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USE OF LOW CONSTANT THRUST TO ACCOMPLISH ORBITAL TRANSFER

By Quintin Peasley and Lester Katz
Space Sciences Laboratory

September 16, 1969

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Marshall Space Flight Center, Alabama

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ABSTRACT

A fuel-minimized trajectory is generated, satisfying initial and final boundary conditions for the problem of a constantly thrusting space vehicle which is transferred from an initial elliptical orbit around Mars to a circular orbit of lower energy. The constant-thrust solution is amenable to electric ion-propelled spacecraft having an initial elliptical or circular planetocentric orbit which must then be powered to a circular orbit of desired altitude.
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<tr>
<td>( \phi )</td>
<td>range angle (deg)</td>
</tr>
<tr>
<td>( r )</td>
<td>radial distance of spacecraft to center of mass of planet (km)</td>
</tr>
<tr>
<td>( u )</td>
<td>radial velocity (km/sec)</td>
</tr>
<tr>
<td>( v )</td>
<td>velocity perpendicular to radial velocity (km/sec)</td>
</tr>
<tr>
<td>( T )</td>
<td>thrust force (newton)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>steering control angle (deg)</td>
</tr>
<tr>
<td>( a )</td>
<td>radius of Mars (km)</td>
</tr>
<tr>
<td>( g_e )</td>
<td>gravitational acceleration at surface of Earth ((9.80665 \times 10^{-3} \text{ km/sec}^2))</td>
</tr>
<tr>
<td>( g_m )</td>
<td>gravitational acceleration at surface of Mars ((3.837 \times 10^{-3} \text{ km/sec}^2))</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Mars gravitational constant ((\text{km}^3/\text{sec}^2) = g_m a^2)</td>
</tr>
<tr>
<td>( I_{sp} )</td>
<td>engine specific impulse (sec)</td>
</tr>
<tr>
<td>( m_0 )</td>
<td>initial mass (kg)</td>
</tr>
<tr>
<td>( m )</td>
<td>constant time rate of change of mass ((\text{kg/sec}))</td>
</tr>
<tr>
<td>( t )</td>
<td>time (sec)</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
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<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>m</td>
<td>mass at time t (kg) = m₀ - \dot{m}(t - t₀)</td>
</tr>
<tr>
<td>λ₁</td>
<td>first Lagrangian variable (sec)</td>
</tr>
<tr>
<td>λ₂</td>
<td>second Lagrangian variable (sec/km)</td>
</tr>
<tr>
<td>λ₃</td>
<td>third Lagrangian variable (sec²/km)</td>
</tr>
<tr>
<td>λ₄</td>
<td>fourth Lagrangian variable (sec³/km)</td>
</tr>
<tr>
<td>A</td>
<td>semimajor axis of starting ellipse (km)</td>
</tr>
<tr>
<td>J₂</td>
<td>coefficient of the second harmonic of Mars's potential function J₂ = 0.002001</td>
</tr>
</tbody>
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USE OF LOW CONSTANT THRUST TO ACCOMPLISH ORBITAL TRANSFER

SUMMARY

A fuel-minimized trajectory is generated, satisfying initial and final boundary conditions for the problem of a constantly thrusting space vehicle which is transferred from an initial elliptical orbit around Mars to a circular orbit of lower energy. The constant-thrust solution is amenable to electric ion-propelled spacecraft having an initial elliptical or circular planetocentric orbit which must then be powered to a circular orbit of desired altitude.

INTRODUCTION

The Advanced Reconnaissance Electric Spacecraft (ARES) is a mission designed principally to map the neighboring planets of Venus, Mars, and Jupiter. To map the planets in a minimum length of time, high data transmission rates (on the order of 2 megabits/sec) are required, which, in turn, necessitate large quantities of electric power. The on-board power source is already required to accomplish the mission mapping objective once an orbit is established; hence, it appears practicable to use the available electric power source for main propulsion to achieve the desired orbit. For cartographic purposes, the most desirable orbits are circular, as opposed to elliptic, because the problem of changing altitude (and, hence, resolution) as a function of time is eliminated. In the ARES mission, once the planet is mapped from a high circular orbit, the altitude is decreased to a lower circular orbit so that a higher-resolution map may be made for easy reference to the previous map. The optimal solution of the equations of motion for a minimum transfer time from one circular orbit to another is desired which includes low continuous thrust and the precessional effects of planetary oblateness. The results of such a solution for a typical problem are presented in this report. However, for purposes of generality the initial orbit is chosen as an ellipse, as if the spacecraft has been captured in an initial elliptical orbit and then powered to a circular orbit.
DISCUSSION

The Differential Equation System

The second order Newton differential equations of motion in plane polar coordinates are used. Included are the acceleration components from continuous thrust (T) along and perpendicular to the radius vector:

\[
\begin{align*}
    \ddot{r} &= r\dot{\phi}^2 + \frac{T}{m} \cos \beta - \mu/r^2 \left[ 1 - \frac{3}{2} J_2 \left( \frac{a}{r} \right)^2 (3 \sin^2 \phi - 1) \right] \\
    \frac{d(r\dot{\phi})}{dt} &= -r\ddot{\phi} + \frac{T}{m} \sin \beta - \mu/r^2 \left[ 3J_2 \left( \frac{a}{r} \right)^2 \sin \phi \cos \phi \right]
\end{align*}
\]  

(1a)  

(1b)

where the terms involving \(J_2\) within the brackets of (1) account for the oblateness of Mars [1].

Introduction of the variables \(u = \dot{r}\) and \(v = r\dot{\phi}\) into (1) yields the following first order differential equation system:

\[
\begin{align*}
    \dot{\phi} &= \frac{v}{r} \\
    \dot{r} &= u \\
    \dot{u} &= \frac{v^2}{r} + \frac{T}{m} \cos \beta - \frac{\mu}{r^2} \left[ 1 - \frac{3}{2} J_2 \left( \frac{a}{r} \right)^2 (3 \sin \phi - 1) \right] \\
    \dot{v} &= -\frac{uv}{r} + \frac{T}{m} \sin \beta - \frac{\mu}{r^2} \left[ 3J_2 \left( \frac{a}{r} \right)^2 \sin \phi \cos \phi \right]
\end{align*}
\]  

(2a)  

(2b)  

(2c)  

(2d)

The variable \(\beta\) of set (2) is not associated with any particular differential equation so it is removed by the definitions:
\[
\tan \beta = \frac{\lambda_4}{\lambda_3}, \quad \cos \beta = \frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}}, \quad \sin \beta = \frac{\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}}
\]

where \( \lambda_3, \lambda_4 \) are functions of time. Recourse is made to the calculus of variations to supply the Euler-Lagrange (E-L) differential equations for \( \lambda_3 \) and \( \lambda_4 \), thus linking the steering angle \( \beta \) to the concept of minimal time. In this example, the solution satisfying the boundary conditions and the E-L equations yields a relative minimum. The E-L equations furnish the necessary criteria for a minimum to exist but are not sufficient to guarantee an absolute minimum. Another trajectory history might exist that uses less time than this example.

The Appendix constitutes a brief exposition of the isoperimetric problem solved here. The integral to be extremalized specializes to the total time of flight: \( I = \int_{t_0}^{t_1} dt \), \( f = 1 \) via equation (A1). The constraining differential equations are:

\[
G_1 = \dot{\phi} - \frac{v}{r} = 0, \quad G_2 = \dot{u} - \frac{v^2}{r} - \frac{T}{m} \frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}}
\]

\[
+ \frac{\mu}{r^2} \left[ 1 - \frac{3}{2} J_2 \left( \frac{a}{r} \right)^2 (3 \sin^2 \phi - 1) \right] = 0
\]

\[
G_2 = \dot{r} - u = 0, \quad G_4 = \dot{v} + \frac{uv}{r} - \frac{T}{m} \frac{\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}}
\]

\[
+ \frac{\mu}{r^2} \left[ 3 J_2 \left( \frac{a}{r} \right)^2 \sin \phi \cos \phi \right] = 0
\]

The isoperimetric integrand function, equation (A5), becomes
\[ F = 1 + \lambda_1 \left( \dot{\phi} - \frac{V}{r} \right) + \lambda_2 \left( \dot{r} - u \right) + \lambda_3 \left( \dot{u} - \frac{V^2}{r} - \frac{T}{m} \frac{\lambda_3}{\sqrt{\lambda_4^2 + \lambda_3^2}} \right) \]

\[ + \frac{\mu}{r^2} \left[ 1 - \frac{3J_2}{J_2} \left( \frac{a}{r} \right)^2 \left( 3 \sin^2 \phi - 1 \right) \right] \]

\[ + \lambda_4 \left\{ \dot{\phi} + \frac{uv}{r} - \frac{T}{m} \frac{\lambda_4}{\sqrt{\lambda_4^2 + \lambda_3^2}} + \frac{\mu}{r^2} \left[ 3J_2 \left( \frac{a}{r} \right)^2 \sin \phi \cos \phi \right] \right\} . \]

Application of the Euler-Lagrange equations (A-9) gives

\[
\frac{\partial F}{\partial \phi} = -9 \lambda_3 \frac{\mu^2 a^2}{r^4} \sin \phi \cos \phi + 3 \lambda_4 \frac{J_2 \mu a^2}{r^6} \left( \cos^2 \phi - \sin^2 \phi \right) 
\]

\[
\frac{d}{dt} \left( \frac{\partial F}{\partial \phi} \right) = \lambda_1 
\]

\[
\frac{\partial F}{\partial r} = \lambda_1 \frac{V}{r^2} + \lambda_3 \left[ \frac{V^2}{r^2} - \frac{2u}{r} + 6J_2 \frac{\mu a^2}{r^6} \left( 3 \sin^2 \phi - 1 \right) \right] 

- \lambda_4 \left[ \frac{uv}{r^2} + \frac{12J_2 \mu a^2}{r^6} \sin \phi \cos \phi \right] 
\]

\[
\frac{d}{dt} \left( \frac{\partial F}{\partial r} \right) = \lambda_2 
\]

\[
\frac{\partial F}{\partial u} = -\lambda_2 + \lambda_4 \frac{V}{r} 
\]

\[
\frac{d}{dt} \left( \frac{\partial F}{\partial u} \right) = \lambda_3 
\]
\[
\frac{\partial F}{\partial v} = -\lambda_1 \frac{1}{r} - 2\lambda_3 \frac{v}{r} + \lambda_4 \frac{u}{r}
\]

\[
\frac{d}{dt}\left(\frac{\partial F}{\partial \phi}\right) = \dot{\lambda}_4
\]

The complete differential equation system is:

\begin{align*}
\dot{\phi} &= \frac{v}{r} \quad (3a) \\
\dot{r} &= u \quad (3b) \\
\dot{u} &= \frac{v^2}{r} + \frac{T}{m} \frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}} - \frac{\mu}{r^3} \left[ 1 - \frac{3}{2} J_2 \left(\frac{a}{r}\right)^2 (3 \sin^2 \phi - 1) \right] \quad (3c) \\
\dot{v} &= -\frac{uv}{r} + \frac{T}{m} \frac{\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}} - \frac{\mu}{r^3} \left[ 3 J_2 \left(\frac{a}{r}\right)^2 \sin \phi \cos \phi \right] \quad (3d) \\
\dot{\lambda}_1 &= -9 \lambda_3 \frac{J_2 \mu a^2}{r^4} \sin \phi \cos \phi + 3 \lambda_4 \frac{J_2 \mu a^2}{r^4} (\cos^2 \phi - \sin^2 \phi) \quad (3e) \\
\dot{\lambda}_2 &= \lambda_1 \frac{v}{r^2} + \lambda_3 \left[ \frac{v^2}{r^2} - \frac{2\mu}{r^3} + 6 \frac{J_2 \mu a^2}{r^5} (3 \sin^2 \phi - 1) \right] \\
&\quad - \lambda_4 \left[ \frac{uv}{r^3} + 12 \frac{J_2 \mu a^2}{r^5} \sin \phi \cos \phi \right] \quad (3f) \\
\dot{\lambda}_3 &= -\dot{\lambda}_2 + \lambda_4 \frac{v}{r} \quad (3g) \\
\dot{\lambda}_4 &= -\lambda_1 \frac{1}{r} - 2\lambda_3 \frac{v}{r} + \lambda_4 \frac{u}{r} \quad (3h)
\end{align*}
Boundary Conditions

The curves of Figure 1 illustrate to scale the elliptic starting orbit and the final circular orbit. Mars is located at a focus of the ellipse and the center of the circle. The radius of Mars is 3332.15 km. The semimajor and semiminor axes of the ellipse are 18500 km and 16000 km. The eccentricity is 0.502; the perihelion distance is 9212.91 km; and the aphelion distance is 27787.09 km. The radius of the circle is 13332.15 km, or at 10000 km altitude above the surface of Mars. The initial spacecraft mass is 2950 kg, and the engine I sp is 3500 sec. The ionic thrust (T) is 0.4 newton. The constant time rate of change of mass (mdot) is $1.165 \times 10^{-6}$ kg/sec. Initial and final conditions and derived quantities are given in Table I.

<table>
<thead>
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<th>Conditions Variables</th>
<th>Initial Conditions and Derived Quantities</th>
<th>Final Conditions and Derived Quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>0</td>
<td>78 orbits + 195°</td>
</tr>
<tr>
<td>$r$</td>
<td>9212.91</td>
<td>13332.15</td>
</tr>
<tr>
<td>$u$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v$</td>
<td>2.6355</td>
<td>1.7876</td>
</tr>
<tr>
<td>$t$</td>
<td>0</td>
<td>64 days 4.6 hours</td>
</tr>
<tr>
<td>$m$</td>
<td>2950.0</td>
<td>2885.37</td>
</tr>
<tr>
<td>$g$</td>
<td>$501.9 \times 10^{-6}$</td>
<td>$239.7 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
FIGURE 1. TYPICAL ORBIT GEOMETRIES
Numerical Integration

The procedure used was a powerful seventh order Runge-Kutta-type method with automatic step size control developed by Dr. Erwin Fehlberg of the MSFC Computation Laboratory. A tolerance assuring eight-place accuracy throughout the history for all variables allowed a minimum step size of 227 sec and a maximum of 1355 sec. The first orbit required 115 time steps, or about 78,000 sec (21.7 hr); the intervals are shown in Figure 1. The 78th orbit required 45 time steps each of 1085 sec, or about 47,000 sec (13.1 hr). The IBM 7094 computer was used with double precision (16-digit word length), and machine time was 12 minutes for the trajectory history.

Remarks

Figures 1 depicts the trajectory history with motion in a counterclockwise direction from the ellipse to the final circle. The first 50 orbits change the initial ellipse to a circular orbit about 14,700 km above the surface of Mars (Figs. 1A through 1F). The thrust vectors (small arrows) of Figures 1A through 1E never create more than a 90° angle with the longitudinal axis of a nonrotating spacecraft until the 51st orbit is reached. Orbits 51 through 78 shrink the radius of the circular orbit to the desired altitude of 10,000 km above the surface of Mars. The thrust vectors of Figures 1F through 1H show continuous braking action, and they are nearly aligned with the tangent of the trajectory curve. The final boundary conditions are met about halfway through the 79th orbit. There is no evidence of precession as the ellipse is turned into the circle.

Figures 2A through 2D give radius and steering angle as functions of time. Figure 2C shows the continuous braking effect of the thrust vector that oscillates about the angle \( \beta = 270^\circ \).

The effects of oblateness were not significant for this example. The total time difference between two comparison runs for \( J_2 = 0 \) and \( J_2 = 0.002001 \) was slightly over 1000 sec, with about a one degree change in the range angle \( \phi \).
FIGURE 2A. RADIAL DISTANCE AND STEERING ANGLE $\beta$ AS A FUNCTION OF TIME
FIGURE 2B. RADIAL DISTANCE AND STEERING ANGLE $\beta$ AS A FUNCTION OF TIME
FIGURE 2C. RADIAL DISTANCE AND STEERING ANGLE $\beta$ AS A FUNCTION OF TIME
FIGURE 2D. RADIAL DISTANCE AND STEERING ANGLE $\beta$ AS A FUNCTION OF TIME
CONCLUSIONS

The problem of changing the planetocentric orbit of a long-term, constantly low-thrusting spacecraft from an initial elliptic to a final circular orbit, so as to obtain a trajectory which requires a minimum of fuel is shown to be tractable. The effect of the oblateness of the planet Mars, which contributes perturbations to the equations of motion in the form of harmonics to the gravitational field representation, was shown to be negligible for the case considered.
REFERENCES


APPENDIX

The integral

\[ I = \int_{t_0}^{t_1} f(x_1, x_2, \ldots, x_K, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_K, t) \, dt \]  \hspace{1cm} (A-1)

is to be extremalized where the functions eligible for the extremalization must obey \( N \leq K \) differential equations as side conditions. The problem is defined as isoperimetric.

A set of comparison functions \( X_i = x_i + \epsilon \eta_i(t) \) \( i = 1, 2, \ldots, K \) is selected with \( \epsilon \) an arbitrary parametric variable and \( \eta_i(t) \) \( i = 1, 2, \ldots, K \) arbitrary functions of \( t \) subject to the restrictions \( \eta_i(t_0) = \eta_i(t_1) = 0, i = 1, 2, \ldots, K \).

The integral becomes

\[ I(\epsilon) = \int_{t_0}^{t_1} f(X_1, X_2, \ldots, X_K, \dot{X}_1, \dot{X}_2, \ldots, \dot{X}_K, t) \, dt, \]

and for \( I(\epsilon) \) to be an extremum for \( \epsilon = 0 \), it is necessary that \( I'(0) = 0 \). Differentiation under the integral sign with respect to \( \epsilon \) produces

\[ I'(\epsilon) = \int_{t_0}^{t_1} \left[ \frac{\partial f}{\partial X_1} \frac{\partial X_1}{\partial \epsilon} + \cdots + \frac{\partial f}{\partial X_K} \frac{\partial X_K}{\partial \epsilon} + \frac{\partial f}{\partial \dot{X}_1} \frac{\partial \dot{X}_1}{\partial \epsilon} + \cdots + \frac{\partial f}{\partial \dot{X}_K} \frac{\partial \dot{X}_K}{\partial \epsilon} \right] \, dt \]

and for \( \epsilon = 0 \) this becomes

\[ I'(0) = \int_{t_0}^{t_1} \left[ \frac{\partial f}{\partial X_1} \eta_1(t) + \cdots + \frac{\partial f}{\partial X_K} \eta_K(t) + \frac{\partial f}{\partial \dot{X}_1} \dot{\eta}_1(t) + \cdots + \frac{\partial f}{\partial \dot{X}_K} \dot{\eta}_K(t) \right] \, dt \]

\[ = 0. \]  \hspace{1cm} (A-2)
N of the arbitrary functions \( \eta_i(t) \), \( i = 1, 2, \ldots, K \) are dependent because of the N differential equations of constraint

\[
G_j(X_1, X_2, \ldots X_K, \dot{X}_1, \dot{X}_2, \ldots \dot{X}_K, t) = 0 \quad j = 1, 2, \ldots, N.
\]

Each of these N equations is differentiated with respect to \( \epsilon \) and multiplied by a Lagrangian variable to give

\[
\lambda_j(t) \left[ \frac{\partial G_j}{\partial x_1} \eta_1(t) + \ldots + \frac{\partial G_j}{\partial x_K} \eta_K(t) + \frac{\partial G_j}{\partial \dot{x}_1} \dot{\eta}_1(t) + \ldots + \frac{\partial G_j}{\partial \dot{x}_K} \dot{\eta}_K(t) \right] = 0
\]

\( j = 1, 2, \ldots, N \)

and, for \( \epsilon = 0 \), this becomes

\[
\lambda_j(t) \left[ \frac{\partial G_j}{\partial x_1} \eta_1(t) + \ldots + \frac{\partial G_j}{\partial x_K} \eta_K(t) + \frac{\partial G_j}{\partial \dot{x}_1} \dot{\eta}_1(t) + \ldots + \frac{\partial G_j}{\partial \dot{x}_K} \dot{\eta}_K(t) \right] = 0
\]

\( j = 1, 2, \ldots, N \) \hspace{1cm} (A-3)

The equations of (A-3) do not disturb the numerical value of the integrand of (A-2), so they may be added to it to form

\[
\Gamma'(0) = \int_{t_0}^{t_1} \left\{ \left[ \frac{\partial f}{\partial x_1} + \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial x_1} \right] \eta_1(t) + \ldots + \left[ \frac{\partial f}{\partial x_K} + \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial x_K} \right] \eta_K(t) \\
+ \left[ \frac{\partial f}{\partial \dot{x}_1} + \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial \dot{x}_1} \right] \dot{\eta}_1(t) + \ldots \\
+ \left[ \frac{\partial f}{\partial \dot{x}_K} + \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial \dot{x}_K} \right] \dot{\eta}_K(t) \right\} dt = 0
\]

\( A-4 \)
A new integrand function is defined to be

\[ F = f + \sum_{j=1}^{N} \lambda_j(t) G_j \]  \hspace{1cm} (A-5)

which when differentiated partially with respect to \( x_i \) and \( x_j \) \( i = 1, 2, \ldots K \) gives

\[
\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial x_i}, \quad \frac{\partial F}{\partial x_i} = \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial x_i}
\]

\( i = 1, 2, \ldots K \quad j = 1, 2, \ldots N \)

The right side of these equations is identical to the bracketed terms of (A-4) so that

\[
\Gamma'(0) = \int_{t_0}^{t_1} \left( \frac{\partial F}{\partial x_1} \eta_1(t) + \ldots + \frac{\partial F}{\partial x_K} \eta_K(t) \right)
\]

\[
+ \frac{\partial F}{\partial x_1} \eta_1(t) + \ldots + \frac{\partial F}{\partial x_K} \eta_K(t) \right) \, dt = 0 \]  \hspace{1cm} (A-6)

Integration by parts on the right half of the integrand of (A-6) is performed using the formula

\[
\int_{t_0}^{t_1} u \, dv = u v \bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} v \, du
\]
\[ u_i = \frac{\partial F}{\partial x_i} \quad dv_i = \eta_i(t) \, dt \] 
\[ du_i = \left(\frac{\partial F}{\partial x_i}\right) v_i = \eta_i(t) \] 
\[ i = 1, 2, \ldots K \]

\[ d\eta_i = \frac{d}{dt} \left(\frac{\partial F}{\partial x_i}\right) dt \]

\[ du_i = d\eta_i \]

\[ \int_{t_0}^{t_1} u_i v_i \, dt = \left. \frac{\partial F}{\partial x_i} \eta_i(t) \right|_{t_0}^{t_1} + \left. \int_{t_0}^{t_1} \eta_i(t) \, d\left(\frac{\partial F}{\partial x_i}\right) dt \right. \]

Equation (A-6) becomes

\[ I'(0) = \left[ \frac{\partial F}{\partial x_1} \eta_1(t) + \ldots + \frac{\partial F}{\partial x_K} \eta_K(t) \right]_{t_0}^{t_1} + \left. \int_{t_0}^{t_1} \left[ \frac{\partial F}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial F}{\partial x_i}\right) \right] \eta_i(t) + \ldots \right. + \left. \frac{\partial F}{\partial x_K} - \frac{d}{dt} \left(\frac{\partial F}{\partial x_K}\right) \eta_K(t) \right] dt = 0 \quad \text{(A-7)} \]

The first term of (A-7) is zero because of the restrictions \( \eta_i(t_0) = \eta_i(t_1) = 0 \) \( i = 1, 2, \ldots K \) so that (A-7) becomes

\[ I'(0) = \int_{t_0}^{t_1} \left[ \frac{\partial F}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial F}{\partial x_i}\right) \right] \eta_i(t) + \ldots + \left[ \frac{\partial F}{\partial x_K} - \frac{d}{dt} \left(\frac{\partial F}{\partial x_K}\right) \right] \eta_K(t) \, dt = 0 \quad \text{(A-8)} \]
The Lagrangian variables \( \lambda_j(t) \) \( j = 1, 2, \ldots, N \) can be algebraically assigned to insure that the coefficients of the first \( N \) terms of (A-8) will vanish. (The \( K - N \) terms can be unconditionally brought to zero because \( \eta_i(t) \) \( i = N + 1, N + 2, \ldots, K \) are independent and arbitrary.) The Euler-Lagrange differential equations of (A-8)

\[
\frac{\partial F}{\partial x_1} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}_1} \right) = 0 \quad i = 1, 2, \ldots, N
\] (A-9)

provide the differential equations for the Lagrangian variables.