NON-GAUSSIAN NOISE

by

Kursad Fevzi Tuncer, B.S.

A Progress Report

Submitted to

National Aeronautics and Space Administration

Washington, D. C.

NASA Research Grant NBR 19-003-003

Electrical Engineering Department

Louisiana Polytechnic Institute

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ABSTRACT

Non-Gaussian noise is the subject of this study. The probability density functions for quantization noise, continuous wave interference, atmospheric noise, and impulse noise are presented and discussed in detail. Impulse noise is described by the hyperbolic and Pareto distributions and quantization noise is represented by the uniform error distribution. Both the continuous wave interference and atmospheric noise follow Rayleigh and lognormal distributions.

FOREWORD

This thesis attempts to satisfy a need which has become apparent in recent years with the development of data transmission systems. This need is for a presentation of types of noise which are not described by a Gaussian process. Though most of the results obtained here have appeared elsewhere and have become well known in recent years they have not been classified and discussed as to their probability density functions. Thus, all types of noise discussed here have in common the probability density functions which are not Gaussian. However, old and well-known topics such as multipath interference and atmospheric noise are very closely related to the Gaussian process. These topics were included here because under transformations they cease to be Gaussian. For example, the Rayleigh distribution is a non-Gaussian distribution which has Gaussian orthogonal components. Also, the lognormal distribution of atmospheric noise amplitudes is a non-Gaussian distribution. It is obtained by the transformation e^{Δ}, where Δ is a Gaussian random variable. On the other hand, impulse noise is in no way related to the Gaussian process. Hyperbolic and Pareto distributions were used to describe the behavior of impulse noise. Another type of noise which is not related to Gaussian is quantization noise in PCM and which has uniform distribution. It is a fact that non-Gaussian noise occurs often in data transmission. Non-Gaussian noise is thus important because the usage of data transmission techniques is extensive and rapidly increasing.

ACKNOWLEDGEMENT

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INTRODUCTION

Noise can be broadly defined as any unwanted disturbance within the useful frequency range. It is distinguished from distortion in that noise is a random process.

One can classify noise into three categories in the frequency domain. Two of these are single frequency noise and impulsive noise. Between these extremes lies Gaussian noise. Naturally, the most commonly encountered noise is Gaussian. This distribution results where the noise is the sum of many independent noises with similar statistical characteristics as predicted by the central limit theorem. Examples of Gaussian distributed noise are thermal and shot noise. However, types of interferences encountered in data transmission, such as quantization noise in Pulse Code Modulation (PCM) and impulse noise in telephone channels, do not follow Gaussian distribution. The same is true for atmospheric noise produced by thunderstorms. The non-Gaussian probability density function is encountered when radio waves at medium range of frequencies are scattered by the ionosphere and then interfere with the direct wave.

Another way of classifying noise is to compare how it behaves with the signal strength. For example, impulsive noise peaks in data transmission caused by "hits" and "dropouts" are often multiplicative in character. That is, noise multiplies the instantaneous value of the signal wave amplitude by some function. Thus, multiplicative noise modulates the signal. Usually the multiplying function

mentioned is 1 and sometimes 0.[1] On the other hand, there exists additive noise, and the effect of this type of noise is reduced by an increase in signal strength.

Impulse noise is characterized by non-overlapping transient disturbances in the time domain. Common sources of impulse noise in wire transmission are dirty switch contacts, defective filters in power supplies, crosstalk through coupling from signaling circuits, improper shielding and grounding, unbalanced circuits, ignition, loose connections, etc.[2] This type of noise is also caused by external or directly connected selector switches in telephone channels. Therefore, during the busy hours of telephone communication impulse noise is increased since switching is increased. However, impulse noise is not so disturbing in voice or continuous communication systems. But, in data transmission systems when the noise pulse has high amplitude, so that it can become comparable in magnitude with the threshold level, it can eliminate or add bits of information which are not present in the original signal. If the impulses occur very frequently, then they can become disturbing even in voice communication.

In PCM systems, even when impulse noise is not present, there is an inherent noise existing. This is quantization noise and it is caused by the random difference between the actual waveform and the quantized approximation. It is clear that quantized noise may be minimized by using as many quantized steps as possible. But this decrease in the noise is accomplished at the expense of increasing the number of code pulses required to transmit a signal sample.

In Chapter 3, an example of continuous wave interference is discussed where the sky wave is scattered in the ionosphere. Scattering of waves is caused by the

inhomogeneities of index of refraction which results from the turbulence in the atmosphere. The scattering becomes more pronounced when the wavelength of the signal is of the same order of magnitude as the dimensions of the atmospheric "blobs".[3] When the scattered signal, as it is picked up by the receiving antenna, interferes with the ground wave, the result is fading. In other words the resultant wave has fluctuating parameters. Due to the fact that this fluctuation is not Gaussian, it is discussed in this presentation.

Atmospheric noise, like impulse noise, often has impulsive character in the time domain. Atmospheric noise is caused by lightning flashes, which radiate electromagnetic energy. It may produce an electric field in the ionosphere much greater than 0.1 v/m and a magnetic field which is comparable to that of earth's.[4] These fields may, in addition to the direct interference, play an important part in the mechanism of reflection of signals from the ionosphere, at points close to the flash. This is so because absorption characteristics are altered by the presence of the fields. Ultraviolet light, which causes ionization in the ionosphere, also causes a change in the absorption characteristics of that medium. This is shown in Figure 1.[5]

It is obvious from Figure 1 that at night, when electron density is less, absorption is less. Thus, atmospheric noise by being subject to propogation conditions just like any other EM wave, causes more direct interference at night. This is because noise can propogate long distances without being absorbed by the ionosphere. However, at higher frequencies this daily situation is somewhat reversed. At high frequencies, the ionosphere will support propogation only during the day-

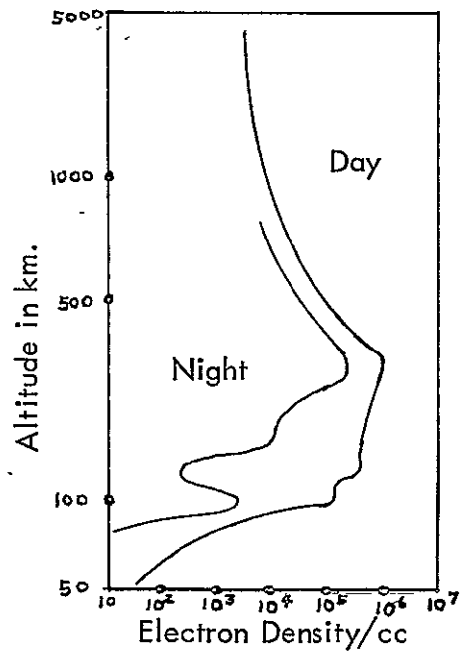

FIGURE 1. Electron density in the ionosphere

light hours. At night when absorption is low, the wave penetrates. In addition to the daily cycles there are seasonal variations in the frequency of occurrence of thunderstorms, which is a common experience witnessed by everyone.

Chapter 1

IMPULSE NOISE

Errors in data transmission do not follow the laws of purely random events.[6] That is, Gaussian noise, which is purely random, does not describe the random process in this case. A characteristic of a Gaussian noise process is that it tends to deliver energy at a uniform rate.[7,8] But a noise process delivers energy at a uniform rate if its standard deviation σ is very small. Obviously, impulse noise, as its name suggests, is impulsive in its character. This means that deviations in amplitude from the mean are highly probable or σ is large. Then, it does have less uniform noise power and the process is considerably different from the Gaussian.

One way to describe errors is to find their amplitude probability density function. Another way is to find the error occurrences in the time domain. Error occurrences can be described either by a distribution of error rates measured in small fixed time intervals or by a distribution of inter-error spacings. In any of the cases different distribution laws are obtained for short-duration and long-duration tests.[9] However, qualitative descriptions of errors in telephone circuits show that they appear to be comprised of bursts of errors, in fact, bursts of bursts of errors in addition to single, independent error events.[10]

The distribution for the amplitudes of impulse noise follows an empirical higher order hyperbolic law.[11]

Namely,

$$p(n) = k/(n+b)^{m+1} \tag{1.1}$$

where

(n) is the amplitude of impulse noise,

(k) is the constant to be determined,

(m) is the order of the hyperbolic equation,

and (b) is an added small constant (bias) to keep the frequency of occurrence of smallest amplitudes finite.

If P (n) is defined as the cumulative probability of amplitude being n or greater, then

$$P(n) = \int_n^{\infty} k/(x+b)^{m+1} \, dx$$

$$= k/m\,(n+b)^m \tag{1.2}$$

k in (1.2) can be determined by using the fact that $\int_o^{\infty} k/(n+b)^{m+1} = 1$. Then let $n \longrightarrow o$ in (1.2) which gives k to be

$$k = mb^m$$

Equation (1.1) becomes

$$p(n) = mb^m/(n+b)^{m+1} \tag{1.3}$$

and (1.2) becomes

$$P(n) = b^m/(n+b)^m \tag{1.4}$$

The average of n is easily obtained by

$$\bar{n} = \int_o^{\infty} n\,p(n)\,d\,n$$

$$= \int_o^{\infty} m\,b^m\,n/(n+b)^{m+1}\,d\,n$$

$$= b/m - 1 \tag{1.4.1}$$

similarly

$$n_{rms}^2 = \int_0^{\infty} n^2 \, p(n) \, d\, n$$
$$= 2\, b^2 / \left[(m - 1)\, (m - 2) \right] \qquad (1.5)$$

It is observed from (1.5) that for m = 1 or m = 2, n_{rms}^2 is not finite. Since p (n) in (1.1) is higher order hyperbolic, i.e. $m \neq 1$, then $m > 2$ is necessary. In fact, $2 < m \leq 5$ is the range of m.[12]

The plots of the cumulative distributions of the amplitudes of impulse noise for different m = 3, 4, 5 are shown and compared with the Gaussian and Rayleigh distributions in Figure 1.1.[13] Bias b = 1 is assumed.

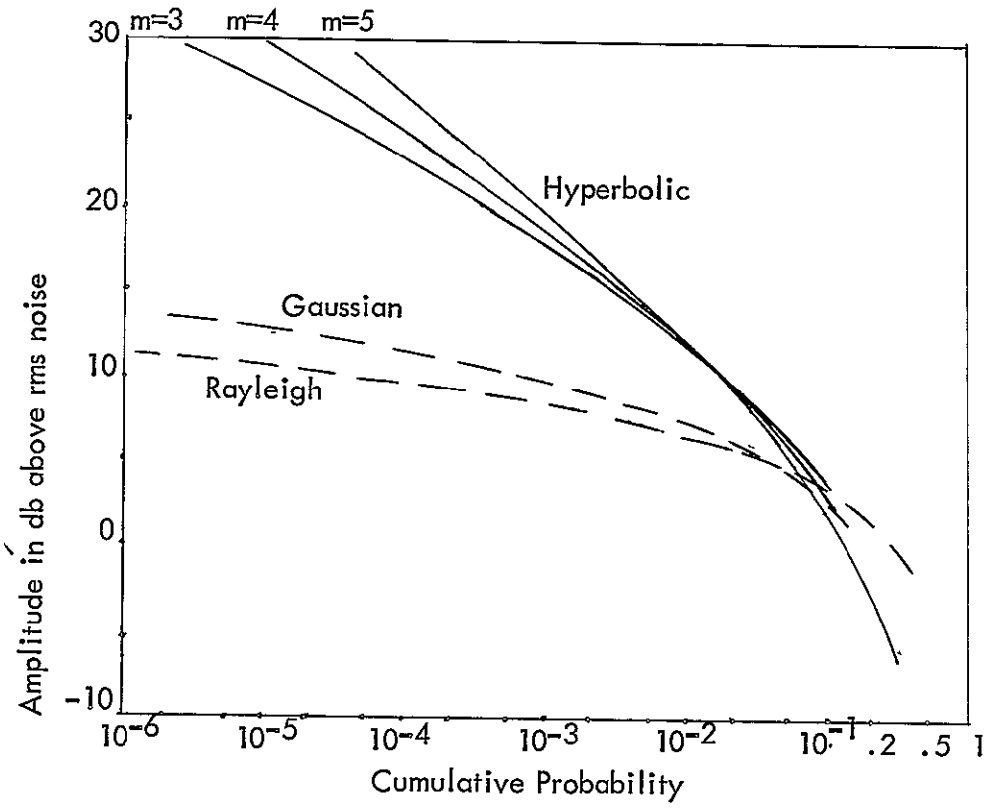

FIGURE. 1.1. Amplitude distribution of impulse noise

It is seen from Figure 1.1 that at low probabilities, the hyperbolic distributions show much larger amplitudes than do the Gaussian or Rayleigh distributions. Also, the root mean square value for the hyperbolic distribution is meaningless if b and m are not known, whereas in the Gaussian distribution the root mean square value is obtained directly by measurements of the amplitudes.

It is proposed that the distribution of error rates, measured in long duration tests follow the first order hyperbolic law.[14] That is, putting m = 1 in Equations (1.3) and (1.4) gives

$$p\,(n) = b/(n + b)^2 \tag{1.6}$$

and

$$p\,(n) = b/(n + b) \tag{1.7}$$

It is observed from (1.4.1) that n has no finite long-time average for m = 1. However, by running the integration to a finite boundary it is possible to derive a long time average. Thus

$$\bar{n}_a = \int_0^a n\,p\,(n)\,d\,n, \tag{1.8}$$

where subscript "a" in $\bar{n}_a$ denotes the upper limit of integration. Equations (1.6) and (1.8) give

$$\bar{n}_a = \int_0^a \left[n\,b/(n + b)^2\right] d\,n \tag{1.9}$$

Let v = n + b and y = a + b in (1.9). Then

$$\bar{n}_a = b\int_b^y (1/v)\,d\,v - b^2\int_b^y (1/v^2)\,d\,v \tag{1.10}$$

or

$$\bar{n}_a = b\left[\ln\,(y/b) - 1 + (b/y)\right]$$

Let q express a quantile boundary. That is, the upper limit of the integration in (1.10) is y = qb.

Then

$$\bar{n}_a = b\left[\ln q - 1 + 1/q\right]$$

or

$$\bar{n}_a = b\left[f(q)\right] \tag{1.11}$$

where $f(q) = \ln q - 1 + 1/q$ and it is a numeric which relates the long-time average $\bar{n}_a$ to the bias b. Now, (n) is no longer continuous by definition. Then it is possible to find the probability of exactly (e) events having the long time average $\bar{n}_a$.

That is,

$$\begin{aligned} p\left[e, \bar{n}_a\right] &= \int_e^{e+1} b/(n+b)^2 \, dn \\ &= b\left[1/(e+b) - 1/(e+b+1)\right] \\ &= b/\left[(e+b)(e+b+1)\right] \end{aligned} \tag{1.12}$$

Replacing the value of b in (1.12) from (1.11) gives $p\left[e, \bar{n}_a\right]$ in terms of the long-time average $\bar{n}_a$. Thus

$$p\left[e, \bar{n}_a\right] = \bar{n}_a(f)/\left[(ef + \bar{n}_a)(ef + f + \bar{n}_a)\right]$$

and let e = 0, 1, 2 etc.

$$p\left[0, \bar{n}_a\right] = f/(f + \bar{n}_a)$$

$$p\left[1, \bar{n}_a\right] = \bar{n}_a(f)/\left[(f + \bar{n}_a)(2f + \bar{n}_a)\right]$$

$$p\left[2, \bar{n}_a\right] = \bar{n}_a(f)/\left[(2f + \bar{n}_a)(3f + \bar{n}_a)\right], \text{ etc.}$$

These probabilities for different e, have been plotted in Figure 1.2, with $q = 1000$.[15]

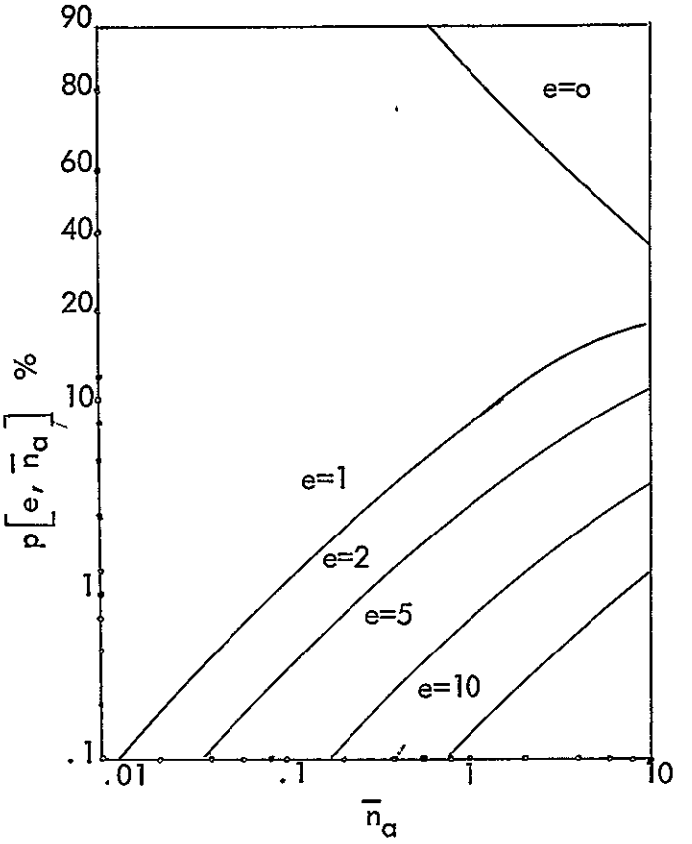

FIGURE 1.2. Error density distribution

To find the probability of at least e events having the long-time average $\bar{n}_a$ one proceeds as follows:

$$P(e, \bar{n}_a = \int_e^{\infty} b/(n+b)^2 \, d\, n$$

$$= b/(e+b)$$

and again using (1.11) one gets

$$P(e, \bar{n}_a) = \bar{n}_a / (ef + \bar{n}_a)$$

Plots for q = 1000 and different e are shown in Figure 1.3 and compared with the Poisson distribution.[16]

It is observed from Figure 1.3 that Poisson probabilities are higher than the hyperbolic toward the larger $\bar{n}_a$.

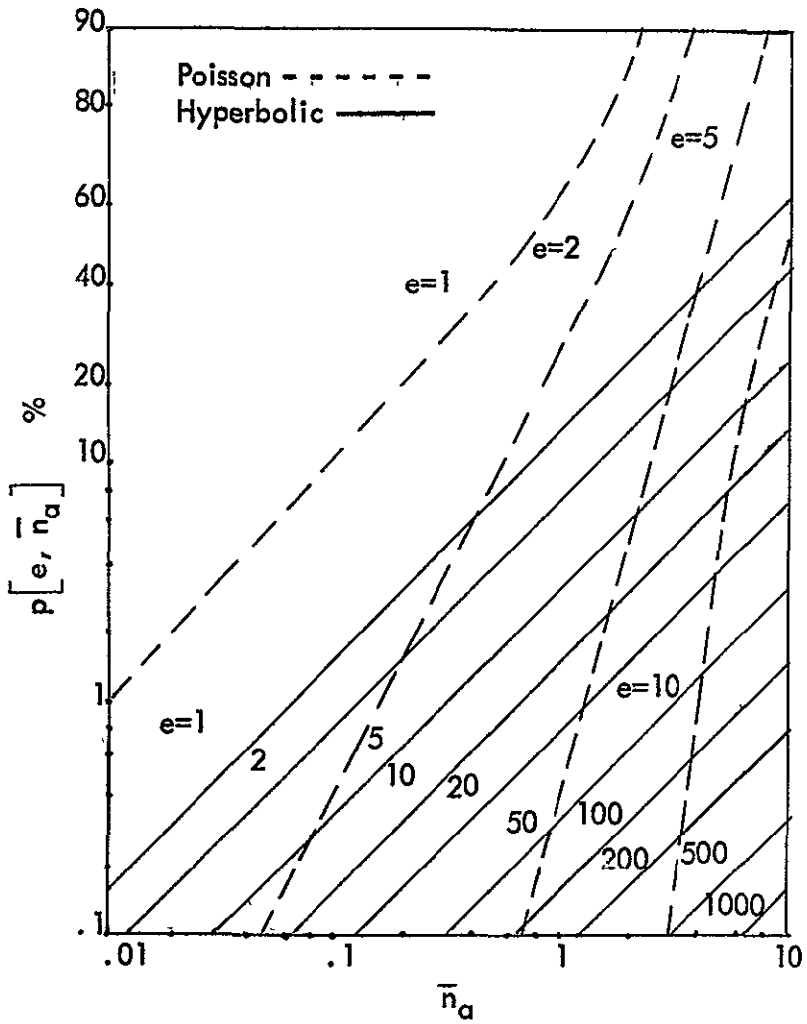

FIGURE 1.3. Cumulative error density distribution

Now, it is of interest to find the cumulated number of error-free intervals having lengths (u) or greater. u is the interval duration with the same units as that of the test time T.

Consider Equation (1.12); here the bias b has a fixed value. However, the amount of bias necessary is proportional to the interval duration u, i.e., $b = b_1 u$. Then letting e = o and replacing b by $b_1 u$ in (1.12) one obtains

$$p(\bar{n}_a, o, u) = 1/(1 + b_1 u)$$

or if $b_o = 1/b_1$

$$p(\bar{n}_a, o, u) = b_o/(b_o + u) \qquad (1.13)$$

Now, $p(\bar{n}_a, o, u)$ represents the fraction of the total number of intervals of duration u in the test that contain no events. If u is allowed to vary, then the total time $d\tau$ devoted to event-free intervals of length between u and u+du is given by

$$d\tau = -T\left[d/du\ p(\bar{n}_a, o, u)\right]du \tag{1.14}$$

where T is the total test time. The negative sign is used in (1.14) because p diminishes as u is increased. From (1.13) and (1.14) one gets

$$d\tau = \left[T\ b_o/(b_o + u)^2\right]du,$$

or

$$d\tau/u = (1/u)\left[T\ b_o/(b_o + u)^2\right]du\ .$$

$d\tau/u$ is the number of error-free intervals in du. Thus, when u varies from u to infinity the cumulated number of intervals M(u) is given by

$$M(u) = \int_u^{\infty}(1/v)\left[T\ b_o/(b_o + v)^2\right]d\,v$$

$$M(u) = T\left[-\frac{1}{b_o+u} + \frac{1}{b_o}\ \ln\ \frac{b_o + u}{u}\right]\ .$$

Thus far only the experimental results obtained from long-term tests were stated. It is now of interest to find the inter-error spacings for short-time tests. The distribution describing this case is called the Pareto distribution.[17] The Pareto distribution is a hyperbolic distribution without bias and of order m, where m is a parameter varying with the overall density of error incidence.. Namely,

$$Q(u) = u^{-m} \tag{1.15}$$

where Q(u) is the probability of an interval of at least u. The interval from one error to the next is u. Distribution (1.15) plotted with logarithmic coordinates is shown in Figure 1.4.

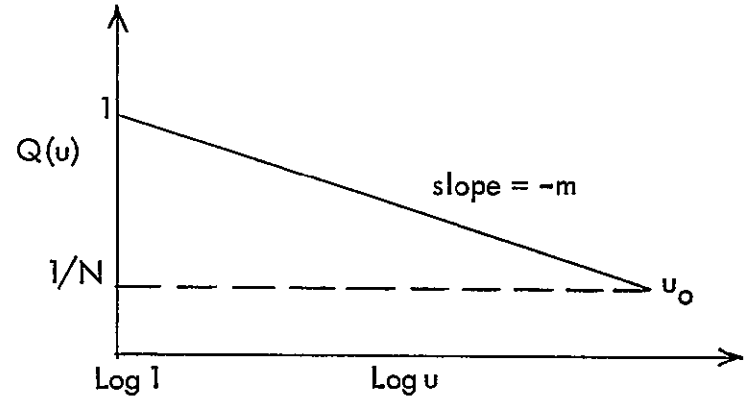

FIGURE 1.4. Logarithmic plot of interval distribution

If the number of errors in the test is N, then the probability of the single longest error u_o is

$$Q(u_o) = 1/N \ .$$

This is shown in Figure 1.4.

Since there are N errors and therefore assuming $N-1 \approx N$ intervals one can redraw Figure 1.4 in a linear scale with N discrete steps. Let the ordinate Q(u) be replaced by NQ(u). Then each unit step in the scale represents one interval between errors. This is shown in Figure 1.5.

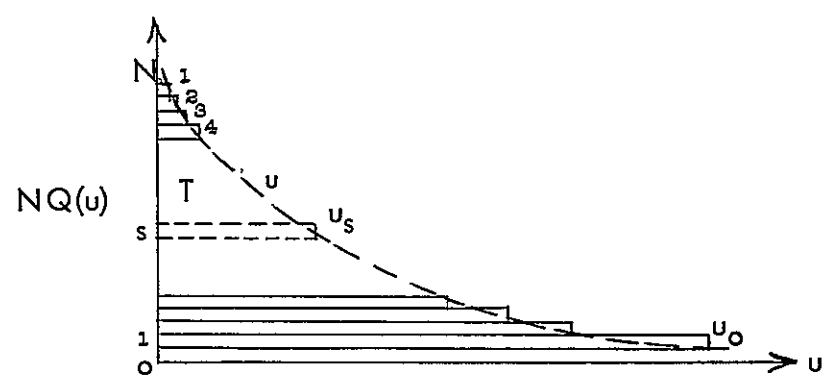

FIGURE 1.5. Linear plot of interval distribution

In Figure 1.5 the sth interval u_s has the probability

$$Q(u_s) = s/N \tag{1.16}$$

Equation (1.15) also gives $Q(u_s)$ as,

$$Q(u_s) = u_s^{-m} \tag{1.17}$$

Then from (1.16) and (1.17)

$$u_s = (N/s)^{1/m} \tag{1.18}$$

Considering Figure 1.5, the total test duration T is equal to the sum of all the intervals u.

Thus

$$T = u_1 + u_2 + u_3 + \ldots + u_s \ldots + u_o \ .$$

Then making use of Equation (1.18)

$$T = N^{1/m}\left[(1/1)^{1/m} + (1/2)^{1/m} + \ldots (1/s)^{1/m} \ldots + (1/N)^{1/m}\right] \tag{1.19}$$

The expression in the brackets in (1.19) is called the zeta function.[18]

$$Z(N, p) = \sum_{s=1}^{N} (1/s)^p \tag{1.20}$$

where $p = 1/m$.

Tables and graphs are given for the zeta function (see Appendix 1.1).[19] Now, once $Z(N, p)$ is computed the test duration T can be found by

$$T = N^{1/m} Z(N, 1/m) \ .$$

Now consider an ensemble of tests, each of duration T bits, but having a different number of errors N_i. In general, the longest spacing u_{o_i} in each test will be shorter when there are more errors or longer when there are less errors. This is illustrated in Figure 1.6.

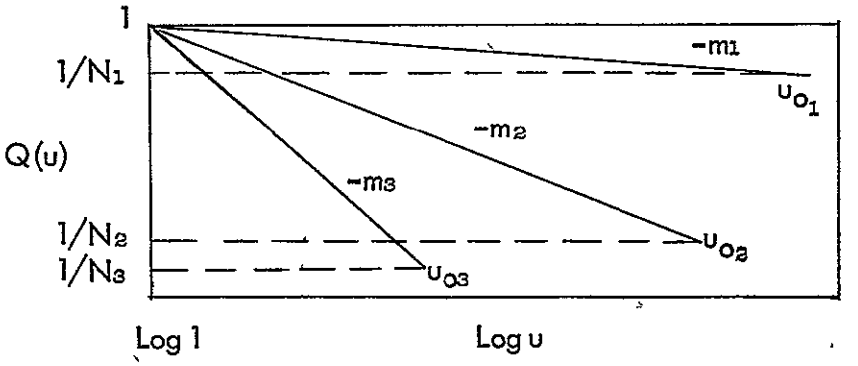

FIGURE 1.6. Variation of maximum interval (u_o)

It is also seen that the magnitude of the slopes ($-m_i$) increases with the increasing number of errors N_i. A better plot showing the relation between the slopes m and number of errors N for a given test time T is shown in Figure 1.7.[20]

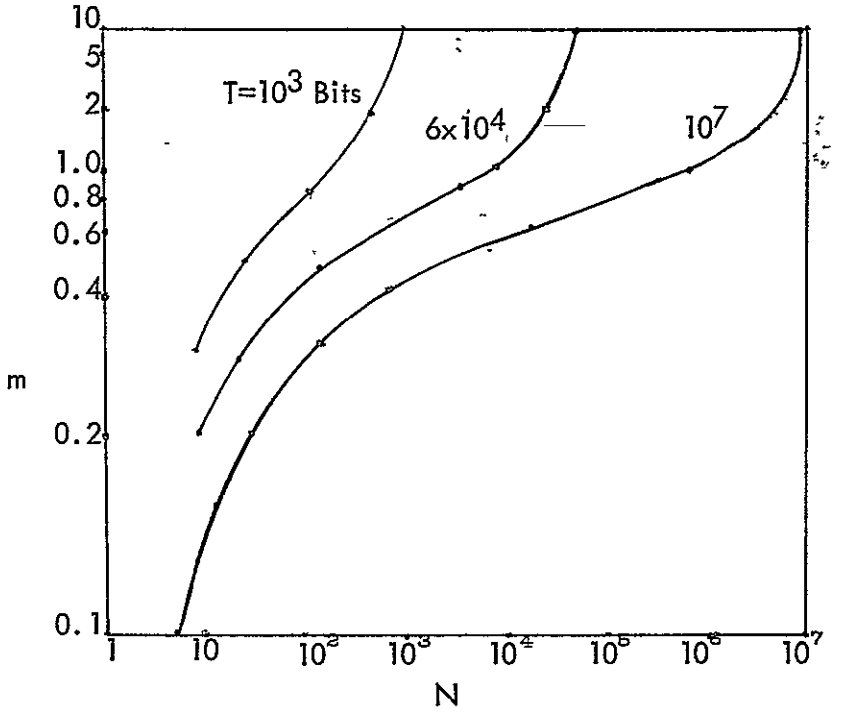

FIGURE 1.7. Variation of order (m) with error incidence

If one determines m from Figure 1.7 for a given number of errors N and test time T, then it is possible to obtain another plot showing maximum inter-error interval u_o. This is shown in Figure 1.8.

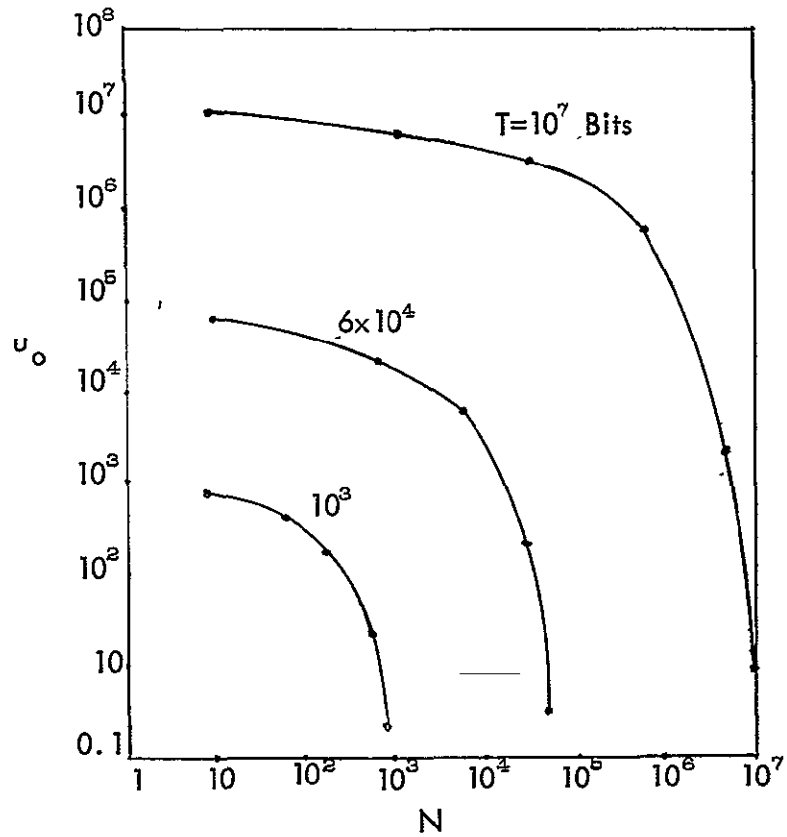

FIGURE 1.8. Variation of Maximum interval with error incidence

It is observed from Figure 1.8 that the maximum intervals show a faster drop over a narrow range of errors. However, it is noted that this rate of fall is independent of test duration time T.

Chapter 2

QUANTIZATION NOISE

Quantization noise is a form of distortion due to "rounding-off" or quantization of a continuous signal into discrete steps. However, the quantization process allows digital encoding which is a very desirable data transmission technique because of its ability to combat the effects of noise. In PCM, a major source of error is due to quantization.

The typical quantized transmission scheme is shown in Figure 2.1.[21]

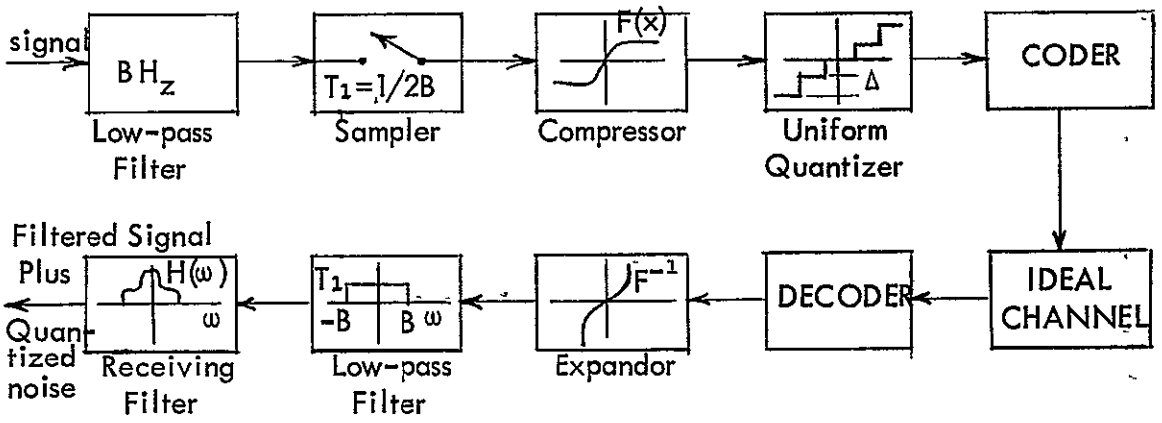

FIGURE 2.1. Quantized transmission scheme

Let the signal to be sampled and quantized be represented by s(t) and the sampling wave as $\sum_k \delta\ (t - k\,T_1 - \tau)$.

This is shown in Figure 2.2.

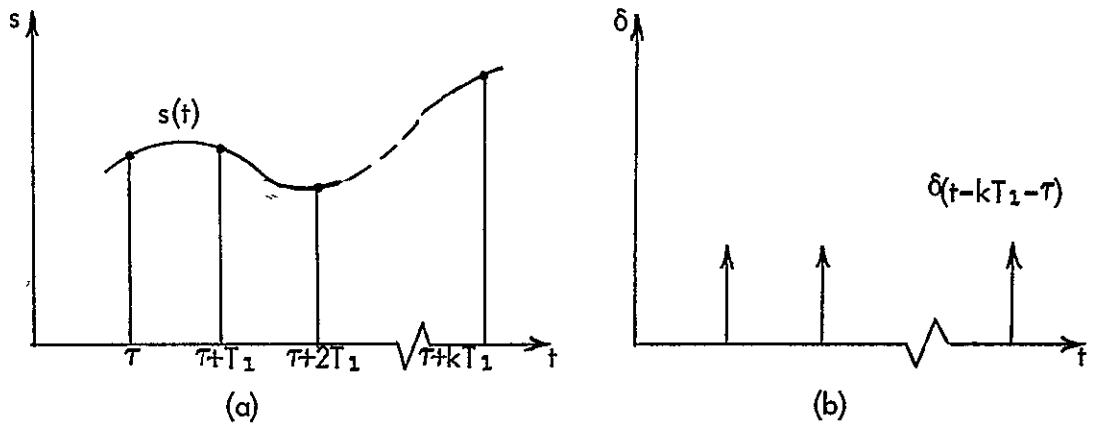

FIGURE 2.2. (a) Sampling of the signal
(b) Impulse train

τ is the timing phase and it is uniformly distributed over the interval $0 \leq \tau \leq T_1$.[22]

s (t) can be represented with the pulse trains after sampling, compression, quantization and expansion, in that order. Thus, if $s(kT_1 + \tau)$ is the sample value of s (t) at time $kT_1 + \tau$, then one obtains

$\sum_k s(kT_1 + \tau)\, \delta(t - kT_1 - \tau)$ as the sampled signal,

$\sum_k s_{comp}(kT_1 + \tau)\, \delta(t - kT_1 - \tau)$ as the sampled and compressed signal,

$\sum_k \hat{s}_{comp}(kT_1 + \tau)\, \delta(t - kT_1 - \tau)$ as the sampled, compressed and quantized signal, and

$\sum_k \left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp} \delta(t - kT_1 - \tau)$ as the sampled, compressed, quantized and expanded signal.

If F(s) is the compandor (in this case compression) characteristics, then $F(s) = F\left[s(kT_1 + \tau)\right] = s_{comp}(kT_1 + \tau)$ is the compressed sample value and $\hat{s}_{comp}(t)$ is one of the $(2^n - 1)$ quantized levels that the quantizer output approximates for

$s_{comp}(t)$. Similarly $F^{-1}\left[\hat{s}_{comp}(kT_1 + \tau)\right] = \left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp}$

Note that from Figure 2.1 the signal entering the expandor is already quantized.

At the input of the low-pass reconstruction filter the impulse associated with time $t = kT_1 + \tau$ is

$$\left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp} \delta\,(t - kT_1 - \tau)$$

and the area of this impulse can be expressed as

$$\left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp} = s(kT_1 + \tau) + \epsilon\,(kT_1 + \tau)\,. \tag{2.1}$$

$\epsilon(kT_1 + \tau)$ in Equation (2.1) is the quantization error defined in an interval $(-\Delta/2) \leq \epsilon(kT_1 + \tau) \leq \Delta/2$ as shown in Figure 2.3.

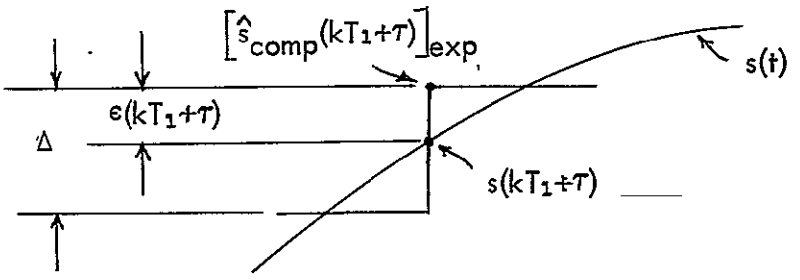

FIGURE 2.3. Error in quantization

$\epsilon(kT_1 + \tau)$ is also the sample value of $\epsilon(t)$ at $t_k = kT_1 + \tau$.

When companding is present the quantization step size Δ varies according to the companding characteristics $F(s)$. But since $s = s(t)$, a function of time, then $F(s)$ and Δ would be functions of time. For $\Delta(t)$ one has

$$\Delta(t) = \frac{\Delta_{uniform}}{F'\ s(t)}$$

where $F'[s(t)]$, in this case, is the derivative of the compressor characteristics and is evaluated at that input amplitude of the signal at the time of the sampling.[23]

To find the quantization noise $\epsilon(t)$ at the output of the reconstruction filter one proceeds as follows. Let

$$\epsilon(t) = \left[\hat{s}_{comp}(t)\right]_{exp} - s(t) \tag{2.2}$$

where $\left[\hat{s}_{comp}(t)\right]_{exp}$ is a compressed, quantized and expanded continuous signal, i.e. defined at any time t. This signal is obtained by passing $\sum_k \left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp} \delta(t - kT_1 - \tau)$ through the low pass filter with the transfer function $T_1 G_{2B}(w)$, where G_{2B} is the gate function with bandwidth 2B. Thus, one obtains (see Appendix 2.1):

$$\left[\hat{s}_{comp}(t)\right]_{exp} = \sum_k \left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp} Sa\left[B(t - kT_1 - \tau)\right] \tag{2.3}$$

Similarly s(t) is obtained by reconstructing it from its samples $s(kT_1 + \tau)$.

$$s(t) = \sum_k s(kT_1 + \tau)\, Sa\left[B(T - kT_1 - \tau)\right]. \tag{2.4}$$

In Equations (2.3) and (2.4) $Sa\left[B(t - kT_1 - \tau)\right]$ is equal to $\left[\sin B(t - kT_1 - \tau)\right] / B(t - kT_1 - \tau)$ and is called the sampling function. Now, replacing $\left[\hat{s}_{comp}(t)\right]_{exp}$ and s(t) in (2.2) by their equals in (2.3) and (2.4) one obtains

$$\epsilon(t) = \sum_k \left\{\left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp} - s(kT_1 + \tau)\right\} Sa\, B(t - kT_1 - \tau) \tag{2.5}$$

or by making use of Equation (2.1)

$$\epsilon(t) = \sum_k \epsilon(kT_1 + \tau)\; Sa\left[B(t - kT_1 - \tau)\right]. \tag{2.5.1}$$

From Equation (2.5) one can obtain $\overline{\epsilon^2(t)}$ (see Appendix 2.2) to be

$$\overline{\epsilon^2(t)} = \frac{1}{2f_s T} \sum_k \left\{\left[\hat{s}_{comp}(kT_1 + \tau)\right]_{exp} - s(kT_1 + \tau)\right\}^2 \tag{2.6}$$

Now, T is the interval over which averaging is performed and f_s is the signal frequency. But, since according to the sampling theorem the minimum sampling rate must be $2f_s$, then $2f_sT$ is the total number of samples in the interval T.

Therefore the right hand side of Equation (2.6) is nothing but the mean square value of the samples at times $t_k = kT_1 + \tau$. Therefore,

$$\overline{\epsilon^2(t)} = \overline{\epsilon^2(kT_1 + \tau)} . \tag{2.7}$$

It is seen from the equality in (2.7) that the problem of finding $\overline{\epsilon^2(t)}$ is merely reduced to finding $\overline{\epsilon^2(kT_1 + \tau)}$ of the band limited signal s(t). This can be found from the following argument.[24]

The quantized levels are Δ volts apart and $\epsilon(kT_1 + \tau)$ must lie in the range $\left[-\Delta/2, \Delta/2\right]$, where the midpoint of quantized interval is taken as reference. The amplitude distribution of the signal s(t) is assumed to be uniform in the range $\left[o, (M-1)\Delta\right]$, where M is the number of quantization levels. Thus, the distribution of $\epsilon(kT_1 + \tau)$ will also be uniform in the range $\left[-\Delta/2, \Delta/2\right]$. The probability density function for $\epsilon\left[kT_1 + \tau\right]$ would be,

$$p\left[\epsilon(kT_1 + \tau)\right] = \begin{cases} 1/\Delta & , \left[-\Delta/2, \Delta/2\right] \\ o & , \text{ otherwise.} \end{cases}$$

Then, the mean square value of $\epsilon(kT_1 + \tau)$ can be found by

$$\begin{aligned} \overline{\epsilon^2(kT_1 + \tau)} &= \int \epsilon^2(kT_1 + \tau)\ p\left[\epsilon(kT_1 + \tau)\right] d\left[\epsilon(kT_1 + \tau)\right] \\ &= \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \epsilon^2(kT_1 + \tau)\, d\left[\epsilon(kT_1 + \tau)\right] \\ &= \Delta^2/12 \end{aligned}$$

Therefore, from Equation (2.7)

$$\overline{\epsilon^2(t)} = \Delta^2/12 \tag{2.8}$$

Equation (2.8) represents the quantization noise power at the output of the reconstruction low-pass filter.

The upper bound for this error can be found by applying the bounding technique of Chernoff.[5] This bounding technique states that if a probability density function p (x) has a moment generating function M (v), then

$$\int_a^{\infty} p(x)\, dx \leq M(v)\, e^{-va} \tag{2.9}$$

where a is a constant.

Now, $\epsilon(t)$ as defined in Equation (2.5.1) is the sum of uniformly distributed functions. Thus, applying the Chernoff bounding technique to uniformly distributed functions one gets

$$\begin{aligned} \left[M(v) \right]_{unif} &= \left[\sinh v(\Delta/2)\right]/v(\Delta/2) \\ &= \sum_{n=o}^{\infty} \left[v(\Delta/2)\right]^{2n} \left[1/(2n+1)!\, \right] \end{aligned} \tag{2.10}$$

One can find the characteristic function of the sum by obtaining the product of the characteristic functions of the individual sum terms.

Instead of going ahead and applying this statement to uniformly distributed variables an interesting fact will be derived. This fact is that the quantization noise power $\overline{\epsilon^2(t)}$ is also upper bounded by $G(v)e^{-va}$, where G(v) is the characteristic function of the Gaussian process. For this to be true one must have $G(v) \geq \left[M(v)\right]_{unif}$. In fact this is true (see Appendix 2.3).

$$G(v) = \exp\left[mv + v^2\sigma^2/2\right] \tag{2.11}$$

where m, σ are the mean and rms values respectively. However, since $\epsilon(t)$ was chosen to have zero mean and variance equal to $\Delta^2/12$, then in accordance with this one has $m = o$, $\sigma^2 = \Delta^2/12$ for the Gaussian process. This statement is justified since one is directly comparing $\epsilon(t)$ and the corresponding Gaussian variable. Therefore from (2.11):

$$G(v) = \exp\left[(\Delta^2/12)\ (v^2/2)\right] \tag{2.12}$$

One can now find the characteristic function of the sum by

$$G_s(v) = \prod_{i=o}^{k} G_i(v)$$

where the subscript s in $G_s(v)$ stands for "sum", and from (2.12)

$$G_s(v) = \exp\left[(k/2)(\Delta^2/12)\ v^2\right]$$

Then

$$G_s(v)\ e^{-va} = \exp\left[(k/2)\ (\Delta^2/12)\ v^2 - av\right] \tag{2.13}$$

In Equation (2.13) v can be chosen such that this upper bound is minimized. Thus, this value of v is found to be (see Appendix 2.4):

$$v = (a/k)\ (12/\Delta^2) \tag{2.14}$$

Placing the value of v in (2.13) from (2.14) one gets

$$\left[G_s(v)\ e^{-va}\right]_{min} = \exp\left[-\ (a^2/2k)\ (12/\Delta^2)\right]$$

The final result is obtained from (2.9) by placing $\exp\left[-(a^2/2k)(12/\Delta^2)\right]$ for $M(v)\ e^{-av}$. Then,

$$P\left[\epsilon(t) > a\right] \leq \exp\left[-\ 6a^2/k\Delta^2\right].$$

Chapter 3

AN EXAMPLE OF CW INTERFERENCE

Continuous wave interference is in contrast to other types of noise discussed, because they were impulsive and/or discrete time domain occurrences. In addition to this, it is assumed CW does not originate from a random source, but from a transmitter which transmits a deterministic continuous carrier wave.

However, an uncertainty is witnessed at the receiving end. This is, among other factors, due to scattering of the transmitted wave which takes different propogation paths to reach the receiving antenna. The probability density functions obtained to describe this uncertainty are very much the same as for the atmospheric noise.[26] This is the main reason CW interference is discussed here along with other non-Gaussian disturbances. However, CW interference differs from atmospheric noise, in addition to the differences described above, in that it does not add noise power to the signal.

A probability density function for the resultant wave is to be found. The resultant wave is obtained by the sum of the direct wave and the wave reflected from the ionosphere. The direct wave will have amplitudes defined by[27]

$$S_o = \exp\left[- \sum_{j=1}^{l} \delta_j d_j \right]. \tag{3.1}$$

This is the equation describing the attenuation of a wave due to the penetration of a wall or other obstacles where

d_j is the thickness of the jth obstacle

δ_j is the attenuation caused by that obstacle.

The distribution of $\sum_{j=1}^{J} \delta_j d_j$ is found by applying the central limit theorem and it is Gaussian.[28] By a transformation of variables using (3.1) one obtains the probability density function for the amplitudes S_o (see Appendix 4.2)

$$p(S_o) = \frac{1}{S_o \sigma \sqrt{2\pi}} \exp\left[-\frac{(\ln S_o - m)^2}{2\sigma^2}\right]. \tag{3.1.1}$$

This is called the lognormal distribution. The phase of the direct wave is assumed to be uniformly distributed at the receiving end.[29]

The sky wave is scattered at the ionosphere before it is reflected back to earth's surface. This well-known scattering problem will yield a waveform described by a Rayleigh phasor. That is, this wave will have a Rayleigh amplitude distribution and uniform phase distribution.[30] A Rayleigh phasor has Gaussian orthogonal components.[31]

The resultant wave will also have uniform phase distribution due to the fact that one has the sum of uniformly distributed phasors (see Appendix 3.1). The problem to be solved in this chapter is to find the amplitude distribution of this resultant wave. Thus, the problem is stated as follows. Let the received signal be

$$S e^{j\theta} = S_o e^{j\theta_o} + S_1 e^{j\theta_1} \tag{3.2}$$

where

S, S_o, S_1 are the amplitudes of the resultant, direct, and reflected waves respectively. θ, θ_o, θ_1 are phases of these waves. Now, the probability

density function p(S) is to be found. Figure 3.1 shows the phasors in (3.2)

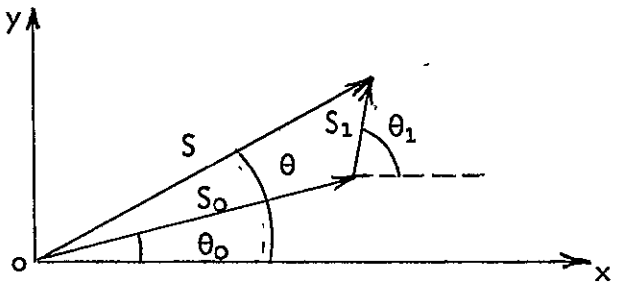

FIGURE 3.1. The random phasors and their resultant (S)

To find the distribution of S, first the conditional distribution $p_c(S) = p(S/S_o)$ is found and then the application of the theorem of total probability yields p(S). Thus, S_o is held constant for the moment. Also, to make the calculations easier θ_o will be assumed a reference phase. This assumption merely rotates x, y to put S_o on the x-axis. Physically this may be achieved by some arrangement of phase-lock system.[32] Then the components of $se^{j\theta}$ are

$$S_x = S_o + S_1 \cos \theta_1 \tag{3.4}$$

$$S_y = S_1 \sin \theta_1 \tag{3.5}$$

It was stated earlier that $S_1 e^{j\theta_1}$ is a Rayleigh phasor and has Gaussian components. Therefore, $S_1 \cos\theta_1$ and $S_1 \sin\theta_1$ are Gaussian. Then,

$p(S_y)$ is Gaussian from (3.5) and

$p(S_x)$ is also Gaussian but shifted to the right by S_o in (3.4).

The joint distribution of S_x and S_y is given by

$$p(S_x, S_y) = \frac{1}{\alpha} \exp\left[\frac{-(S_x - S_o)^2}{\alpha} - \frac{S_y^2}{\alpha} \right], \tag{3.6}$$

where $\alpha = 2\sigma^2$.

Equation (3.6) is transformed into polar coordinates by

$$p_c(S,\theta) = (S)\, p(S_x, S_y)$$

where

$p_c(S,\theta)$ is the conditional probability,

$S_x = S\cos\theta$,

$S_y = S\sin\theta$ and

$S^2 = S_x^2 + S_y^2$.

Thus

$$p_c(S,\theta) = \frac{S}{\pi\alpha} \exp\left[-\frac{(S+S_o)^2}{\alpha} + \frac{(2SS_o\cos\theta)}{\alpha}\right] \tag{3.7}$$

One can now obtain $p_c(s) = p(S/S_o)$ from (3.7) by

$$p_c(S) = \int_o^{2\pi} p_c(S,\theta)\, d\theta$$

$$= \frac{S}{\pi\alpha} e^{-(S^2+S_o^2)/\alpha} \int_o^{2\pi} e^{(2SS_o/\alpha)\cos\theta}\, d\theta. \tag{3.8}$$

But $\int_o^{2\pi} e^{(2SS_o/\alpha)\cos\theta}\, d\theta = I_o\left(\frac{2SS_o}{\alpha}\right)$.

$I_o(x)$ is modified Bessel function of x.

Then equation (3.8) becomes

$$p_c(S) = \frac{2S}{\alpha}\left[e^{-(S^2+S_o^2)/\alpha}\right] I_o\left(\frac{2SS_o}{\alpha}\right). \tag{3.9}$$

Applying the theorem of total probability to (3.9) one gets

$$p(S) = \int_o^{\infty} p_c(S)\, p(S_o)\, dS_o \tag{3.10}$$

or by replacing $p_c(S)$ from (3.8) and $p(S_o)$ from (3.1.1)

$$p(S) = \frac{2S\sqrt{2}}{\alpha\sqrt{\alpha\pi}} \int_o^{\infty} \frac{1}{S_o} \exp\left\{-\frac{\left[S^2+S_o^2+(\ln S_o - m)^2\right]}{\alpha}\right\} I_o\left(\frac{2SS_o}{\alpha}\right) dS_o$$

The evaluation of this integral is difficult.[33] However, the following practical results are obtained:

Case 1.

If S and S_o are much smaller than $\sqrt{\alpha}$, the rms value of S, then $\left[e^{-S_o^2/\alpha}\right] I_o\left(\frac{2SS_o}{\alpha}\right)$ approaches unity. Thus, from Equations (3.9) and (3.10) one gets

$$p(S) = \frac{2S}{\alpha} e^{-S^2/\alpha} \int_o^\infty p(S_o)\, dS_o$$

$$p(S) = \frac{2S}{\alpha} e^{-S^2/\alpha} \text{ if } S_o > 0 \text{ for all time.}$$

This is the Rayleigh distribution.

Case 2.

A somewhat more difficult argument is followed to obtain p(S) when $S \gg \sqrt{\alpha}$.[34] The result is that, (3.10) approaches $p(S_o)$ given by (3.1.1). In other words, p(S) becomes lognormally distributed.

Chapter 4

ATMOSPHERIC NOISE

The lightning discharges in storms all over the world are the main cause of atmospheric noise. The model that will be proposed here does conform with the experimental results plotted in Figure 4.1.[35]

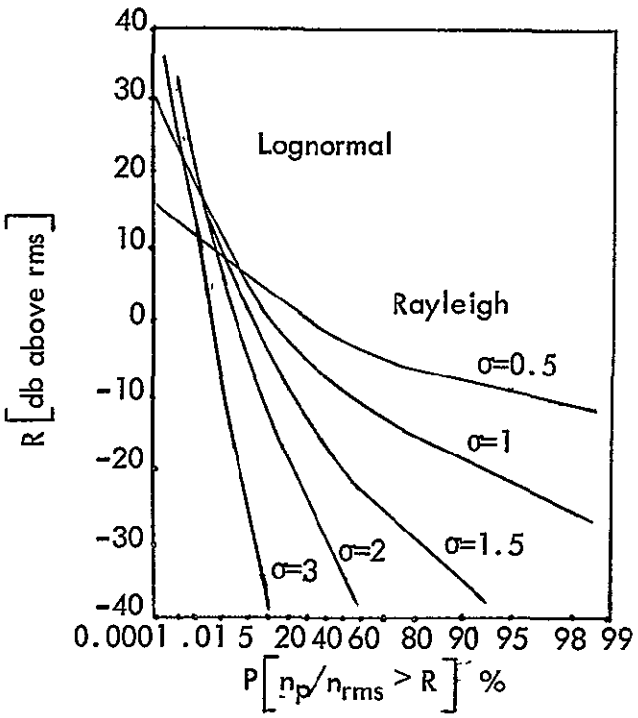

FIGURE 4.1. Amplitude distribution of atmospheric noise plotted on Rayleigh paper.

Distributions approach Rayleigh for small amplitudes (high probabilities) and lognormal for large amplitudes (low probabilities).[36] A lightning discharge is transmitted by radiation and therefore superposition applies at a particular point or time. For example, atmospheric noise is the sum of all the other atmospherics propogated to that point. The model proposed here describes each atmospheric as upsurging and decaying exponentials in time.[37, 38] This model is shown in Figure 4.2.

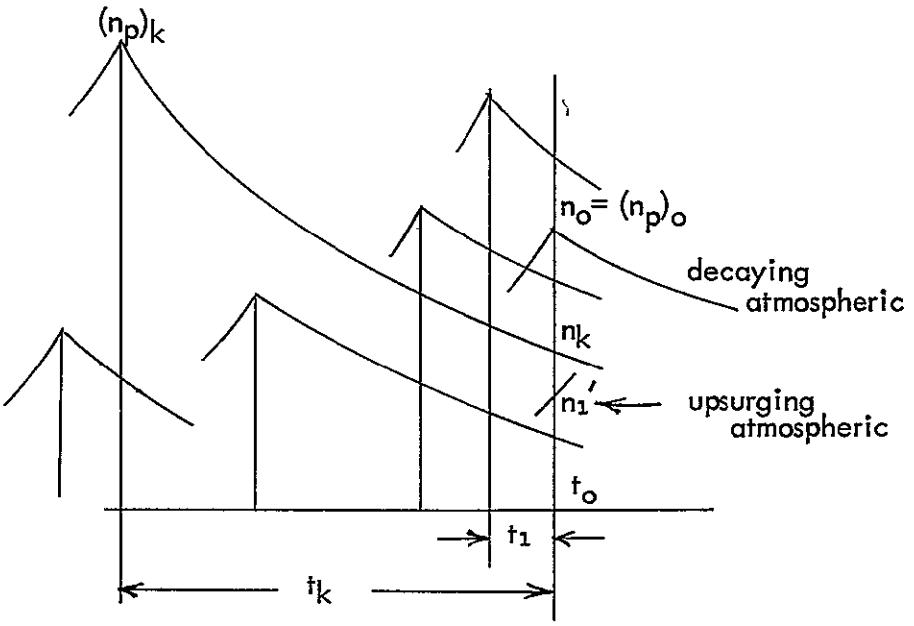

FIGURE 4.2. Random sequence of atmospherics

If $(n_p)_o$ is the peak value of the atmospheric received at time t_o and α is the exponential decay time constant, then the atmospheric $n_o(t)$ is given by

$$n_o(t) = \begin{cases} (n_p)_o \exp\left(\frac{-t - t_o}{\alpha}\right) & \text{for } t > t_o \\ (n_p)_o \exp\left(\frac{t - t_o}{\beta}\right) & \text{for } t < t_o \end{cases} .$$

For $t < t_o$, β is the rise constant.

The amplitude of the kth previous atmospheric n_k at time t_o is given by

$$n_k = (n_p)_k \; e^{-t_k/\alpha} \tag{4.1.1}$$

At a particular time t_o the total atmospheric noise is

$$ne^{j\theta} = n_o e^{j\varphi_o} + \sum_{k=1}^{\infty} n_k e^{j\varphi_k} + \sum_{k=1}^{\infty} (n')_k \, e^{j\varphi_k} \tag{4.1}$$

where (n') is used for upsurging atmospheric.

Since t_k, shown in Figure 4.2, is a random variable, the phases of the atmospherics at the receiving point are also random. Thus one has random phasors. In fact they are uniformly distributed and each component in (4.1) is called a uniformly distributed phasor (UDP).[39]

Now, at any time t the atmospheric noise N(t), when the uprising atmospherics are neglected, is

$$N(t) = n e^{-t/\alpha} \text{ for } t_o < t < t_1, \tag{4.2}$$

Here, it is assumed that there are no new atmospherics occurring in the time interval t, defined in (4.2). The next atmospheric occurs at $t = t_1$. But in this time interval the total amplitude n is fixed by the set of atmospherics that have already occurred. Therefore, N(t) is a deterministic function of time. Conversely, if time is held constant and n varied randomly, then N will have the same distribution as for n, of course differing by a multiplication constant.

The question arises as to what the distribution of N would be if N were to vary randomly, not at a fixed time chosen in the interval $t_o < t < t_1$ but throughout this interval. The distribution in this interval can be found once the conditional distribution at a given time is found. Then the theorem of total probability is

applied to find the general distribution.[40] However, no attempt is made here to find this general distribution because it is very involved. Instead, at a given time the cumulative distribution $P(N/N_{rms} > R)$ is found. Even before finding this, one must find the distribution of n which in turn is determined by the distributions of n_o, n_k and $(n')_k$.

n_o has the same distribution as the distribution of the peak values (n_p) of the atmospherics but differing by a multiplying constant (see Appendix 4.1). The peak values follow the field strength formula

$$n_p = \frac{K\sqrt{P_n}}{d} \exp\left(-\sum_{j=1}^{l} \delta_j d_j\right) , \tag{4.3}$$

where P_n is the peak power at a particular frequency,

d is the distance of discharge from the receiving point, in general

$$d = \sum_{j=1}^{l} d_j ,$$

K is a constant of proportionality, and

δ_j is the path of propogation of jth section.

One can express Equation (4.3) as

$$n_p = e^{\Delta}$$

where $\Delta = -\sum_{j=1}^{l} \delta_j d_j + 1/2 \ln P_n + \ln K - \ln d$. , (4.4)

All quantities in (4.4), except K, are random. Also the fluctuation in the first term of Δ is more dominant. Thus the distribution of Δ would approximate the distribution of $\sum_{j=1}^{l} \delta_j d_j$. However, the central limit theorem applies to $\sum_{j=1}^{l} \delta_j d_j$ which makes it Gaussian. Therefore, Δ is Gaussian as well.

One can now find the probability density function of the peak noise amplitudes as (see Appendix 4.2)

$$p(n_p) = \left[1/(n_p)\ \sigma\sqrt{2\pi}\right] \exp\left\{-\frac{\left[\ln(n_p) - m\right]^2}{2\sigma^2}\right\} \tag{4.5}$$

The distribution described by (4.5) is called lognormal distribution. As was previously stated, $p(n_o)$ differs from this distribution by a multiplying constant.

Now, one must obtain $p(n_k)$. From Equation (4.1.1) one has

$$n_k = u/v \tag{4.6}$$

where $u = (n_p)_k$ and $v = \exp\left[t_k/\alpha\right]$. (4.7)

The distribution of u is given by (4.5). However, before the distribution of v can be found the distribution of t_k must be obtained. One way of finding $p(t_k)$ is to divide t_k into k intervals of length T_i, each having exponential distribution.[41,42] Then, $p(T_i) = \lambda e^{-\lambda t}$ and $t_k = \sum_{i=1}^{k} T_i$ where $i = 1,2,3\ldots k$ and λ is the number of atmospherica per unit time. Now $p(t_k)$ is found by making use of these (see Appendix 4.3), and

$$p(t_k) = \left[\lambda^k t_k^{k-1}\ e^{-\lambda t_k}\right]/(k-1)\,! \quad . \tag{4.8}$$

It is interesting to note that, one could have just as well obtained (4.8) by assuming that the number of atmospherics in a given interval is Poisson distributed. Then conversely, under this assumption, one finds the interval distribution given the number of atmospherics in that interval.[43]

If one proceeds to find p(v) by using (4.7) and (4.8), (See Appendix 4.4), one gets

$$p(v) = \frac{\lambda^k \alpha^k\ (\ln v)^{k-1}}{(k-1)\,!\ v^{\lambda\alpha+1}} \tag{4.8.1}$$

Now, since the distributions of u and v are known, one can find the distribution of n_k from Equation (4.6), (see Appendix 4.5).

$$p(n_k) = \frac{\lambda^k \alpha^k}{(k-1)!\, \sigma n_k \sqrt{2\pi}} \int_0^\infty x^{k-1} \exp\left[-\alpha\lambda x - \frac{(\ln n_k + x - m)^2}{2\sigma^2}\right] dx \quad (4.9)$$

where $\ln v = x$.

Returning to Equation (4.1) one can prove (see Appendix 4.6) that,

$$\langle n^2 \rangle = \langle n_o^2 \rangle + \sum_{k=1}^{\infty} \langle n_k^2 \rangle + \sum_{k=1}^{\infty} \langle (n_k')^2 \rangle. \quad (4.10.1)$$

The mean square of n_k is given by:

$$\langle n_k^2 \rangle = \int_0^\infty n_k^2\, p(n_k)\, d\, n_k \quad (4.10)$$

Placing the equal of $p(n_k)$ in (4.10) from (4.9) one gets

$$\langle n_k^2 \rangle = \int_0^\infty n_k^2 \frac{\lambda^k \alpha^k}{(k-1)!\, \sigma n_k \sqrt{2\pi}} \int_0^\infty x^{k-1} \exp\left[-\alpha\lambda x - \frac{(\ln n_k + x - m)^2}{2\sigma}\right] dx\, dn_k \quad (4.11)$$

When the integral in Equation (4.11) is evaluated (see Appendix 4.7) one gets

$$\langle n_k^2 \rangle = \frac{\lambda^k \alpha^k \exp 2(\sigma^2 + m)}{(\lambda\alpha + 2)^k} \quad (4.12)$$

Also, consider the upsurging atmospherics (n_k') attaining their peak values at a time $t_o + t_k$. The second moment for this is obtained in the same way as that for decaying atmospheric, except that the α time constant is replaced by β.

$$\langle (n_k')^2 \rangle = \frac{\lambda^k \beta^k \exp 2(\sigma^2 + m)}{(\lambda\beta + 2)^k}$$

If one assumes at the time of observation that there are no upsurging atmospherics, then one can neglect $\langle (n_k')^2 \rangle$ in Equation (4.10.1). This assumption is

especially true if the set of atmospherics is occurring away from the observation point. Now, replacing $<n^2>$ in (4.10.1) by its equal in (4.12) one obtains

$$<n^2> = \sum_{k=o}^{\infty} <n_k^2> = e^{2(\sigma^2+m)} \sum_{k=o}^{\infty} \left(\frac{\lambda\alpha}{\lambda\alpha+2}\right)^k . \qquad (4.13)$$

$\sum_{k=o}^{\infty}\left(\frac{\lambda\alpha}{\lambda\alpha+2}\right)^k$ in (4.13) is a converging geometric series which is equal to $\lambda(\alpha/2) + 1$.

Equation (4.13) becomes

$$<n^2> = \left[(\lambda\alpha/2) + 1\right] e^{2(\sigma^2+m)} \qquad (4.14)$$

Case 1

If $\lambda\alpha/2 \gg 1$, then $\lambda\alpha/(\lambda\alpha+2)$ will approach unity and the geometric series will thus tend to converge less rapidly. The first term $<n_o^2>$ in (4.13) would be negligible as compared to the sum.

In other words

$$<n_o^2> \ll \sum_{k=o}^{\infty} <n_k^2> \approx (\lambda\alpha/2)\, e^{2(\sigma^2+m)} \quad . \qquad (4.15)$$

From Appendix 4.6

$$\sum_{k=o}^{\infty} <n_k^2> = <\left(\sum_{k=o}^{\infty} n_k e^{j\varphi_k}\right)^2> . \qquad (4.16)$$

The sum $\sum_{k=o}^{\infty} (n_k e^{j\varphi_k})$ in (4.16) has a Rayleigh amplitude distribution since it represents the sums of uniformly distributed phasors $n_k e^{j\varphi_k}$.[44] Consequently, for $\lambda\alpha/2 \gg 1$, the random variable n will approach the Rayleigh distribution at time t_o. However, experiments have contradicted this result.[45] Therefore, one rejects the possibility $\lambda\alpha/2 \gg 1$ and seeks the possibility $\lambda\alpha/2 \ll 1$.

Case 2

If $\lambda\alpha/2 \ll 1$, then $\sum_{k=0}^{\infty}\left(\frac{\lambda\alpha}{\lambda\alpha+2}\right)^k$ will converge to $2/(2-\lambda\alpha)$. Since the first term is not negligible as compared to $2/(2-\lambda\alpha)$, one cannot neglect any term in the sum (4.13). From Appendix 4.1, the distribution of n_o was found to be lognormal. Also, for the reasons stated in Case 1, $\sum_{k=1}^{\infty} n_k e^{i\varphi_k}$ is still a Rayleigh phasor. Then one has the sum of a lognormal and a Rayleigh phasor, the distribution of which was derived in Chapter 3. Thus

$$p(n) = \begin{cases} (1/n\sigma\sqrt{2\pi})\exp\left[\frac{-(\ln n-m)^2}{2\sigma^2}\right], & \text{for } n \gg \sqrt{M} \\ (2n/M)\ \exp\left[-n^2/M\right], & \text{for } n \ll \sqrt{M} \end{cases} \tag{4.17}$$

where $M = (\alpha\lambda/2)\, e^{2(\sigma^2+m)}$.

As a final solution one uses (4.2) to find p(N) at a given time or more practically $p\left[N/N_{rms}\right]$. It was said before that at any given time the values of N would differ from n by some multiplying constant C. Then

$$N = Cn$$

or $$(N_{rms}^2) = C^2(n_{rms}^2)$$

and $$N/N_{rms} = n/n_{rms} \tag{4.18.1}$$

where $$n_{rms} = \left[\sqrt{1+\lambda\alpha/2}\right]e^{\sigma^2+m} . \tag{4.18.2}$$

From (4.18.1) one obtains

$$p\left[N/N_{rms}\right] = n_{rms}\, p(n) \tag{4.18}$$

Equations (4.17), (4.18.1) and (4.18) are used to obtain

$$p(N/N_{rms}) = \begin{cases} \dfrac{1}{(N/N_{rms})\sigma\sqrt{2\pi}} \exp\left\{-\left[\dfrac{\ln(N/N_{rms}) + \sigma^2}{\sqrt{2}\ \sigma}\right]^2\right\} \\ \qquad\qquad \text{for } n \gg \sqrt{M} \text{ and } N/N_{rms} \gg \lambda\alpha/2 \\ (N/N_{rms})\left(\dfrac{2n_{rms}^2}{M}\right)\exp\left[-(N/N_{rms})^2 \dfrac{n_{rms}^2}{M}\right] \\ \qquad\qquad \text{for } n \ll \sqrt{M} \text{ and } N/N_{rms} \ll \lambda\alpha/2. \end{cases} \tag{4.19}$$

One can find the cumulative distribution $P(N/N_{rms})$ from (4.19). It is also noted from (4.18.1) that $P(N/N_{rms} > R) = P(n/n_{rms} > R)$ and is given by [46]

$$P(N/N_{rms} > R) = \begin{cases} 1/2\left[1 - \mathrm{erf}\left(\dfrac{\ln R + \sigma^2}{\sigma\sqrt{2}}\right)\right], \text{ for } R \gg \lambda\alpha/2 \\ \exp\left[-R^2/(M/n_{rms}^2)\right], \text{ for } R \ll \lambda\alpha/2 \ . \end{cases}$$

Note that $M/n_{rms}^2 = \lambda\alpha/2\ (\lambda\alpha/2 + 1)$.

APPENDICES

Appendix 1.1

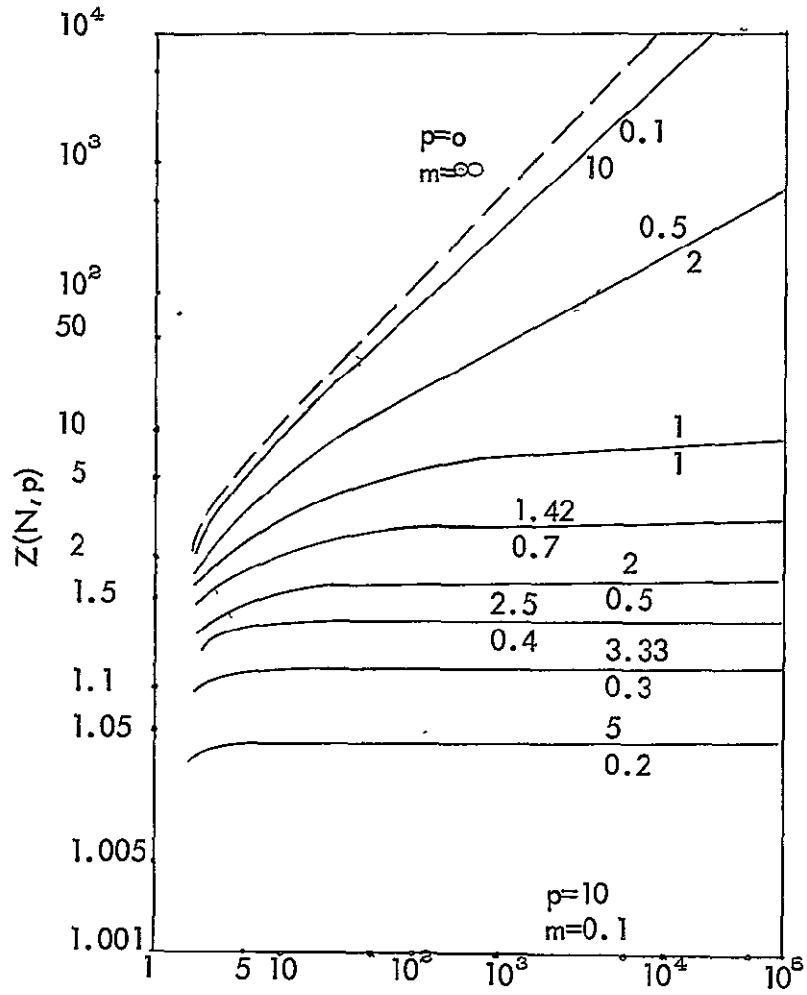

FIGURE 1.1A. Zeta function

Figure 1.1A is the plot of incomplete ($N \neq \infty$) zeta function versus number of errors. Note that it is practically unity for $m = 1/p = 0.1$.

Appendix 1.1 (Continued)

Table 1

INCOMPLETE ZETA FUNCTION $Z(N,p)$

$p = 0.100000$, $m = 10$			$p = 0.500000$, $m = 2$	
$Z(N,p)$	N^p	N	$Z(N,p)$	N^p
1.933033	1.071773	2.00	1.707107	1.414214
4.550882	1.174619	5.00	3.231671	2.236068
8.619334	1.258925	10	5.020998	3.162278
1.623687×10	1.349283	2×10	7.595255	4.472136
3.730404×10	1.478758	5.10	1.275237×10	7.071068
6.981874×10	1.584893	10^2	1.858960×10	1.000000×10
1.305143×10^2	1.698646	2×10^2	2.685924×10	1.414214×10
2.980869×10^2	1.861646	5×10^2	4.328328×10	2.236068×10
4.036227×10^2	1.925350	7×10^2	5.147344×10	2.645751×10
5.565207×10^2	1.995262	10^3	6.180081×10	3.162278×10
1.038811×10^3	2.138469	2×10^3	8.799798×10	4.472136×10
2.370094×10^3	2.343673	5×10^3	1.399766×10^2	7.071068×10
4.423059×10^3	2.511886	10^4	1.985553×10^2	1.000000×10^2
8.254026×10^3	2.692173	2×10^4	2.813980×10^2	1.414214×10^2
1.882878×10^4	2.950509	5×10^4	4.457689×10^2	2.236068×10^2
3.513606×10^4	3.162278	10^5	6.310108×10^2	3.162278×10^2
6.556652×10^4	3.389245	2×10^5	8.929825×10^2	4.472136×10^2
1.495648×10^5	3.714471	5×10^5	1.412769×10^3	7.071068×10^2
2.790981×10^5	3.981072	10^6	1.998555×10^3	1.000000×10^3
5.208158×10^5	4.266807	2×10^6	2.826982×10^3	1.414214×10^3
1.188038×10^6	4.676242	5×10^6	4.470691×10^3	2.236068×10^3
2.216958×10^6	5.011872	10^7	6.323110×10^3	3.162278×10^3
4.136989×10^6	5.371592	2×10^7	8.942827×10^3	4.472136×10^3
9.436923×10^6	5.887040	5×10^7	1.414069×10^4	7.071068×10^3
1.760992×10^7	6.309573	10^8	1.999856×10^4	1.000000×10^4
3.286128×10^7	6.762433	2×10^8	2.828283×10^4	1.414214×10^4
7.496015×10^7	7.411344	5×10^8	4.471992×10^4	2.236068×10^4
1.398806×10^8	7.943282	10^9	6.324411×10^4	3.162278×10^4

Table 1 (Continued)

p = 1.000000, m = 1			p = 1.428561, m = 0.7	
Z(N,p)	N^p	N	Z(N,p)	N^p
1.500000	2.000000	2.00	1.371499	2.691800
2.283333	5.000000	5.00	1.818010	9.966177
2.928968	1.000000×10	10	2.089265	2.682696×10
3.597739	2.000000×10	2×10	2.301445	7.221281×10
4.499205	5.000000×10	5×10	2.506341	2.673622×10^2
5.187376	1.000000×10^2	10^2	2.617318	7.196856×10^2
5.878027	2.000000×10^2	2×10^2	2.700206	1.937250×10^3
6.792811	5.000000×10^2	5×10^2	2.778246	7.172514×10^3
7.128992	7.000000×10^2	7×10^2	2.800060	1.159914×10^4
7.485444	1.000000×10^3	10^3	2.819999	1.930698×10^4
8.1784	2.000000×10^3	2×10^3	2.851059	5.197053×10^4
9.0945	5.000000×10^3	5×10^3	2.880221	1.924167×10^5
9.7876	1.000000×10^4	10^4	2.895803	5.179474×10^5
10.4806	2.000000×10^4	2×10^4	2.907381	1.394211×10^6
11.3967	5.000000×10^4	5×10^4	2.918252	5.161955×10^6
12.0896	1.000000×10^5	10^5	2.924060	1.389495×10^7
	2.000000×10^5	2×10^5	2.928376	3.740244×10^7
	5.000000×10^5	5×10^5	2.932428	1.384795×10^8
	1.000000×10^6	10^6	2.934593	3.727593×10^8
	2.000000×10^6	2×10^6	2.936202	1.003394×10^9
	5.000000×10^6	5×10^6	2.937713	3.714985×10^9
	1.000000×10^7	10^7	2.938520	9.999996×10^9
	2.000000×10^7	2×10^7		
	5.000000×10^7	5×10^7		
	1.000000×10^8	10^8		
	2.000000×10^8	2×10^8		
	5.000000×10^8	5×10^8		
	1.000000×10^9	10^9		

Table 1 (Continued)

$\bar{p} = 2.000000$, $m = 0.5$		$\bar{p} = 2.500000$, $m = 0.4$		
$Z(N, \bar{p})$	$N^{\bar{p}}$	N	$Z(N, \bar{p})$	$N^{\bar{p}}$
1.250000	4.000000	2.00	1.176777	5.656854
1.463611	2.500000x10	5.00	1.290065	5.590170x10
1.549768	1.000000×10^{2}	10	1.321921	3.162278×10^{2}
1.596163	4.000000×10^{2}	2x10	1.334307	1.788854×10^{3}
1.625132	2.500000×10^{3}	5x10	1.339629	1.767767×10^{4}
1.634983	1.000000×10^{4}	10^{2}	1.340825	9.999999×10^{4}
1.639945	4.000000×10^{4}	2×10^{2}	1.341251	5.656854×10^{5}
1.642932	2.500000×10^{5}	5×10^{2}	1.341424	5.590170×10^{6}
1.643501	4.900000×10^{5}	7×10^{2}	1.341446	1.296418×10^{7}
1.643927	1.000000×10^{6}	10^{3}	1.341459	3.162278×10^{7}
1.644427	4.000000×10^{6}	2×10^{3}	1.341472	1.788854×10^{8}
1.644727	2.500000×10^{7}	5×10^{3}	1.341478	1.767767×10^{9}
1.644827	1.000000×10^{8}	10^{4}	1.341479	9.999998×10^{9}
1.644877	4.000000×10^{8}	2×10^{4}	1.341480	5.656853×10^{10}
1.644907	2.500000×10^{9}	5×10^{4}	1.341480	5.590169×10^{11}
1.644917	1.000000×10^{10}	10^{5}	1.341480	3.162279×10^{12}
1.644922	4.000000×10^{10}	2×10^{5}	1.341480	1.788855×10^{13}
1.644925	2.500000×10^{11}	5×10^{5}	1.341480	1.767767×10^{14}
1.644926	1.000000×10^{12}	10^{6}	1.341480	1.000000×10^{15}
1.644927	4.000000×10^{12}	2×10^{6}	1.341480	5.656853×10^{15}
1.644927	2.500000×10^{13}	5×10^{6}	1.341480	5.590168×10^{16}
1.644927	1.000000×10^{14}	10^{7}	1.341480	3.162277×10^{17}

Table 1 (Continued)

p = 3.333333, m = 0.3		p = 5.000000, m = 0.2		
Z(N, p)	N	N	Z(N, p)	N^p
1.099213	1.007937×10	2.00	1.031250	3.200000×10
		3.00	1.035365	2.43×10^2
		4.00	1.036342	1.024×10^3
1.139414	2.137470×10^2	5.00	1.036662	3.125×10^3
		6.00	1.036790	7.776×10^3
		7.00	1.036850	1.6807×10^4
		8.00	1.036880	3.2768×10^4
1.45586	2.154435×10^3	10	1.036907	1.000000×10^5
1.46984	2.171534×10^4	2×10	1.036926	3.200000×10^6
1.147310	4.605039×10^5	5×10	1.036927	3.124999×10^8
1.147346	4.641588×10^6	10^2	1.036927	1.000000×10^{10}
1.147353	4.678427×10^7	2×10^2	1.036927	3.200000×10^{11}
1.147353	9.921255×10^8	5×10^2	1.036927	3.125000×10^{13}
1.147353	3.045510×10^9	7×10^2	1.036927	1.680700×10^{14}
1.147353	9.999998×10^9	10^3	1.036927	1.000000×10^{15}
1.147353	1.007937×10^{11}	2×10^3	1.036927	3.200000×10^{16}
1.147353	2.137469×10^{12}	5×10^3	1.036927	3.125000×10^{18}
1.147353	2.154434×10^{13}	10^4	1.036927	1.000000×10^{20}
1.147353	2.171533×10^{14}	2×10^4	1.036927	3.200000×10^{21}
1.147353	4.605039×10^{15}	5×10^4	1.036927	3.125000×10^{23}
1.147353	4.641588×10^{16}	10^5	1.036927	1.000000×10^{25}
1.147353	4.678428×10^{17}	2×10^5	1.036927	3.200000×10^{26}
1.147353	9.921253×10^{18}	5×10^5	1.036927	3.125000×10^{28}
1.147353	9.999997×10^{19}	10^6	1.036927	1.000000×10^{30}
1.147353	1.007937×10^{21}	2×10^6	1.036927	3.200000×10^{31}
1.147353	2.137469×10^{22}	5×10^6	1.036927	3.125000×10^{33}
1.147353	2.154434×10^{23}	10^7	1.036927	1.000000×10^{35}

p = 10, m = 0.1

p(N,p)	N^p	N
1.000977	1.024×10^3	2.00
1.000993	5.9049×10^4	3.00
1.000994	1.048576×10^6	4.00
1.000995	9.765625×10^6	5.00
1.000995	6.046619×10^7	6.00
1.000995	2.824752×10^8	7.00
1.000995	1.073742×10^9	8.00

Appendix 1.2

The plot of number of errors N versus the total test time T as the slope m is the parameter is given in Figure 1.2A.

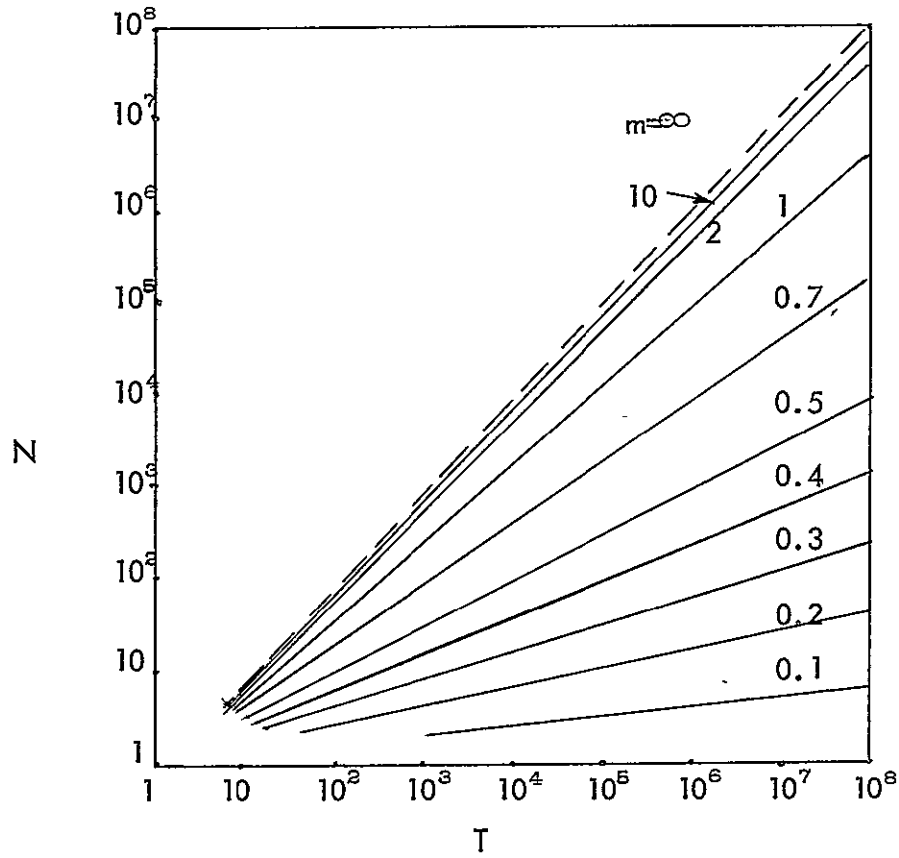

FIGURE 1.2A. Total test time T in Bits

Appendix 2.1

Consider the low-pass filter shown in Figure 2.1A

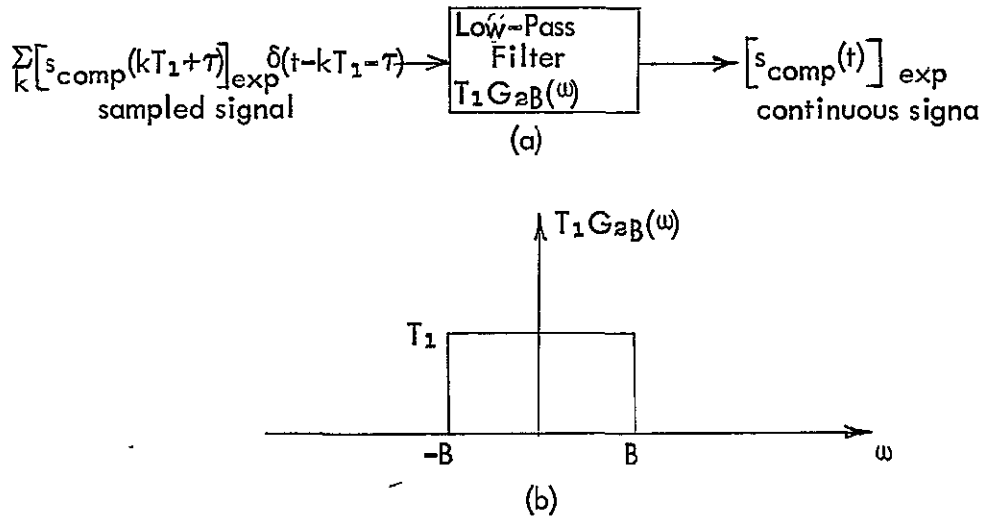

FIGURE 2.1A (a) Reconstruction filter
(b) Gate function

To obtain the output one takes the inverse transform of the transfer function $[T_1 G_{2B}(w)]$ and convolves with the input. Thus

$$[\hat{s}_{comp}(t)]_{exp} = \sum_k [\hat{s}_{comp}(kT_1+\tau)]_{exp}\delta(t-kT_1-\tau)*Sa(Bt) \qquad (2.1A)$$

where

*means convolution and $Sa(Bt) = \sin Bt/Bt$ is the inverse transform of the transfer function or the gate function. Now, since

$$\delta(t-kT_1-\tau)*Sa(Bt) = Sa[B(t-kT_1-\tau)]$$

Equation (2.1A) becomes

$$[\hat{s}_{comp}(t)]_{exp} = \sum_k [\hat{s}_{comp}(kT_1+\tau)]_{exp} Sa[B(t-kT_1-\tau)]$$ which is

Equation (2.3).

Appendix 2.2

Find the mean square value of $\epsilon(t)$, where

$$\epsilon(t) = \sum_k \left\{\left[\hat{s}_{comp}(kT_1+\tau)\right]_{exp} - s(kT_1+\tau)\right\} Sa\left[B(t-kT_1-\tau)\right].$$

$$\overline{\epsilon^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_k\left\{\left[\hat{s}_{comp}(kT_1+\tau)\right]_{exp} - s(kT_1+\tau)\right\} Sa\left[B(t-kT_1-\tau)\right]\right)^2 dt . \qquad (2.2A)$$

Various sampling functions appearing in the summation in Equation (2.2A) are orthogonal, i.e.

$$\int_{-\infty}^{\infty} Sa\left[B(t-mT_1-\tau)\right] Sa\left[B(t-nT_1-\tau)\right] dt = \begin{cases} \pi/B & \text{for } m=n \\ 0 & \text{for } m\neq n \end{cases} \qquad (2.2.1A)$$

Therefore, after changing the operation of integration and summation in (2.2A) one obtains

$$\overline{\epsilon^2(t)} = \sum_k \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left\{\left[\hat{s}_{comp}(kT_1+\tau)\right]_{exp} - s(kT_1+\tau)\right\}^2 Sa^2\left[B(t-kT_1-\tau)\right] dt$$

$$\overline{\epsilon^2(t)} = \sum_k \lim_{T\to\infty} \frac{1}{T}\left\{\left[\hat{s}_{comp}(kT_1+\tau)\right]_{exp} - s(kT_1+\tau)\right\}^2 \int_{-T/2}^{T/2} Sa^2\left[B(t-kT_1-\tau)\right] dt \qquad (2.2.2A)$$

But from (2.2.1A)

$$\int_{-T/2}^{T/2} Sa^2\left[B(t-kT_1-\tau)\right] dt = \pi/B$$

then (2.2.2A) becomes

$$\overline{\epsilon^2(t)} = \sum_k \left[\pi/BT\right]\left\{\left[\hat{s}_{comp}(kT_1+\tau)\right]_{exp} - s(kT_1+\tau)\right\}^2 .$$

Appendix 2.2 (Continued)

But since $B = 2\pi f_s$

$$\overline{\varepsilon^2(t)} = \frac{1}{2f_sT} \sum_k \left\{ \left[\hat{s}_{comp}(kT_1+\tau) \right]_{exp} - s(kT_1+\tau) \right\}^2, \text{ which is (2.6).}$$

Appendix 2.3

Show that $G(v) \geq [M(v)]_{unif}$, where $G(v)$ and $M(v)$ are the moment generating functions of Gaussian and uniform distributions respectively.

Proof.

$$\text{mgf of } (x) = \int_{-\infty}^{\infty} [\exp(vx)]\, p(x)\, dx \tag{2.3.1A}$$

thus from (2.3.1A)

$$G(v) = \exp\left[mv + v^2\sigma^2/2\right]$$

and for m=o, $\sigma^2 = \Delta^2/12$

$$G(v) = \exp\left[(\Delta^2/12)(v^2/2)\right]$$

$$= \sum_{k=o}^{\infty} \left[(v\Delta/2)^2\ 1/6\right]^k\ 1/k!\ , \tag{2.3.2A}$$

also

$$\left[M(v)\right]_{unif} = \sinh v(\Delta/2)\ /v(\Delta/2)$$

$$= \sum_{k=o}^{\infty} \left[v(\Delta/2)\right]^{2k}\ 1/(2k+1)!\ , \tag{2.3.3A}$$

Now, if one compares the sums (2.3.2A) and (2.3.3A) it is obvious that

$$\left[v(\Delta/2)\right]^{2k}\ 1/(2k+1)!\ \leq \left[(v\Delta/2)^2\ 1/6\right]^k\ 1/k!$$

thus

$$G(v) \geq \left[M(v)\right]_{unif}$$

Appendix 2.4

Find the value of v that would minimize $G_s(v)\ e^{-va} = \exp\left[(k/2)(\Delta^2/12)\ v^2 - av\right]$. (2.4A)

Solution:

Minimizing $\ln\left[G_s(v)\ e^{-va}\right]$ will also minimize $G_s(v)\ e^{-va}$.

Thus

$$\ln\left[G_s\ (v)\ e^{-va}\right] = \ln\left[G_s(v)\right] - va \qquad (2.4.1A)$$

taking the derivative of (2.4.1A)

$$d/dv\left[\ln G_s(v)\ e^{-va}\right] = \left[1/G_s(v)\right] d/dv\left[G_s(v)\right] - a \qquad (2.4.2A)$$

Equating (2.4.2A) to zero.

$$d/dv\left[G_s(v)\right] = a\ G_s(v) \qquad (2.4.3A)$$

v must be chosen such that (2.4.3A) is satisfied. Putting $G_s(v) = \exp\left[(k/2)(\Delta^2/12)\ v^2\right]$ into (2.4.3A)

one gets

$$k(\Delta^2/12)v\ \exp\left[(k/2)(\Delta^2/12)v^2\right] = a\ \exp\left[(k/2)(\Delta^2/12)v^2\right]$$

and $\quad k\ (\Delta^2/12)\ v = a$

or $\quad v = (a/k)\ (12/\Delta^2)$

which is (2.14).

Appendix 3.1

Prove that the sum of uniformly distributed phase phasors is itself a uniformly distributed phase phasor.

Proof:

$$\text{Let } R e^{j\theta} = \sum_{k=o}^{n} A_k e^{j\theta_k}$$

where

(A_k) are amplitude distributions such that the central limit theorem is satisfied. That is, no one of the distributions of A_k is dominant, and (θ_k) are the uniformly distributed phases.

If n is large, then by the central limit theorem $R_x = \sum_{k=o}^{n} A_k \cos\theta_k$ and $R_y = \sum_{k=o}^{n} A_k \sin\theta_k$ are Gaussian with zero mean and variance $\left(\frac{1}{2} n < A_k^2 >\right)$. Since R_x and R_y are orthogonal they are uncorrelated. The joint distribution is found as follows:

$$p(R, \theta) = (R)\, p(R_x, R_y). \tag{3.1.1A}$$

The two-dimensional Gaussian distribution is given by

$$p(R_x, R_y) = \frac{1}{2\pi\sigma^2} e^{-(R_x^2 + R_y^2)/2\sigma^2}$$

where $\sigma^2 = (1/2) n < A_k^2 >$.

Now, from (3.1.1A) one gets

$$p(R, \theta) = \frac{R}{2\pi\sigma^2} e^{-R^2/2\sigma^2} \tag{3.1.2A}$$

(3.1.2A) is integrated with respect to R to obtain $p(\theta)$.

$$p(\theta) = \int_0^\infty p(R, \theta)\, dR = \frac{1}{2\pi\sigma^2} \int_0^\infty R e^{-(R^2/2\sigma^2)}\, dR$$

$$p(\theta) = \frac{1}{2\pi} \tag{3.1.3A}$$

Equation (3.1.3A) shows that $Re^{j\theta}$ is a uniformly distributed phase phasor.

Appendix 4.1

Let $n_o = n_p \exp\left[-\frac{(t-t_o)}{\alpha}\right]$.

Find the distribution of n_o at a given time and if the distribution of n_p is known.

Solution:

At a given time let $\exp\left[-\frac{(t_x-t_o)}{\alpha}\right] = k$ where k is a constant and let $p(n_p)$ be defined by Equation (4.5). Now then,

$$n_o = k\, n_p$$

but

$$p(n_o)\, d(n_o) = p(n_p)\, d(n_p)$$

or

$$p(n_o) = p(n_p) \left| \frac{d(n_p)}{d(n_o)} \right|$$

then

$$p(n_o) = 1/k\; p(n_p)$$

which proves the statement made in page 32.

Appendix 4.2

Given $n_p = e^{\Delta}$, where Δ is Gaussian, find the distribution of n_p.

Solution:

$$p(\Delta) = \frac{1}{(\sigma)\sqrt{2\pi}} \exp - \left[\frac{(\Delta-\mu)^2}{2\sigma^2}\right] \tag{4.2.1A}$$

and since $\quad p(n_p)\, d(n_p) = p(\Delta)\, d\Delta$

or $$p(n_p) = p(\Delta)\ d\Delta/d(n_p) \tag{4.2.2A}$$

but $\quad d(n_p) = e^{\Delta} d\Delta = \left[e^{\ln(n_p)}\right] d\Delta = n_p\, d\Delta$

then (4.2.2A) becomes

$$p(n_p) = (1/n_p)\ p(\Delta)$$

$$p(n_p) = (1/n_p) p_{\Delta}(\ln n_p) \tag{4.2.3A}$$

or from (4.2.1A) and (4.2.3A)

$$p(n_p) = \frac{1}{(n_p)\sigma\sqrt{2\pi}} \exp - \left\{\frac{\left[\ln(n_p) - m\right]^2}{2\sigma^2}\right\},$$

which is Equation (4.5).

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Appendix 4.3

Let $t_k = \sum_{i=1}^{k} T_i$, where T_i are distributed identically and $p(T_i) = \lambda e^{-\lambda t}$. Then find $p(t_k)$.

Solution:

The characteristic function of the random variable $t = T_i$ is

$$\int_0^\infty (e^{j\omega t})\,(\lambda e^{-\lambda t})\,d\,t = \frac{\lambda}{\lambda - j\omega}$$

The characteristic function of t_k is the product of the characteristic functions of the sum terms $t = T_i$. Then the characteristic function of t_k is

$$\text{chf. of } t_k = \frac{\lambda^k}{(\lambda - j\omega)^k} \tag{4.3.1A}$$

Taking the inverse transform of (4.3.1A) one obtains

$$p(t_k) = \frac{1}{2\pi} \int_0^\infty \frac{\lambda^k}{(\lambda - j\omega)^k} \quad e^{-j\omega t_k}\,d\omega$$

$$p(t_k) = \frac{\lambda^k t_k^{\,k-1} e^{-\lambda t_k}}{(k-1)\,!}$$

which is Equation (4.8)

Appendix 4.4

Let $v = \exp\left[t_k/\alpha\right]$, and the distribution of t_k is $p(t_k)$. Find the distribution of v, p(v).

Solution:

$$p(t_k) = \lambda^k t_k^{k-1} e^{-\lambda t_k} / (k-1)!$$

and

$$p(v) = p_{t_k}(v) \left| dt_k/dv \right| \tag{4.4.1A}$$

since $t_k = \alpha \ln v$, then $dt_k/dv = \alpha/v$.

The $p(t_k)$ in terms of v is given by

$$p_{t_k}(v) = \frac{\lambda^k (\alpha \ln v)^{k-1} e^{-\lambda\alpha \ln v}}{(k-1)!}$$

$$p_{t_k}(v) = \frac{\lambda^k (\alpha \ln v)^{k-1}}{(k-1)!\ v^{\lambda\alpha}} \tag{4.4.2A}$$

Then from (4.4.1A) and (4.4.2A)

$$p(v) = (\alpha/v) \frac{\lambda^k (\alpha \ln v)^{k-1}}{(k-1)!\ v^{\lambda\alpha}}$$

$$p(v) = \frac{\lambda^k \alpha^k (\ln v)^{k-1}}{(k-1)!\ v^{\lambda\alpha+1}}$$

which is Equation (4.8.1).

Appendix 4.5

Given $n_k = u/v$, $p(u)$ and $p(v)$ find $p(n_k)$. Assume u and v are independent.

Solution: Let $n_k = n_k(u,v)$, $m_k = v$ and their inverse functions are

$$u = u(n_k, m_k), \; v = v(n_k, m_k) .$$

Since the following relation holds,

$$p(u,v)\, d\,u\, d\,v = p(n_k, m_k)\, d\,n_k\, d\,m_k$$

then

$$p(n_k, m_k) = p(u, v) \left| \frac{d\,u\,d\,v}{d\,n_k\,d\,m_k} \right| , \qquad (4.5.1A)$$

but

$$\left| \frac{dudv}{dn_k\, dm_k} \right| = \left| \frac{\partial\,(u,v)}{\partial(n_k, m_k)} \right| .$$

Therefore one first finds the Jocobian to be:

$$\frac{\partial(u,v)}{\partial(n_k, m_k)} = \begin{vmatrix} v & o \\ -u/n_k^2 & 1 \end{vmatrix} = v .$$

The Equation (4.5.1A) becomes

$$p(n_k, m_k) = vp\,(u,v)$$

$$= vp_{uv}(n_k, m_k) \qquad (4.5.2A)$$

Now, since u and v are independent

$$p(u, v) = p(u)\, p\,(v).$$

Also noting that $dv = dm_k$, upon integrating both sides of Equation (4.5.2A) one gets

$$\int p(n_k, m_k)\, dm_k = \int vp(u)\, p(v)\, dv \qquad (4.5.3A)$$

Appendix 4.5 (Continued)

Left side of Equation (4.5.3A) is $p(n_k)$. Also replace $u = n_k v$. Now, if $v = \exp\left[t_k/\alpha\right]$ and $o \leq t_k \leq \infty$ then $1 \leq v \leq \infty$. Therefore Equation (4.5.3A) becomes

$$p(n_k) = \int_1^{\infty} v\, p(v)\, p_u(n_k, v)\, dv \qquad (4.5.4A)$$

From Equation (4.5) and (4.8.1) $p_u(n_k v)$ and $p(v)$ are obtained. But first the following transformations are made

$\ln v = x$, then $d\,v = e^x dx$ and when $v = 1$, $x = o$ and $v = \infty$, $x = \infty$.

Then

$$p(v) = \frac{\lambda^k \alpha^k x^{k-1}}{(k-1)!\; e^{x(\lambda\alpha+1)}} \qquad (4.5.5A)$$

and

$$p_u(n_k v) = \frac{1}{\sigma n_k e^x \sqrt{2\pi}} \exp\left[-\frac{(\ln n_k + x - m)^2}{2\sigma^2}\right] \qquad (4.5.6A)$$

Placing the values of $p(v)$ and $p_u(n_k, v)$ in (4.5.4A) from (4.5.5A) and (4.5.6A) one gets

$$p(n_k) = \frac{\lambda^k \alpha^k}{(k-1)!\; \sigma n_k \sqrt{2\pi}} \int_0^{\infty} x^{k-1} \exp\left[-\alpha\lambda x - \frac{(\ln u_k + x - m)^2}{2\sigma^2}\right] dx$$

this is Equation (4.9).

Appendix 4.6

Show that $<\left(\sum_{k=o}^{\infty} n_k e^{j\varphi_k}\right)^2> = \sum_{k=o}^{\infty} <n_k^2>$.

Proof:

$$\text{Let } S = Re^{j\theta} = \sum_{k=o}^{n} n_k e^{j\varphi_k} \tag{4.6.1A}$$

The mean-square value of the complex random variable S is <SS*>. Therefore, from (4.6.1A) one has

$$<R^2> = <\sum_{k=o}^{n} n_k e^{j\varphi_k} \sum_{l=o}^{n} n_l e^{-j\varphi_l}> . \tag{4.6.2A}$$

If one assumes n_k and n_l are independent and uses the fact that the average of the sum is equal to the sum of the averages, then

$$<R^2> = \sum_{k=o}^{n} \sum_{l=o}^{n} <n_k n_l e^{j(\varphi_k-\varphi_l)}> \tag{4.6.3A}$$

If the amplitudes of the phasors are independent of their phases, then (4.6.3A) becomes

$$<R^2> = \sum_{k=o}^{n} \sum_{l=o}^{n} <n_k n_l><e^{j(\varphi_k-\varphi_l)}> \tag{4.6.4A}$$

But, if the phases are uniformly distributed then

$$<e^{j(\varphi_k-\varphi_l)}> = \frac{1}{4\pi^2} \int_o^{2\pi} \int_o^{2\pi} e^{j(\varphi_k-\varphi_l)}\, d\varphi_k d\varphi_l = \begin{cases} o, & \text{for } k \neq l \\ 1, & \text{for } k=l \end{cases}$$

Thus (4.6.4A) is now

$$<R^2> = \sum_{k=o}^{n} <n_k^2> \tag{4.6.5A}$$

or from (4.6.2A) and (4.6.5A) letting $n = \infty$ one gets

$$\sum_{k=1}^{\infty} <n_k^2> = <\left(\sum_{k=o}^{\infty} n_k e^{j\varphi_k}\right)^2> .$$

Appendix 4.7

Evaluate the following integral

$$<n_k^2> = \int_0^{\infty} n_k^2 \frac{\lambda^k \alpha^k}{(k-1)!\, \sigma n_k \sqrt{2\pi}} \int_0^{\infty} x^{k-1} \exp\left[-\alpha\lambda x - \frac{(\ln n_k + x - m)^2}{2\sigma^2}\right] dx\, dn_k \,. \qquad (4.7.1A)$$

Set $\ln n_k = v$ and interchange integrals over x and n_k, then integrate with respect to n_k. Since $dn_k = e^v dv$ and when $n_k = o$, $v = -\infty$ and $n_k = \infty$, $v = \infty$, the Equation (4.7.1A) becomes

$$<n_k^2> = \frac{\lambda^k \alpha^k}{(k-1)!\, \sigma\sqrt{2\pi}} \int_{x=o}^{\infty} x^{k-1} e^{-\lambda\alpha x} dx \int_{v=-\infty}^{\infty} e^v \exp\left[-\frac{(v+x-m)^2}{2\sigma^2}\right] e^v dv \,,$$

or

$$<n_k^2> = \frac{\lambda^k \alpha^k}{(k-1)!\, \sigma\sqrt{2\pi}} \int_{x=o}^{\infty} x^{k-1} e^{-\lambda\alpha x} dx \int_{v=-\infty}^{\infty} \exp\left[\frac{-v^2 + 2v(x-m-2\sigma^2) + (x-m)^2}{2\sigma^2}\right] dv \,. \qquad (4.7.2A)$$

Now, if one adds and subtracts $(x-m-2\sigma^2)^2$ from the numerator of the exponent of the integral with respect to (v) and considering that integral only, then one gets

$$\int_{-\infty}^{\infty} \exp\left[\frac{-\left\{[v+(x-m-2\sigma^2)]^2 + 4\sigma^2[x-m-\sigma^2]\right\}}{2\sigma^2}\right] dv$$

$$= \exp\left[\frac{-4\sigma^2(x-m-\sigma^2)}{2\sigma^2}\right] \int_{-\infty}^{\infty} \exp\left\{\frac{-[v+(x-m-2\sigma)^2]^2}{2\sigma^2}\right\} dv \qquad (4.7.3A)$$

The integral in (4.7.3A) is easy to evaluate and it is equal to $\sigma\sqrt{2\pi}$. Thus (4.7.3A) is now equal to

$$\sigma\sqrt{2\pi}\, \exp\left[\frac{-4\sigma^2(x-m-\sigma^2)}{2\sigma^2}\right] \qquad (4.7.4A)$$

Appendix 4.7 (Continued)

Returning to Equation (4.7.2A) and replacing the integral with respect to (v) by its value from (4.7.4A) one obtains

$$\langle n_k^2 \rangle = \frac{\lambda^k \alpha^k}{(k-1)!} \int_{x=o}^{\infty} x^{k-1} e^{-\lambda\alpha x} \exp\left[\frac{-4\sigma^2 (x-m-\sigma^2)}{2\sigma^2}\right] dx$$

or

$$\langle n_k^2 \rangle = \frac{\lambda^k \alpha^k}{(k-1)!} \left[e^{2(m+\sigma^2)}\right] \int_{x=o}^{\infty} x^{k-1} e^{-(\lambda\alpha+2)x} \, dx \, . \quad (4.7.5A)$$

Let $(\lambda\alpha+2)x = \tau$ and $d\tau = (\lambda\alpha+2)\, dx$. When $\tau = o$, $x = o$ and $\tau = \infty$, $x = \infty$.

Equation (4.7.5A) becomes

$$\langle n_k^2 \rangle = \frac{\lambda^k \alpha^k}{(k-1)!} \left[e^{2(m+\sigma^2)}\right] \int_{\tau=o}^{\infty} \left[\frac{\tau}{(\lambda\alpha+2)}\right]^{k-1} e^{-\tau} \frac{d\tau}{\lambda\alpha+2}$$

or

$$\langle n_k^2 \rangle = \frac{\lambda^k \alpha^k e^{2(m+\sigma^2)}}{(k-1)!\,(\lambda\alpha+2)^k} \int_{\tau=o}^{\infty} \tau^{k-1} e^{-\tau} \, d\tau \, . \quad (4.7.6A)$$

But the integral in (4.7.6A) is the Gamma function $\Gamma(k)$ and since $\Gamma(k) = (k-1)!$, one has

$$\langle n_k^2 \rangle = \frac{\lambda^k \alpha^k e^{2(m+\sigma^2)}}{(\lambda\alpha+2)^k}$$

which is (4.12).

REFERENCES

1. P. Mertz, Impulse Noise and Error Performance in Data Transmission, AD 614416 U.S. Department of Commerce, Clearing House for Scientific and Technical Information, April 1965, p. 12.

2. D. H. Hamsher, Communication System Engineering Handbook, McGraw-Hill Book Co., 1967, pp. 1-23, 6-23, 11-22, 15-2.

3. Angelakos & Everhart, Microwave Communications, McGraw-Hill Book Co., 1968, pp. 174-175.

4. K. G. Budden, Radio Waves in Ionosphere, Cambridge, 1961, pp. 29.

5. Jordan & Balmain, EM Waves and Rad. Sys., Prentice Hall, 1968, p. 669.

6. P. Mertz, op. cit., pp. 1.

7. P. Beckmann, Probability in Communication Engineering, Harcourt, Brace & World, Inc., New York, 1967, p. 234.

8. Baghdady, E. J., Lectures on Comm Systems Theory, McGraw-Hill, 1961, p. 260.

9. P. Mertz, op. cit., chpt. 4, 5.

10. Berger, J. M. and B. Mendelport, A New Model for Error Clustering in Telephone Circuits, IBMJ.R. and D Vol. 7, July 1963, pp. 224.

11. P. Mertz, op. cit., p. 8.

12. P. Mertz, Ibid., p. 10.

13. P. Mertz, Ibid., p. 11.

14.-20. P. Mertz, Ibid., Chpt. 4, 5.

21. J. E. Mazo, Quantization Noise and Data Transmission, Bell System Technical Journal, Vol. 47, October 1968, No. 8, pp. 1739.

22. Ibid., p. 1739.

23. Ibid., p. 1741.

24. B. P. Lathi, Communication Systems, John Wiley & Sons, Inc., New York, 1968, pp. 356-357.

25. J. E. Mazo, op. cit., pp. 1742.

26. Peter Beckman, op. cit., p. 152.

27. Peter Beckman, op. cit., p. 152.

28. Ibid., pp. 108, 152.

29. Ibid., pp. 124, 151.

30. Ibid., pp. 146-150.

31. Ibid., pp. 118-120.

32. F. M. Gardner, Phase-Lock Techniques, John Wiley & Sons, Inc., New York, 1966.

33. Peter Beckman, op. cit., pp. 126-127, 152.

34. Ibid., pp. 125-127.

35. P. Beckman, Amplitude Probability Distribution of Atmospheric Radio Noise, J. Res. Natl. Bur. Std. 68D June 1964, pp. 723.

36. Ibid., pp. 723.

37. Ibid.

38. P. Beckman, op. cit., pp. 154.

39. Ibid., pp. 116.

40. P. Beckman, NBS, op. cit.

41. P. Beckman, "Prob. in Comm. Eng.", op. cit., pp. 80.

42. A. Papoulis, Prob. Rand. Var. and Stoch. Proc., McGraw-Hill Book Co., 1965, pp. 558.

43. B. W. Lindgren, Statistical Theory, MacMillan Co., 1968, pp. 166-67.

44. Peter Beckman, op. cit., pp. 118-120.

45. Ibid., pp. 157.

46. Ibid., pp. 158.

BIBLIOGRAPHY

Baghdady, E. J., Lectures on Communication System Theory, McGraw-Hill Book Company, New York, 1960.

Beckmann, P., Probability in Communication Engineering, Harcourt, Brace & World, Inc., New York, 1967.

Beckmann, P., "Amplitude-Probability Distribution of Atmospheric Radio Noise," J. Res. Natl. Bur. Std., Vol. 68D, June, 1964, p. 723.

Bennett, W. R., Electrical Noise, McGraw-Hill Book Company, New York, 1960.

Bennett, W. R., "Methods of Solving Noise Problems," Proc. IRE, Vol. 44, May 1956.

Bennett, W. R., "Spectra of Quantized Signals," B.S.T.J. Vol. 27, pp. 446-472, July, 1948.

Berger, J. M., and B. Mandelbrot, "A New Model for Error Clustering in Telephone Circuits," IBM J. Res. & Dev. Vol. 7, pp. 224, 1963.

Blackman, N. M., Noise and Its Effect on Communication, McGraw-Hill Book Company, New York, 1966.

Bodonyi, A. B. "Effects of Impulse Noise on Digital Data Transmission," IRE Trans. on Comm. Sys. Vol. 9, December 1961, p. 355.

Budden, K. G., Radio Waves in the Ionosphere, Cambridge University Press, London, 1961.

Carlson, A. B., Communication Systems; an Introduction to Signals and Noise in Elec. Comm., New York, McGraw-Hill Co., 1968.

Chrichow, W. Q., C. J. Roubique, A. D. Spaulding, W. M. Beery, "Determination of the Amplitude Probability Distribution of Atmospheric Radio Noise from Statistical Moments," J. Res. NBS 64D (Radio Prop.) 1960, pp. 49-56.

Downing, J. J., Modulation Systems and Noise, Prentice-Hall, Inc., N.J. 1964.

Engel, J. S., "Digital Transmission in the Presence of Impulsive Noise," B.S.T.J. Vol. 44, 1965, pp. 1699-1743.

Fennick, J. H., "Amplitude Distributions of Telephone Channel Noise and a Model of Impulse Noise," B.S.T.J. Vol. 48, Dec. 1969, pp. 3243.

Fennick, J. H., "A Report on Some Characteristics of Impulse Noise in Telephone Communication," IEEE Trans. On Comm. and Elec. Vol. 83, 1964, pp. 700-705.

Furutsu, K., T. Ishida, "On the Theory of Amplitude Distribution of Impulsive Random Noise," Journal of Applied Physics, New York, Vol. 32, July 1961, pp. 1206-21.

Gilbert, E. N., "Capacity of Burst-Noise Channel," BSTJ, Vol. 39 1960, p. 1253.

Hamsher, D. H., Communication System Engineering Handbook, McGraw-Hill Book Company, New York, 1967.

Jordan, E. C., and K. G. Balmain, Electromagnetic Waves and Radiating Systems, Prentice-Hall, Inc., New Jersey, 1968.

Lathi, B. P., Communication Systems, Wiley, New York, 1968.

Lucky, R. W., J. Salz, E. J. Weldon, Jr., Principles of Data Comm., McGraw-Hill Book Company, New York, 1968.

Mazo, J. E., "Quantizing Noise and Data Transmission," BSTJ, Vol. 47, Oct. 1968, p. 1737.

Mertz, P., "Impulse Noise and Error Performance in Data Transmission," AD 614 416, U.S. Department of Commerce, Clearing House for Scientific and Technical Information, April 1965.

Mertz, P., "Statistics of Hyperbolic Error Distribution in Data Transmission," IRE Trans. On Comm. Sys., Vol. 9, December 1961, p. 377.

Mertz, P., "Model of Impulsive Noise for Data Transmission," IRE Trans. on Comm. Sys., Vol. 9, June 1961, pp. 130.

Panter, P. F., and W. Dite, "Quantization Distortion in Pulse-Count Modulation with Nonuniform Spacing of Levels," Proc. IRE, Vol. 39, Jan. 1951, pp. 44-48.

Papoulis, A., The Fourier Integral and its Applications, McGraw-Hill Book Co., New York, 1962.

Papoulis, A., Prob., Rand. Var., and Stoch. Proc., McGraw-Hill Book Co., 1965.

Rowe, H. E., Signals and Noise in Communication Systems, D. Van Nostrand Co., Princeton, N.J., 1965.

Sakrison, D. J., Communication Theory, Trans. of Waveforms and Dig. Inf., New York, Wiley, 1968.

Schwartz, M., W. R. Bennett and S. Stien, Comm. Sys. and Tech., McGraw-Hill Book Co., New York, 1966.

Shepelavey, B., "Non-Gaussian Atmospheric Noise in Binary-Data, Phase-Coherent Communication Systems," IEEE Trans. Comm. Sys. Vol. 11, September 1963, pp. 280=284.

Splitt, F. G., "Comparative Performance of Digital Data Transmission Systems in the Presence of CW Interference," IEEE Trans. Comm. Sys., Vol. 10, June, 1962, pp. 169-176.

Smith, B., "Instantaneous Companding of Quantized Signals," BSTJ, Vol. 36 1957, pp. 653-709.

Sussman, S. M., "Analysis of the Pareto Model for Error Statistics on Telephone Circuits," IEEE Trans. on Comm. Sys., Vol. CS-11, 1963, p. 213.

Wax, N., Selected Papers on Noise and Stochastic Processes, Dover Publications, Inc., New York, 1954.

Wozencraft, J. M., and I. M. Jacobs, Principles of Communication Engineering, John Wiley & Sons, Inc., New York, 1965.

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