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COSMIC RAY DIFFUSION AND THE "LEAKAGE LIFETIME" APPROXIMATION: AN INVESTIGATION OF ITS VALIDITY BY THE EIGENFUNCTION EXPANSION TECHNIQUE

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ABSTRACT

The diffusion of cosmic ray particles in a finite volume of space with a simultaneous diffusion and/or transport in energy is considered. The solution of the appropriate differential equation may be expressed as an expansion in eigenfunctions of the differential operator. If one approximates the solution by keeping only the lowest or fundamental eigenfunction one obtains the common "leakage lifetime" approximation. In some situations this approximation can be justified but in others (e.g., synchrotron or inverse Compton losses) it cannot. The reason for the failure in this case can be seen from the point of view of the expansion. The solution of the general case of Fermi acceleration, synchrotron losses, and energy fluctuation acting together is also obtained by this method.

INTRODUCTION

There has been recent discussion in the literature as to the correct method of treating the loss of particles from a region of space where spatial diffusion and energy transport and/or diffusion are occurring simultaneously a common method of treating this situation when it has arisen in the field of cosmic physics has been to describe it by an inhomogeneous, partial differential equation

$$\frac{\partial \rho(E,t)}{\partial t} + \mathcal{L}_E \rho(E,t) + \frac{\rho(E,t)}{\tau} = q(E,t). \qquad (1)$$

In equation 1 $\mathcal{L}_E$ is a differential operator in energy that describes the various energy changing processes at work within the region, τ is the average lifetime of a particle against a variety of loss mechanisms including leakage from the boundary, and q(E, t) describes the energy distribution of the particles when they are introduced into the region; the inhomogeneous term q is often referred to as the injection spectrum.

Solutions of this equation are usually sought for the steady state case, $\partial\rho/\partial t = 0$, for a variety of energy transport mechanisms and injection spectra. In his now classic papers[1,2] Fermi in essence solved this equation for the case $\mathcal{L}_E \rho = \frac{\partial}{\partial E}(aE\rho)$. In his first paper[1] he considered $\tau = \tau_c$ the lifetime against nuclear collisions of the cosmic ray particles. In his second paper[2], he had come to the opinion that diffusive leakage from the galaxy was the most significant loss mechanism and hence considered $\tau = \tau_e$ or the "leakage lifetime". At present it is not believed to be very likely that Fermi's mechanism offers the correct explanation of cosmic rays, however, it is generally believed

that he presented a correct treatment of a plausible process that should, in fact, occur even though it might not produce cosmic rays.

The time independent form of equation 1 has been used extensively to calculate equilibrium spectra for a wide variety of problems in cosmic physics. Energy loss as well as energy gain has been included in $\mathcal{L}_E$ and many different injection spectra have been considered. The use of this approach has been too widespread to give any references that would be complete, however, recently a particular application has been made to the case of cosmic-ray electrons[3-12] where it is assumed that the injection spectrum is of the form $q(E) = kE^{-\rho}$ and that synchrotron radiation and inverse Compton scattering are the main contributers to the energy transport term, i.e., $\mathcal{L}_E \rho = \frac{\partial}{\partial E}(-bE^2\rho)$. This analysis predicts a solution of the form $\rho(E) \propto E^{-\rho}$ for $E \ll E_c$ and $\rho(E) \propto E^{-\rho-1}$ for $E \gg E_c$ where $E_c = (b\tau_e)^{-1}$. The significance of this analysis lies in the fact that an observation of the "break" in the spectrum at $E = E_c$ (although the term "break" should be used with great caution as I shall show in the next section) will yield the product $b\tau_e$ with the result that an assumption about the value of b (the strength of the magnetic field or the energy density of radiation) will give the properties of the diffusing region through the relation $\tau_e \approx R^2/D$, where R is the characteristic dimension of the diffusing region and D is the diffusion coefficient.

In all of the treatments of cosmic ray electrons the inhomogeneous term is included but in some treatments of cosmic radio sources[13, 14] and X-ray sources[15] the homogeneous equation is used. It is not appropriate for cosmic ray electrons for as Kardashev[13] points out one can not obtain steady flat spectra in the presence of inverse Compton or synchrotron losses. Melrose[14] asserts the contrary and obtains solutions of the homogeneous

equation that resemble the cosmic ray electron spectrum. There appear to be serious errors[16] in Melrose's paper, however, which invalidate his conclusion so I shall consider only equations with the injection spectrum included hereafter.

In spite of its wide spread use equation 1 has been recently questioned by Jokipii and Meyer[17] who quite correctly point out that the analogy between a collision lifetime and a leakage lifetime is not a valid one. The probability of loss by a catastrophic event such as a nuclear collision may properly be characterized by a uniform probability per unit time but the probability that a particle will be lost by leakage at the boundary depends on the position of observation and even then can not be characterized by a uniform probability per unit time.

At any point in the diffusing region the "ages" (time since injection) of the particles will be distributed exponentially with a mean age if collision is the dominant loss mechanism. If diffusive loss is dominant, on the other hand, the age distribution can not, in general, be considered an exponential even with a variable mean age $\tau_e = \tau_e(\underset{\sim}{r})$. Jokipii and Meyer point out that the correct equation to solve is not (1) but rather

$$\mathcal{L}_E \rho(E,\underset{\sim}{r}) - \underset{\sim}{\nabla}\cdot(D\underset{\sim}{\nabla}\rho(E,\underset{\sim}{r})) + \rho(E,\underset{\sim}{r})/\tau_c = q(E,\underset{\sim}{r}) \qquad (2)$$

where D is the diffusion coefficient and once again we consider the equilibrium case, $\partial P/\partial t = 0$. The solution of this equation with the appropriate boundary conditions will yield the correct energy spectrum at any point.

Earlier Shen[11] had pointed out that the diffusion term was important at high energies since here the leakage lifetime could be much longer than the lifetime against synchrotron and inverse Compton losses and the spectrum could therefore depend on the spatial distribution of the sources. Nevertheless Shen continued to treat the boundary conditions by means of the leakage lifetime approximation.

Jokipii and Meyer consider a flat disk source of electrons embedded in a diffusing medium of infinite extent. They find that rather than one "break" of one power in the exponent they obtain two "breaks" of one-half power each at energies $E_1 = D/bR_1^2$, $E_2 = D/bR_2^2$ where R_1 and R_2 are the diameter thickness and thickness of the disk respectively. A similar calculation had been performed earlier by Syrovatskii[18] but his results were not presented in such a way as to make comparison with the leakage lifetime approximation very easy.

More recently Dogel and Syrovatskii[19] have considered a similarly shaped source distribution but instead of an infinite diffusing medium they consider a spherical region with free escape at the boundary of radius R_o where $R_o \approx R_1$. They obtain results essentially identical to those of Jokipii and Meyer. Further departures from the leakage lifetime approximation are indicated in a paper by Longair and Sunyaev[20].

Furthermore Berkey and Shen[21] claim and amply demonstrate that in general the lifetime of a cosmic ray electron in a general diffusing region has little or nothing to do with the equilibrium spectrum when sychrotron and inverse Compton losses are important. Rather the

structure of the sources is the determining factor.

Clearly the leakage lifetime approach does not work at all well for cosmic-ray electrons. This brings to mind the question of whether the theory of Fermi acceleration might also be in error. It turns out that this, and other, questions may be invetigated from a general point of view. If equation 2 can be solved by means of an eigenfunction expansion of the Green's function the leakage lifetime approximation may be seen in its natural mathematical setting and the question of its validity may be more easily investigated.

In Section II the method of eigenfunction expansion will be described and the leakage lifetime approximation will be placed in its setting. In Section II Fermi's theory and the theory of cosmic ray electrons undergoing synchrotron and inverse Compton losses will be compared. We will see that Fermi's theory turns out to be a very good approximation and also why the method fails in the latter case. In Section IV we shall see how the method of eigenfunction expansion may be applied to more general case where $\mathcal{L}_E$ is a Fokker-Planck operator of the form

$$\mathcal{L}_E \rho = \frac{\partial}{\partial E}(a_1 E - a_2 E^2)\rho - \frac{1}{2}\frac{\partial^2}{\partial E^2}(a_3 E^2)\rho$$

In the above expression the term proportional to a_1 describes both Fermi acceleration and bremstrahlung losses and the term proportional to a_2 describes synchrotron and inverse Compton losses. The second order term proportional to a_3 describes a statistical spreading or diffusion in energy space. This term was first considered in a cosmic ray setting by Terletski and Luganov[22] and later in a more general treatment by Davis[23].

Further discussion of the importance of this term may be found in work by Morrison[24] and it has been employed in many later works[13, 14, 15, 25].

Although all of the processes discussed above have been treated by many authors to the knowledge of the present author the combination of the above mentioned Fokker-Planck energy operator with spatial diffusion, particle losses, and injection has never been treated before. It is therefore believed that the results of Section IV are essentially new.

II. MATHEMATICAL METHOD

The equation we wish to solve is

$$\mathcal{L}_E \rho(E,\underset{\sim}{r}) - \underset{\sim}{\nabla}\cdot\left(D\underset{\sim}{\nabla}\rho(E,\underset{\sim}{r})\right) + \rho(E,\underset{\sim}{r})/\tau_c = q(E,\underset{\sim}{r}) \quad (2)$$

If we define the total operator $\mathcal{L}$ by

$$\mathcal{L} = \mathcal{L}_E - \underset{\sim}{\nabla}\cdot D\underset{\sim}{\nabla} + 1/\tau_c$$

we may write (2) as

$$\mathcal{L}\rho(E,\underset{\sim}{r}) = q(E,\underset{\sim}{r}) \quad (2)$$

The solution may be written in terms of the Green's function

$$\rho(E,\underset{\sim}{r}) = \int d^3r' dE' \, G(E,\underset{\sim}{r};E',\underset{\sim}{r}')\, q(E',\underset{\sim}{r}') \quad (3)$$

where

$$\mathcal{L}G(E,\underset{\sim}{r};E',\underset{\sim}{r}') = \delta(E-E')\,\delta(\underset{\sim}{r}-\underset{\sim}{r}') \quad (4)$$

and the Green's function is constructed to fit the proper boundary conditions. If the operator $\mathcal{L}$ and its adjoint $\mathcal{L}^T$ have "eigen-solutions"

$$\mathcal{L}\rho_n = \lambda_n \rho_n$$

$$\mathcal{L}^{+}\rho_n^{+} = \lambda_n \rho_n^{+} \tag{5}$$

where n represents all of the parameters required to specify the solutions, and if the delta function may be expanded in these functions (i.e., they form a complete basis set)

$$\delta(E-E')\delta(\underset{\sim}{r}-\underset{\sim}{r}') = \sum_n \rho_n(E,\underset{\sim}{r})\rho_n^{+}(E',\underset{\sim}{r}') \tag{6}$$

where $\sum_n$ represents summation over discrete parameters and integration over continuous ones, then the Green's function may be also expanded by[26, 27]

$$G(E,\underset{\sim}{r};E',\underset{\sim}{r}') = \sum_n \frac{\rho_n(E,\underset{\sim}{r})\rho_n^{+}(E',\underset{\sim}{r}')}{\lambda_n} \tag{7}$$

In many cases the notion of "eigensolution" must be taken with a grain of salt since the solutions ρ_n will not fit the boundary conditions. Nevertheless, much of the time (and in particular in the cases we shall consider) an expansion of the form (6) can be defined in terms of transform theory and the analysis follows through.

At this point we shall make the assumption that the problem may be separated into a spatial part and an energy part. That is to say that

the diffusion coefficient D is independent of energy and the energy operator $\mathcal{L}_E$ has coefficients that are independent of position. If this is the case, the solutions are separable

$$\rho_{\underset{\sim}{k},\nu} = R_{\underset{\sim}{k}}(\underset{\sim}{r})\, f_\nu(E)$$

and (5) becomes

$$\mathcal{L}\rho_{\underset{\sim}{k},\nu} = R_{\underset{\sim}{k}}(\underset{\sim}{r})\,\mathcal{L}_E f_\nu(E) - f_\nu(E)\,\underset{\sim}{\nabla}\cdot D\,\underset{\sim}{\nabla} R_{\underset{\sim}{k}}(\underset{\sim}{r})$$

$$+ R_{\underset{\sim}{k}}(\underset{\sim}{r})\, f_\nu(E)\,\tau_c^{-1} = \lambda_{\underset{\sim}{k},\nu}\, R_{\underset{\sim}{k}}(\underset{\sim}{r})\, f_\nu(E)$$

and dividing by $\rho_{\underset{\sim}{k},\nu}$ gives

$$\frac{\nabla\cdot D\nabla R_{\underset{\sim}{k}}(\underset{\sim}{r})}{R_{\underset{\sim}{k}}(\underset{\sim}{r})} - \frac{\mathcal{L}_E f_\nu(E)}{f_\nu(E)} = \frac{1}{\tau_c} - \lambda_{\underset{\sim}{k},\nu} \qquad (8)$$

Since $\underset{\sim}{r}$ and E are independent variables the first and second terms are functions of $\underset{\sim}{r}$ only and E only respectively and hence must be equal to constants so we have

$$\underset{\sim}{\nabla}\cdot D\,\underset{\sim}{\nabla} R_{\underset{\sim}{k}}(\underset{\sim}{r}) + k^2 R_{\underset{\sim}{k}}(\underset{\sim}{r}) = 0 \qquad (9)$$

$$\mathcal{L}_E f_\gamma(E) - \gamma f_\gamma(E) = 0 \tag{10}$$

and

$$\lambda_{\underset{\sim}{k},\gamma} = k^2 + \gamma + 1/\tau_c \tag{11}$$

At this point we shall consider equation (9) only briefly. Solutions to (9) may be found for a variety of shapes of diffusing volumes. The usual boundary conditions are $R_{\underset{\sim}{k}} = 0$ on the boundary; however, any homogeneous boundary conditions may be applied. Although the differential operator is formally self-adjoint the <u>system</u> will be self-adjoint only for certain types of boundary conditions. So in general $R_{\underset{\sim}{k}}^{+} \neq R_{\underset{\sim}{k}}$ Since the volume we are considering is finite, equation (9) will have solutions only for a discrete set of values of k^2. The smallest value k^2 can have will be of the order

$$k_0^2 \simeq \pi^2 D/L^2 \simeq \pi^2 c\ell/L^2$$

if the boundary is freely penetrated or $k^2 \simeq \alpha c/L$ where α is the boundary transmission coefficient $(\alpha \ll \ell/L)$ and L is the linear dimension of the diffusing region. In either case we see

that $k_0^2 \approx 1/\tau_e$ where τ_e is the time required to random walk a distance L in the first case or the time required to penetrate the boundary barrier in the second case.

Since for higher modes one usually has $k_n \approx n k_0$ we see that particles remain in the nth mode a time of the order $\tau_n \approx \tau_e / n^2$ where τ_e is the random walk time, not the boundary penetration time.

We may now see what the essence of the "leakage time" approximation is. In this approach only the <u>lowest</u> or <u>fundamental mode</u> is considered. To determine whether or not this is a valid approximation we must turn to an examination of (10), the energy equation.

Since the domain of equation (10) is semi-infinite, $0 \leq E < \infty$, the concept of "boundary" conditions must be modified somewhat since setting a value of $f(E)$ at the two end points is usually not relevant, i.e., $f(0) =$ const. is too restrictive and $f(\infty) \to 0$ is usually not restrictive enough. The physical condition that we must apply in place of boundary conditions is simply that the particle density be finite or that the function $f(E)$ be integrable

$$\int_0^\infty |f(E)|\, dE < \infty . \tag{12}$$

With this condition imposed on the solutions it often turns out[27] (and it will indeed be the case in the following sections) that there are <u>no</u> eigenfunctions of equation (10) that also fulfill condition (12). In this case we must abandon our notion of a solution in terms of "a complete set of orthonormal eigenfunctions."

On the other hand, one can often find solutions for the adjoint equation such that

$$\int_0^\infty f^+(E)\, g(E)\, dE < \infty$$

for any function $g(E)$ such that

$$\int_0^\infty |g(E)|\, dE < \infty .$$

These solutions can be labled by a continuous parameter s and

$$\mathcal{L}_E^+ f_s^+(E) - \gamma(s)\, f_s^+(E) = 0$$

$$\mathcal{L}_E f_s(E) - \gamma(s)\, f_s(E) = 0$$

The delta function $\delta(E-E')$ may then be expressed in terms of the integral

$$\delta(E-E') = \int_C ds\; f_s(E)\, f_s^+(E') \tag{13}$$

where the integral is over some contour in the complex s plane. In this situation one usually says that the operator $\mathcal{L}_E$ has a continuous spectrum of eigenvalues $\gamma(s)$ and calls the function $f_s(E)$ an "eigenfunction" even though this is not strictly true since

$$\int_0^\infty |f_s(E)|\, dE = \infty .$$

To understand what it means to say that an equation like (13) is a spectral expansion of the delta function we must consider the usual case of expansions of a function space in its basis functions.

Using the Dirac notation[26] we say that a function may be expanded in a basis function set and write

$$|f\rangle = \sum_n |n\rangle\langle n|f\rangle \tag{14}$$

where $|n\rangle = u_n(x)$ the basis functions and the expansion coefficients are given by

$$\langle n|f\rangle = \int_D u_n^+(x')\, f(x')\, dx' \tag{15}$$

where the integration is taken over the domain of definition of the function space. Since the expansion is supposed to hold true for any function in the space we may define the unit operator $\underset{\sim}{1}$ and write

$$f = \underset{\sim}{1} f \; ; \quad \underset{\sim}{1} = \sum_n |n\rangle\langle n|$$

The delta function <u>is</u> the unit operator since

$$f(x) = \int \delta(x-x')\, f(x')\, dx \, .$$

In the case of a continuous spectrum we can replace the summation by an integration and write

$$|f\rangle = \int ds\, |s\rangle\langle s|f\rangle$$

and

$$\underset{\sim}{1} = \delta(x-x') = \int ds\, |s\rangle\langle s|$$

What we now have of course is an integral transform where

$$\langle s|f\rangle = \tilde{f}(s) = \int u_s^{+}(x')\, f(x')\, dx'$$

and the inverse is given by

$$|f\rangle = \int ds\, |s\rangle\langle s|f\rangle = \int ds\, \tilde{f}(s)\, u_s(x) \, .$$

The theory of the Fourier, Laplace, Mellin and other transforms is of this type and such things as the "continuum wave functions" of quantum mechanical scattering problems are to be understood from this point of view. In Section III we will be able to construct the delta function by comparing our results to known transforms such as the Mellin and Fourier transforms. In Section IV, however, we shall be forced to consider a non-standard transform based on confluent hypergeometric functions. This transform will be established in the appendix.

III. FERMI ACCELERATION VERSUS SYNCHROTRON AND INVERSE COMPTON LOSSES

Let us first consider the problem of Fermi acceleration.[1,2] In this case we have the equation

$$\mathcal{L}_E f_\nu(E) = \frac{\partial}{\partial E}(aEf) = \nu f$$

$$\text{or} \quad \frac{\partial f}{\partial E} = \frac{(\nu/a - 1)}{E} f \tag{16}$$

The solution to (16) is

$$f = \text{Const.}\, E^{(\nu/a - 1)} . \tag{17}$$

$$\mathcal{L}_E^+ f_\nu^+(E) = aE\frac{\partial f^+}{\partial E} = \nu f^+ , \tag{18}$$

$$f^+ = \text{const.}'\, E^{-\nu/a} \tag{.9}$$

It is easy to see that there is no value of ν for which f_ν is integrable from 0 to ∞. However, it is also easy to see that the relevant transform theory is that of the Mellin transform;

$$\int_0^\infty f_\nu^+ q(E')\,dE' = \int_0^\infty E'^{-\nu/a} q(E')\,dE' < \infty$$

for $R_e(\nu/a) < \epsilon$ and an integrable q(E'). We may therefore immediately write

$$\delta(E - E') = \frac{1}{2\pi i E}\int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E}{E'}\right)^{\nu/a} d(\nu/a) \tag{20}$$

where $\sigma < \epsilon$.

Remembering equation (11), we have from equation (7)

$$G(E,\underset{\sim}{r};E',\underset{\sim}{r}') = \sum_{\underset{\sim}{k}} R_{\underset{\sim}{k}}(\underset{\sim}{r}) R^{+}_{\underset{\sim}{k}}(\underset{\sim}{r}') \, g_k(E,E') \tag{21}$$

where

$$g_k(E,E') = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E^{\nu/a-1} \, E'^{-\nu/a}}{\nu + k^2 + 1/\tau_c} \, d(\nu/a) \tag{22}$$

For $E < E'$ we may close the contour on the right hand infinite semi-circle without adding anything to the integral. There are no singularities in the right hand plane so

$$g_k(E,E') = 0 \quad ; \; E < E'.$$

For $E > E'$ we may similarly close the contour to the left and pick up the pole at $\nu = -1/\tau_k \equiv -(k^2 + 1/\tau_c)$ to obtain

$$g_k(E,E') = E'^{1/a\tau_k} \left[a E^{1+1/a\tau_k}\right]^{-1} \text{ for } \quad E > E'$$

If all of the particles are injected at a single, low energy E_o

i.e., $q(E,\underset{\sim}{r}) = q(\underset{\sim}{r})\,\delta(E-E_o)$

we will have

$$\rho(E,\underset{\sim}{r}) = \sum_{\underset{\sim}{k}} C_{\underset{\sim}{k}} R_{\underset{\sim}{k}}(\underset{\sim}{r})\, E^{-(1+1/2\,\tau_k)} \tag{23}$$

where

$$C_{\underset{\sim}{k}} = \frac{E_o}{2}^{1/2\,\tau_k} \int d^3r'\, R^{+}_{\underset{\sim}{k}}(\underset{\sim}{r}')\, q(\underset{\sim}{r}'). \tag{24}$$

Since, as we have seen in Section II, the mode lifetimes are monotonicly decreasing functions of n and hence the spectrum index $-(1+1/2\,\tau_k)$ is too.

For this reason the higher the mode, the steeper the spectrum, and for some energy $E \gg E_0$ one can neglect all but the fundamental mode. This is illustrated in Fig. 1 where log $\rho(E)$ versus log E is sketched for the first three modes. In the case of Fermi's origin theory, the next mode to be considered (the first harmonic) would be of the form $E^{-\Gamma}$ where $\Gamma \approx 4$ so one may say that the leakage lifetime approach was quite justified in this case.

On the other hand, if we consider the case of electrons injected into the region with a power low spectrum, $q(E,\underline{r}) = q(\underline{r})E^{-p}$ and subsequently undergoing energy loss by synchrotron radiation[25,28] or inverse Compton scattering[29], we have

$$\mathcal{L}_E f = \frac{\partial}{\partial E}(-bE^2 f) = \nu f \tag{25}$$

whose solution is

$$f_\nu(E) = \text{Const. } E^{-2} \exp[\nu/bE]$$

The adjoint solution is

$$f^+_\nu(E) = \text{Const. } \exp[-\nu/bE]$$

If we consider the variable $y = E^{-1}$, we see that the transformation in this case is equivalent to the Laplace transform and we have

$$\delta(E-E') = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E^{-2}\exp[\nu/bE - \nu/bE']\, d\left(\frac{\nu}{b}\right).$$

Using the same analysis as before, we obtain for $g_k(E,E')$

$$g_k(E,E') = (bE^2)^{-1} \exp\left[(\tau_k bE')^{-1} - (\tau_k bE)^{-1}\right] \tag{26}$$

Continuing, we obtain for $\rho(E,\underset{\sim}{r})$

$$\rho(E,\underset{\sim}{r}) = \sum_{\underset{\sim}{k}} C_{\underset{\sim}{k}} R_{\underset{\sim}{k}}(\underset{\sim}{r}) f_k(E)$$

where

$$C_{\underset{\sim}{k}} = \int d^3r' R^{+}_{\underset{\sim}{k}}(\underset{\sim}{r}') g(\underset{\sim}{r}')$$

and

$$f_k(E) = \frac{\exp\left[-(\tau_k bE)^{-1}\right]}{bE^2} \int_E^{\infty} E'^{-p} \exp\left[-(\tau_k bE')^{-1}\right] dE' \tag{27}$$

This expression is well known; it has been pointed out many times that $f_k(E)$ has two distinct asymptotic forms. For $E \ll E_c \equiv (\tau_k b)^{-1}$

$$f_k(E) \simeq \tau_k E^{-\rho}$$ and for $E \gg E_c$

$$f_k(E) \simeq E^{-(\rho+1)} / b(\rho-1)$$

This is the spectrum (with $\tau_k = \tau_e$) that appears in the leakage lifetime approximation and it has led many authors[3, 5-10] to refer to a "break" in the spectrum at the critical energy, E_c. This is, in fact, rather unrealistic, in Fig. 2 we see plotted the value of the effective spectral index increment

$$\delta = -\frac{d \log f}{d \log x} - \rho \quad \text{vs.} \quad \log(\rho-1)x$$

where

$$f = x^{-2} \exp[-1/x] \int_x^\infty x'^{-\rho} \exp[1/x'] \, dx'$$

and $\rho = 2.5$. We can see that the spectral index does not sharply break, but changes from 2.5 to 3.5 very smoothly over about <u>two decades</u> of the parameter $X = E/E_c$. For comparison one should check the spectrum curves in figures of references 3,5, 8, 9, and 10. At this point it should also be pointed out that the spectrum of photons produced by these particles in the inverse Compton or synchrotron process will change its slope over about <u>four</u> decades, a fact that should be kept in mind when interpreting "breaks" in X-ray or gamma ray spectra.

In this situation, the effect of including higher modes is somewhat more complicated. At first glance one might think that there would

always be a steepening at the critical energy of the fundamental mode $E_c = (\tau_0 b)^{-1} \approx \frac{D\pi^2}{bR^2}$ where R is the linear dimension of the diffusing volume and for the moment we have neglected τ_c, assuming it to be very long compared to diffusion times. As a matter of fact, just such an argument was made[6] in attempting to explain the shape of the galactic cosmic ray electron spectrum. Such a conclusion would seem to be borne out by the calculations of Dogel and Syrovatskii[19] who consider a spherical diffusing region of radius R_o and an ellipsoidal source region of axes R_1 and R_2 where $R_1 \approx R_o$ and $R_1/R_2 = 10^2$. They obtain a spectrum that steepens by 1/2 in the exponent at an energy of $D\pi^2/bR_o^2$ and again by a factor 1/2 at an energy $D\pi^2/bR_2^2$. This would appear to indicate that R_o is a dimension that determines the first "break" in the spectrum. This is misleading; it is, in fact, the dimension R_1, which in this problem is equal to R_o, that determines the position of the first break. The size of the diffusing region has very little effect on the shape of the equilibrium spectrum, this is almost entirely determined by the size and shape of the source region.

This point has been made by Berkey and Shen[21] and has been illustrated by them in a model calculation where a spherical source of radius 13 in a concentric diffusing volume of radius R is considered to inject electrons with a spectrum $q(E') \propto E'^{-2}$. For this injection spectrum $f_R(E)$ simplifies and may be expressed in closed form, i.e.

$$f_R(E) = \tau_R E^{-2}\left(1 - \exp\left[-1/\tau_R bE\right]\right)$$

For completeness we shall consider essentially the same case as Berkey and Shen but instead of limiting our consideration to an injection spectrum proportional to E^{-2} we shall consider a general power law and make use of a convenient approximate form of $f_m(E)$. We shall also see that to an observer inside the source region, the outer diffusing volume serves only as a "quantization volume" for the eigenfunctions and for this reason has relatively little effect on the solution.

Consider a spherical diffusing region of radius R and a concentric source region of radius B with $B < R$. Due to the spherical symmetry of the problem, the eigenfunctions are $j_0(k_m r)$ where $k_m = \pi m / R$ and $j_0(k_m r) = (k_m r)^{-1} \sin(k_m r)$. If the source strength is η particles per unit volume, the resulting density will be

$$\rho(E, r) = \sum_{m=1}^{\infty} C_m j_0(k_m r) f_m(E) \tag{28}$$

where

$$C_m = \frac{\eta B}{R}\left(\frac{\sin(k_m B)}{k_m B} - \cos(k_m B) \right) \tag{29}$$

and $f_m(E)$ is given by equation (27).

A good approximation to (27) is given by

$$\tilde{f}_m(E) = \frac{E^{-p}}{(\tau_m)^{-1} + (p-1) b E} = \frac{E^{-p}}{D \pi^2 m^2 / R^2 + (p-1) b E} \tag{30}$$

It is easy to see that $\tilde{f}_m$ has the same asymptotic properties as f_m and we can see in Fig. 2 that the spectral index is a similar function of E. The major difference between $\tilde{f}_m$ and f_m is that $\tilde{f}_m$ is overwhelmingly simpler.

Now using $\tilde{f}_m$ instead of f_m in (28), we may write

$$\rho(E,r) = \frac{\eta B}{RD} \sum_{m=1}^{\infty} \left(\frac{\sin(k_m B)}{k_m B} - \cos(k_m B) \right) \frac{\sin(k_m r)}{k_m r} \frac{E^{-p}}{(k_m^2 + k_E^2)} \tag{31}$$

where $k_E = [(p-1)bE/D]^{1/2}$. Since $\Delta m = 1$, $\pi/R = \Delta m \pi/R = \Delta k_m$ we may write (30) as

$$\rho(E,r) = \frac{\eta B E^{-p}}{\pi D} \sum_{m=1}^{\infty} \Delta k_m \left(\frac{\sin(k_m B)}{k_m B} - \cos(k_m B) \right) \frac{\sin(k_m r)}{k_m r (k_m^2 + k_E^2)} \tag{31'}$$

The only role that R plays in the above expression is to determine how close the spacing is of the levels k_m. In this sense it acts only as a "quantization volume" as in quantum mechanics and does not have a profound effect on the system (of size $\sim$B) provided $B \ll R$. In fact, if $B \ll R$ there will be many levels contributing to a small variation of $k_m B$ or k_m and we may approximate the sum in (31') by an integral

$$\rho(E,r) \simeq \frac{\eta B E^{-p}}{\pi D} \int_0^{\infty} dk \left(\frac{\sin(kB)}{kB} - \cos(kB) \right) \frac{\sin(kr)}{kr(k^2 + k_E^2)} \tag{32}$$

and R no longer appears in the expression at all. This holds true for

$f_m(E)$ as well as for $\tilde{f}_m(E)$. In fact, it would hold for any spectrum that is a function of k such that the integral or sum is not dominated by the lowest values of k. If it is so dominated, the approximation $k_{min} \approx 0$ is not valid and the result will be a function of k_{min} and hence R since $k_{min} = k_1 = \pi/R$ The Fermi acceleration spectrum $f_k(E) \propto E^{-1-k^2/a}$ can be seen to be of the kind that is dominated by the smallest values of k.

If we note that $C_k \approx \frac{1}{3}(kB)^2$ for $kB << 1$ and oscillates for $kB >> 1$ we may approximate the integral by simply cutting it off at $k = 1/B$ and writing $(kr)^{-1}\sin(kr) \approx 1$ (assuming $r < B$). We obtain

$$\rho(E,r) \approx \frac{\eta B^3 E^{-p}}{3\pi D} \int_0^{1/B} \frac{k^2 dk}{k^2 + k_E^2}$$

$$= \frac{\eta B^2 E^{-p}}{3\pi D}\left[1 - k_E B \,\mathrm{Tan}^{-1}(1/k_E B)\right]. \qquad (33)$$

There are two asymptotic forms of (33) depending on the magnitude of $k_e B$.

$$\rho(E) \approx \frac{\eta B^2 E^{-p}}{3\pi D}, \quad k_E B << 1 \qquad (34)$$

$$\rho(E) \simeq \frac{\eta E^{-(p+1)}}{9\pi(p-1)b}, \quad k_E B >> 1 \tag{34'}$$

The critical energy is therefore seen to be determined by

$$k_E B \simeq 1 \quad \text{or} \quad E_c = D/(p-1)bB^2 .$$

Expression (31) may in fact be summed exactly giving

$$\rho(E,r) = \frac{\eta E^{-(p+1)}}{(p-1)2b} \left\{ 1 + \left[k_E B \sinh(k_E B) - \cosh(k_E B) + \frac{\sinh(k_E B) - k_E B \cosh(k_E B)}{\tanh(k_E R)} \right] \frac{\sinh(k_E r)}{k_E r} \right\}, \quad r < B; \tag{35}$$

$$= \frac{\eta E^{-(p+1)}}{(p-1)2b} \left\{ \left[k_E B \cosh(k_E B) - \sinh(k_E B) \right] \left[\frac{\cosh(k_E r)}{k_E r} - \frac{\sinh(k_E r)}{k_E r \tanh(k_E R)} \right] \right\}, \quad r > B . \tag{35'}$$

If $k_E B << 1$ we may write (35) and (35')

$$\rho(E,r) \approx \frac{\eta B^2 E^{-p}}{4D}\left[1 - \frac{1}{3}(r/B)^2 - \frac{2}{3}\frac{k_E B}{\tanh(k_E R)}\right], \quad r < B; \tag{36}$$

$$\approx \frac{\eta B^3 E^{-p}}{6Dr}\left[\cosh(k_E r) - \frac{\sinh(k_E r)}{\tanh(k_E R)}\right], \quad r > B. \tag{36'}$$

For $k_E B >> 1$ we have

$$\rho(E,r) \approx \frac{\eta E^{-(p+1)}}{(p-1)2b}\left[1 - (1+k_E B)\frac{\sinh(k_E r)}{k_E r}\exp(-k_E B)\right], \quad r < B; \tag{37}$$

$$\simeq \frac{\eta E^{-(p+1)}}{(p-1)2b} \frac{(k_E B - 1)}{2 k_E r} \exp\left[-k_E(r-B)\right], \quad r > B. \tag{37'}$$

We can see at once that for $r < B$ the leading terms are just those, apart from a numerical factor, obtained by the simple calculation of expressions (34) and (34'). The correction terms are small and do not affect the spectrum appreciably. If $R \simeq B$ the third term in (36) is of the order unity but in this case the two critical energies, $D/(p-1)bR^2$ and $D/(p-1)bB^2$ are approximately equal and one still sees only one "break" at a critical energy that is determined by the size of the source region rather than the whole diffusing volume.

In Fig. 3 the spectral index increment, δ has been plotted as a function of $X = E/E_c$ where $E_c = D/(p-1)bB^2$. p is given by equation 35, $r = 1/2\ B$ and $R/B = 1$, 2, 10 and 100.

The primary effect of increasing R from B to B x 10^2 is to broaden the transition region to some degree. However, the transition remains in the vicinity of $E_c(B)$ and no partial break appears at a different characteristic energy $E_c(R)$

From this we can see why the results of Dogel and Syrovatskii[19] agree so well with those of Jopipii and Meyer [17] even though the latter considered the case $R_0 = \infty$.

IV. SOLUTION FOR FULL FOKKER-PLANCK ENERGY OPERATOR

We now want to consider the equation

$$\mathcal{L}_E f_\gamma(E) = \frac{\partial}{\partial E}(a_1 E - a_2 E^2) f_\gamma - \frac{1}{2}\frac{\partial^2}{\partial E^2}(a_3 E^2) f_\gamma = \gamma f_\gamma \tag{38}$$

This equation may be written in the form

$$E^2 f'' + (bE + \delta E^2) f' + (C + 2\delta E) f = 0 \tag{38'}$$

where $b = 4 - (a_1/a_3)$, and

$C = 2 + (\gamma - a_1)/a_3$ $\quad \delta = a_2/a_3$.

If we make the substitution

$$f(E) = E^\alpha u(\alpha, E)$$

we obtain an equation for $u(\alpha, E)$

$$E^2 u'' + E[2\alpha + b + \delta E] u' + [\alpha(\alpha-1) + b\alpha + c + (\alpha+2)\delta E] u = 0 \tag{39}$$

If we choose α such that

$$\alpha(\alpha+1) + b\alpha + c = 0$$

i.e. $\alpha_\pm = \frac{1}{2}\left[1 - b \pm \sqrt{(1-b)^2 - 4c}\right]$

$$= \frac{a_1/a_3 - 1}{2} - 1 \pm \sqrt{\left(\frac{a_1/a_3 - 1}{2}\right)^2 - \frac{\gamma}{a_3}}$$

we may make the transformation

$z = -\delta E$; $A = \alpha + 2$; $B = 2\alpha + b$

and obtain the equation

$$zu'' + (B - z)u' - Au = 0 \tag{40}$$

This is Kummer's equation whose solutions are the confluent hypergeometric functions[30];

$$u_1 = {}_1F_1(A; B; z) = {}_1F_1(\alpha + 2; 2\alpha + b; -\delta E). \tag{41}$$

These functions are defined by the series

$${}_1F_1(A; B; z) \equiv \sum_{n=0}^{\infty} \frac{(A)_n z^n}{(B)_n n!} \tag{42}$$

which is convergant for all finite z if $B \neq -1, -2, -3, etc.$

where

$$(A)_n \equiv A(A+1)(A+2) \cdot \cdot \cdot \cdot \cdot (A+n-1)$$

and

$$(A)_0 \equiv 1$$

An independent solution is

$$u_2 = z^{1-B}\, {}_1F_1(1 + A - B; 2 - B; z)$$

It may be seen by straightforward substitution that although there are two independent solutions to equation 40 and two values of α, namely α_+ and α_- the solutions of equation 38 have the property

$$z^{\alpha_+} u_1(\alpha_+, z) = z^{\alpha_-} u_2(\alpha_-, z)$$

$$z^{\alpha_-} u_1(\alpha_-, z) = z^{\alpha_+} u_2(\alpha_+, z) \tag{43}$$

so there are still only two independent solutions of equation 38.

The adjoint equation is,

$$E^2 f^{+\prime\prime} + (b^+ E + \delta^+ E^2) f^{+\prime} + C^+ f^+ = 0 \tag{44}$$

where $b^+ = a_1/a_3$ and $\delta^+ = -a_2/a_3 = -\delta$.

$C^+ = \gamma/a_3$

By the same procedure as before, we obtain solutions

$$f^+(E) = E^{-1-\alpha_\pm} {}_1F_1(-1-\alpha_\pm; 2+2\alpha_\pm - b; \delta E) \tag{45}$$

The solutions depend on the eigenvalue γ through $\alpha_\pm = \alpha_\pm(\gamma)$. The functional relationship of α on γ is rather complicated so instead of γ we will define a new variable, s, with which we will label our solutions.

Define: $\alpha_\pm = \beta \mp s$

$$\beta = \tfrac{1}{2}(a_1/a_3) - \tfrac{3}{2}$$

$$s = -\sqrt{\left(\frac{a_2/a_3 - 1}{2}\right)^2 - \gamma/a_3} \tag{46}$$

We now have

$$f_{\pm s}(E) = E^{\beta \pm s} {}_1F_1(2+\beta \pm s; 1 \pm 2s; -\delta E) \tag{47}$$

and

$$f^+_{\pm s}(E) = E^{-2-\beta \pm s} {}_1F_1(-1-\beta \pm s; 2 \pm 2s; \delta E) \tag{48}$$

We must now determine whether or not the solutions we have chosen form a complete set (and if so, in what sense) and if not we must find the proper combination of solutions that will have this property. Making use of the asymptotic forms of the confluent hypergeometric function[30]

$$ {}_1F_1(A;B;z) \to 1 \quad , \quad z \to 0 $$

and

$$ {}_1F_1(A;B;z) \to \frac{\Gamma(B)}{\Gamma(B-A)} z^{-A} , \; |z| \to \infty , \; Re\, z < 0 $$

$$ \frac{\Gamma(B)}{\Gamma(A)} z^{A-B} \exp z , \; |z| \to \infty , \; Re\, z > 0 $$

we see at once that the adjoint solution has the property

$$ f^{+}(E) \to \exp[\delta E] \quad \text{as} \quad E \to \infty . $$

This means that $\int_0^\infty f^{+}(E)\, q(E)\, dE$

will be defined only for a very restricted class of injection spectra.

It turns out that the functions we want are the linear combinations

$$ f_s(E) = (\delta E)^{\beta+s} \exp(-\delta E)\, U(s, \delta E) \tag{49} $$

$$ f_s^{+}(E) = (\delta E)^{-1-\beta-s}\, U(-s, \delta E) \tag{50} $$

where

$$U(s,z) \equiv \frac{\Gamma(-2s)}{\Gamma(-1-\beta-s)} \, {}_1F_1(-1-\beta+s;\, 1+2s;\, z)$$

$$+ z^{-2s} \frac{\Gamma(2s)}{\Gamma(-1-\beta+s)} \, {}_1F_1(-1-\beta-s;\, 1-2s;\, z) \tag{51}$$

Using the Kummer transformation[30]

$$\exp(z) \, {}_1F_1(A;\, B;\, z) = {}_1F_1(B-A;\, B;\, -z)$$

it is straightforward to verify that 49, and 50 are indeed solutions of equations 38 and 44 respectively. One may also verify from equation 51 that these solutions are symmetric in s, i.e., $z^{s} U(s,z) = z^{-s} U(-s,z)$.

The asymptotic forms of U(s, z) are[30]

$$U(s,z) \rightarrow z^{1+\beta-s} \quad , \quad |z| \rightarrow \infty$$

so we see that $f_s^+(E) \to \text{Const.} \quad , |E| \to \infty$

$$\to E^{-1-\beta \pm s} \quad , |E| \to 0$$

since $-1-\beta = \frac{1}{2}(1 - a_1/a_3)$, if $a_1 < a_3$

and $|\mathrm{Re}\, s| < \frac{1}{2}(1 - a_1/a_3)$

then we have $\int_0^\infty f_s^+(E) q(E)\, dE < \infty$

if $\int_0^\infty |q(E)|\, dE < \infty$ and we have a well defined transform. It is not clear that there is any physical reason to require $a_1 < a_3$, however, if we require this to be formally true we will see that our final Green's function may be quite trivially continued from $a_1 < a_3$ to $a_1 \geq a_3$ so no real restriction is imposed.

It is shown in the appendix that the proper form of the unit operator appropriate to this transform is,

$$\underset{\sim}{1} = \delta(x - x')$$

$$= \frac{x^\beta e^{-x}}{4\pi i (x')^{1+\beta}} \int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta+s)\Gamma(-1-\beta-s)}{\Gamma(2s)\,\Gamma(-2s)} \left(\frac{x}{x'}\right)^s u(s,x)\, u(-s,x')\, ds \tag{52}$$

where $x = \delta E$.

To obtain the Green's function, we note that $\gamma = a_3[(1+\beta)^2 - s^2]$

and since $\lambda_{k,\gamma} = k^2 + 1/\tau_c + \gamma$ we have

$$\lambda = -a_3(s-s_1)(s+s_1$$

where

$$s_1 = [(1+\beta^2 + (k^2 + 1/\tau_c)/a_3]^{1/2}$$

therefore

$$g(x,x') = \frac{-x^{\beta} e^{-x}}{4\pi i\, a_3 (x')^{1+\beta}}$$

$$\times \int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta+s)\Gamma(-1-\beta-s)}{\Gamma(2s)\Gamma(-2s)} \frac{x^{s} U(s,x)\, x'^{-s} U(-s,x')}{(s-s_1)(s+s_1)}\, ds \qquad (53)$$

To evaluate the integral in equation 53 we use the definition of the function U(s, x), equation 51 to expand the integrand as,

$$g(x,x') = \frac{-x^{\beta} e^{-x}}{4\pi i\, a_3 (x')^{1+\beta}} \int_{-i\infty}^{i\infty} (I + II + III + IV)\, ds$$

where

$$\mathrm{I} = \left(\frac{x}{x'}\right)^{s} \frac{F(s,x)\,F(-s,x')}{(s-s_1)(s+s_1)}$$

$$\mathrm{II} = \left(\frac{x}{x'}\right)^{-s} \frac{F(-s,x)\,F(s,x')}{(s-s_1)(s+s_1)}$$

$$\mathrm{III} = \frac{\Gamma(-2s)\,\Gamma(-1-\beta+s)}{\Gamma(2s)\,\Gamma(-1-\beta-s)} \frac{(xx')^{s}\,F(s,x)\,F(s,x')}{(s-s_1)(s+s_1)} \tag{54}$$

$$\mathrm{IV} = \frac{\Gamma(2s)\,\Gamma(-1-\beta-s)}{\Gamma(-2s)\,\Gamma(-1-\beta-s)} \frac{(xx')^{-s}\,F(-s,x)\,F(-s,x')}{(s-s_1)(s+s_1)}$$

where we have used the notation

$$F(s,x) \equiv {}_2F_1(-1-\beta+s;\,1+2s;\,x).$$

We may now determine the behavior of each of the four terms as $s \to \infty$ from the fact[30] that as $s \to \infty$, $F(s,x) \longrightarrow$ const. We see immediately that the first two terms are dominated by the value of the ratio (x/x') and that we may close the contour of integration in I to the

right or left as (x/x') is less than or greater than one respectively. The opposite is true for II. The third and fourth terms, however, are dominated as $s \to \infty$ by the gamma functions since

$$\left| \frac{\Gamma(2s)\,\Gamma(-1-\beta-s)}{\Gamma(-2s)\,\Gamma(-1-\beta+s)} \right| \to \left(\frac{4s}{e}\right)^{2s} \qquad \text{as } s \to \infty$$

so we see that III may always be closed to the right and IV may always be closed to the left regardless of the value of (x/x').

If we designate by C_R and C_L the contours that run up the imaginary axis and are closed on the right and left infinite semi-circles respectively, we may write if $x < x'$

$$g(x,x') = \frac{-x^{\beta} e^{-x}}{4\pi i\, a_3 (x')^{1+\beta}}$$

$$\times \left\{ \int_{C_R} \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)} \left(\frac{x}{x'}\right)^{s} \frac{F(s,x)\,U(-s,x')}{(s-s_1)(s+s_1)}\, ds \right. \tag{55}$$

$$\left. + \int_{C_L} \frac{\Gamma(-1-\beta-s)}{\Gamma(-2s)} \left(\frac{x}{x'}\right)^{-s} \frac{F(-s,x)\,U(s,x')}{(s-s_1)(s+s_1)}\, ds \right\}$$

and if $x > x'$

$$g(x,x') = \frac{x^{\beta} e^{-x}}{4\pi i a_3 (x')^{1+\beta}}$$

$$\times \left\{ \int_{C_R} \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)} \left(\frac{x'}{x}\right)^{s} \frac{F(s,x')\,U(-s,x)}{(s-s_1)(s+s_1)}\, ds \right. \tag{56}$$

$$\left. + \int_{C_L} \frac{\Gamma(-1-\beta-s)}{\Gamma(-2s)} \left(\frac{x'}{x}\right)^{s} \frac{F(-s,x')\,U(s,x)}{(s-s_1)(s+s_1)}\, ds \right\}.$$

We must now investigate the positions of the various singularities of the integrands in the s plane. We first note that the gamma functions will contribute no singularities, within their respective contours as long as $-(1+\beta)>0$, $(a_2<a_3)$. Furthermore U(s, x) is an entire function of s[30] and F(s,x) has only simple poles at $s=-\frac{1}{2}(n+1)$ [30]. We see, therefore, that in each of the integrals of equations 55 and 56 the only poles that contribute to the integrals are those of the denominator $(s-s_1)(s+s_1)$. Evaluation of the residues is straightforward from this point on and if we adopt the notion, $x_>$ $(x_<)$ is the larger (smaller) of the two variables x and x' we may write the result in the simple form

$$g(x,x') = \frac{x^{\beta} e^{-x}}{2a_3 s_1 (x')^{1+\beta}} \frac{\Gamma(-1-\beta+s_1)}{\Gamma(2s_1)} \left(\frac{x_<}{x_>}\right)^{s_1} F(s_1, x_<)\, U(-s_1, x_>) \tag{57}$$

In terms of E we have

$$g(E,E') = \frac{E^{\beta} e^{-\delta E}}{2a_2 s_1 E'^{1+\beta}} \frac{\Gamma(-1-\beta+s_1)}{\Gamma(2s_1)} \left(\frac{E_<}{E_>}\right)^{s_1} F(s_1, \delta E_<)\, U(-s_1, \delta E_>) \tag{58}$$

At this point we may note that since $s_1 = \sqrt{(1+\beta)^2 + (k^2 + 1/\tau_c)/a_3} > |1+\beta|$, $-1-\beta+s_1 > 0$ for any value of a_1/a_3 so the restriction $a_1 < a_3$ may be relaxed and our Green's function remains perfectly regular.

The Green's function of equation 58 is quite complicated and one would not expect simple spectra to result in general; however, in the case of an inverse power low injection $q(E) \propto E^{-p}$ one simple result can be obtained. If we examine the case of large E we may investigate the asymptotic form of the Green's function. If $E > E'$ we have

$$g(E,E') \approx \frac{\Gamma(-1-\beta+s_1)}{2a_2 s_1 \Gamma(2s_1)\,\delta} \frac{E^{1+2\beta}}{E'^{3+2\beta}} e^{-\delta(E-E')} \tag{59}$$

The exponential quickly damps any contribution from values of E' much removed from E so little contribution is obtained from energies $E' < E$ On the other hand, if $E' > E$ we have

$$g(E,E') \approx \frac{\Gamma(-1-\beta+s_1)}{2a_2 s_1 \Gamma(2s_1)\,\delta} E^{-2}. \tag{60}$$

The resulting spectrum is approximately

$$\begin{aligned} f(E) &\approx \int_E^\infty g(E,E') E'^{-p} dE' \\ &\approx \frac{\Gamma(-1-\beta+s_1)}{2a_2 s_1 \Gamma(2s_1)\,\delta} E^{-2} \int_E^\infty E'^{-p} dE' \\ &= \text{Const.}\ E^{-(p+1)} \end{aligned} \tag{61}$$

This result is the same as for the case of synchrotron and inverse Compton

losses alone. This is indeed reasonable since if one assumes that f(E) is a power low the original differential equation, equation 38, shows that as $E \to \infty$ the synchrotron or Compton loss term proportional to a_2 dominates the entire process.

A more detailed examination of the asymptotic forms of F(s,z) and U(s,z) shows that the asymptotic forms are valid if we have

$$\frac{s_1^2-(1+\beta)^2}{\delta E} << 1$$

Inserting the expressions for s_1, β and δ we obtain the condition

$$E >> \frac{k^2+1/\tau_c}{a_2} = \frac{1}{a_2 \tau_n}$$

where τ_n is the total lifetime for the nth mode. This is the same condition as for synchrotron and inverse Compton losses alone so we may conclude that inclusion of energy diffusion and Fermi acceleration does not substantially alter the form of cosmic ray electron spectra at high energies.

V. SUMMARY

We have investigated the validity of the leakage lifetime approximation that has had wide use in cosmic physics. We have seen that this approximation has its natural setting in the eigenfunction expansion of the differential equation describing spatial diffusion and energy transport. The spectra that are derived in the leakage lifetime approximation are the energy spectra of the various eigenmodes of the diffusion operator plus spatial boundary conditions. Each mode has its own leakage lifetime with that of the nth mode being approximately $\tau_n \simeq \tau_0/n^2$ where τ_0 is the random walk time across the diffusing region of linear dimension L, $\tau_0 \simeq L^2/\pi D$ D being the diffusion coefficient.

We saw that for Fermi acceleration the lowest mode dominates and hence the leakage lifetime approximation is quite good. On the other hand for the case of inverse power law injection of electrons with subsequent synchrotron or inverse Compton losses all higher modes contribute significantly with the fundamental lifetime τ_0 being of no particular significance. The resulting spectrum in this case is determined almost wholly by the spatial distribution of the sources.

Finally we were able to obtain a solution to the more general Fokker-Planck energy operator in terms of confluent hypergeometric functions. These solutions are, in general, quite complex, however, for inverse power law injection they were seen to have the familiar property of steepening the spectrum by one power at high energy.

All in all the eigenfunction expansion of the Green's function is seen to be an effective method for putting the leakage lifetime approximation in its natural mathematical setting and thereby making it possible to investigate the limits of its validity.

Appendix

We wish to consider the transform

$$\hat{f}(s) = \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)} \int_0^\infty x'^{-1-\beta-s} \, u(-s,x') \, F(x') \, dx'$$

and its "inverse"

$$\tilde{F}(x) = \frac{e^{-x}}{4\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)}{\Gamma(-2s)} \, x^{\beta+s} \, u(s,x) \, \hat{f}(s) \, ds$$

and discover to what extent we may say $\tilde{F}(x) = F(x)$

First we will rewrite the transform formula as a Lebesgue-Stielijes ($L-S$) integral.

$$\hat{f}(s) = \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)} \int_{Q(0)}^{Q(\infty)} F(x') \, dQ(s,x')$$

where

$$Q(s,x') = \int_{(x'')}^{x'} {}^{-1-\beta-s} \, U(-s,x'')\, dx''$$

$$= \frac{\Gamma(2s)}{\Gamma(-1-\beta+s)} x'^{-\beta-s} \mathcal{J}(-s,x') + \frac{\Gamma(-2s)}{\Gamma(-1-\beta-s)} x'^{-\beta+s} \mathcal{J}(s,x') \; ;$$

$$\mathcal{J}(s,x') \equiv \sum_{n=0}^{\infty} \frac{(-1-\beta+s)_n \quad (x')^n}{(1+2s)_n \; (-\beta+s+n) \; n!}$$

We note the following properties of $\mathcal{J}(s,x')$

as $s \to \beta - n$, $\mathcal{J}(s,x') \longrightarrow \dfrac{(-1-n)_n \; x'^{n}}{(1+2\beta-2n)_n \; n!} \left(\dfrac{1}{s-\beta+n} \right)$

as $1 \pm 2s \to -m$

$$\mathcal{J}(\pm s, x') \longrightarrow \Gamma(-m) \, x'^{m+1} \frac{\left(-1-\beta-\frac{m+1}{2}\right)_{m+1}}{(m+1)!} \mathcal{J}\left(-\frac{m+1}{2}, x'\right).$$

If $F(x')$ is of unbounded variation and/or of unbounded range in x' the proper definition of the $L\text{-}S$ integral is

$$\int_{Q(0)}^{Q(\infty)} F(x')\,dQ = \lim_{\substack{N\to\infty \\ X\to\infty}} \int_{Q(0)}^{Q(X)} F_N(x')\,dQ$$

where $F_N(x') = F(x')$ if $|F(x')| \leq N$ and $F_N(x') = \pm N$ if $|F(x')| > N$ or $< -N$ respectively. It is assumed that the limits exist. We may now define

$$\hat{f}_{N,X}(s) \equiv \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)} \int_{Q(0)}^{Q(X)} F_N(x')\,dQ$$

and

$$\hat{f}(s) = \lim_{\substack{N\to\infty \\ X\to\infty}} \hat{f}_{N,X}(s)$$

Since $Q(-s,x') = Q(s,x')$ we see that for pure imaginary s Q is real. We may therefore introduce two auxiliary functions of s,

$$\sigma_{N,X}(s) = \sum_{r=0}^{R} l_r(s)\, q_r(s)$$

and

$$\Sigma_{N,X}(s) = \sum_{r=0}^{R} L_r(s)\, q_r(s)$$

where $q_r(s)$ is the variation of $Q(s,x')$ over the set E_r of points x', $E_r = \{x' : f_r \le F_N(x') < f_{r+1},\ x' \le X\}$ and $l_r = f_r$ if $q_r(s) \ge 0$ and $l_r = f_{r+1}$ if $q_r(s) < 0$. Likewise $L_r = f_{r+1}$ if $q_r(s) \ge 0$ and $L_r = f_r$ if $q_r(s) < 0$ ($q_r(s)$ is real), where we have divided the ordinate between -N and +N into R intervals. From the theory of the $L-S$ integral we know that

$$\frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\,\sigma_{N,X}(s) \le \zeta_{N,X}(s) \le \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\,\Sigma_{N,X}(s)$$

We now note the fact that since for s pure imaginary $x'^{-1-\beta-s}\,U(-s,x')$ is bounded uniformly in x' and s, $|x'^{-1-\beta-s}\,U(-s,x')| < M$ where M is a constant independent of x' and s. Therefore we have

$$|Q(s,x_2) - Q(s,x_1)| = \left| \int_{x_1}^{x_2} x'^{-1-\beta-s} u(-s,x')\,dx' \right|$$

$$\leq \int_{x_1}^{x_2} \left| x'^{-1-\beta-s} u(-s,x') \right| dx' < M(x_2 - x_1)$$

so $|q_r(s)| < M\, mE_r$ where mE_r is the measure of the set E_r. If we now let R be large enough so that the maximum separation of ordinate subdivisions $(f_{r+1} - f_r)_{Max} = \epsilon$ we can consider $\Sigma_{N,X}(s) - \sigma_{N,X}(s)$.

We have

$$\Sigma_{N,X}(s) - \sigma_{N,X}(s) = \sum_r \left(L_r(s) - l_r(s)\right) q_r(s)$$

$$= \sum_r |f_{r+1} - f_r|\, |q_r(s)| < \epsilon M \sum_r mE_r$$

$$= \epsilon M X.$$

We see that as we let $R \to \infty$ so that $\epsilon \to 0$ the two functions $\Sigma_{N,X}(s)$ and $\sigma_{N,X}(s)$ converge <u>uniformly in s</u> to one another. We may define another auxiliary function

$$\mu_{N,\Sigma}(s) = \sum_r f_r q_r(s)$$

and readily see that

$$\sigma_{N,\Sigma}(s) \leq \mu_{N,\Sigma}(s) \leq \Sigma_{N,\Sigma}(s)$$

Since the functions $\sigma_{N,\Sigma}(s)$ and $\Sigma_{N,\Sigma}(s)$ bracket <u>both</u> the functions $\mu_{N,\Sigma}(s)$ and $\hat{\xi}_{N,\Sigma}(s)\dfrac{\Gamma(2s)}{\Gamma(-1-\beta+s)}$ we see that as $R \to \infty$ and $\epsilon \to 0$ we have

$$\mu_{N,\Sigma}(s) \xrightarrow[\text{uniformly in } s]{} \hat{\xi}_{N,\Sigma}(s)\frac{\Gamma(2s)}{\Gamma(-1-\beta+s)}$$

Also from the uniform boundedness of $x'^{-1-\beta-s}\,U(-s,x')$ we note that

$$\left|\frac{\Gamma(2s)}{\Gamma(-1-\beta+s)}\left(\hat{\xi}(s) - \hat{\xi}_{N,\Sigma}(s)\right)\right|$$

$$= \left|\int_0^\infty \left[F(x') - F_N(x')\,S(x';0,\Sigma)\right] x'^{-1-\beta-s}\,U(-s,x')\,dx'\right|$$

(where $S(x;a,b) \equiv 1$ if $a \leq x \leq b$, $\equiv 0$ otherwise)

$$\leq \int_0^\infty \left|F(x') - F_N(x')\,S(x';0,\Sigma)\right| \left|x'^{-1-\beta-s}\,U(-s,x')\right| dx'$$

$$< M \int_0^\infty |F(x') - F_N(x') S(x'; 0, \bar{X})| \, dx'$$

$$= M \left\{ \int_0^{\bar{X}} |F(x') - F_N(x')| \, dx' + \int_{\bar{X}}^\infty |F(x')| \, dx' \right. .$$

Since we have assumed that $\int_0^\infty |F(x')| \, dx' < \infty$

we have

$$\lim_{\substack{N \to \infty \\ \bar{X} \to \infty}} \left| \frac{\Gamma(2s)}{\Gamma(-1-\beta+s)} \left(\mathcal{F}(s) - \mathcal{F}_{N,\bar{X}}(s) \right) \right| = 0$$

and the limit is uniform in s. We may now write

$$\tilde{F}(x) = \frac{e^{-x}}{4\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)}{\Gamma(2s)} x^{\beta+s} U(s,x) \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)} \left\{ \lim_{\substack{N \to \infty \\ \bar{X} \to \infty}} \lim_{\substack{\lambda \to \infty \\ \epsilon \to 0}} \mu_{N,\bar{X}}(s) \right\} ds$$

The limits in the brackets are uniform in s, the remainder of the integrand outside of the brackets is uniformly bounded in s, hence the entire integrand approaches its limiting value uniformly in s so we may write

$$\tilde{F}(x) = \lim_{\substack{N\to\infty \\ \bar{X}\to\infty}} \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left\{ \frac{e^{-x}}{4\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)} x^{\beta+s} U(s,x)\, \mu_{N,\bar{X}}(s)\, ds \right\}$$

Since $\mu_{N,\bar{X}}(s) = \sum_{r=0}^{R} f_r g_r(s)$ we may write

$$\tilde{F}_{N,\bar{X},R}(x) = \frac{e^{-x}}{4\pi i} \sum_r f_r \int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)} x^{\beta+s} U(s,x)\, g_r(s)\, ds$$

and

$$\tilde{F}(x) = \lim_{\substack{N\to\infty \\ \bar{X}\to\infty}} \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \tilde{F}_{N,\bar{X},R}(x) .$$

Since the set E_r may be covered by a series of intervals

$$_mE_r = \sum_i |I_{r,i}|$$

where $|I_{r,i}|$ is the lenght of the interval $I_{r,i}$

$|I_{r,i}| = X_{2,r,i} - X_{1,r,i}$ where $X_{2,r,i}$ and $X_{1,r,i}$ are the upper and lower end points respectively of the (r,i) th interval. We have then $q_r = \sum_i q_{r,i}$

where

$$q_{r,i} = Q(X_{2,r,i}) - Q(X_{1,r,i})$$

Since the sum over i may be at times an infinite series we must question whether its convergence (which is assumed) is also uniform in s. We have

$$\left| q_r - \sum_{i=1}^{n} q_{r,i} \right| = \left| \sum_{i=n+1}^{\infty} q_{r,i} \right| \leq \sum_{i=n+1}^{\infty} |q_{r,i}| < M \sum_{i=n+1}^{\infty} |I_{r,i}|$$

This may be made as small as desired by manking n sufficiently large independently of s so we may write.

$$\tilde{F}_{N,\Sigma,\beta}(x) = \frac{e^{-x}}{4\pi i} \sum_{r,i} f_n \int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)} x^{\beta+s} U(s,x)\, q_{r,i}(s)\, ds.$$

We must now evaluate integrals of the type

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)} x^{\beta+s} U(s,x)\, \mathcal{G}(s,x_n)\, ds$$

$$= \int_{-i\infty}^{i\infty} ds \Big\{ \left(\frac{x}{x_n}\right)^{\beta+s} F(s,x)\, \mathcal{G}(-s,x_n) + \left(\frac{x}{x_n}\right)^{\beta-s} F(-s,x)\, \mathcal{G}(s,x_n)$$

$$+ \frac{\Gamma(-2s)\Gamma(-1-\beta+s)}{\Gamma(2s)\Gamma(-1-\beta-s)} \left(\frac{x}{x_n}\right)^{\beta} (x x_n)^{s} F(s,x)\, \mathcal{G}(s,x_n)$$

$$+ \frac{\Gamma(2s)\Gamma(-1-\beta-s)}{\Gamma(-2s)\Gamma(-1-\beta+s)} \left(\frac{x}{x_n}\right)^{\beta} (x x_n)^{-s} F(-s,x)\, \mathcal{G}(-s,x_n) \Big\}$$

It can be readily seen that the third (and fourth) term makes no cotribution to the integral for any value of x and x_n. This is because all of the poles of this term are in the left (right) hand plane and the combination of Gamma gunctions always insures that the contour may be closed in the right (left) hand infinite semi-circle thereby enclosing no poles. We may therefore drop these terms and consider simply

$$\left(\frac{x}{x_n}\right)^{\beta}\int_{-i\infty}^{i\infty} ds\left\{\left(\frac{x}{x_n}\right)^{s}F(s,x)\,\mathcal{G}(-s,x_n)+\left(\frac{x}{x_n}\right)^{-s}F(-s,x)\,\mathcal{G}(s,x_n)\right\}.$$

We now note that as $s\to\infty$, $F(\pm s,x)\to e^{x/2}$ and $\mathcal{G}(\pm s,x_n)\to \pm e^{x_n/2}s^{-1}$ Therefore the behavior on the infinite circle is dominated by the factors $(x/x_n)^{\pm s}$ if $x<x_n$ we may close the first (second) term to the left (right) and vice-versa if $x>x_n$. If $x=x_n$ we may close the contours either way but the contribution if the infinite semi circle is finite.

Next we note that the poles at $1\pm 2s=-m$ make no contribution as long as the first and second terms are closed in opposite directions. To see this observe that when $1\pm 2s\to -m$ the first term approaches

$$(xx_n)^{\frac{m+1}{2}}\frac{\Gamma(-m)\left(-1-\beta-\frac{m+1}{2}\right)_{m+1}}{(m+1)!}F\left(\mp\frac{m+1}{2},x\right)\mathcal{G}\left(\pm\frac{m+1}{2},x_n\right).$$

The second term approaches the same value for $1\pm 2s\to -m$ and since the two sets of poles would be encircled in opposite directions the contributions would cancel, term for term. So that we may ignore these poles we will always close the two terms in opposite directions even when $x=x_n$.

We next turn to the poles at $s = \pm(m-\beta)$. The first term has the poles along the positive axis at $s = m-\beta$ and the second term has the poles along the negative axis at $s = -m+\beta$. We see at once that when $X > X_n$ the integral is zero since in this case each term will be closed to the side where it has no poles.

If $X < X_n$ we close the first term to the right and the second to the left. Each term will give a contribution when $s = \pm(m-\beta)$ equal to

$$2\pi i\left(\frac{X}{X_n}\right)^{-\beta} \frac{X^m(-1-m)_m}{(1+2\beta-2m)_m\, m!} F(m-\beta, X)$$

Adding these together we obtain for the integral

$$4\pi i \sum_{m=0}^{\infty} \frac{(-1-m)_m}{(1+2\beta-2m)_m\, m!} X^m F(m-\beta, X)$$

This series may be evaluated simply by evaluating the case $X = X_n$. If we close the contours in the same manner as for $X < X_n$ we will obtain the same series but with a correction to compensate for the contributions from the infinite circle. We obtain for the integral

$$4\pi i \sum_{m=0}^{\infty} \frac{(-1-m)_m}{(1+2\beta-2m)_m\, m!} X^m F(m-\beta, X) - 2\pi i\, e^{X}$$

However, we may also close the contours in the opposite manner as for the case $X > X_n$ and get no contributions from the poles. The contribution from the infinite circle is of opposite sign so we obtain a value of $2\pi i e^{X}$ for the integral. The two methods of evaluation must yeild equal results so we have

$$4\pi i\,(\text{Series}) - 2\pi i e^{X} = 2\pi i e^{X}$$

or

$$\sum_{m=0}^{\infty} \frac{(-1-m)_m}{(1+2\beta-2m)_m\, m!} X^m F(m-\beta, X) = e^{X}$$

Combining these results we see that the integral

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)} X^{\beta+s} U(s,X)\, Q(s,X_n)\, ds$$

$$= 0 \qquad X > X_n ,$$

$$= 4\pi i\, e^{x} \qquad x < x_n ,$$

$$= 2\pi i\, e^{x} \qquad x = x_n .$$

If now instead of $Q(s, x_n)$ we insert $q(r,i) = Q(s, x_{2,r,i}) - Q(s, x_{1}, r, i)$ into the integral we will obtain 0 if X is not in or an end point of the interval $I_{r,i}$, $4\pi i e^{x}$ if X is in the interval $I_{r,i}$, and $2\pi i e^{x}$ if X is an end point of the interval $I_{r,i}$.

We now note that the intervals $I_{r,i}$ do not overlap and must cover the entire interval 0 to $\bar{X}$. Therefore if we assume that X is in the interval 0 to $\bar{X}$ the point X will fall in one of three categories; (i) it will lie in an interval $I_{r,i}$. (ii) it will be a boundary point of two adjacent intervals (with different valuses of r) or (iii) it will be a point of accumulation of an infinite sequence of intervals.

In the case (iii) the point X can not be unambigously assigned to any interval and hence the value of the integral is indeterminite. It should be noted, however, that the only way an infinite sequence of intervals can arise is for the function $F(x)$ to have a point of infinite oscillation and at this point the function itself is indeterminite, eg., $\sin(\frac{1}{x - x_p})$ at the point $x = x_p$.

For the case (i) we have $\tilde{F}_{N,\bar{X},R}(x) = f_r$ and for the case (ii) we have $\tilde{F}_{N,\bar{X},R}(x) = \frac{1}{2}(f_r + f_{r'})$ where r and r' are the values of r for the two intervals that have X as a common boundary point.

We must now inquire as to the effect of taking the limit $R \to \infty, \epsilon \to 0$. A little reflection will show that for both cases (i) and (ii) the limit

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left\{ f_r \text{ or } \tfrac{1}{2}(f_r + f_{r'}) \right\} = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} F_N(x')\,dx'$$

This may or may not be equal to $F_N(x)$ but it will be equal for "almost all" values of x (values taken by $F_N(x)$ on sets of measure zero are ignored). It is also easy to see that case (i) corresponds to a point of continuity and case (ii) may (but not necessarily) correspond to a point of discontinuity of the function $F_N(x)$ (once again ignoring values taken on sets of measure zero). If x is a point of discontinuity the transform will take on a value half way between the values on either side.

Taking the limit $X \to \infty$ is now trivial since we have assumed that X was large enough so that $x < X$ and increasing it further has no effect so

$$\lim_{X\to\infty} \tilde{F}_{N,X}(x) \equiv \tilde{F}_N(x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} F_N(x')\,dx'$$

If $\lim_{N\to\infty} F_N(x)$ exists so will $\lim_{N\to\infty} \tilde{F}_N(x)$ and

$$\lim_{N\to\infty} \tilde{F}_N(x) \equiv \tilde{F}(x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} F(x')\,dx' .$$

We therefore have $\tilde{F}(x) = F(x)$ almost everywhere.

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Figure Captions

Figure 1 - Log $\rho(E)$ vs. log E for the first three modes of Fermi acceleration showing how the fundamental mode $(N=0)$ dominates at high energy.

Figure 2 - Plot of spectral index increment

$$\delta = -d\log f/d\log x - \rho \qquad \text{vs. } \log x = \log(E/E_c)$$

for the functions

$$f = x^{-2}\exp[-(\rho-1)/x]\int_{x/(\rho-1)}^{\infty} x'^{-\rho}\exp[(\rho-1)/x']\,dx'$$

and $\tilde{\delta}$ for the approximate function $\tilde{f} = x^{-\rho}/(1+x)$

where $E_c = [(\rho-1)T_n b]^{-1}$

and $\rho = 2.5$

Figure 3 - Plot of the spectral index increment $\delta = -d\log_c\rho/d\log x - \rho$ vs. $\log x = \log(E/E_c)$. ρ is given by equation 35 in the text, $\alpha \equiv r/B = .5$, $\beta \equiv R/B =$ 1, 2, 10, and 100, $E_c = D[(\rho-1)bB^2]^{-1}$, and $\rho = 2.5$

FIGURE 1

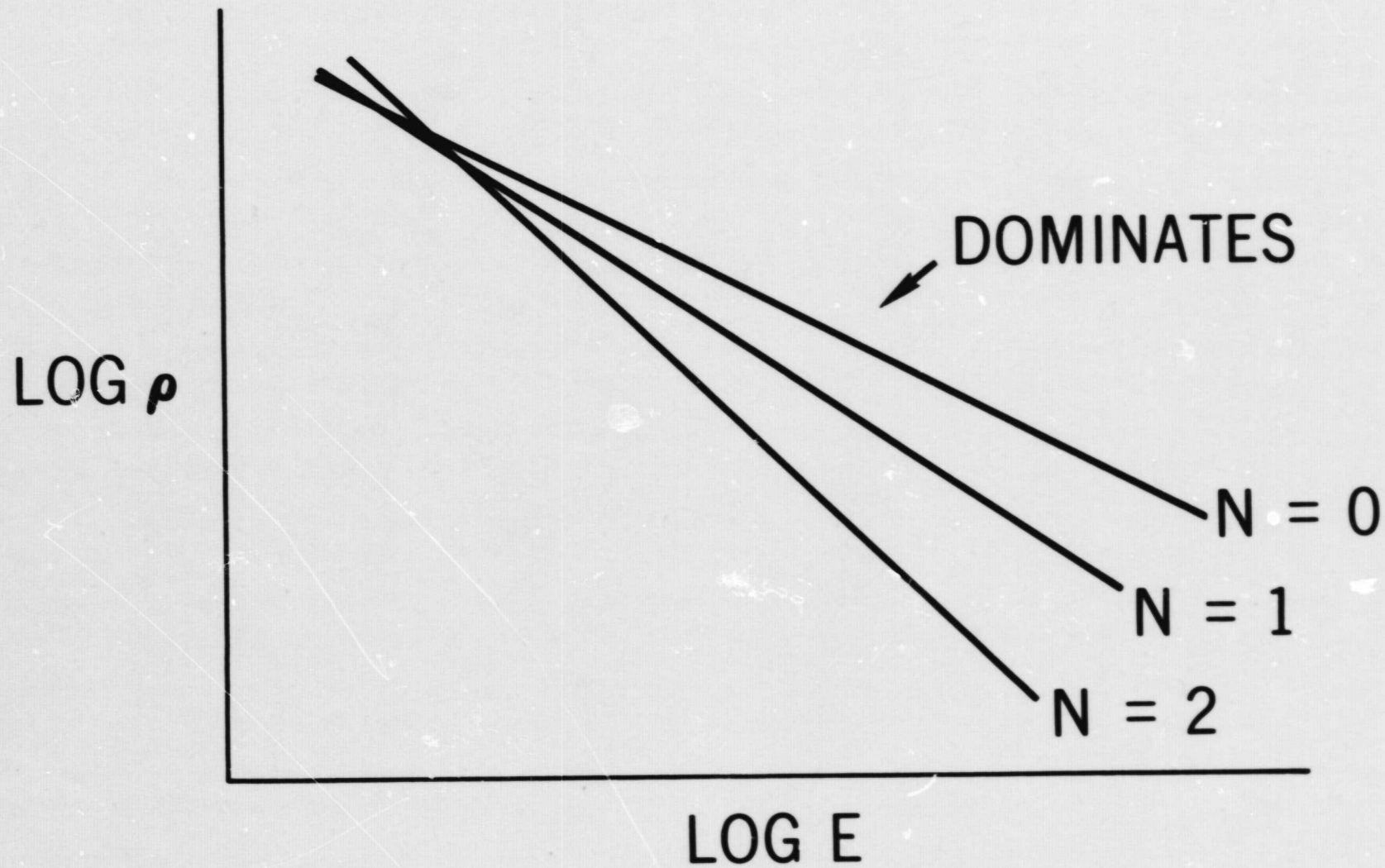


FIGURE 2

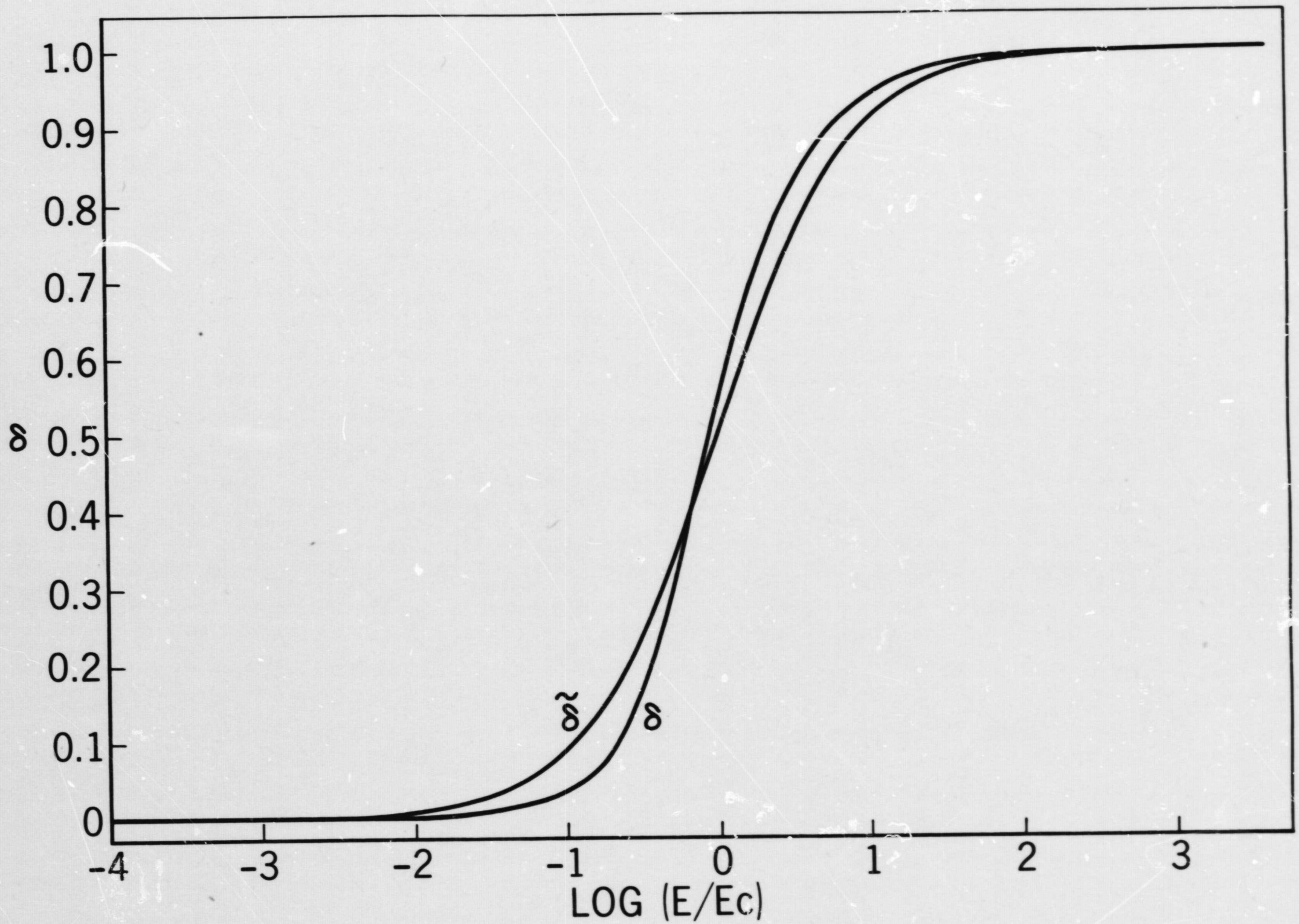
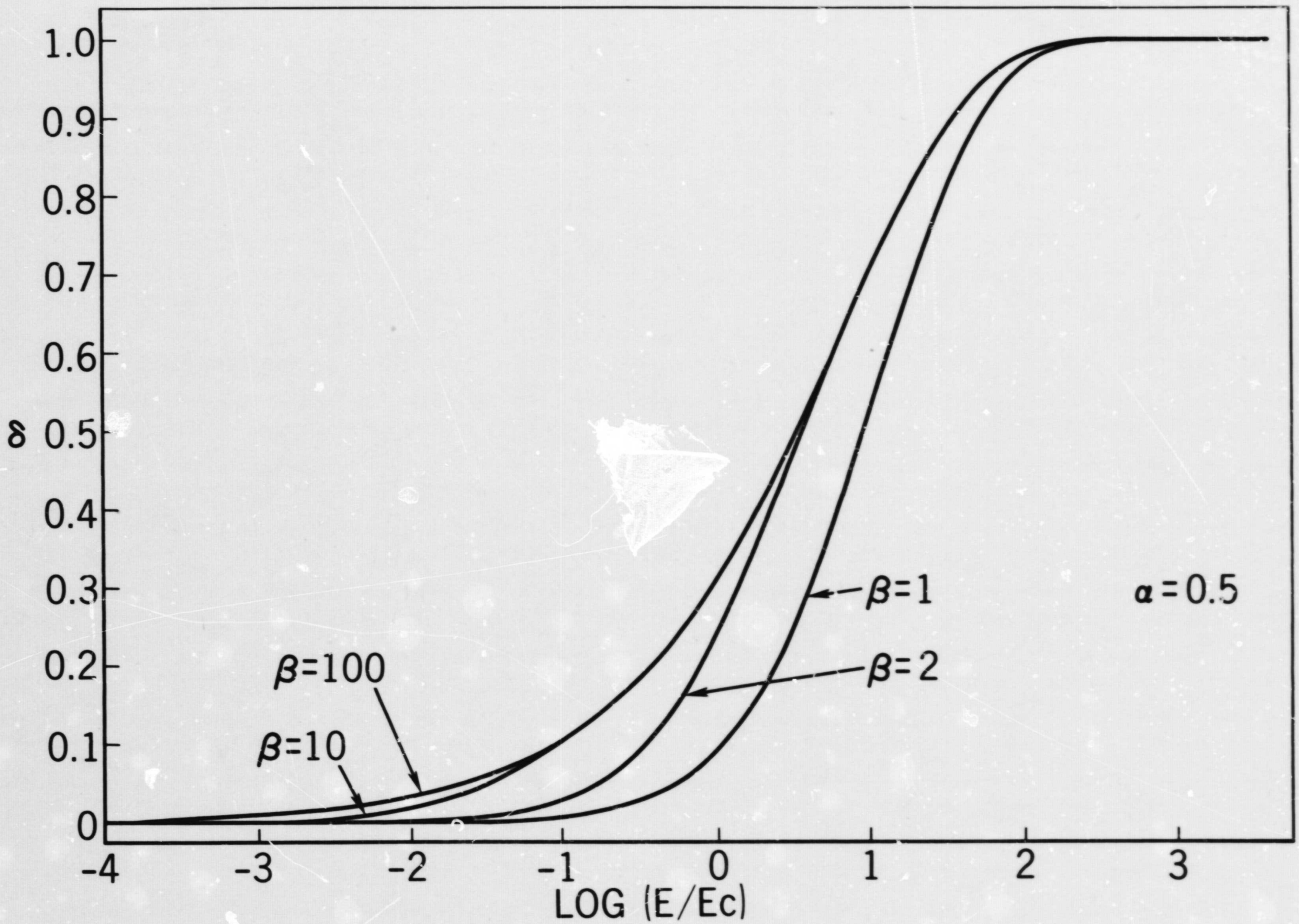


FIGURE 3


1.0
0.9
0.8
0.7
0.6
δ 0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
LOG (E/Ec)
β=100
β=10
β=1
β=2
α=0.5