT,V,W
1006 format (2x,'PHIT='F9.5,3x,'LAM3DA-1='F10.5,3x,
1'LAM3DA-2='F10.5,3x,'V='F9.5,3x,'W='F9.5)
125 DEL=PHI-DPHI
if (DEL .GE. DPHI3) go to 130
DPHI=-DEL
IND=4
GO TO 140
130 IND=3
DPHI=DPHI3
140 V=DPHI*(XLO-XLT+W)+V
CALL RK4B
COT=COS(PHI)/SIN(PHI)
W=(V+COT)+XLO-XLT
if (IND .EQ. 4) go to 150
write (6,1011) PHI,XLO,XLT,V,W
GO TO 125
150 CONTINUE
PHIO=PHI
write (6,1007) PHIO,XLO,XLT,V,W
1007 format (2x,'PHIO='F9.5,3x,'LAMBD-1='F10.5,3x,
1'LAMBD-2='F10.5,3x,'V='F9.5,3x,'W='F9.5)
TANO=SIN(PHIO)/COS(PHIO)
COTU=1.0/TANO
SECO=1.0/COS(PHIO)
A1=TANO*(XLO+XLT)
A2=SIN(PHIO)*((AX*COTU*A1)-(2.0*XLO))
A3=(W+SECO)+(A2*TANO)
PHI=PHI-DPHI1
DO 200 M=1,500
101 CSC=1.0/SIN(PHI)
COT=COS(PHI)/SIN(PHI)
XLO=0.5*((AX*A1+COT)-(A2*CSC))
XLT=0.5*((BX*A1+COT)+(A2*CSC))
V=(A3*SIN(PHI))+((A2*COS(PHI))-(A1*ALP))
W=(A3*COS(PHI))-(A2*SIN(PHI))
IF (IND .EQ. 5) go to 250
IF (PHI .LE. 0.001) go to 175
write (6,1011) PHI,XLO,XLT,V,W
GO TO 200
175 PHI=0.001
IND=5
GO TO 101
200 PHI=PHI-DPHI1
250 WRITE (6,1011) PHI,XLO,XLT,V,W
XINEQ1=(AX*A1)-A2
XINEQ2=(BX*A1)+A2
write (6,1008) XINEQ1,XINEQ2,A1,A2
1008 format (2x,'(1+ALP)A1-A2='F9.5,3x,'(1-ALP)A1+A2='F9.5,
13X,'A1='F9.5,3X,'A2='F9.5)
IF (ID .EQ. 0) GO TO 2
950 CONTINUE
STOP
END

SUBROUTINE RKINT
COMMON AX,PHI,DPHI,XN,XNP,SM,SNM,XNT,XMTP,CX,BX,
COT,XLT
COT=COS(PHI)/SIN(PHI)
SEC=1.0/COS(PHI)
TAN=1.0/COT
CSC=1.0/SIN(PHI)
F=2.0*SEC*CSC
G=(2.0*SEC*SEC)+(0.5*AX)
R=(0.5*COT*COT)-(P*G)
XNP=COT*((0.5*XM)-(0.5*XN)+((XN*AX)*TAN*TAN))
H=-(F*XNP)+(G*XN)+R
B1=DPHI*H
PHII=PHI+(0.5*DPHI)
XN2=XN+(0.5*DPHI*XNP)
XNP2=XNP+(0.5*B1)
COT=COS(PHI)/SIN(PHI)
SEC=1.0/COS(PHI)
TAN=1.0/COT
CSC=1.0/SIN(PHI)
F=2.0*SEC*CSC
G=(2.0*SEC*SEC)+(0.5*AX)
R=(0.5*COT*COT)-(P*G)
H=-(F*XNP2)+(G*XN2)+R
B2=DPHI*H
XN3=XN+(0.5*DPHI*XNP)+(0.25*DPHI*B1)
XNP3=XNP+(0.5*B2)
H=-(F*XNP3)+(G*XN3)+R
B3=DPHI*H
PHII=PHI+DPHI
XN4=XN+(DPHI*XNP)+(0.5*DPHI*B2)
XNP4=XNP+B3
COT=COS(PHI)/SIN(PHI)
SEC=1.0/COS(PHI)
TAN=1.0/COT
CSC=1.0/SIN(PHI)
F=2.0*SEC*CSC
G=(2.0*SEC*SEC)+(0.5*AX)
R=(0.5*COT*COT)-(P*G)
H=-(F*XNP4)+(G*XN4)+R
B4=DPHI*H
XN=XN+(DPHI*XNP)+((DPHI*(B1+B2+B3))/6.0)
XNP=XNP+((B1+(2.0*B2)+(2.0*B3)+B4))/6.0
PHII=PHI
XH=(2.0*XNP*TAN)+XN+(2.0*(XN*AX)*TAN*TAN)
SNM=XN+XM
XNT=0.5*SNM
\[ XMT = 1.0 + XNT \]
RETURN
END

SUBROUTINE RK3B
COMMON AX, PHI1, DPHI1, XN, XNP, XM, XMP, SNM, DNM, XNT, XMT, P, CX, BX,
XL0, XLT

\[ \tan = \sin(\phi1) / \cos(\phi1) \]
\[ \cot = 1.0 / \tan \]
\[ F1 = -(2.0 \times \cot) - (0.5 \times AX \times \tan) \]
\[ F2 = -(0.5 \times AX \times \tan) \]
\[ F3 = -(0.5 \times BX \times \tan) \]
\[ F4 = -(2.0 \times \cot) - (0.5 \times BX \times \tan) \]
\[ E1 = (F1 \times XL0) + (F2 \times XLT) \]
\[ E2 = (F3 \times XL0) + (F4 \times XLT) \]
\[ AK1 = DPHI1 \times E1 \]
\[ Q1 = DPHI1 \times E2 \]
\[ PHI1 = PHI1 + (0.5 \times DPHI1) \]
\[ XL02 = XL0 + (0.5 \times AK1) \]
\[ XLT2 = XLT + (0.5 \times Q1) \]
\[ \tan = \sin(\phi1) / \cos(\phi1) \]
\[ \cot = 1.0 / \tan \]
\[ F1 = -(2.0 \times \cot) - (0.5 \times AX \times \tan) \]
\[ F2 = -(0.5 \times AX \times \tan) \]
\[ F3 = -(0.5 \times BX \times \tan) \]
\[ F4 = -(2.0 \times \cot) - (0.5 \times BX \times \tan) \]
\[ E1 = (F1 \times XL02) + (F2 \times XLT2) \]
\[ E2 = (F3 \times XL02) + (F4 \times XLT2) \]
\[ AK2 = DPHI1 \times E1 \]
\[ Q2 = DPHI1 \times E2 \]
\[ XL03 = XL0 + (0.5 \times AK2) \]
\[ XLT3 = XLT + (0.5 \times Q2) \]
\[ E1 = (F1 \times XL03) + (F2 \times XLT3) \]
\[ E2 = (F3 \times XL03) + (F4 \times XLT3) \]
\[ AK3 = DPHI1 \times E1 \]
\[ Q3 = DPHI1 \times E2 \]
\[ PHI1 = PHI1 + DPHI1 \]
\[ XL04 = XL0 + AK3 \]
\[ XLT4 = XLT + Q3 \]
\[ \tan = \sin(\phi1) / \cos(\phi1) \]
\[ \cot = 1.0 / \tan \]
\[ F1 = -(2.0 \times \cot) - (0.5 \times AX \times \tan) \]
\[ F2 = -(0.5 \times AX \times \tan) \]
\[ F3 = -(0.5 \times BX \times \tan) \]
\[ F4 = -(2.0 \times \cot) - (0.5 \times BX \times \tan) \]
\[ E1 = (F1 \times XL04) + (F2 \times XLT4) \]
\[ E2 = (F3 \times XL04) + (F4 \times XLT4) \]
\[ AK4 = DPHI1 \times E1 \]
\[ Q4 = DPHI1 \times E2 \]
\[ XL0 = XL0 + (AK1 + (2.0 \times AK2) + (2.0 \times AK3) + AK4) / 6.0 \]
\[ XLT = XLT + ((Q1 + (2.0 \times Q2) + (2.0 \times Q3) + Q4) / 6.0) \]
\[ PHI = PHI1 \]
RETURN
END

SUBROUTINE RK4B
COMMON AX, PHI, DPHI, XN, XNP, XM, XMP, SNM, DNM, XNT, XMT, P, CX, DX,
1XLO, XLT
TAN = SIN(PHI) / COS(PHI)
COT = 1.0 / TAN
F5 = -COT - (0.5 * AX * TAN)
F6 = -(0.5 * AX * TAN)
F7 = -(0.5 * BX * TAN)
F8 = -COT - (0.5 * BX * TAN)
E3 = (F5 * XLO) + (F6 * XLT)
E4 = (F7 * XLO) + (F8 * XLT)
AK1 = DPHI * E3
Q1 = DPHI * E4
PHII = PHI + (0.5 * DPHI)
XLO2 = XLO + (0.5 * AK1)
XLT2 = XLT + (0.5 * Q1)
TAN = SIN(PHIII) / COS(PHIII)
COT = 1.0 / TAN
F5 = -COT - (0.5 * AX * TAN)
F6 = -(0.5 * AX * TAN)
F7 = -(0.5 * BX * TAN)
F8 = -COT - (0.5 * BX * TAN)
E3 = (F5 * XLO2) + (F6 * XLT2)
E4 = (F7 * XLO2) + (F8 * XLT2)
AK2 = DPHI * E3
Q2 = DPHI * E4
XLO3 = XLO + (0.5 * AK2)
XLT3 = XLT + (0.5 * Q2)
E3 = (F5 * XLO3) + (F6 * XLT3)
E4 = (F7 * XLO3) + (F8 * XLT3)
AK3 = DPHI * E3
Q3 = DPHI * E4
PHII = PHI + DPHI
XLO4 = XLO + AK3
XLT4 = XLT + Q3
TAN = SIN(PHIII) / COS(PHIII)
COT = 1.0 / TAN
F5 = -COT - (0.5 * AX * TAN)
F6 = -(0.5 * AX * TAN)
F7 = -(0.5 * BX * TAN)
F8 = -COT - (0.5 * BX * TAN)
E3 = (F5 * XLO4) + (F6 * XLT4)
E4 = (F7 * XLO4) + (F8 * XLT4)
AK4 = DPHI * E3
Q4 = DPHI * E4
XLO = XLO + ((AK1 + (2.0 * AK2) + (2.0 * AK3) + AK4) / 6.0)
XLT = XLT + ((Q1 + (2.0 * Q2) + (2.0 * Q3) + Q4) / 6.0)
PHI = PHI
RETURN
END
## APPENDIX H

### Table H.1

**DIMENSIONLESS PRESSURES FOR SMALL CAP ANGLES**

<table>
<thead>
<tr>
<th>Uniform Load</th>
<th>$\varphi_L = 0.20$ rad.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_0$</td>
<td>$\bar{P}$</td>
</tr>
<tr>
<td>0.08578</td>
<td>0.91746</td>
</tr>
<tr>
<td>0.09214</td>
<td>0.91713</td>
</tr>
<tr>
<td>0.10016</td>
<td>0.91657</td>
</tr>
<tr>
<td>0.11073</td>
<td>0.91586</td>
</tr>
<tr>
<td>0.12555</td>
<td>0.91469</td>
</tr>
<tr>
<td>0.14287</td>
<td>0.92924</td>
</tr>
<tr>
<td>0.15362</td>
<td>0.92686</td>
</tr>
<tr>
<td>0.20145</td>
<td>0.91748</td>
</tr>
<tr>
<td>0.27804</td>
<td>0.91028</td>
</tr>
</tbody>
</table>

The values in Table H.1 are calculated from (6.47).
APPENDIX I

STRESS AND VELOCITY DISTRIBUTIONS FOR CLAMPED CAP

Stress and velocity distributions for the uniformly loaded cap with $p^* = 1.420$ are presented in Figures I.1 and I.3. Stress and velocity distributions for the partially loaded cap, $p^* = 1.715$ and $\bar{\varphi}_L = 0.20$ rad., are presented in Figures I.2 and I.4.
Figure I.1. Stress distribution for $p^* = 1.420$, $k = 1/50$ (uniform load).
Figure 1.2. Stress distribution for $p^* = 1.715$, $k = 1/50$, $\varphi_L = 0.20$ rad.
Figure 1.3. Velocity distribution for $p^* = 1.420$, $k = 1/50$ (uniform load).
Figure I.4. Velocity distribution for $p^* = 1.715$, $k = 1/50$, $\varphi_L = 0.20$ rad.
LIST OF REFERENCES


