General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)

INSTITUTE FOR SPACE STUDIES

DUPLICATE

EXPLODING SPHERES OF DUST

Jeffrey M. Cohen

RECEIVED
A.I.A.A.
'68 JAN 22 AM 8:31
T.I.S. LIBRARY

FACILITY FORM 602
N69-38628 (ACCESSION NUMBER) (THRU)
24 (PAGES) 1 (CODE)
CR-106096 (NASA CR OR TMX OR AD NUMBER) 13 (CATEGORY)

MAR 1969
RECEIVED
NASA STI FACILITY
INPUT BRANCH

GODDARD SPACE FLIGHT CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Exploding Spheres of Dust

Jeffrey M. Cohen

Institute for Space Studies
Goddard Space Flight Center, NASA
New York, New York

The expansion and gravitational collapse of spherical balls of dust are investigated. Three cases are considered: positive, zero and negative curvature of the space within the cloud of dust. In each of these three cases the interior geometry is matched to an exterior Schwarzschild geometry. Also, in each case, the mass m seen by an observer at infinity is found to be positive definite and related to the density ρ and radius r of the dust via $m = (4\pi/3)\rho r^3$. The radius and density of the dust are functions of time. The method developed in this paper can be applied to thick shells of dust surrounding a collapsing or stationary body, giving simple models of supernovae. The envelope expands leaving the central body as a remnant. These results are compared to those of Newtonian physics. Some effects of rotation are discussed.

I. INTRODUCTION

In Newtonian mechanics one investigates the expansion or collapse of a homogenous spherical ball of dust by integrating Newton's equations. The same problem in general relativity is more complicated; it usually involves solving Einstein's field equations in two coordinate patches and then fitting the geometries together. The patching problem is often the most difficult part of the problem. However, in the case of expanding or collapsing dust, an interesting theorem due to Beckedorff and Misner[1] and Lindquist and Wheeler[2] reduces the patching problem almost to the simplicity of Newtonian mechanics. This theorem[3] states that: a necessary and sufficient condition for two 4-geometries to join smoothly (at the interfact) is that particles at the interface follow simultaneously the geodesic laws of motion for the two separate geometries. The method is applied in ref. 3 for the case of positive curvature of the space within the dust. Through the use of this theorem one can match the geometries without matching the metrics.

In this paper we consider the cases of zero and negative curvature as well as (for completeness) positive curvature. One objective is to write the solution in such a way that it can be used in a subsequent paper concerning

the gravitational collapse of rotating balls of dust.[4]

In section II, Newton's equations for the gravitational collapse of a ball of dust are written down and integrated. This is doen to facilitate the interpretation of the relativistic results for the cases of zero and negative curvature. For the case of positive curvature, it is often more convenient to interpret the solution via the initial value equations at the point of time symmetry.[4] In sections III and IV geodesics are investigated. The geometries are matched and the results are discussed physically in sections V and VI. In section VII, the effect of rotation is considered.

II. NEWTONIAN COLLAPSE

To facilitate the interpretation of later results, a short Newtonian investigation of the expansion or collapse of a dust ball is given here. Newton's equaaions for the outermost dust particle of the dust ball are

$$M \ddot{r} = -Mm\, r^{-2}. \tag{1}$$

Here M denotes the geometrized mass[3] of the dust particle m, the mass of the entire dust ball r, the radius of the ball, and the dot denotes differentiation with respect to time. Integration of Eq. (1) yields

$$\dot{r}^2 = -K_n + 2m\, r^{-1} \tag{2}$$

where $-K_n$ is integration constant equal to twice the total energy of the particle per unit mass.

If the cases of positive, zero and negative K_n are investigated separtely, the solution takes the simple form

$$r = r_o \sin^2 (\eta/2)$$
$$t = (\tfrac{1}{2})(r_o^3/2m)^{\frac{1}{2}} \quad (\eta - \sin \eta) \tag{3a}$$

for

$$0 < K_n = 2m/r_o \text{ (def. of } r_o),$$

$$r = \beta t^{2/3}$$
$$\beta^3 = \varsigma\pi/2 \text{ for } K_n = 0, \tag{3b}$$

and

$$r = r_o \sinh^2 (\eta/2)$$
$$t = (\tfrac{1}{2})(r_o^3/2m)^{\frac{1}{2}} (\sinh \eta - \eta) \tag{3c}$$

for

$$0 > K_n = -2m/r_o.$$

For $K_n > 0$ the dust expands and then recontracts. In the other two cases the dust only expands or contracts. In general relativity, the cases of K_n positive, zero and negative correspond to positive, zero and negative curvature

of the space within the dust. Also the expressions (3) become identical with those of general relativity if the time is replaced by the proper time.

III. INTERIOR SOLUTION

The space interior to a homogeneous distribution of dust is often described by the Robertson-Walker[5] Metric:

$$ds^2 = -dT^2 + a^2(T)\left[1 + (k/4)U^2\right]^{-2}\left[dU^2 + U^2\, d\theta^2 + U^2 \sin^2\theta d\varphi^2\right]. \tag{4}$$

Here K =1, 0, -1 corresponds to positive, zero negative curvature of the space within the dust.

The quantity a, which is proportional to the luminosity radius of the dust ball, must satisfy the field equation

$$G^{oo} = 8\pi T^{oo}.$$

In a synchronous frame 6 on a three dimensional spacelike surface, this equation takes the form

$$^{(3)}R + K_2 = 16\pi\rho$$

where $^{(3)}R$ is the scalar curvature of the spacelike surface, K_2 is the extrinsic curvature related to the second fundamental form K_{ij} via

$$K_2 = \left(K^i{}_i\right)^2 - K^{ij} K_{ij},$$

and ρ is the energy density of the dust.

Direct calculation yields

$$^{(3)}R = 3k\ a^{-2} \text{ and } K_2 = 3\ \dot{a}^2\ a^{-2},$$

Here $\dot{a}$ denotes differentiation with respect to T. Consequently the above field equation takes the form

$$k\ a^{-2} = (16\ \pi\ \rho/3) - (\dot{a}/a)^2$$

The Robertson-Walker[5] Metric (4) is often used to describe a homogeneous and isotropic universe (a Friedman universe). The average density ρ necessary to yield a given Hubble constant $H = \dot{a}/a$ depends on the curvature constant k. A larger density is necessary for positive curvature than for zero or negative curvature as can be seen from inspection of the above equation.

When these three cases are treated separately, we obtain results very similar to those of Eq. (3):

$$\begin{aligned} a &= a_o \sin^2 (\eta/2) \\ T &= (a_o/2)(\eta - \sin \eta) \\ a_o &= (8\pi/3)\ \rho a^3 \end{aligned} \tag{5a}$$

for k = 1,

$$\begin{aligned} a &= \alpha T^{2/3} \\ \alpha^3 &= 6\ \pi\rho a^3 \end{aligned} \tag{5b}$$

for k = 0,

$$a = a_o \sinh^2 (\eta/2)$$

$$T = (a_o/2)(\sinh \eta - \eta) \qquad (5c)$$

$$a_o = (8\pi/3)\, \rho a^3$$

for $k = -1$,

Here ρ is the time dependent density of the dust while a_o and α are integration constants.

In the space described by this metric there are geodesics for which U = constant and T is the proper time along the geodesic. This is not surprising since the dust was assumed to be co-moving in the deviation of this metric.

IV. EXTERIOR SOLUTION

The space exterior to a sphere of dust can be described by the Schwarzschild metric:

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2,$$

where

$$A^2 = B^{-2} = 1-2\, m\, r^{-1}.$$

Here the constant m is the mass seen by an observer at infinity. The connection between this mass and the density of the dust can be obtained by matching the interior and exterior solutions (patching).

An especially simple way to obtain the geodesic equations for radial motion of a dust particle is this geometry (6) is via the variational principle

$$o = \delta \int ds \tag{7}$$

These geodesic equations are

$$\left[A^2 t'\right]' = 0 \tag{8}$$

and

$$\left[A^{-2} r'\right]' + (A^2)_r \left[(t')^2\right] + A^{-4} (r')^2 = 0 \tag{9}$$

where the prime denotes differentiation with respect to the proper time
$T(dT^2 = -ds^2)$ and $A^2 = 1 - 2m\, r^{-1}$. Integration of Eq. (8) yields

$$A^2 t' = K \tag{10}$$

where K is an integration constant. From Eq. (9), t' can be eliminated via substitution of Eq. (10) yielding

$$\left[A^{-2} (r')^2\right]' + K^2 A^{-4} (A^2)' = 0 \tag{11}$$

after some manipulation.

Integration of Eq. (11) yields

$$(r')^2 = K^2 + \bar{K} A^2 = K^2 + \bar{K} - 2m \bar{K} r^{-1}, \tag{12}$$

where $\bar{K}$ is an integration constant. For large r and small velocities, Eq. (2) should reduce to a Newtonian expression (2). The two expressions (12) and (2) agree in this limit if $\bar{K} = -1$. Hence, Eq. (12) like Eq. (2) contains only one adjustable constant K. The first integral (12) can also be obtained directly from the line element by dividing all terms by the proper time and eliminating t′ via Eq. (10). This latter method gives $\bar{K} = -1$ automatically.

The solution of the general relativistic equations of motion takes a simple form when the three cases $(1-K^2)$ positive, zero and negative are treated seaprately. The solution is very similar to that for the Newtonian case:

$$r = r_o \sin^2 (\eta/2)$$

$$T = (\tfrac{1}{2})(r_o^3 /2m)^{\frac{1}{2}} (\eta - \sin \eta) \tag{13a}$$

for $0 < 1 - K^2 = 2m\, r_o^{-1}$

$$r = \beta\, T^{2/3}$$

$$\beta^3 = 9m/2 \tag{13b}$$

for $K^2 = 1$,

and

$$r = r_o \sinh^2 (\eta/2)$$

$$T = (\tfrac{1}{2})(r_o^3 /2m)^{\frac{1}{2}} (\sinh \eta - \eta) \tag{13c}$$

for $0 > 1 - K^2 = -2m\, r_o^{-1}$.

The expressions[7] (3) and (13) differ only in that the time in Eq. (3) is replaced by the proper time in Eq. (13).

The solution (13) describes the motion of a test particle in gravitational field of a spherically symmetric body of mass m. When the massive body is composed of gravitationally collapsing dust, there is a connection between the mass seen by an observer at infinity and the density of the dust. This connection is obtained by matching the interior and exterior solutions.

V. MATCHING OF GEOMETRIES AND DISCUSSION

Application of the theorem discussed in section I, reduces the problem of matching four-geometries of that of matching eodesics. In section III and IV the geodesics can be matched by inspection. The motion of the same particle is describable in two different coordinate patches. Thus, in each description, the proper time of a particle at the inteface between the two geometries and the proper circumference of the ball of dust should agree.

In view of the above discussion, comparision of Eqs. (5) and (13) yields

$$\left[1 + (k/4)\ U_o^{\ 2}\right]^{-1} a_o U_o = r_o$$

$$a_o = (r_o^3\ /2m)^{\frac{1}{2}} \tag{14a}$$

for k = 1 or -1

and

$$\alpha U_o = \beta$$

for (14b)

$$k = 0.$$

Thus, if the conditions (14) are satisfied, the four-geometries will join smoothly at the interface. These conditions (14) when combined with Eqs. (5) and (13) yield[8]

$$m = (4\pi/3)\ \rho\ r^3 \tag{15}$$

for each case. Hence, the general relativistic expression for the mass of a homogeneous ball of dust is identical with the well known Newtonian expression in terms of the density ρ and radius r of the dust ball. Also, in each case the proper time required for collapse from a finite radius to the gravitational radius is finite.

To an observer at infinity, the radius of the dust ball always appears to be greater than the Schwarzschild radius. Thus, an observer is unable to see the later stages of the collapse because of the time dilation. Since no energy escapes from the dust ball, the mass seen by this observer is constant. On the other hand, if the ball of dust (with $r > 2m$) is initially expanding it will recontract only for the case $k = 1$.

Using the method described above there can be constructed a universe which has sections with negative as well as positive curvature and with all of these sections fitting together smoothly. A simple example can be constructed as follows: In the above calculations we have obtained the geometry exterior and interior to a collapsing dust ball with negative curvature within it. If a positive curvature Friedman universe is then patched to the large r portion of the Schwarzschild region, we have a universe. The method used to patch these two regions together is identical with that used in this paper to patch the dust ball to the Schwarzschild region. It is not known whether the resulting universe is open or closed.

As on aid to visualizing the situation, consider a closed universe at the instant of time symmetry containing a Schwarzschild region with a positive curvature Friedman region at each end. Such a universe is shown in Fig. 1.

The universe with a negative curvature section is very similar, it has a Schwarzschild region with a positive curvature Friedman region on one end and a negative curavature Friedman region on the other. Such a universe can expand in one region and contract in another.

Other results of physical interest can be obtained via differentiation of Eq. (13) with respect to the proper time, or substitution of Eq. (13) into Eq. (12), yielding

$$r' = (2m/r)^{\frac{1}{2}} (1 - r\, r_o^{-1})^{\frac{1}{2}} \tag{16a}$$

for $k = 1$

$$r' = (2m/r)^{\frac{1}{2}} \tag{16b}$$

for $k = 0$,

and

$$r' = (2m/r)^{\frac{1}{2}} (1 + r\, r_o^{-1})^{\frac{1}{2}} \tag{16c}$$

for $k = -1$.

This velocity r' is related to the velocity $\dot{r} = dr/dt$ via the line element (6) $\dot{r}^2 = A^4 (r')^2 \left[A^2 + (r')^2\right]^{-1}$. Thus, the velocity seen by an observer at infinity (who remains at a constant distance from the center of the dust ball) is

$$\dot{r} = A^2\left[2mr^{-1} - 2mr_o^{-1}\right]^{\frac{1}{2}} \left[1 - 2mr_o^{-1}\right]^{-\frac{1}{2}} \tag{17a}$$

for $k = 1$,

$$\dot{r} = A^2 (2m\, r^{-1})^{\frac{1}{2}} \tag{17b}$$

for $k = 0$

and

$$\dot{r} = A^2\left[2m\, r^{-1} + 2m\, r_o^{-1}\right]^{\frac{1}{2}} \left[1 + 2m\, r_o^{-1}\right]^{\frac{1}{2}} \tag{17c}$$

for $k = -1$.

In Newtonian mechanics, there are three cases: (1) the dust expands and then recontracts, (2) the dust expands

continuously with the expansion velocity $\dot{r}$ approaching zero as r approached infinity, (3) the dust expands continuously but the expansion velocity approaches a finite value as a r approaches infinity. Inspection of Eq. (17) shows that in general relativity these three cases correspond to positive[9], zero, and negative curvature respectively.. In the latter case $\dot{r}$ approaches $(2m\, r_o^{-1})^{\frac{1}{2}}$ $(1 + r_o^{-1}\, 2m\quad)^{\frac{1}{2}}$ as r approaches infinity.

When combined with Eq. (17), Eq. (10) has a simple physical interpretation. For k = 1 and $r = r_o$, Eq. (10) takes the form

$$dT = (1 - 2m\, r_o^{-1})^{\frac{1}{2}}\ dt \tag{18}$$

which is the well known relation between time intervals of the Schwarzschild solution. For k = 0 and k = -1, the dust expands to infinite radius. Consequently, the expression (10) can be compared with the results of special relativity since the Schwarzschild geometry is asymptomatically flat. According to special relativity, the time interval dT measured by an observer sitting on a moving particle is shorter than the time interval dT seem by a stationary observer i.e.

$$dT = (1 - \dot{r}^2)^{\frac{1}{2}}\ dt. \tag{19}$$

Substitution of the value of $\dot{r}$ for r approaching infinity yields

$$t' = (dt/dt) = 1 \tag{20}$$

for k = 0,

and

$$t' = (1 + 2m\, r_o^{-1})^{\frac{1}{2}} \tag{21}$$

for k = -1 where $t' = dt/dt$.

These expressions are identical with Eq. (10) for $r \to$ infinity and k = 0 and -1 respectively. Thus, in certain regions the results of Eq. (10) can be obtained by simple physical arguments.

VI. DISCUSSION OF ASTROPHYSICAL APPLICATIONS

The results of this paper have <u>astrophysical applications</u>. These results allow a general relativistic description of astrophysical systems which expand and never recontract as well as those which recontract e.g. exploding stars (supernovae), galaxies, or clusters of galaxies at the stage where the pressure and rotation are negligible. The negative curvature case corresponds to an unbound system in Newtonian mechanics, one which never recontracts if it is initially expanding. The zero curvature case corresponds to the transition between bound and unbound systems (the zero total energy case); it just barely expands to infinity using up all

its kinetic energy in the process. Although it is possible, it is very unlikely in the $k = 0, -1$ case that the velocity vectors will be arranged such that the system will collapse. Consequently, in a physical situation, the $k = 0, -1$ cases are more likely to be expanding than contracting.

Using the method discussed in this paper, one can construct general relativistic supernovae models. The expanding envelope of the supernovae corresponds to the expanding negative curvature solution given here. For example, to construct such a model of a supernovae, (collapsing or static) to an exterior Schwarzschild solution, then patch this to an expanding negative curvature solution, and finally patch this to another exterior Schwarzschild solution. For such a model, the envelope blows off, leaving a remnant behind. The mass of this remnant m_r is given by

$$m_r = (4\pi/3)\ \rho r_1^{\ 3}$$

where ρ is the density and r_1 is the inner radius of the envelope. In this way, we can treat the expansion or collapse of thick shells of dust.

VII. EFFECT OF ROTATION

An important question in astrophysics is: Does rotation stop collapse or does collapse crush rotation[11]? In order to study this question, consider a test particle on the surface of the collapsing ball of dust. In Newtonian mechanics one learns that if the particle has sufficient tangential velocity, it will remain in orbit around the collapsing dust. If the particle remains in orbit, it has a point of closest approach to the center of the collapsing dust ball. At this point the outward radial acceleration is equal to or greater than zero. The former case corresponds to circular motion.

The general relativistic equations of motion of the test particle can be obtained from the variational principle (7) yielding

$$(r^2 \varphi')' = 0 \tag{22a}$$

$$(A^2 t')' = 0 \tag{22b}$$

$$2B^2 r'' + (B^2)_r (r') = 2r(\varphi')^2 - (A^2)_r (t')^2. \tag{22c}$$

Integration of Eqs. (22a) and (22b)

$$r^2 \varphi' = \ell \tag{23a}$$

$$A^2 t' = K \tag{23b}$$

and substitution into Eq. (22c) yields

$$2B^2 r'' + (B^2)_r (r')^2 = 2\ell^2 r^{-3} - 2mr^{-2} K^2 A^{-4}. \tag{23c}$$

At the test particle's point of closest approach to the center of the dust ball, the radial velocity r' vanishes and the radial acceleration r'' is equal or greater than zero. This implies that

$$\ell^2 \geq m\, r\, K^2 A^{-4} \tag{24}$$

On the other hand, the world line of the particle must remain within the light cone. This is true only if

$$\ell^2 \leq K^2 r^2 A^{-2} \tag{25}$$

at the point of closest approach. These two conditions (24) and (25) are consistent only for $r \geq 3m$.

Thus, for $r \geq 3m$, rotation of the test particle about the center of the dust ball can stop the collapse of the particle[12]. The particles can remain in orbit abound the collapsing dust ball. However, for $3m > r > 2m$ the gravitational attraction overpowers the centrifugal force.

Thus, in this region the rotation cannot stop the collapse. This is because the expression for centrifugal force deviated from the Newtonian one since the particle world line must remain within the light cone; the local velocity of the particle cannot exceed that of light.

A collapsing star can shed some of its angular momentum via the above mechanism and collapse toward its Schwarzschild radius leaving planets in orbit about it. But for $3m > r > 2m$ the planets cannot remain in orbit. The centrifugal force is overpowered by the gravitational attraction and everything collapses.

Acknowledgement

For helpful discussion I am indebted to Professors A.G.W. Cameron and R. Kantowski. Part of this research was done while the author held a NAS-NRC Research Associateship at the NASA Institute for Space Studies, New York.

Footnotes

1. Beckedorff and Misner (unpublished) their results are given in reference 3.

2. Lindquist and Wheeler, (1957). Rev. Mod. Phys. 29, 432.

3. Harrison, Thorne, Wakano, Wheeler, (1965). Gravitation Theory and Gravitation Collapse, University of Chicago Press, Chicago.

4. Cohen, (to be published).

5. H.P. Robertson, (1935). Astrophys. J. 82, 284
   A.G. Walker, (1937). Proc. Lon. Math. Soc., Ser. 2, 42, 90.

6. L. Landau and E. Lifshitz, (1962) The Classical Theory of Field, Addison-Wesley, Mass.

7. The origin and sign of the time in the above expression are arbitrary. Thus the above solutions are valid if the time t is replaced by $\pm (t-t_o)$. This is also true of the previous solutions. The substitution $t \to -t$ gives an initially collapsing solution.

8. For the case where k = 1 or -1 this result is obtained as follows: $1 = (8\pi/3)\ \rho a^3/a_o = (8\pi/3)\ \rho\ r^3(1 + (k/4)\ (U_o^2)^3/a_o U_o^3 = (8\pi/3)\ \rho r^3\ a_o^2\ r_o^{-3} = (4\pi/3)\ \rho r^3/m$.

9. For a discussion of a ball of dust with positive curvature via the initial value equations, see ref. 3. An extensive bibliography of previous work on the subject is also given there.

10. A similar problem was considered by Wahlquist and Estabrook (Phys. Rev. 156, 1359 (1967); these authors consider only the case of positive curvature of the space within the dust as did Oppenheimer and Snyder (Phys. Rev. 56, 455 (1939)).

11. Hoyle, Fowler, Burbidge, and Burbidge, (1964). Ap. J. 139, 909.

12. For a complete discussion of such orbits see Y. Hagihara (1931). Jap. J. of Ast. and Geoph. VIII 67 see also e.g. B. Kuchowicz, (1966). Acta Phys. Polom. 30, 981 and the references cited there.

Figure Caption

Figure 1: Closed inhomogeneous universe at the moment of of time symmetry. This universe is obtained by patching positive curvature Friedman regions to both ends of a Schwarzschild region.

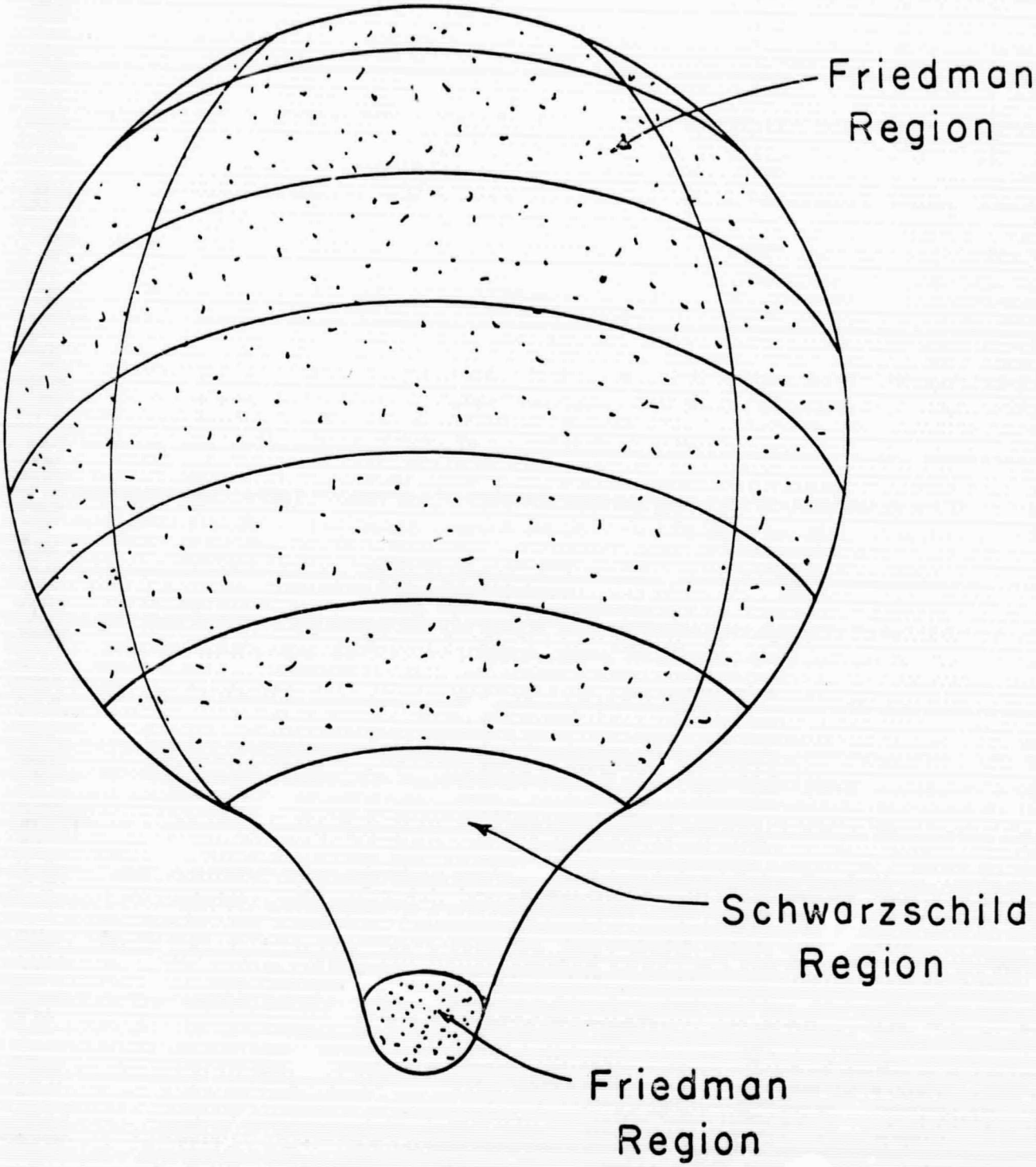

Figure 1