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CLOSED-FORM APPROXIMATE SOLUTIONS TO THE OSCILLATORY PLASMA CONTINUITY EQUATIONS

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16. Abstract

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CLOSED-FORM APPROXIMATE SOLUTIONS TO THE OSCILLATORY PLASMA CONTINUITY EQUATIONS

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SUMMARY

Recent experimental observation of continuity-equation plasma oscillations has made desirable an improved closed-form approximation to the nonlinear differential equations which describe them. Such an approximation has been found, and is compared with the less satisfactory approximations previously obtained, and also with the exact solutions to these equations obtained by numerical integration.

INTRODUCTION

When the mean free paths of electrons and neutrals in a partially ionized gas are comparable to or larger than its dimensions, Fick's law of diffusion is inappropriate. One therefore cannot approximate the divergence terms of the continuity equations by the product of a diffusion coefficient and the Laplacian of the number density. In this case, the continuity equations assume the general form discussed in references 1 to 3. The results of experimental measurements of the continuity-equation oscillation (refs. 4 and and 5) can be described by equations (7) and (8) of reference 1. This specialized form of the continuity equations may be written

$$\frac{dx}{d\tau} = \epsilon x \frac{(\eta - y)}{\eta} \tag{1}$$

$$\frac{dy}{d\tau} = -\epsilon^{-1} y(1 - x) \tag{2}$$

where, in terms of the parameters used in reference 1, $x = N/N_o$ is the dimensionless

neutral number density, $y = N_e/N_{eo}$ is the dimensionless ratio of electron number density to its initial (extremal) value, the amplitude index $\eta \equiv \hat{N}_e/N_{eo}$ is the ratio of the average to the extremal electron number density, $\epsilon = \sqrt{\hat{N}_e/N_o}$ is the square root of the average fraction ionized, and the dimensionless time $\tau = \omega_o t = \sqrt{\hat{N}_e N_o}\langle\sigma v\rangle_{ne} t$, where $\langle\sigma v\rangle_{ne}$ is the ionization rate parameter for electron-neutral ionization. The initial conditions of the problem at $\tau = 0$ are $x(0) = 1.0$ and $y(0) = 1.0$.

The first terms in the parentheses on the right side of equations (1) and (2) follow from the divergence term of the continuity equations when Fick's law is inappropriate; (ref. 4, appendix B) the second (interaction) terms in the parentheses account for the production and loss of each species by the ionization process.

Approximate solutions to equations (1) and (2) were obtained by Lotka (ref. 6) and Volterra (ref. 7) in the linearized limit for which the peak-to-peak amplitude of the fluctuations of both x and y are small compared with unity. In references 1 to 3 approximate closed-form solutions to equations (1) and (2) were obtained in terms of Jacobian elliptic functions for the more general case in which only the fluctuations in $x(\tau)$ were assumed small compared to unity. While the method of approximation in references 1 to 3 may be applied to equations of continuity more general than equations (1) and (2), the closed-form solutions it provides to these equations are defective in several respects. The approximation discussed in references 1 to 3 predicts that oscillatory solutions to equations (1) and (2) occur for the range $2/3 \leq \eta \leq 1.0$, while exact solutions to these same equations show that oscillatory solutions exist for all $\eta > 0$, except the steady-state solution at $\eta = 1.0$. The approximations of reference 1 reproduced the exact solutions reasonably well over the range $0.80 \leq \eta \leq \infty$, but a better approximation was clearly desirable.

A NEW APPROXIMATION TO $y(\tau)$

One can obtain an exact relation between $x(\tau)$ and $y(\tau)$ by taking the ratio of equation (1) to (2), integrating the resulting equation, and applying the initial condition that $x = 1$ when $y = 1$. This yields a very simple "phase" equation,

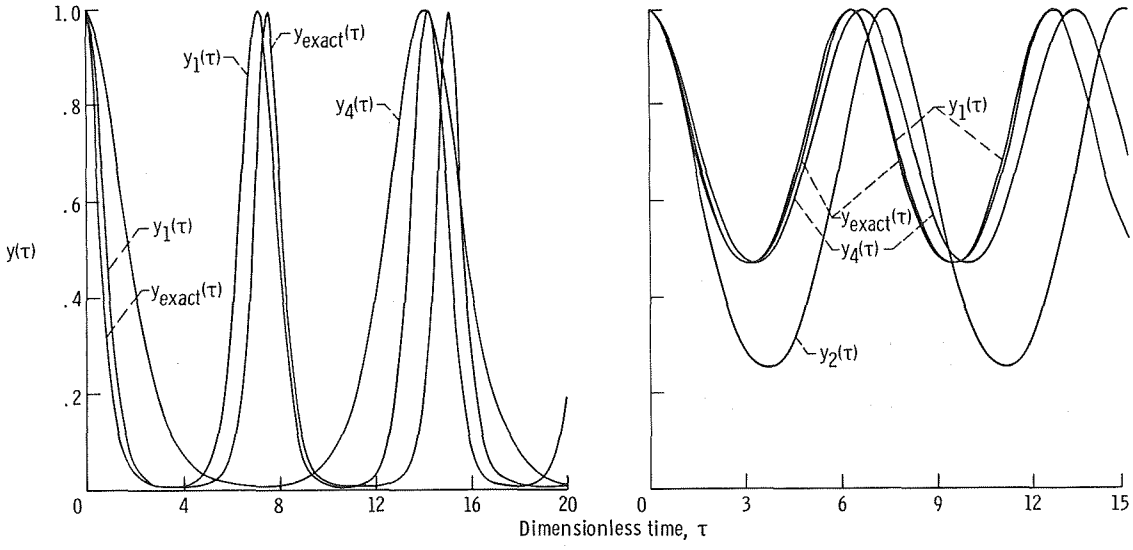
$$(x - 1) - \ln x = \frac{\epsilon^2}{\eta}(1 - y + \eta \ln y) \tag{3}$$

Note that a closed curve, indicating oscillating solutions, can be obtained for all $\eta > 0$, and that $x = 1$ at both the maximum and minimum values of y. One of these extremal

values of y is the initial value $y = 1$. If the other extremal value is denoted y_o, it is apparent from equation (3) that y_o is related to the amplitude index η by

$$\eta = \frac{y_o - 1}{\ln y_o} \tag{4}$$

The parameter y_o is the ratio of the extremal values of $y(\tau)$. If equation (4) is used to replace η in equation (1), it becomes evident that if $y(y_o, \eta)$ and $x(y_o, \eta)$ are solutions to equations (1) and (2), then so are $y(1/y_o, \eta/y_o)/y_o$ and $x(1/y_o, \eta/y_o)$. As a result of this transformation, the solutions for $n = \eta > 1$ (and $y_o > 1$) can be obtained from the solutions for $\eta < 1$ (and $y_o < 1$), and vice versa. The discussion below will be confined to the case for which $\eta < 1$, and $y_o = y_{min}/y_{max} < 1$.



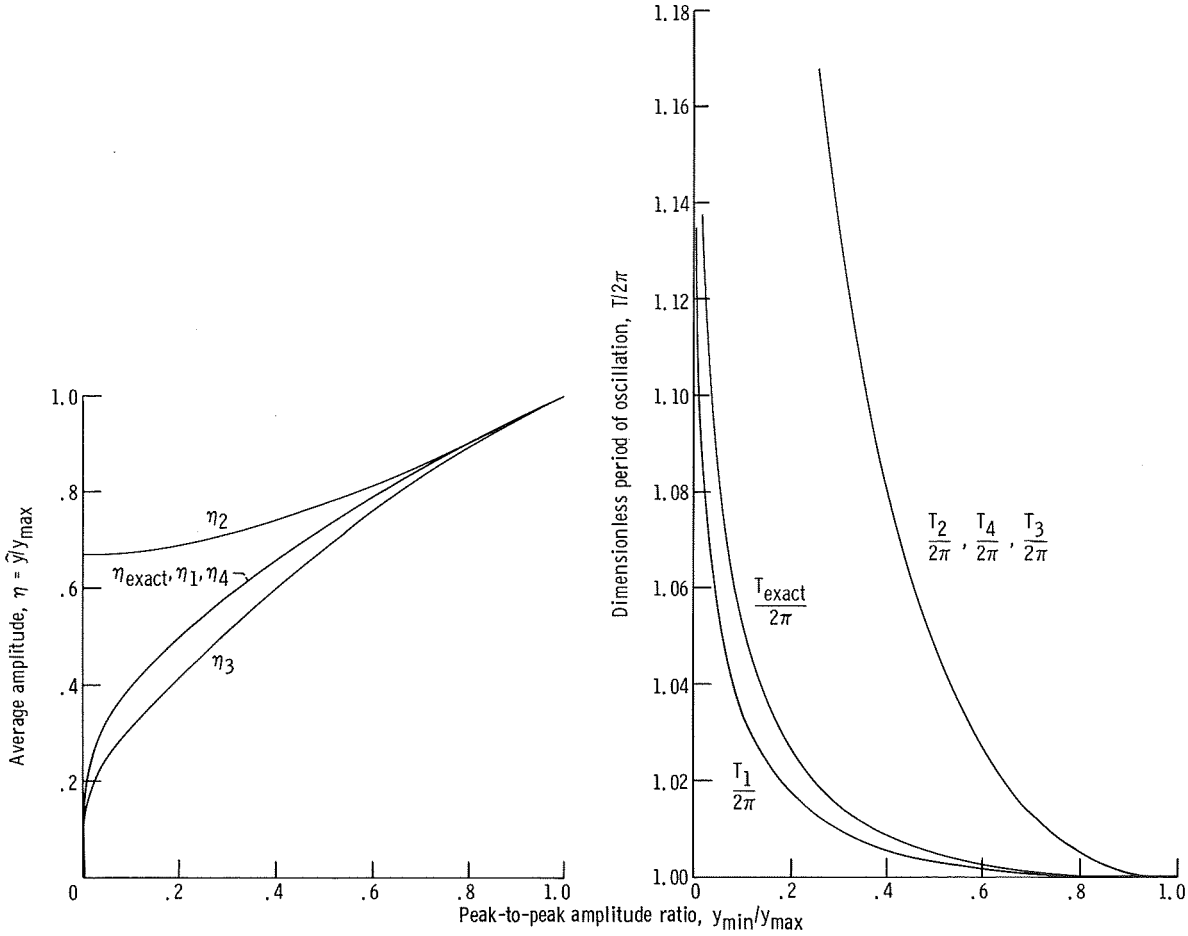
(a) $\eta = 0.20$.

(b) $\eta = 0.70$.

Figure 1. - Comparison of exact solutions $y_{exact}(\tau)$, obtained from the nonlinear pair of continuity equations (eqs. (1) and (2)), with the approximations $y_1(\tau)$ (from eq. (10)), $y_2(\tau)$ (from eq. (12) for η), and $y_4(\tau)$ (from eq. (17) or $\bar{\eta} = \eta/y_0$). $\epsilon = 0.01$.

The exact solutions to equations (1) and (2) for $y(\tau)$ with $\epsilon = 0.01$ are shown in figure 1 for two values of η. These solutions were obtained by numerical integration on a computer. In figure 2 are plotted the frequency and the average value of the solution η as functions of the peak-to-peak ratio y. These exact solutions will be compared with several approximate solutions for $y(\tau)$ given subsequently.

If it is assumed that the neutral density fluctuates only very little compared to its average value, then x is near 1, and the left side of equation (3) approaches



(a) Average amplitude $\eta = \hat{y}/y_{max}$. (Amplitudes of y_{exact}, y_1, and y_4 are equal.)

(b) Dimensionless period of oscillation.

Figure 2. - Average amplitude and period of oscillation as functions of peak-to-peak amplitude ratio for the exact and approximate solutions.

$1/2(x - 1)^2$. This requirement of small amplitude oscillations of $x(\tau)$ can be guaranteed by proper choice of ϵ and η. The right side of equation (3) takes its maximum value at $y = \eta$. Therefore, if one chooses ϵ and η such that

$$\epsilon\left[\frac{2(1 - \eta + \eta \ln \eta)}{\eta}\right]^{1/2} << 1 \tag{5}$$

then one is assured that

$$(x - 1) - \ln x << 1 \tag{6}$$

which implies that $x - 1 << 1$. After expanding the left side of equation (3), one may solve for $(x - 1)$ and obtain

$$(x - 1) \approx \pm\epsilon \left[\frac{2(\eta \ln y - y + 1)}{\eta}\right]^{1/2} = \pm\epsilon \left[\frac{2}{\eta} f(\eta, y)\right]^{1/2} \tag{7}$$

Substituting this approximation into equation (2) yields an approximate equation in terms of the single variable $y(\tau)$,

$$\frac{1}{y^2}\left(\frac{dy}{d\tau}\right)^2 \approx \frac{2}{\eta}\left[\eta \ln y - (y - 1)\right] = \frac{2}{\eta} f(\eta, y) \tag{8}$$

This approximate equation for y may be integrated by adopting an integrable approximation to $f(y_o, y)$:

$$f_1(y_o, y) = \frac{4(1 - \eta + \eta \ln)}{(\ln y_o)^2} (\ln y_o - \ln y)\ln y \tag{9}$$

which satisfies the requirement that f_1 have the same maximum as f, and be zero at $y = 1$ and $y = y_o$. The solution obtained by substituting equation (9) into equation (8) is

$$y_1(\tau) = y_o^{(1-\cos \omega_1 \tau)/2} \tag{10}$$

where the parameter ω_1 is

$$\omega_1(\eta) = \left[\frac{8(1 - \eta + \eta \ln \eta)}{\eta(\ln y_o)^2}\right]^{1/2} = \omega_1(\bar{\eta}) \tag{11}$$

Note that $y_1(\tau)$ reproduces the peak-to-peak amplitude of the exact solutions and is the same function regardless of whether η or $\bar{\eta}$ is used. This approximation is plotted in figures 1 and 2. It does not appear separately on figure 2(a) because equation (9) was chosen so that the relation between η and y_{min}/y_{max} is exact. The approximate frequency given by equation (11) differs from the exact value by no more than a few percent over the range shown.

OTHER APPROXIMATIONS TO $y(\tau)$

The preceding approximation may be compared to the two "small amplitude" approximations discussed in reference 1. The approximation using η may be written in the dimensionless form

$$y_2(\tau) = 1 + Z_3 SN^2 \omega_2 \tau \tag{12}$$

where SN is the Jacobian elliptic sine, and

$$\omega_2 = \left(-\frac{Z_2}{6\eta}\right)^{1/2} \tag{13}$$

$$Z_2(\eta) \equiv -\frac{3}{4}\left[(2 - \eta) + \sqrt{\left[\eta - (2/2)\right](\eta + 2)}\right] \tag{14}$$

$$Z_3(\eta) \equiv -\frac{3}{4}\left[(2 - \eta) - \sqrt{\left[\eta - (2/3)\right](\eta + 2)}\right] \tag{15}$$

with the elliptic modulus given by

$$k^2 = \frac{Z_3}{Z_2} \tag{16}$$

This approximation is shown in figure 2, and is seen to fail for $\eta < 2/3$. A second approximation taken from reference 1 is obtained by using $\overline{\eta} = \eta/y_o$ in the expressions

$$y_3(\tau) = 1 + \frac{\overline{Z}_2 \overline{Z}_3 SN^2 \overline{\omega}_3 \tau}{\overline{Z}_2 - \overline{Z}_3 + \overline{Z}_3 SN^2 \overline{\omega}_3 \tau} \tag{17}$$

where

$$\overline{\omega}_3(\overline{\eta}) = \frac{1}{2}\left\{\frac{\left[\overline{\eta} - (2/3)\right](\overline{\eta} + 2)}{\overline{\eta}^2}\right\}^{1/4} \tag{18}$$

$$\overline{Z}_2 \equiv Z_2(\overline{\eta}) \tag{19}$$

$$\overline{Z}_3 \equiv Z_3(\overline{\eta}) \tag{20}$$

$$k^2 = \frac{Z_3}{\overline{Z}_3 - \overline{Z}_2} \tag{21}$$

This approximation exists over the entire range of η. However, the relation between η and y_o differs from the exact value by a maximum of about 20 percent and the frequency differs by a maximum of about 15 percent.

The approximations $y_2(\tau)$ and $y_3(\tau)$ can be improved by defining an equivalent small amplitude index η_s, which is that value of η which gives the correct peak-to-peak oscillation amplitude. This value of $\eta = \eta_s$ is then substituted in equation (12) for an improved approximation $y_4(\tau)$, which is shown in figure 11.

CONCLUDING REMARKS

In summary, it has been shown that the approximations given in reference 1 can be modified to improve their usefulness and accuracy. A new approximation $y_1(\tau)$ has also been developed which uses trigonometric functions and gives the frequency as a function of the peak-to-peak amplitude ratio y_o. This approximation is the most accurate of those considered, does not require the use of Jacobian elliptic functions, and provides sufficient accuracy for most applications.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, November 7, 1969,
129-02.

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