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DECOUPLING AND POLE ASSIGNMENT BY DYNAMIC COMPENSATION

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INTRODUCTION

In a previous article [1] the authors defined in geometric terms the decoupling problem for a constant linear multivariable system: namely, the problem of achieving independent control of specified outputs by means of suitably combined inputs and of suitable linear state variable feedback. Necessary and sufficient conditions for decoupling to be possible were found in two important cases; but the general problem is unsolved. However, if in addition to state feedback, dynamic (integrator) compensation may be utilized, it becomes possible to state general necessary and sufficient conditions for decoupling in a simple and constructive way. Geometrically the decoupling synthesis amounts to extending the state space of the original system to a larger space, the increase in dimension being the number of integrators used in dynamic compensation. In addition, state space extension can be used to achieve a desired pole distribution for the closed loop system transfer matrix.

In the present article we state and solve the extended decoupling problem (§ 1). Under certain restrictions, the problem of minimizing the order of dynamic compensation (i.e., the dimension of the extended state space) is solved in § 2. This solution is actually the best possible if the number of scalar inputs is equal to the number of output blocks to be decoupled (§ 3). In § 4 the role of state space extension in pole assignment is determined. It is shown that, with dynamic compensation of high enough order, any pole distribution can be synthesized for the decoupled system, whenever decoupling is possible at all. An example is given in § 5. In conclusion (§ 6) a more general view of decoupling is taken, with the restriction to linear compensation relaxed. The resulting open loop decoupling problem is shown to be equivalent, however, to the extended decoupling problem of § 1.

In the sequel, the material in [1] is assumed to be known.

NOTATION

Script letters $\mathcal{E},\mathcal{E}',\mathcal{R},\mathcal{N},\ldots$ denote vector spaces over the reals, with elements $x,y,\ldots$; $d(\mathcal{E})$ is the dimension of $\mathcal{E}$; $\mathcal{U} \approx \mathcal{V}$ means $\mathcal{U}$, $\mathcal{V}$ are isomorphic, i.e., $d(\mathcal{U}) = d(\mathcal{V})$. $A,B,C,\ldots$ are linear maps; $A|\mathcal{R}$ is the restriction of A to $\mathcal{R}$; $\mathcal{B}$ or $\{B\}$ is the range of B. <u>Spectrum</u> means complex spectrum. A <u>symmetric</u> set of complex numbers is one of the form

$$\{\alpha_1,\alpha_2,\ldots\ ;\ \beta_1,\bar{\beta}_1\ ;\ \beta_2,\bar{\beta}_2\ ;\ \ldots\}$$

where the α_i are real and $\bar{\beta}_i$ is the complex conjugate of β_i. $\mathcal{N}(H)$ is the kernel (null space) of H.

With $A,B,\mathcal{E}'$ fixed, $\underline{C}(\mathcal{V})$ is the set of maps C such that $(A+BC)\mathcal{V} \subset \mathcal{V}$, $\underline{C}'(\mathcal{V})$ the set of C such that $(A+(B+E')C)\mathcal{V}\subset\mathcal{V}$. $\underline{I}$ (resp. $\underline{I}'$) is the class of $\mathcal{V}$ such that $\underline{C}(\mathcal{V})\neq\phi$ (resp. $\underline{C}'(\mathcal{V})\neq\phi$). If $d(\mathcal{E}) = n$, $A : \mathcal{E}\rightarrow\mathcal{E}$ and $\mathcal{B}\subset\mathcal{E}$, then

$$\{A \mid \mathcal{B}\} \equiv \sum_{j=1}^{n-1} A^{j-1}\mathcal{B}$$

$\mathcal{R}\subset\mathcal{E}$ is a <u>controllability subspace</u> (c.s.) for the pair (A,B), written $\mathcal{R}\in \underline{C}$, if $\underline{C}(\mathcal{R})\neq\phi$ and if, for some $C\in \underline{C}(\mathcal{R})$,

$$\mathcal{R} = \{A+BC \mid \mathcal{B}\cap\mathcal{R}\}\ ;$$

$\mathcal{R}$ is determined uniquely, as written, by any $C\in \underline{C}(\mathcal{R})$. Similarly $\mathcal{S}$ is a c.s. for $(A,B+E')$, written $S \in \underline{C}'$, if $\underline{C}'(S) \neq \phi$ and

$$S = \{A+(B+E')C \mid (B+E') \cap S\}, \quad C \in \underline{C}'(S)$$

The maximal (i.e., largest) element of $\underline{I}$ (resp. $\underline{C}$) contained in a subspace T is denoted by max $(\underline{I},T)$ (resp. max $(\underline{C},T)$), and similarly for $\underline{I}'$, $\underline{C}'$. It is known from [1] that these maximal elements exist and are unique for each fixed T and that, if V = max $(\underline{I},T)$, then

$$\max(\underline{C},T) = \{A + BC \mid B \cap V\}, \quad C \in \underline{C}(V)$$

J is the set of integers $(1,\ldots,k)$. Unless otherwise noted, all summations and intersections are over J. If R_i, $i \in J$, is a family of subspaces,

$$R_i^* \equiv \sum_{j \neq i} R_j, \qquad R^* \equiv \bigcap_i R_i^*$$

$$\Delta[R_i, J] \equiv \sum_i d(R_i) - d(\sum_i R_i)$$

Certain auxiliary results needed are collected in the Appendix.

1. EXTENDED DECOUPLING PROBLEM

As in [1] the control system is specified by the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

and output relations

$$y_i(t) = H_i x(t) \qquad i \in J \tag{1.2}$$

The state vector $x \in E$, $d(E) = n$; the control vector $u \in U$, $d(U) = m$;

the output vector $y_i \varepsilon Y_i$, $d(y_i) = q_i$. The maps A, B, H_i are independent of t; in fact, (1.1) qua differential equation plays no role until § 6, as our problem is purely algebraic.

Write $N_i \equiv N(H_i)$, $i\varepsilon J$; as in [1] we assume $N_i \neq E$, $i\varepsilon J$. In [1] we discussed the restricted decoupling problem (RDP): Find $R_i \varepsilon \underline{C}$ $(i\varepsilon J)$ such that

$$\bigcap_i \underline{C}(R_i) \neq \phi \tag{1.3}$$

$$R_i \subset \bigcap_{j\neq i} N_j \qquad i\varepsilon J \tag{1.4}$$

$$R_i + N_i = E \qquad i\varepsilon J \tag{1.5}$$

A family of c.s. $R_i \; \varepsilon \; \underline{C}$ $(i\varepsilon J)$ which satisfies (1.4) (but not necessarily (1.3) or (1.5)) is admissible. Let R_i^M be the maximal admissible c.s. (R_i^M was denoted by $\overline{R}_i$ in [1]). It is clear that RDP is solvable only if

$$R_i^M + N_i = E \qquad i\varepsilon J \tag{1.6}$$

In general (1.6) is not sufficient for solvability of RDP because (1.3) may fail for the R_i^M, i.e., there may not exist any C such that $(A + BC)R_i^M \subset R_i^M$, $i\varepsilon J$. To avoid this difficulty we introduce an xtended decoupling problem as follows.

Adjoin to (1.1) the equation of a new dynamic element:

$$\dot{x}'(t) = I'u'(t) \tag{1.7}$$

where $x'\varepsilon E'$, $u'\varepsilon U'$, $d(E') = d(U') = n'$, and $I':U'\approx E'$;

the input $u'(\cdot)$ can be freely chosen. For the system (1.1) extended by (1.7) define the state space

$$E^e = E \oplus E'$$

and the extended input space

$$U^e = U \oplus U'$$

Define extensions A^e, B^e, E' of A,B,I' as follows:

$$\left.\begin{aligned} A^e &: E^e \to E^e \ ; \quad A^e(x + x') \equiv Ax \ (x \varepsilon E,\ x' \varepsilon E') \\ B^e &: U^e \to E^e \ ; \quad B^e(u + u') \equiv Bu \ (u \varepsilon U,\ u' \varepsilon U') \\ E' &: U^e \to E^e \ ; \quad E'(u + u') \equiv I'u' (u \varepsilon U,\ u' \varepsilon U') \end{aligned}\right\} \qquad (1.8)$$

Below we write A,B for A^e,B^e ; x for vectors in E^e ; and P for the projection $E \oplus E' \to E$. Observe that PA = AP = A, PB = B, $PE' = 0$. The combined system (1.1), (1.7) is now specified by the pair (A,B+E').

The <u>extended decoupling problem</u> (EDP) is the following: <u>Given the original maps</u> $A: E \to E$, $B: U \to E$, <u>and</u> $N_i \subset E$ ($i \varepsilon J$), <u>find</u>: (i) E' (<u>that is</u>, n'), (ii) <u>extensions</u> A,B,E' <u>as in (1.8)</u>, (iii) $S_i \ \varepsilon \ \underline{C}'$ ($i \varepsilon J$), <u>with the properties</u>

$$\bigcap_i \underline{C}'(S_i) \neq \phi \qquad (1.9)$$

$$S_i \subset \bigcap_{j \neq i} (N_j \oplus E') \qquad i \varepsilon J \qquad (1.10)$$

$$S_i + (N_i \oplus E') = E \oplus E' \qquad i \varepsilon J \qquad (1.11)$$

It is clear that the choice of isomorphism I', and so of E' in (1.8), can be arbitrary after n' is fixed: for instance, I' = n'×n' identity matrix, in the coordinates selected.

EDP has the same structure in E^e as RDP has in E, but flexibility is gained from the special form of the new system map A and constraint spaces $N_i \oplus E'$. Justification of EDP as the correct description of decoupling by dynamic compensation is clear: the output relations (1.2) are preserved on replacing N_i by $N_i \oplus E'$ (equivalently by defining extensions H_i^e of H_i to be zero on E') ; no additional control inputs (B) to the original system (1.1) are postulated; subject to the latter constraint, full linear coupling is allowed between (1.1) and (1.7).

Our main result (Th. 1.1) states that decoupling by dynamic compensation is possible if and only if the maximal admissible c.s. R_i^M of RDP are sufficiently large.

<u>Theorem 1.1</u>

<u>For the RDP of (1.3) - (1.5), let R_i^M be the maximal admissible c.s. in $\underline{C}$. The corresponding EDP of (1.9) - (1.11) is solvable if and only if</u>

$$R_i^M + N_i = E \qquad i\varepsilon J \qquad (1.6\text{ bis})$$

<u>Proof</u>

1. (Only if) We show first that $S \varepsilon \underline{C}'$ implies $R \equiv PS \varepsilon \underline{C}$. Since $\underline{C}'(S) \neq \phi$, $AS \subset S+B+E'$, and $AR = PAS \subset R+B$, so that $\underline{C}(R) \neq \phi$. Also, by Th. 2.1 of [1], $S = \lim S^\mu$ ($\mu = 0,1,2,\ldots$) where $S^0 = 0$, $S^{\mu+1} = S \cap (AS^\mu + B + E')$. Write $R^\mu \equiv PS^\mu$. Since $PE' = 0$, Prop. A.4 implies

$$R^{\mu+1} = PS^{\mu+1} = R \cap (AR^\mu + B) \ ;$$

again by [1], Th. 2.1, $R = \lim R^\mu \varepsilon \underline{C}$. Thus $S_i \varepsilon \underline{C}'$ ($i\varepsilon J$) implies $PS_i \varepsilon \underline{C}$ ($i\varepsilon J$) ; and (1.10), (1.11) yield

$$PS_i \subset \bigcap_{j\neq i} N_j \qquad i\varepsilon J \qquad (1.12)$$

$$PS_i + N_i = E \qquad i\varepsilon J \qquad (1.13)$$

By (1.12) and maximality of the R_i^M, $PS_i \subset R_i^M$; this and (1.13) imply (1.6).

2. (If) Assuming (1.6) holds, define $n' = \sum_i d(R_i^M)$. With n' so large, there clearly exist maps $M_i : E^e \to E'$ with the properties:

$$R_i^M \cap N(M_i) = 0, \quad \{M_i\} = M_i R_i^M \qquad i\varepsilon J$$

and the ranges $\{M_i\}$ ($i\varepsilon J$) independent. Define $S_i = (P + M_i)R_i^M$ ($i\varepsilon J$). Then

$$AS_i = AR_i^M \subset R_i^M + B \subset S_i + B + E', \qquad i\varepsilon J \quad ;$$

and since the S_i are clearly independent, there exists $C: E^e \to U^e$ with $C\varepsilon \bigcap_j \underline{C}(S_j)$. It will be shown that $S_i \varepsilon \underline{C}'$. Dropping the subscript i, suppose $R\varepsilon\underline{C}$, so that the relations

$$R^o \equiv 0, \quad R^{\mu+1} \equiv R \cap (AR^\mu + B) \qquad (\mu = 0,1,..)$$

imply $R_\mu \uparrow R$. Let $\{M\} \subset E'$ and

$$S \equiv (P + M)R, \quad S_o^0 \equiv 0, \quad S^{\mu+1} \equiv S \cap (AS^\mu + B + E')$$

Then $S^o \supset (P + M)R^o$; and if $S^\mu \supset (P + M)R^\mu$,

$$S^{\mu+1} \supset [(P + M)R] \cap [A(P + M)R^\mu + B + E']$$

$$= [(P + M)R] \cap [AR^\mu + B + E']$$

$$\supset (P + M)[R \cap (AR^{\mu} + B + E')]$$

$$= (P + M)R^{\mu+1}$$

By induction $S \supset S^{\mu} \supset (P + M)R^{\mu} \uparrow (P + M)R = S$, i.e., $S^{\mu} \uparrow S$; so $S \varepsilon \underline{C}'$. Application of this argument to the R_i^M and S_i yields the desired result.

The relation $PS_i = R_i^M$ implies

$$S_i \subset R_i^M \oplus E' \subset (\bigcap_{j \neq i} N_j) + E' = \bigcap_{j \neq i} (N_j \oplus E') \qquad (1.10 \text{ bis})$$

By (1.6)

$$S_i + (I + M_i)N_i = (I + M_i)E$$

and addition of E' to both sides yields (1.11). ∎

Remark 1 The proof reveals the symmetry between a c.s. and its extension: if $R \varepsilon \underline{C}$ and $S \equiv (I + M)R$ with $\{M\} \subset E'$ then $S \varepsilon \underline{C}'$. Conversely if $S \varepsilon \underline{C}'$ then $R \equiv PS \varepsilon \underline{C}$.

Remark 2 In part 2 of the proof the S_i were constructed to be independent. By [1], Th. 2.2, $C \varepsilon \bigcap_i \underline{C}(S_i)$ can be chosen such that, for each i, the spectrum of $(A + (B + E')C)|S_i$ is any symmetric set of $d(S_i)$ complex numbers.

Remark 3 Condition (1.6) is not implied by controllability of ,B), i.e., by the condition $\{A|B\} = E$. For example let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_1 = (1,0,0), \quad H_2 = (0,1,0)$$

By the methods of [1] one finds

$$R_1^M = R_2^M = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

and (1.6) fails for i = 1, 2.

The following description of the structure of a decoupled system will be applied later to solutions of EDP. The result is stated for RDP for simplicity of notation. Let R_i, $i \in J$, be any solution of RDP, write $R \equiv \sum_i R_i$, and write $A_c \equiv A + BC$ for $C \in \underline{C} \equiv \bigcap_i \underline{C}(R_i)$. Let $\bar{x}$ be the coset of x in E/R^*. Noting that $A_c R^* \subset R^*$, we define the induced map $\bar{A}_c : E/R^* \to E/R^*$ by $\bar{A}_c \bar{x} \equiv \overline{A_c x}$.

<u>Theorem 1.2</u>

<u>There exist</u> $\hat{R}_i \subset R_i$, $i \in J$, <u>independent of</u> $C \in \underline{C}$ <u>such that</u>

$$R \equiv R^* \oplus \hat{R}_1 \oplus \dots \oplus \hat{R}_k \tag{1.15}$$

<u>and</u>

$$\hat{R}_i \approx \bar{R}_i \equiv (R_i + R^*)/R^* \tag{1.16}$$

<u>The</u> $\hat{R}_i$ <u>satisfy</u>

$$\hat{R}_i + N_i = E \qquad i \in J \tag{1.17}$$

$$A_c \hat{R}_i \subset \hat{R}_i \oplus R^* \qquad i \in J,\ C \in \underline{C} \tag{1.18}$$

<u>The spectrum of</u> $\bar{A}_c | \bar{R}_i$ ($i \in J$) <u>can be assigned as any symmetric set of</u> $d(\bar{R}_i)$ <u>complex numbers by suitable choice of</u> $C \in \underline{C}$.

<u>Proof</u> Let $\hat{R}_i$ be any subspace such that $R_i = \hat{R}_i \oplus R_i \cap R^*$. Independence of R^*, $\hat{R}_i$ ($i \in J$) follows by Prop. A.1; hence (1.15) is true, and (1.16) is clear. Since

$$R^* \subset \bigcap_i \sum_{j\neq i} \bigcap_{\alpha\neq j} N_\alpha = \bigcap_i N_i,$$

(1.17) follows from (1.4) and (1.5). Since $A_c R_i \subset R_i$, (1.18) is clear.

Let $C_o \varepsilon \underline{C}$ be fixed; write $A_o \equiv A_{c_o}$, $\bar{A}_o \equiv \bar{A}_{c_o}$; and let Q be the projection: $E \to E/R^*$; thus $\bar{A}_o Q = QA_o$. Let $B_i : U \to E$ be any map with range $B \cap R_i$, and write $\bar{B}_i \equiv QB_i$, $\bar{B}_i \equiv Q(B \cap R_i)$. It will be shown that $\bar{R}_i$ is a c.s. for the pair $(\bar{A}_o, \bar{B}_i)$. In fact

$$\bar{R}_i = QR_i = Q\{A_o \mid B \cap R_i\}$$

$$= \{\bar{A}_o \mid Q(B \cap R_i)\}$$

$$= \{\bar{A}_o \mid \bar{B}_i\}$$

and the assertion follows. By Prop. A.1 the $\bar{R}_i$ are independent. Hence (cf. [1] §§ 4,5) there exist $\bar{D}_i : E/R^* \to U$ $(i\varepsilon J)$ such that $\bar{D}_i \bar{R}_j = 0$ $(i,j\varepsilon J \;;\; j\neq i)$, $(\bar{A}_o + \bar{B}_i\bar{D}_i)\bar{R}_i \subset \bar{R}_i$ $(i\varepsilon J)$ and $(\bar{A}_o + \bar{B}_i\bar{D}_i)|\bar{R}_i$ $(i\varepsilon J)$ has any pre-assigned spectrum. Define $D_i = \bar{D}_i Q$ $(i\varepsilon J)$. Then $D_i(R_j + R^*) = 0$ $(i,j\varepsilon J \;;\; j\neq i)$ and $B_i D_i R_i \subset B_i \subset R_i$. Let $D: E \to U$ be any map such that $BD = \sum_i B_i D_i$; D exists since $\{B_i\} \subset B$ $(i\varepsilon J)$. Then the map $C: E \to U$ defined by $C = C_o + D$ has the properties required. ∎

2. MINIMAL STATE SPACE EXTENSION

Theorem 1.1 shows that if (1.6) holds, EDP can always be solved by dynamic compensation of order $n' \leq \sum_i d(R_i^M)$. There is then a least integer $n_o \geq 0$ for which EDP is solvable with $n' = n_o$; in case $n_o = 0$, the corresponding EDP reduces to RDP. From a practical viewpoint it is of interest to find n_o: we call this the problem of minimal state space extension, or of minimal solution of EDP.

The general problem of minimal extension includes the general solvability problem for RDP, and is unsolved. However, suppose the additional constraint is imposed, that

$$PS_i = \mathcal{R}_i^M \qquad i \varepsilon J \tag{2.1}$$

where the $\mathcal{R}_i^M$ are the maximal admissible c.s. in $\underline{C}$. In this case it will be shown how to compute the minimal n', say n_M. In general $n_M > n_o$, because (2.1) rules out extension of any $\mathcal{R}_i \varepsilon \underline{C}$ which is properly contained in $\mathcal{R}_i^M$, but which still may be large enough to satisfy (1.5). However, if $d(\mathcal{B}) = k$, it will be shown in § 3 that (2.1) holds for any solution of EDP, hence $n_M = n_o$, and so this case will be solved completely.

It is convenient for later purposes to adjoin to (2.1) the additional constraint

$$PS^* \subset \mathcal{V} \tag{2.2}$$

where $\mathcal{V}$ is a subspace such that

$$\mathcal{V} \varepsilon \underline{I}, \ \mathcal{V} \subset (\mathcal{R}^M)^* \tag{2.3}$$

In (2.3) $(\mathcal{R}^M)^*$ is the $*$ space (see Notation) of the family $\mathcal{R}_i^M$, $i \varepsilon J$. Relations of form (2.2) arise in the synthesis of pole distributions (§ 4). With $\mathcal{V}$ fixed, let

$$\mathcal{R}_o^M(\mathcal{V}) \equiv \bigcap_i \sum_{j \neq i} (\mathcal{R}_j^M \cap \mathcal{V}) \tag{2.4}$$

In the remainder of this section we write $\mathcal{R}_i \equiv \mathcal{R}_i^M$, $i \varepsilon \{0\} \cup J$.

<u>Theorem 2.1</u>

<u>For the RDP of (1.3) - (1.5) let</u> $\mathcal{R}_i$ $(i \varepsilon J)$ <u>be the maximal</u>

admissible c.s. in C, and assume (1.6) is true. If V satisfies (2.3) and if

$$d(E') \geq n_M(V) \equiv \Delta[(R_i + R_o(V))/R_o(V), J] \tag{2.5}$$

then a solution S_i $(i\varepsilon J)$ of EDP exists such that $PS_i = R_i$ $(i\varepsilon J)$ and $S^* \subset R_o(V)$.

Conversely if EDP has a solution S_i $(i\varepsilon J)$ such that $PS_i = R_i$ $(i\varepsilon J)$ then

$$PS^* \varepsilon \underline{I} \;, \quad PS^* \subset R^* \tag{2.6}$$

If for some V, $PS^* \subset V$ then $PS^* \subset R_o(V)$ and (2.5) is true. If equality holds in (2.5) then $PS^* = R_o(V)$ and $(S_i + S^*) \cap E' = 0$, $i\varepsilon J$.

Corollary

For the RDP of (1.3) - (1.5), suppose (1.6) is true and let $V^M = \max(\underline{I}, R^*)$. Under the constraint (2.1) there exists a solution $\{E', S_i, i\varepsilon J\}$ of EDP if and only if $d(E') \geq n_M(V^M)$.

Existence of S_i will be proved by a refinement of the construction used in the proof of Th. 1.1. For this we need Lemmas 2.1 - 2.3. Of these the first two assert general properties of extensions.

Lemma 2.1

Let $U_i \subset E$ $(i\varepsilon J)$. If $d(E') \geq \delta \equiv \Delta[U_i, J]$ there exist maps $M_i : E^e \to E'$ $(i\varepsilon J)$ such that the subspaces $V_i \equiv (P + M_i)U_i$ $(i\varepsilon J)$ are independent.

Proof Write $W_1 = 0$, $W_i = U_i \cap \sum_{j=1}^{i-1} U_j$ $(i = 2,\dots,k)$. Then

$$\sum_{i=1}^{k} d(W_i) = \sum_{i=2}^{k} \Big[d(U_i) + d\Big(\sum_{j=1}^{i-1} U_j\Big) - d\Big(\sum_{j=1}^{i} U_j\Big)\Big]$$

$$= \delta \;;$$

hence there exist M_i such that $N(M_i) \cap W_i = 0$, $\{M_i\} = M_i W_i$ and the $\{M_i\}$, $i \varepsilon J$, are independent. Suppose the V_i are not independent and let $i \geq 2$ be the greatest integer such that $V_i \cap V_i^* \neq 0$. There is $x \neq 0$ such that

$$x = (P + M_i)u_i = \sum_{j=1}^{i-1} (P + M_j)u_j,$$

where $u_j \varepsilon U_j$ $(1 \leq j \leq i)$, so that

$$Pu_i = u_i = \sum_{j=1}^{i-1} Pu_j = \sum_{j=1}^{i-1} u_j$$

and $u_i \varepsilon W_i$. By independence of the $\{M_j\}$, $M_i u_i = 0$, hence $u_i = 0$ and $x = 0$, a contradiction. ∎

<u>Lemma 2.2</u>

<u>Let</u> V, $R_i \subset E$ $(i \varepsilon J)$ <u>and define</u>

$$R_o \equiv \bigcap_i \sum_{j \neq i} (R_j \cap V)$$

$$\delta \equiv \Delta[(R_i + R_o)/R_o, J]$$

<u>If</u> $d(E') \geq \delta$ <u>there exist maps</u> $M_i : E^e \to E'$ $(i \varepsilon J)$ <u>such that, if</u> $V_i \equiv (P + M_i)R_i$ $(i \varepsilon J)$, <u>then</u> $V^* = R_o$.

<u>Proof</u> Write $\bar{R}_i \equiv (R_i + R_o)/R_o$ $(i \varepsilon J)$ and let $\bar{P}$ be the projection: $E \oplus E' \to (E/R_o) \oplus E'$. By Lemma 2.1 there exist $\bar{M}_i : (E/R_o) \oplus E' \to E'$ such that $\bar{V}_i \equiv (\bar{P} + \bar{M}_i)\bar{R}_i$ $(i \varepsilon J)$ are independent subspaces of $(E/R_o) \oplus E'$. Let $M_i = \bar{M}_i \bar{P}$; then V_i is well defined. Since $\bar{V}_i = (V_i + R_o)/R_o$, it follows by independence of the $\bar{V}_i$ $(i \varepsilon J)$ and Prop. A.1 that $R_o \supset V^*$. For the reverse inclusion observe that, by (A.1) and (A.3),

$$R_o = \sum_i (R_i \cap R_o \cap \sum_{j \neq i} (R_j \cap R_o))$$

and that $M_i R_o = 0$ $(i \varepsilon J)$. Then $x \varepsilon R_o$ implies $x = \sum_i x_i$, with

$$x_i = \sum_{j \neq i} x_{ij} \; ; \; x_i \varepsilon R_i \cap R_o \; ; \; x_{ij} \varepsilon R_j \cap R_o$$

and

$$x_i = (P + M_i)x_i \; \varepsilon \; V_i \; ; \; x_{ij} = (P + M_j)x_{ij} \; \varepsilon \; V_j$$

Thus $x_i \varepsilon V_i \cap V_i^*$, so $x \; \varepsilon \; V^*$. ∎

Lemma 2.3

Let R_i $(i \varepsilon J)$ satisfy the hypotheses of Th. 2.1 and let V satisfy (2.3). If R_o is defined by (2.4) then $R_o \varepsilon \underline{I}$.

Proof Since $R^* \subset \bigcap_{j \neq i} N_j$ $(i \varepsilon J)$ there follows $V \subset V_i$ $(i \varepsilon J)$ where $V_i \equiv \max\,(\underline{I}, \bigcap_{j \neq i} N_j)$. Hence for each $i \varepsilon J$ there exists $C_i : U \rightarrow E$ in $\underline{C}(V_i) \cap \underline{C}(V)$. Since $R_i = \max\,(\underline{C}, V_i)$, there follows $R_i = \{A + BC_i \mid B \cap V_i\}$, so that $C_i \varepsilon \underline{C}(R_i) \cap \underline{C}(V) \subset \underline{C}(R_i \cap V)$. That is, $R_i \cap V \varepsilon \underline{I}$ $(i \varepsilon J)$, hence $\sum_{j \neq i} (R_j \cap V) \varepsilon \underline{I}$ $(i \varepsilon J)$. Now apply the same argument to the pair of subspaces R_i, $\tilde{V}_i \equiv \sum_{j \neq i} (R_j \cap V)$ to get that $R_i \cap \tilde{V}_i \; \varepsilon \; \underline{I}$ $(i \varepsilon J)$. Finally, use (A.1), (A.3) to obtain $R_o = \sum_i R_i \cap \tilde{V}_i \; \varepsilon \; \underline{I}$.

Proof of Theorem 2.1 (direct statement) Lemmas 2.2 and 2.3 provide E' and $V_i \subset E \oplus E'$ $(i \varepsilon J)$ with the properties: $d(E') = n_M(V)$, and

$$PV_i = R_i \; (i \varepsilon J), \; V^* = R_o, \; V^* \; \varepsilon \; \underline{I} \qquad (2.7\text{ a,b,c})$$

Since $\underline{I} \subset \underline{I}'$, $V^* \; \varepsilon \; \underline{I}'$. Also, by (2.7a)

$$AV_i \subset A(R_i + E') \subset R_i + B \subset V_i + E' + B, \qquad i \varepsilon J;$$

hence $V_i \; \varepsilon \; \underline{I}'$, and

$$V_i + V^* \in \underline{I}' \qquad i \in J \qquad (2.8)$$

Because the factor spaces $(V_i + V^*)/V^*$ are independent, there exists $C \in \bigcap_i \underline{C}'(V_i + V^*)$. Define

$$S_i = \{A + (B + E')C \mid (B + E') \cap (V_i + V^*)\} \quad i \in J \qquad (2.9)$$

It will be shown that $PS_i = R_i$ and $S^* \subset R_o$. By Remark 1 after Th. 1.1, $PS_i \in \underline{C}$; also

$$PS_i \subset P(V_i + V^*) = R_i + R_o \subset R_i + R^* \subset \bigcap_{j \neq i} N_j$$

Since R_i is maximal, $PS_i \subset R_i$. For the reverse inclusion, by Prop. A.5,

$$PS_i \supset P[(B + E') \cap (V_i + V^*)] = B \cap (R_i + V^*) \supset B \cap R_i$$

Since $PS_i \subset R_i$ there exists $C_i \in \underline{C}(PS_i) \cap \underline{C}(R_i)$. Thus

$$PS_i = \{A + BC_i \mid B \cap PS_i\} \supset \{A + BC_i \mid B \cap R_i\} = R_i \ ;$$

and so $PS_i = R_i$ $(i \in J)$. Finally, by (A.2),

$$S^* \subset \bigcap_i \sum_{j \neq i} (V_j + V^*) = V^* = R_o \qquad \blacksquare$$

The idea of this proof was to use (2.9) to manufacture 'compatible' c.s. contained in the $V_i + V^*$. The method works because the $V_i + V^*$ satisfy (2.8). For this one needs (2.7c), which is guaranteed (Lemma 2.3) by maximality of the R_i, and also $V_i \in \underline{I}'$, which follows by $R_i \in \underline{I}$. Maximality ensures also that $R_i \supset PS_i$.

Proof of Theorem 2.1 (converse statement) Since S_i $(i\varepsilon J)$ is a solution of EDP, $S^*\varepsilon \underline{I}'$, hence $PS^*\varepsilon\underline{I}$. Since $PS_i = R_i$ $(i\varepsilon J)$, clearly $PS^* \subset R^*$, so (2.6) is true. Let $PS^* \subset V$. Then $S^* = \sum_{j\neq i} S_j \cap S^*$ $(i\varepsilon J)$ implies $PS^* \subset \sum_{j\neq i}(R_j \cap V)$ $(i\varepsilon J)$ and so $PS^* \subset R_o(V)$. By Prop. A.2 (where S_o is defined)

$$d(E') \geq \delta_1 \equiv \Delta[(S_i + S_o + E')/(S_o + E'), J]$$

$$= \Delta[\Gamma(S_i + S_o)/PS_o\ , J]$$

$$= \Delta[(R_i + R_o)/R_o\ , J]$$

Finally, if $d(E') = \delta_1$, Prop. A.3 implies $S^* + E' = S_o$, hence $PS^* = R_o$; and also $(S_i + S^*) \cap E' = 0$. $\blacksquare$

Proof of Corollary Any solution of EDP subject to (2.1) satisfies (2.6), hence $PS^* \subset V^M$, and by (2.5) $d(E') \geq n_M(V^M)$. Thus $n_M(V^M)$ is the least integer for which EDP is solvable subject to (2.1). $\blacksquare$

3. MINIMAL EXTENSION WHEN d(B) = k

Assume $d(B) = k$ and let S_i $(i\varepsilon J)$ be any solution of EDP. It will be shown that

$$R_i \equiv PS_i = R_i^M \qquad i\varepsilon J \tag{3.1}$$

By Remark 1 after Th. 1.1, $R_i \ \varepsilon\ \underline{C}$ and clearly the R_i satisfy (1.4), (1.5). It is enough to show that

$$d(B \cap R_i^M) = 1 \qquad i\varepsilon J \tag{3.2}$$

In fact, since $N_i \neq E$, (1.5) implies $R_i \neq 0$, hence (3.2) implies

$B \cap R_i = B \cap R_i^M$. Since $R_i \subset R_i^M$ there exists $C_i \varepsilon \underline{C}(R_i) \cap \underline{C}(R_i^M)$.
Thus

$$R_i = \{A + BC_i \mid B \cap R_i\} = \{A + BC_i \mid B \cap R_i^M\} = R_i^M$$

To verify (3.2) start from

$$d\left(B \cap \sum_{i=1}^{j} R_i^M\right) \leq d\left(B \cap \sum_{i=1}^{j+1} R_i^M\right) \quad 1 \leq j \leq k-1 \tag{3.3}$$

If (3.3) holds with equality for $j = \ell$, then

$$B \cap \sum_{i=1}^{\ell} R_i^M = B \cap \sum_{i=1}^{\ell+1} R_i^M \tag{3.4}$$

Write $P \equiv \sum_{i=1}^{\ell} R_i^M$. Then (3.4) implies

$$B \cap (P + R_{\ell+1}^M) = B \cap P + B \cap R_{\ell+1}^M$$

so that (Prop. A.4)

$$P \cap (B + R_{\ell+1}^M) = B \cap P + P \cap R_{\ell+1}^M$$

Then

$$A(P \cap R_{\ell+1}^M) \subset (B + P) \cap (B + R_{\ell+1}^M)$$
$$\subset B + P \cap R_{\ell+1}^M$$

By Lemma (5.1) of [1] there exists

$$C \;\varepsilon\; \underline{C}(P) \cap \underline{C}(R_{\ell+1}^M)$$

so that

$$R_{\ell+1}^M = \{A + BC \mid B \cap R_{\ell+1}^M\} \subset \{A + BC \mid P\}$$
$$\subset P \subset N_{\ell+1}$$

in contradiction to (1.5). Therefore (3.3) holds with inequality at each j. Since

$$d(\mathcal{B} \cap \sum_{i=1}^{k} \mathcal{R}_i^M) \leq d(\mathcal{B}) = k$$

and $d(\mathcal{B} \cap \mathcal{R}_i^M) \geq 1$ $(i \in J)$ the result (3.2) follows. Combining (3.1) with Th. 2.1, Corollary, we obtain:

Theorem 3.1

Let $d(\mathcal{B}) = k$. For the RDP of (1.3) - (1.5) suppose (1.6) is true, and let $\mathcal{V}^M = \max(\underline{I}, (\mathcal{R}^M)^*)$. There exists a solution $\{\mathcal{E}', \mathcal{S}_i, i \in J\}$ of EDP if and only if $d(\mathcal{E}') \geq n_M(\mathcal{V}^M)$, where n_M is given by (2.5).

4. STATE SPACE EXTENSION AND POLE ASSIGNMENT

With the minimal extension of § 2 or § 3 it may happen that some poles of the closed loop transfer matrix are necessarily fixed at unstable, or otherwise 'bad', locations. It is possible to shift the bad poles by additional dynamic compensation. This aim is achieved by choosing the extension such that all the fixed eigenvalues of A + (B + E')C are 'good'.

To identify the fixed eigenvalues we need the following Lemmas.

Lemma 4.1

Let $\mathcal{V} \in \underline{I}$, write $\underline{C} \equiv \underline{C}(\mathcal{V})$, and let $\mathcal{R} = \max(\underline{C}, \mathcal{V})$. Write $A_c \equiv A + BC$, $C \in \underline{C}$, and define $\bar{A}_c$: $\mathcal{V}/\mathcal{R} \rightarrow \mathcal{V}/\mathcal{R}$ as follows: if $\bar{x}$ is the coset of x in $\mathcal{V}/\mathcal{R}$, $\bar{A}_c\bar{x} \equiv \overline{A_c x}$. Then $\mathcal{R}$ and $\bar{A}_c$ are constant with respect to $C \in \underline{C}$. In particular the characteristic polynomial (ch. p.) of $A_c|\mathcal{V}$ has the form $\pi(\lambda)\pi_c(\lambda)$, where π is the ch. p. of $\bar{A}_c$ and is fixed for all $C \in \underline{C}$; π_c is the ch. p. of $A_c|\mathcal{R}$; and the roots of π_c can be assigned arbitrarily by suitable choice of $C \in \underline{C}$.

Proof

By [1], Th. 2.3,

$$\mathcal{R} = \{A + BC \mid \mathcal{B} \cap \mathcal{V}\}, \quad C \in \underline{C}$$

and $\underline{C} \subset \underline{C}(\mathcal{R})$. If $C_1, C_2 \in \underline{C}$ and $x \in \mathcal{V}$ then $A_{c_i} x \in \mathcal{V}$ $(i = 1,2)$ and

$$(A_{c_1} - A_{c_2})x = B(C_1 - C_2)x \in \mathcal{B} \cap \mathcal{V} \subset \mathcal{R},$$

hence $\bar{A}_{c_1} = \bar{A}_{c_2}$. Assignability of the roots of π_c follows by [1], Th. 2.2. $\blacksquare$

Lemma 4.2

Under the conditions of Lemma 4.1, let $\alpha(\lambda)$ be the minimal polynomial of $\bar{A}_c$, and factor $\alpha(\lambda) = \alpha_g(\lambda)\alpha_b(\lambda)$, where the polynomials α_g, α_b are coprime. Then

$$\mathcal{V} = \mathcal{R} \oplus \mathcal{R}_g \oplus \mathcal{R}_b \tag{4.1}$$

where

$$\mathcal{R} \oplus \mathcal{R}_g = \{x : x \in \mathcal{V}, \ \alpha_g(\bar{A}_c)\bar{x} = \bar{0}\} \tag{4.2}$$

and similarly for $\mathcal{R} \oplus \mathcal{R}_b$. The subspaces $\mathcal{R} \oplus \mathcal{R}_g$, $\mathcal{R} \oplus \mathcal{R}_b$ are fixed with respect to $C \in \underline{C}$.

Proof Since α_g, α_b are coprime, $\mathcal{V}/\mathcal{R} = \bar{\mathcal{R}}_g \oplus \bar{\mathcal{R}}_b$, where

$$\bar{\mathcal{R}}_g = \{\bar{x} : \bar{x} \in \mathcal{V}/\mathcal{R}, \ \alpha_g(\bar{A}_c)\bar{x} = \bar{0}\}$$

$$\bar{\mathcal{R}}_b = \{\bar{x} : \bar{x} \in \mathcal{V}/\mathcal{R}, \ \alpha_b(\bar{A}_c)x = \bar{0}\}$$

Since R and $\bar{A}_c$ are constant with respect to $C \varepsilon \underline{C}$, the result follows. ∎

Lemma 4.3

Let $W \varepsilon \underline{I}'$ and let $S = \max(\underline{C}', W)$. Write $V \equiv PW$ and $R \equiv PS$.

Then

1. $V \varepsilon \underline{I}$, and $R = \max(\underline{C}, V)$.
2. $V/R \approx W/S$
3. The fixed eigenvalues of $(A + (B + E')C)|W$, $C \varepsilon \underline{C}'(W)$, coincide with the fixed eigenvalues of $(A + BC_o)|V$, $C_o \varepsilon \underline{C}(V)$.

Proof

1. If $AW \subset W + B + E'$, $AV \subset PAW \subset V + B$, so $V \varepsilon \underline{I}$. Let $R^M \equiv \max(\underline{C}, V)$ and define

$$R^o \equiv 0, \quad R^{\mu+1} \equiv V \cap (AR^\mu + B) \quad (\mu = 0,1,\ldots) \tag{4.3}$$

It will be shown that $T \equiv \lim R^\mu = R^M$. Since $R^\mu \subset V$ and $AV \subset V + B$,

$$AR^\mu \subset (V + B) \cap (AR^\mu + B) = R^{\mu+1} + B,$$

so that $AT \subset T + B$. Since $R^\mu \subset T \subset V$ $(\mu=0,1,\ldots)$, (4.3) implies

$$R^{\mu+1} = T \cap (AR^\mu + B) \qquad (\mu = 0,1,\ldots)$$

By [1], Th. 2.1, $T \varepsilon \underline{C}$ and $T \subset V$, hence $T \subset R^M$. On the other hand $R^M = \lim \hat{R}^\mu$, where $\hat{R}^o = 0$ and

$$\hat{R}^{\mu+1} = R^M \cap (A\hat{R}^\mu + B) \qquad (\mu = 0,1,\ldots)$$

Since $R^M \subset V$, by induction on μ we have $\hat{R}^\mu \subset R^\mu$, hence $R^M \subset T$. Thus the rule (4.3) computes $\max(\underline{C}, V)$.

Applying this result to the pair S,W we have $S = \lim S^\mu$, where $S^o = 0$ and

$$S^{\mu+1} = W \cap (AS^\mu + B + E') \quad (\mu = 0,1,\ldots)$$

Thus (Prop. A.4) $PS^{\mu+1} = V \cap (APS^{\mu}+B)$, and comparison with (4.3) yields

$$R^M = \lim R^{\mu} = \lim PS^{\mu} = PS$$

2. In general

$$W/S \approx (W + E')/(S + E') \oplus (W \cap E')/(S \cap E')$$

But

$$\frac{W + E'}{S + E'} \approx \frac{(W + E')/E'}{(S + E')/E'} \approx PW/PS = V/R \tag{4.4}$$

Also, for $C \varepsilon \underline{C}'(W)$,

$$S = \{A + (B + E')C \mid (B + E') \cap W\}$$

so that $W \cap E' \subset S$, hence

$$(W \cap E')/(S \cap E') \approx 0$$

3. Let $C \varepsilon \underline{C}'(W)$. It will be shown that there exist

$$C_1 \varepsilon\ \underline{C}(S + E') \cap \underline{C}(V), \quad C_0 \varepsilon\ \underline{C}(V) \tag{4.5 a,b}$$

such that the diagram commutes. By the isomorphisms shown, the result

$$\begin{tikzcd}
W \arrow[r, "A + (B + E')C"] \arrow[d] & W \arrow[d] \\
W/S \arrow[r, "\overline{A + (B + E')C}"] \arrow[d] & W/S \arrow[d] \\
\frac{W + E'}{S + E'} \arrow[r, "\overline{A + BC_1}"] \arrow[d] & \frac{W + E'}{S + E'} \arrow[d] \\
V/R \arrow[r, "\overline{A + BC_0}"] & V/R
\end{tikzcd}$$

will then follow from Lemma 4.1. In the diagram a bar denotes the induced map in the indicated factor space. Turning to the proof, since

$$(A+(B+E')C)\ \mathcal{S} \subset \mathcal{S},$$

the top square commutes (by definition of bar). Recall that $\mathcal{E}' \cap \mathcal{W} \subset \mathcal{S}$ and write

$$\begin{aligned} \mathcal{W} &= ((\mathcal{S}+\mathcal{E}') \cap \mathcal{W}) \oplus \mathcal{Z} \\ &= \mathcal{S} \oplus \mathcal{Z} \end{aligned} \tag{4.6}$$

ince $A\mathcal{E}' = 0$ and $\mathcal{S} \varepsilon \underline{I}'$, there follows $\mathcal{S} + \mathcal{E}' \varepsilon \underline{I}'$, and there exists $C_1 \varepsilon \underline{C}(\mathcal{S}+\mathcal{E}')$ such that $(C_1-C)\mathcal{Z} = 0$. Then $\overline{A+BC_1}$ is defined, and

$$[A+(B+E')C - (A+BC_1)]\ \mathcal{W} \subset \mathcal{S} + \mathcal{E}',$$

so the middle square commutes. Clearly $(A+BC_1)(\mathcal{W}+\mathcal{E}') \subset \mathcal{W}+\mathcal{E}'$; since $\mathcal{V} \subset \mathcal{W} + \mathcal{E}'$, $(A+BC_1)\mathcal{V} = P(A+BC_1)\mathcal{V} \subset P(\mathcal{W}+\mathcal{E}') = \mathcal{V}$, i.e., $C_1 \varepsilon \underline{C}(\mathcal{V})$ and (4.5a) is true. By (4.4) and (4.6), $P\mathcal{Z} = \mathcal{Z}$, i.e., $\mathcal{V} = \mathcal{R} \oplus P\mathcal{Z}$, and C_o exists such that

$$(C_o-C_1)\mathcal{R} = 0, \quad (C_oP-C_1)\mathcal{Z} = 0 \tag{4.7}$$

Then

$$[(A+BC_o)P - P(A+BC_1)]\mathcal{Z} = 0 \tag{4.8}$$

Also, if $x \varepsilon \mathcal{S}$ then $Px \varepsilon \mathcal{R}$ and

$$\begin{aligned} (A+BC_o)Px &= (A+BC_1)Px \\ &= (A+BC_1)(x+e'), \text{ for some } e' \varepsilon \mathcal{E}' \\ &= (A+BC_1)x+e'', \text{ for some } e'' \varepsilon \mathcal{S}+\mathcal{E}' \end{aligned}$$

so that

$$(A+BC_0)Px = P(A+BC_0)Px$$

$$= P(A+BC_1)x + Pe" \tag{4.9}$$

and $Pe"\varepsilon R$. Then (4.6), (4.8) and (4.9) imply

$$[(A+BC_0)P - P(A+BC_1)]\mathcal{W} \subset R,$$

so the bottom square of the diagram commutes. Finally it is clear from (4.5a) and (4.7) that (4.5b) is true. ∎

We now state a procedure for minimal extension of c.s. R_i^M to achieve both decoupling and an assigned distribution of eigenvalues of $A+(B+E')C$. We write $R_i \equiv R_i^M$ and assume the hypotheses of Th. 2.1.

Extension procedure (EXT)

Here A will denote the original map in E, not its extension, and similarly for B,C. Under the conditions of Th. 2.1, let $V^M \equiv \max \cdot (\underline{I}, R^*)$ $R^M \equiv \max(\underline{C}, V^M)$. For $C\varepsilon\underline{C} \equiv \underline{C}(V^M)$, write $A_c \equiv A+BC$, and let $\alpha(\lambda)$ be the minimal polynomial (mod R^M) of $A_c|V^M$. Factor $\alpha(\lambda) = \alpha_g(\lambda)\alpha_b(\lambda)$, where the roots of $\alpha_g(\alpha_b)$ are good (bad). For arbitrary $C\varepsilon\underline{C}$ determine

$$R^M \oplus R_g \equiv \{x: x\varepsilon V^M,\ \alpha_g(A_c)x\varepsilon R^M\} \tag{4.10}$$

In (2.4) substitute $V = R^M \oplus R_g$, compute $R_o \equiv R_o(V)$, and construct a minimal solution of EDP as in the proof (direct half) of Th. 2.1.

With EXT completed, a solution of EDP is now in hand: symbols A etc. will again denote the extended maps, defined by (1.8). Write $\underline{C}' \equiv \bigcap_i \underline{C}'(S_i)$, $A_c \equiv A+(B+E')C$.

Theorem 4.1

Any solution E', $S_i(i\varepsilon J)$ of EDP determined by EXT has the following properties:

1. $R^M \subset S^* \subset R^M \oplus R_g$ (4.11)

2. If $C\epsilon\underline{C}'$ the ch.p. $\pi_c^*(\lambda)$ of $A_c|S^*$ can be factored as

$$\pi_c^*(\lambda) = \pi_g(\lambda)\pi_c^M(\lambda) \tag{4.12}$$

Here the roots of π_g are fixed for $C\epsilon\underline{C}'$ and each root is a root of α_g; the roots of π_c^M can be assigned as any symmetric set of $d(R^M)$ complex numbers by suitable choice of $C|R^M$, $C\epsilon\underline{C}'$.

3. Write $S = S_1+\ldots+S_k$. The ch.p. $\pi_c(\lambda)$ of $A_c|S$ can be factored as

$$\pi_c(\lambda) = \pi_{1c}(\lambda)\ \ldots\ \pi_{kc}(\lambda)\ \pi_c^*(\lambda) \tag{4.13}$$

where

$$\begin{aligned} d_i &\equiv \deg \pi_{ic} = d((R_i+R_o)/R_o) \qquad i\epsilon J \\ &\deg \pi_c^* = d(R_o) \end{aligned} \tag{4.14}$$

The roots of π_{ic} $(i\epsilon J)$ can be assigned as any symmetric set of d_i complex numbers by suitable choice of $C\epsilon\underline{C}'$, independent of $C|R^M$.

Proof

1. By Th. 2.1, EXT determines the S_i such that

$$S^* = R_o = \bigcap_i \sum_{j\neq i} (R_j \cap (R^M \oplus R_g))$$

Since $R^M \subset R^* \subset \bigcap_i N_i$, maximality of the R_j implies $R_j \supset R^M$ $(j\epsilon J)$ so that $\subset S^*$ and (4.11) follows.

2. If $C\epsilon\underline{C}'$, $A_cS^* \subset S^* \subset V^M \subset E$, so that $A_c|S^* = (A+BC)|S^*$ and $C|S^*$ has an extension $C_1: E\to U$ such that $C_1\epsilon\underline{C}$ and $A_{c1}|S^* = A_c|S^*$. By (4.11) and Lemma 4.1 (with $R=R^M$, $V=V^M$) the ch.p. of $A_{c1}|S^*$ factors as in (4.12), and the roots of π_{c1}^M are freely assignable by suitable choice of $C_1|R^M$, $C_1\epsilon\underline{C}$, hence by suitable choice of $C|S^*$, $C\epsilon\underline{C}'$.

3. The expression (4.13) and assignability of the roots of π_{ic} follow by Th. 1.2 applied to the S_i; (4.14) follows by the fact (Th. 2.1) that $(S_i+S^*) \cap E' = 0$ and $S^* = R_o$, hence

$$(S_i+S^*)/S^* \approx \frac{(S_i+S^*+E')/E'}{(S^*+E')/E'} \approx (R_i+R_o)/R_o. \quad \blacksquare$$

Now suppose $\{E', S_i, i \varepsilon J\}$ is any solution of EDP, not necessarily determined by EXT. Then $S^* \varepsilon \underline{I}'$ and $PS^* \varepsilon \underline{I}$. By Lemma 4.3 the fixed eigenvalues of

$$A_c^* \equiv (A+(B+E')C)|S^*, \; C \varepsilon \underline{C}'(S^*),$$

coincide with the fixed eigenvalues of

$$(A+BC_o)|PS^*, \; C_o \varepsilon \underline{C}(PS^*)$$

As shown in the proof of Th. 1.1, $PS_i \subset R_i (\equiv R_i^M)$, hence $PS^* \subset R^*$, and by maximality of V^M, $PS^* \subset V^M$. Therefore $C_o|PS^*$ has an extension $C_o^M \varepsilon \underline{C}(V^M)$. By Lemma 4.2,

$$V^M = R^M \oplus R_g \oplus R_b$$

with $R^M \oplus R_g$ given by (4.10). Since the fixed eigenvalues of A_c^* coincide with those of $(A+BC_o^M)|PS^*$, it follows, if the fixed eigenvalues of A_c^* are all good, that

$$PS^* \subset V \equiv R^M \oplus R_g \tag{4.16}$$

Since the extension constructed by EXT is minimal with respect to the properties (2.1) and (4.16), we have proved the following.

<u>Theorem 4.2</u>

<u>The construction EXT yields a minimal solution of EDP, subject to (2.1) and the requirement that the fixed eigenvalues of</u>

$$(A+(B+E')C)|S, \quad C\varepsilon\underline{C}'$$

all be good.

Remark. Assuming as in [1] that $\{A|\mathcal{B}\} = \mathcal{E}$, we have that $\{A|\mathcal{B}\oplus\mathcal{E}'\} = \mathcal{E}\oplus\mathcal{E}'$. By the technique used in proving Th. 1.2, it is straightforward to show that $(\mathcal{E}\oplus\mathcal{E}')/S$ can be regarded as a c.s (mod S) for $(A,B+E')$, hence that the only fixed eigenvalues of A_c are those of A_c^*.

5. EXAMPLE

Let $n = d(\mathcal{E}) = 5$ and let $e_i\,(1\le i\le 5)$ be the i^{th} unit column vector, with 1 in the i^{th} row and 0 elsewhere. Let

$$A = [e_4,e_1,e_3,e_3,e_4], \quad B = [e_2,e_1+e_5],$$

H_1 = row e_1, H_2 = row e_2. Writing $\{\cdot\}$ for the span of the vectors bracketed, we have

$$N_1 = \{e_2,e_3,e_4,e_5\}, \quad N_2 = \{e_1,e_3,e_4,e_5\}$$

It is easily checked that

$$R_1^M = N_2, \quad R_2^M = N_1, \quad \mathcal{B}\cap R_1^M = \mathcal{B}\cap R_2^M = \{e_2\}$$

By Th. 5.1 of [1], decoupling by state feedback is not possible. However, since (1.6) is satisfied, namely

$$R_1^M + N_1 = R_2^M + N_2 = \mathcal{E},$$

Th. 1.1 asserts that decoupling is possible by use of dynamic compensation.

In this example $d(\mathcal{B}) = 2 = k$, and according to §3 any solution S_i $(i=1,2)$ of EDP must satisfy

$$PS_i = R_i^M \quad i=1,2 \tag{5.1}$$

By Th. 3.1 a minimal extension has $d(E') = n_M(V^M)$ given by (2.5), where

$$V^M = \max\ (\underline{I}, (R^M)^*)$$

In this example,

$$(R^M)^* = R_1^M \cap R_2^M = \{e_2, e_3, e_4\}$$

and one easily computes $V^M = \{e_3, e_4\}$. By (2.4),

$$R_o(V^M) = R_1^M \cap R_2^M \cap V^M = V^M$$

Then (2.5) gives $n_M(V^M) = 1$, so that just one integrator is needed to achieve decoupling by dynamic compensation.

To determine the spectrum of $A+(B+E')C$ we follow the procedure EXT of §4, and start by finding $R^M = \max\ (\underline{C}, V^M)$. Since $B \cap V^M = 0$, we have $R^M = 0$. Since $AV^M \subset V^M$ we can take $A_c|V^M = A|V^M$; in our coordinate system

$$A|V^M = \begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

so that $\alpha(\lambda) = \lambda(\lambda-1)$. If unstable eigenvalues are considered 'bad', we have $R_g = 0$ and $R_b = V^M$. Both the bad eigenvalues are fixed in the minimal extension determined first. To find the minimal extension subject to the constraint that all fixed eigenvalues be good, we set $V = R^M \oplus R_g = 0$. By (2.4), $R_o(V) = 0$, and (2.5) gives

$$d(E') = n_M(V) = 3$$

Exactly three compensating integrators are needed to achieve decoupling together with stability of the (extended) closed loop system matrix.

The reader may wish to investigate the possibilities with two compensating integrators.

6. DECOUPLING AND OPEN LOOP CONTROL

In previous sections and in [1], the apparently stringent restriction was imposed that feedback and dynamic compensation be linear. In particular the definition of controllability subspace [1] was tied to a specific linear feedback structure. We now show that, as regards decoupling, nothing is gained by considering more general types of control. To this end we show that maximal c.s. can be defined in an open loop sense without any assumptions on controller structure. Consider

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in T \qquad x(0) = 0 \tag{6.1}$$

on the time interval $T = [0,1]$, and let $\mathcal{N} \subset \mathcal{E}$. Let $\underline{U}$ denote the class of m-vector-valued functions $u(.)$, defined and continuous on T. Denote by $\phi: T \times \underline{U} \to \mathcal{E}$ the solution of (6.1), i.e.,

$$\phi(t,u) = \int_0^t e^{(t-s)A} Bu(s)\, ds, \quad t \in T, \ u \in \underline{U}$$

Theorem 6.1

Let $\mathcal{X}$ be the set of states $x \in \mathcal{N}$ such that, for some $u \in \underline{U}$,

$$\phi(t,u) \in \mathcal{N}, \ t \in T; \ \phi(1,u) = x$$

Then $\mathcal{X} = \mathcal{R}^M \equiv \max(\underline{C}, \mathcal{N})$.

Thus $\mathcal{R}^M$ is characterized as the largest set of states in $\mathcal{N}$ which can be reached from the zero state, by any control whatever, without leaving $\mathcal{N}$.

<u>Proof</u> Let

$$R^M = \{A+BC | \{BK\}\}$$

and write $\hat{A} \equiv A+BC$, $\hat{B} = BK$. We claim that

$$R^M = \{R\} \tag{6.2}$$

where

$$R = \int_0^1 e^{(1-t)\hat{A}}\hat{B}\hat{B}'\, e^{(1-t)\hat{A}'}\, dt$$

(here and below, a prime denotes transpose). In fact $z\varepsilon N(R)$ implies $z'e^{(1-t)\hat{A}}\hat{B} = 0$, $t\varepsilon T$, i.e., $z'\hat{A}^{j-1}\hat{B} = 0$ $(j=1,\ldots,n)$, so $z\varepsilon (R^M)^{\perp}$. Thus $N(R) \subset (R^M)^{\perp}$, so that $R^M \subset \{R\}$, and the reverse inclusion is obvious.

To show $R^M \subset X$, let $x\varepsilon R^M$ and note from (6.2) that $x = Rw$ for some $w\varepsilon E$. Set

$$v(t) = \hat{B}'\, e^{(1-t)\hat{A}'} w \qquad t\varepsilon T$$

Then the equation

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}v(t) \qquad t\varepsilon T$$

$$x(0) = 0$$

implies $x(T) \subset R^M \subset N$ and $x(1) = x$, where $x(T) \equiv \{x(t): t\varepsilon T\}$. Put

$$u(t) = Kv(t) - Cx(t), \qquad t\varepsilon T$$

Then $u\varepsilon \underline{U}$; $\phi(t,u)\varepsilon N$, $t\varepsilon T$; $\phi(1,u) = x$; and so $x\varepsilon X$.

To show $X \subset R^M$, let $V = \max(\underline{I}, N)$. By [1], Th. 1.1, $V = V^n$, where $V^o = N$ and

$$V^{\mu+1} = V^{\mu} \cap A^{-1}(V^{\mu}+B) \qquad (\mu=0,1,\ldots) \qquad (A^{-1}V \equiv \{x: Ax\varepsilon V\})$$

If $x\varepsilon X$ then for some $u\varepsilon \underline{U}$ (6.1) yields

$$x(T) \subset N, \; x(1) = x$$

Thus $x(T) \subset \mathcal{V}^0$. If $x(T) \subset \mathcal{V}^\mu$ than $\dot{x}(T) \subset \mathcal{V}^\mu$, so $Ax(T) = (\dot{x}-Bu)(T) \subset \mathcal{V}^\mu + \mathcal{B}$, hence

$$x(T) \subset \mathcal{V}^\mu \cap A^{-1}(\mathcal{V}^\mu + \mathcal{B}) = \mathcal{V}^{\mu+1},$$

and by induction $x(T) \subset \mathcal{V}$. Let $C \epsilon \underline{C}(\mathcal{V})$. Then

$$\dot{x}(t) = (A+BC)x(t) + Bv(t), \quad t \epsilon T$$

where $v(t) = u(t) - Cx(t)$. Thus

$$Bv(T) = (\dot{x} - (A+BC)x)(T) \subset \mathcal{V}$$

so that $\{Bv(t)\} \subset \mathcal{B} \cap \mathcal{V}$, $t \epsilon T$. So, for $t \epsilon T$,

$$\begin{aligned} x(t) &= \int_0^t \exp\,[(t-s)(A+BC)]\, Bv(s)\, ds \\ &\quad \epsilon\ \{A+BC \mid \mathcal{B} \cap \mathcal{V}\} \\ &= \mathcal{R}^M \end{aligned} \qquad \blacksquare$$

We now pose an *open loop decoupling problem* (ODP) as follows. *Given* (6.1), *and* (1.2) *defined for* $t \epsilon T$, *together with arbitrary vectors* $y_i \epsilon \mathcal{H}_i$ $(i \epsilon J)$, *find controls* $u_i \epsilon \underline{U}$ $(i \epsilon J)$ *such that*

$$H_i \phi(1, u_i) = y_i \quad i \epsilon J \tag{6.3}$$

$$H_j \phi(T, u_i) = 0 \quad i, j \epsilon J;\ j \neq i \tag{6.4}$$

Under these conditions each u_i affects only the output $y_i(\cdot)$, and $y_i(1) = y_i$.

Theorem 6.2

Write $\mathcal{N}_i \equiv \mathcal{N}(H_i)$ $(i \epsilon J)$. *ODP is solvable for arbitrary* $y_i \epsilon \mathcal{H}_i$ $(i \epsilon J)$ *if and only if*

$$\mathcal{R}_i^M + \mathcal{N}_i = \mathcal{E} \quad i \epsilon J \tag{6.5}$$

where

$$R_i^M = \max\,(\underline{C},\ \bigcap_{j\neq i} N_j) \quad i\varepsilon J \tag{6.6}$$

<u>Proof</u> If (6.5) is true than $H_i R_i^M = H_i$, and there is $x_i \varepsilon R_i^M$ with $H_i x_i = y_i$. By Th. 6.1 there is $u_i \varepsilon \underline{U}$ such that

$$\phi(1,u_i) = x_i,\ \phi(T,u_i) \subset \bigcap_{j\neq i} N_j$$

i.e.,

$$H_i\phi(1,u_i) = y_i,\ H_j\phi(T,u_i) = 0,\ j\neq i$$

Conversely if (6.5) fails, then for some $i\varepsilon J$ there is $y\varepsilon H_i$ such that $y \notin H_i R_i^M$. Therefore any control $u\varepsilon\underline{U}$, such that $H_i\phi(1,u) = y$, has the property $\phi(1,u) \notin R_i^M$. By Th. 6.1

$$\phi(t,u) \not\subset \bigcap_{j\neq i} N_j$$

for some $t\varepsilon T$; i.e., for this t, $H_j\phi(t,u) \neq 0$ for some $j\varepsilon J$, $j\neq i$, and (6.4) fails. ∎

Comparing Th. 6.2 with Th. 1.1 we have

<u>Corollary</u>

<u>ODP is solvable if and only if EDP is solvable, namely if and only if (6.5) is true.</u>

In the definition of ODP the choice $\underline{U}$ for the class of admissible controls, and the choice in (6.2) of common endpoint t=1, are obviously not crucial. In fact we have shown implicitly that a wide class of dynamic decoupling problems is equivalent to the EDP of §1.

CONCLUDING REMARK

Taken with its predecessor [1], the present article provides effective machinery for the formulation and solution of the decoupling problem. The results prescribe the synthesis of dynamic compensation by which decoupling can be realized, and clarify the conditions under which such compensation exists. Nevertheless, further aspects of the problem remain for investigation. These include computer implementation, sensitivity analysis, and perhaps most important, a deeper account of algebraic structure.

APPENDIX

We collect here some auxiliary results; verifications, when straightforward, are omitted.

1. Let $V_i\,(i\varepsilon J)$ be arbitrary subspaces. Let

$$V_i^* \equiv \sum_{j\neq i} V_j,\ V^* \equiv \bigcap_i V_i^*$$

Then

$$V^* = \sum_i V_i \cap V^* = \sum_i V_i \cap V_i^*$$

$$= \sum_{i\neq j} V_i \cap V_i^*,\ j\varepsilon J \tag{A.1}$$

2. If $U_i \equiv V_i + V^*$ $(i\varepsilon J)$ then $U^* = V^*$ (A.2)

3. If $X \equiv \bigcap_i \sum_{j\neq i} V_j \cap Y$ for some Y, then

$$X = \bigcap_i \sum_{j\neq i} V_j \cap X \tag{A.3}$$

4. By definition the $V_i\,(i\varepsilon J)$ are mutually <u>independent</u> if and only if $V^* = 0$, i.e., $V_i \cap V_i^* = 0$, $i\varepsilon J$. More generally:

<u>Prop. A.1</u>

V^* <u>is the smallest subspace</u> V_o <u>such that the factor spaces</u> $(V_i + V_o)/V_o$ <u>are independent</u>.

<u>Proof</u> Independence of the factor spaces is equivalent to

$$V_o = \sum_i (V_i + V_o) \cap (V_i^* + V_o) \tag{A.4}$$

From (A.4), $V^* = \lim V^\mu (\mu=0,1,\ldots)$, where

$$V^o = 0, \quad V^{\mu+1} = \sum_i (V_i+V^\mu) \cap (V_i^*+V^\mu) \tag{A.5}$$

By (A.2), V^* satisfies (A.4), and (A.5) implies that any solution V_o of (A.4) contains V^*. $\blacksquare$

5. By Prop. A.1,

$$\begin{aligned}\sum_i d(V_i/(V_i \cap V^*)) &= \sum_i d((V_i+V^*)/V^*) \\ &= d(\sum_i (V_i+V^*)/V^*) \\ &= d(\sum_i V_i)/V^*)\end{aligned}$$

so that

$$\Delta[V_i,J] \equiv \sum_i d(V_i) - d(\sum_i V_i) = \sum_i d(V_i \cap V^*) - d(V^*) \tag{A.6}$$

6. If U, V, W are arbitrary subspaces,

$$\frac{(U+W) \cap (V+W)}{U \cap V + W} = \frac{(U+V) \cap W}{U \cap W + V \cap W} \tag{A.7}$$

7. <u>Prop. A.2</u>

<u>Let</u> $S_i (i\varepsilon J)$, V, E' <u>be such that</u> $V \cap E' = 0$, $S^* \subset V \oplus E'$. <u>Define</u> $S \equiv \sum_i S_i$ <u>and</u>

$$S_o \equiv \bigcap_i \sum_{j \neq i} (S_j+E') \cap (V \oplus E')$$

<u>Then</u>

$$d(E') = \delta_1 + \delta_2 + \rho \tag{A.8}$$

<u>where</u>

$$\delta_1 \equiv \Delta[(S_i+S_o+E')/(S_o+E'),\ J] \tag{A.9}$$

$$\delta_2 \equiv \Delta[(S_i+E') \cap (S_o+E')+S^*)/(S^*+E'),\ J] \tag{A.10}$$

$$\begin{aligned}\rho \equiv & \sum_i d[((S_i+S^*) \cap E')/(S_i \cap E'+S^* \cap E')] \\ & + \sum_i d[(S_i \cap E')/(S_i \cap S^* \cap E')] \\ & + d(S^* \cap E') + d(E'/(S \cap E'))\end{aligned} \tag{A.11}$$

<u>Proof</u>. The proof is a direct computation, starting from the easy identity

$$d(E') = d(S) - d((S+E')/E') + d(E'/(S \cap E'))$$

and using (A.1) - (A.7); from (A.3) note especially

$$S_o = \bigcap_i \sum_{j \neq i} (S_j+E') \cap (S_o+E') \quad \blacksquare \tag{A.12}$$

8. <u>Prop. A.3</u>

<u>If in</u> (A.8), $d(E') = \delta_1$, <u>then</u> $S^* + E' = S_o$, $(S_i+S^*) \cap E' = 0$ $(i \varepsilon J)$, <u>and</u> $E' \subset S$.

<u>Proof</u>. $\rho = 0$ implies $S \cap E' = E'$, i.e., $E' \subset S$; also $S^* \cap E' = 0$, hence $S_i \cap E' = 0$ $(i \varepsilon J)$; so that, from the first summation in (A.11), $(S_i+S^*) \cap E$ $= 0$ $(i \varepsilon J)$. Also, $\delta_2 = 0$ implies that the bracketed factor spaces in (A.10) are independent; by Prop. A.1 and (A.12),

$$S^* + E' \supset \bigcap_i \sum_{j \neq i} ((S_j+E') \cap (S_o+E')) = S_o \tag{A.13}$$

By (A.3) and the definitions of S^*, S_o,

$$S_o \supset \bigcap_i \sum_{j \neq i} S_j \cap S^* = S^*,$$

hence the reverse inclusion holds in (A.13), so $S^* + \mathcal{E}' = S_o$. $\blacksquare$

9. <u>Prop. A.4</u>

<u>For arbitrary</u> $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, if

$$\mathcal{U} \cap (\mathcal{V}+\mathcal{W}) = \mathcal{U} \cap \mathcal{V} + \mathcal{U} \cap \mathcal{W}$$

<u>then</u>

$$\mathcal{V} \cap (\mathcal{U}+\mathcal{W}) = \mathcal{U} \cap \mathcal{V} + \mathcal{V} \cap \mathcal{W}$$

10. <u>Prop. A.5</u>

<u>For arbitrary</u> $\mathcal{U}$, $\mathcal{V}$ <u>and a map</u> T,

$$T(\mathcal{U} \cap \mathcal{V}) = (T\mathcal{U}) \cap (T\mathcal{V})$$

<u>if and only if</u>

$$(\mathcal{U}+\mathcal{V}) \cap \mathcal{N}(T) = \mathcal{U} \cap \mathcal{N}(T) + \mathcal{V} \cap \mathcal{N}(T)$$

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