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THE RISK OF SOLAR PROTON EVENTS TO SPACE MISSIONS

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16. Abstract

The total dose from solar protons was tabulated in rads-tissue for weekly time intervals, and the number of weeks that gave a dose above 25 rads behind 10 g/m^2 of aluminum for the active six years of the 19th cycle were called dangerous or large event weeks. Only three such event weeks were found during the past 20 years.

Even though the possibility of smaller events was examined, it was found that for any reasonably high confidence level (95 percent), the smaller events could be ignored. Consequently, the total particle flux for the 19th cycle was divided by a factor of three, and a single large event week was determined.

The Poisson distribution was the most logical choice for predicting the probability of an event occurring. Section III is devoted to examining this conclusion and the methods of using this probability model. Section IV defines the confidence one could use in employing the Poisson process and specifically arriving at confidence levels for the experimental or observed value of the mean of the Poisson distribution function. Several examples are given for different mission lengths, and comparisons to the results of other authors are made.

Section V is an extension of the Poisson process that incorporates the concept of small sample theory and arrives at the expected distribution function that answers the following question. If X_0 events are observed in time T_0, what is the probability of seeing X events in any observation time t?

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Concluded)

LIST OF TABLES

LIST OF TABLES (Concluded)

THE RISK OF SOLAR PROTON EVENTS TO SPACE MISSIONS

SUMMARY

The total dose from solar protons was tabulated in rads-tissue for weekly time intervals, and the number of weeks that gave a dose above 25 rads[1] behind 10 g/cm^2 of aluminum for the active six years of the 19th cycle were called dangerous or large event weeks. Only three such event weeks were found during the past 20 years.

Even though the possibility of smaller events was examined, it was found that for any reasonably high confidence level (95 percent), the smaller events could be ignored. Consequently, the total particle flux for the 19th cycle was divided by a factor of three, and a single large event week was determined.

Using this spectrum, one can calculate the tissue dose in rads at the center of an aluminum spherical shell. To correct for geometric effects and self-shielding, this dose should be reduced by a factor of approximately three.

The Poisson distribution seemed to be the most logical choice for predicting the probability of an event occurring. Section III is devoted to examining this conclusion and the methods of using this probability model. Section IV defines the confidence one could use in employing the Poisson process and specifically arriving at confidence levels for the experimental or observed value of the mean of the Poisson distribution function. Several examples are given for different mission lengths, and comparison to the results of other authors are made.

Section V is an extension of the Poisson process that incorporates the concept of small sample theory and arrives at the expected distribution function that answers the following question. If X_0 events are observed in time T_0, what is the probability of seeing X events in any observation time t?

1. To convert rads to SI Units in joules/kilogram, multiply rads by 0.01.

SECTION I. INTRODUCTION

This study is concerned with the practical treatment of hazards resulting from solar proton events outside the magnetosphere of the earth. It is not concerned with prediction of flares as such or the long-range solar cycle indicators such as sunspots. This distinction is made because these phenomena have been related to solar proton events and it has been assumed that statistical studies of these phenomena give insight into the problem of dangerous solar proton events. In fact, the study of all solar proton events as such is only casually related to the problem of large dose rates inside realistic spacecraft. For example, in Reference 1, 76 events are listed from 1942 through 1963. Since 1963 there are probably an additional 24, which yields a statistical sample of some 100 events. However, the correlation of this large sample with the events that are of real danger to space flight is very poor. For example, only six large events from 1950 through 1969 would have given about 85 percent of the total 20-year proton dose behind a thin wall of only 2 cm of aluminum. In addition, these six flares occurred in three weekly periods. Thus, three of these six large events occurred from July 10 through July 16, 1959; two occurred from November 12 through November 18, 1960; and on February 23, 1956, one large event was observed. The duration of an event is from 1 to 3 days. Even with this small sample it seems that if conditions are suitable for a large event, the chances are very good that it will be followed within hours by another large event. Thus, the treating of such large events as mutually independent seems questionable. (For example, if three tornadoes hit a city within a 24-hour time period and this occurred once in 30 years, one would hesitate to say that the chance for a tornado to hit this city is one in 10 years.) Perhaps a time span of at least 1 week should be used to depict a total solar event; it would be designated as the solar proton flux or dose per week. Under this definition, there were only 3 sample weeks of large solar events from about 1950 through 1969 (1000 weeks). One is not dealing with an ordinary problem of statistical analysis, but with rare large events.

The author has undertaken the awesome task of predicting the improbable not by virtue of his background in solar physics or statistical analysis but because of his concern with protection of man and his radiation-sensitive equipment from space radiations in applications involving realistic spacecraft and missions. However, the reader should ask, "Why not leave this field of statistical astronomy to the experts?" The answer is that the environmental scientists do not have to design or evaluate realistic shields for sensitive film or radiation-conscious astronauts.

The consequence of having several solar proton prediction models (which the author does not wish to evaluate) has led to a wide disparity in results, especially when a reasonably high level of statistical certainty is desired. For example, at the 99-percent probability (percentile) level, the predicted dose behind 20 g/cm^2 of aluminum for a 1-year mission may vary by a factor of 10 or more between different investigators [2]. This may not sound too bad considering the nature of the problem; however, if a mission is planned with the requirement that the astronauts should not receive doses exceeding a 100-rad skin dose with a 99-percent probability and if one solar proton dose prediction model requires a shield of only 15 g/cm^2 while another model requires a shield of 50 g/cm^2, the radiation analyst has to make a vital decision that possibly affects the life of an astronaut. Since no one wishes to be responsible for making a decision that could lead to dire consequences, the most pessimistic model is often chosen. The author does not question this approach except to note that the desire to be on the safe side may readily become uncontrollable and a subsequent loss in mission capability may develop.

As a result of the problems mentioned above, NASA has been an easy target for various academic and industrial contractors who propose to solve certain aspects of this problem. Their proposals may range from early solar flare predictions based on observing the sun to statistical studies of sunspots and recorded flare data, and include special concepts for shielding against large solar proton events. Many of these studies are interesting and even of credible workmanship, but a large majority are of only marginal value to a radiation analyst who is confronted with real problems. This writer feels that most of the studies are of dubious value, because statistical studies of sunspots or flares or all solar proton events do not provide any meaningful guide to predicting large, very hazardous proton events. For example, there were only 3 weeks from the past 1000 weeks during which an astronaut inside a spacecraft having a wall thickness of only 3 cm of aluminum would have received as much as a 20- to 30-rad dose at a depth of 5 cm in the body.

In addition to the statistics problem, there exists the rather common procedure of calculating the radiation hazard from a solar proton event for a point detector at the center of a spherical shell. This may lead to overestimates of the actual dose to an astronaut in a realistic spacecraft by a factor of from approximately three to as much as six. Also, it should be mentioned that the solar proton dose inside the earth's magnetic field will be considerably reduced. The actual reduction depends on the orbital inclination and altitude of the spacecraft, as well as the disturbances in the earth's magnetic field that are caused by magnetic storms accompanying the solar proton event. Treating this aspect of the problem correctly requires a sophisticated mathematical model; for an example, see Reference 3.

SECTION II. A PRIMITIVE DOSE MODEL IS PROPOSED

The most direct measure of the hazard of a given solar proton event is the rads-tissue absorbed dose that would be measured behind various thicknesses of a typical spacecraft material such as aluminum. The simplest method is to find the point tissue dose at the center of a spherical shell of aluminum. However, because of self-shielding by the astronaut (approximately a factor of two, the spacecraft geometry, and onboard equipment, the actual skin dose may be less, by a factor of three or more, than the point dose at the center of a spherical shell. This factor varies depending on the solar proton spectrum, the spacecraft geometry, and the location of the astronaut in the craft. This writer suggests using a reduction factor of approximately three for the solar proton point dose at the center of a spherical shell to estimate the probable skin dose to an astronaut in a real spacecraft of a given average thickness. This factor is believed to be reasonable in the sense that this reduced dose is probably high when the complex geometry and spectrum shape of a solar proton event are considered.[2] To correct for depth dose (bone-marrow depth), a thickness of 5 cm of tissue is often employed. This is approximately equal to 6.5 g/cm^2 of aluminum in equivalent shielding effectiveness.

To clarify the exact assumptions that are to be employed, the following information is pertinent. The only adequate data available at this writing for solar proton predictions are from the 19th cycle (1954 through 1964). However, there have been observations of sunspots (indicators of solar activity) for approximately 200 years. Based on more than 200 years of observations, in which the first cycle dates back to the middle of the 18th century (the average cycle length is about 11 years), the 19th cycle had the highest maximum sunspot count yet recorded. In fact, the average maximum sunspot count is more than a factor of two lower than that of the 19th cycle. The 20th cycle, which we are now well into (past the peak activity), has a sunspot count somewhat above this average. There has not been a large solar proton event in the 20th cycle comparable to the eight largest events of the 19th cycle (dose behind 10 g/cm^2). At this point one may be led to believe that the 19th cycle is a relatively rare cycle. With the present low occurrence of large solar proton events and the rather extensive sunspot counting dating back over 200 years, one might conclude that the probability of a solar cycle being as active as the 19th is on the order of 1/20 or 0.05. This, of course, cannot be

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2. It should be noted that this factor of three may be too large for a very "hard" spectrum such as seen in the trapped radiation belts, where a a factor of 2.5 may be more appropriate.

objectively demonstrated, and will not be, unless considerably more knowledge is obtained about the physics of the sun. One valid objection to the above is that the number of sunspots is a poor indicator of large solar proton events. Also, the sunspot indicator may have changed during the last 50 years because of better observation techniques, etc., so that possibly the first 15 of the 20 observed cycles should not be used. The purpose of this paper is not to attempt to evaluate the above but to present the information for the reader's consideration. At any rate, the reader may not find it difficult to believe that the use of the 19th cycle solar proton flux data may yield a pessimistic estimate of the solar proton hazard. With the above background, one can at least study the proton events characterized by the 19th cycle and infer proton events for future cycles that are similar to the 19th cycle.

Next, consider the observation that during the 19th cycle, the large solar proton events did occur primarily around the most active years of the 11-year cycle, however not necessarily in proportion to sunspot count. It is generally assumed that there were only about 6 years of observed large solar proton events. Thus, the following analysis is based on the so-called active 6 years (300 weeks) of the cycle. The second assumption is that it is fair and logical to lump the actual dose rates over an active week into units of total dose per week (see Section I). This may include up to three proton events in a given week. Thus, instead of using fundamental units of proton events, it is proposed that observed weekly dose rates behind various aluminum thicknesses be used. The data available from various sources gave a total of 24 proton events that were worth considering. (The ten or more weeks discarded gave less than 3 rads total behind 5 g/cm^2 of aluminum.) The results of grouping this data into weekly time periods gave a total of 18 weeks with the following frequencies: 13 weeks, one event; 4 weeks, two events; and 1 week, three events. Most of this data is recorded in NASA TN D-4404 [3]. The only major revision in Reference 3 is in the use of the spectrum of A. J. Masley [4] for the November 12, 1960, flare. The previous calculations were based on the work of W. R. Webber [5]. The differences are shown in Table 1.

One note of explanation should be made regarding the dose rates that are used in this study. They include a correction for secondary particles and thus are the sum of the primary proton dose and the secondary particles (neutrons and protons). This total dose at 20 g/cm^2 and at 10 g/cm^2 is approximately 20 percent and approximately 10 percent, respectively, above the primary proton dose rates.

Now one comes to a very treacherous part of the analysis. How does one choose a rational and sufficient model for an event dose-week for solar

TABLE 1. SKIN DOSES (RADS-TISSUE) BEHIND VARIOUS SHIELDS OF ALUMINUM FOR NOVEMBER 12, 1960, FLARE

Source \ Shield (g/cm^2)	2	5	10	20[a]
Webber	270	106	45	16.2
Masley	1900	400	61	9.5

a. 20 g/cm^2-aluminum = 7.4 cm = 2.9 in.

protons ? Table 2 is presented as a summary of the weekly solar proton doses of the 19th solar cycle that the writer will consider. The absorbed doses in rads-tissue were calculated at the center point of a spherical shield of aluminum, with the shell thickness being designated as the shield.

The data of Table 2 have been grouped into three categories: (1) large weekly doses, (2) medium weekly doses, and (3) small weekly doses. Category 1, consisting of only 3 weeks, gave more than 84 percent of the total 6-year dose. Category 2 with 6 samples gave about 13 percent, and category 3 with 9 samples gave less than 3 percent of the total dose.

The grouping shown is certainly not unique, and the last 2 weeks of category 2 perhaps should be in category 3, or perhaps the largest dose week of category 2 should be in category 1. The following work has attempted to provide for various combinations that the reader may wish to investigate by making the methods of approach more important than the choice of a precise set of data.

Since over 84 percent of the total solar proton dose is grouped into the 3 weeks of the first category, it would seem that one feasible solar proton dose model for an active week could be depicted by dividing the total 6-year dose behind the various shields by three. Since the most active phase of the 19th cycle is about 6 years, it seems sufficient to use 300 weeks as the basic period of major solar activity. With this assumption, which will be used throughout this study, it follows that the estimated weekly expectation of a dangerous large solar proton dose would be $\lambda_0 = 0.01$ (event/week). Thus,

TABLE 2. WEEKLY TISSUE DOSES OF THE 19TH SOLAR CYCLE

Sample Number	Week	Year	Number of Flares	Shield (6-Year Dose) 5(906)[a]	10(251)	20(73)	Remarks
1	Nov 12 – 18	1960	2	456	83.4	17.0	Category 1 – Large Weekly Doses $\lambda_1 = 0.01$
2	July 10 – 16	1959	3	216	75.0	21.2	
3	Feb. 23	1956	1	92	50.0	24.8	
	Summed Dose		(6)	764	208.4	63.0	Approximately 85 Percent of 6-Year Total
	Percent of Total 6-Year Dose			84.3	83	86.3	
1	May 10	1959	1	59.3	18.3	4.4	Category 2 – Medium Weekly Doses $\lambda_2 = 0.02$
2	July 18 – 20	1961	2	22.0	8.2	2.4	
3	March 23	1958	1	10.9	2.5	0.4	
4	July 7	1958	1	10.5	2.3	0.4	
5	Jan. 20	1957	1	8.3	1.8	0.3	
6	Oct. 20	1957	1	4.1	1.8	0.7	
	Summed Dose		(7)	115.1	34.9	8.6	Approximately 13 Percent of 6-Year Total
	Percent of Total 6-Year Dose			12.7	13.9	11.8	
1	Aug. 22 – 26	1958	2	5.9	1.0	0.2	Category 3 – Small Weekly Doses $\lambda_3 = 0.03$
2	Aug. 29	1957	1	4.2	0.8	0.1	
3	Nov. 20	1960	1	3.6	1.5	0.05	
4	Aug. 3	1956	1	2.2	1.0	0.4	
5	Sept. 13	1960	1	2.9	1.2	0.5	
6	Aug. 16	1958	1	1.8	0.4	0.1	
7	July 3	1957	1	1.2	0.43	0.03	
8	July 12	1961	1	1.4	0.30	0.04	
9	April 28 – 29	1960	2	0.77	0.27	0	
	Summed Dose		(11)	23.97	6.90	1.42	Approximately 2 Percent of 6-Year Total
	Percent of Total 6-Year Dose			2.65	2.75	1.95	

a. The first number is the shield thickness in g/cm^2-aluminum; the number in parenthesis is the 6-year dose behind the shield.

the simple expectation for 300 weeks is three events, where each event yields one-third of the 19th solar cycle dose behind various shields. This concept is usable, but as the reader might suspect, the calculation of any fairly sophisticated statistical measure would lead to very large uncertainties because of the uncertainty in the value of the parameter (λ_0), since the actual number of recorded events is very small. Also, it might seem that this concept could not be used for predicting the likelihood of solar proton doses for short exposure periods. In the next section this point will be discussed in more detail. In any case, for a short exposure time, there is a greater chance for an event of category 2 or category 3 to occur than there is for an event of category 1 to occur. For this reason a model will also be developed to reflect medium and small flare doses. Thus, two solar proton dose models will be constructed, and the results will be compared in the following work.

The next part of our approach is to choose the best composite 19th cycle solar proton flare data that are available and to arrive at total tissue dose in rads from primary and secondary particles at the center of spherical shells with varying thicknesses. T. T. White, et al. [6] have developed a composite model (designated as the MSC model) for the total proton flux during the 19th cycle. This model gives a larger dose for energies less than approximately 115 MeV[3] than does a model based on Webber's work [5]. Above 115 MeV a composite flare model based on Webber's work (designated as the MSFC model) gives a larger dose. The 19th-cycle, composite, 6-year proton spectrum that will be used in this work is:

$$\begin{aligned} &\text{Model I (MSC)} \quad J(>p) = 5.28 \times 10^{11}\, e^{-p/73} (30 \le E \le 115 \text{ MeV}) \\ &\text{Model II (MSFC)} \quad J(>p) = 1.14 \times 10^{11}\, e^{-p/100} (E > 115 \text{ MeV}) \end{aligned} \quad , \quad (1)$$

where p is in rigidity units of megavolts and $J(>p)$ is the integral spectrum (proton/cm^2) with energies above p. For protons, the relationship between p(MV) and E(MeV) is given by $p = \sqrt{E^2 + 1876E}$ [3].

Using the spectrum of equation (1), the best estimate of the 6-year (300-week) total dose (primaries and secondaries) is given in Figure 1. From Figure 1 and Table 2 the solar proton 1-week-dose event models are constructed. (Figure 1 gives the magnitude and Table 2 gives the fractions.) Model I will be the dangerous solar event model that is represented by three large events over a period of 300 weeks. The expected number of events

3. To convert electron volts to SI Units in joules, multiply electron volts by 1.60210×10^{-19}.

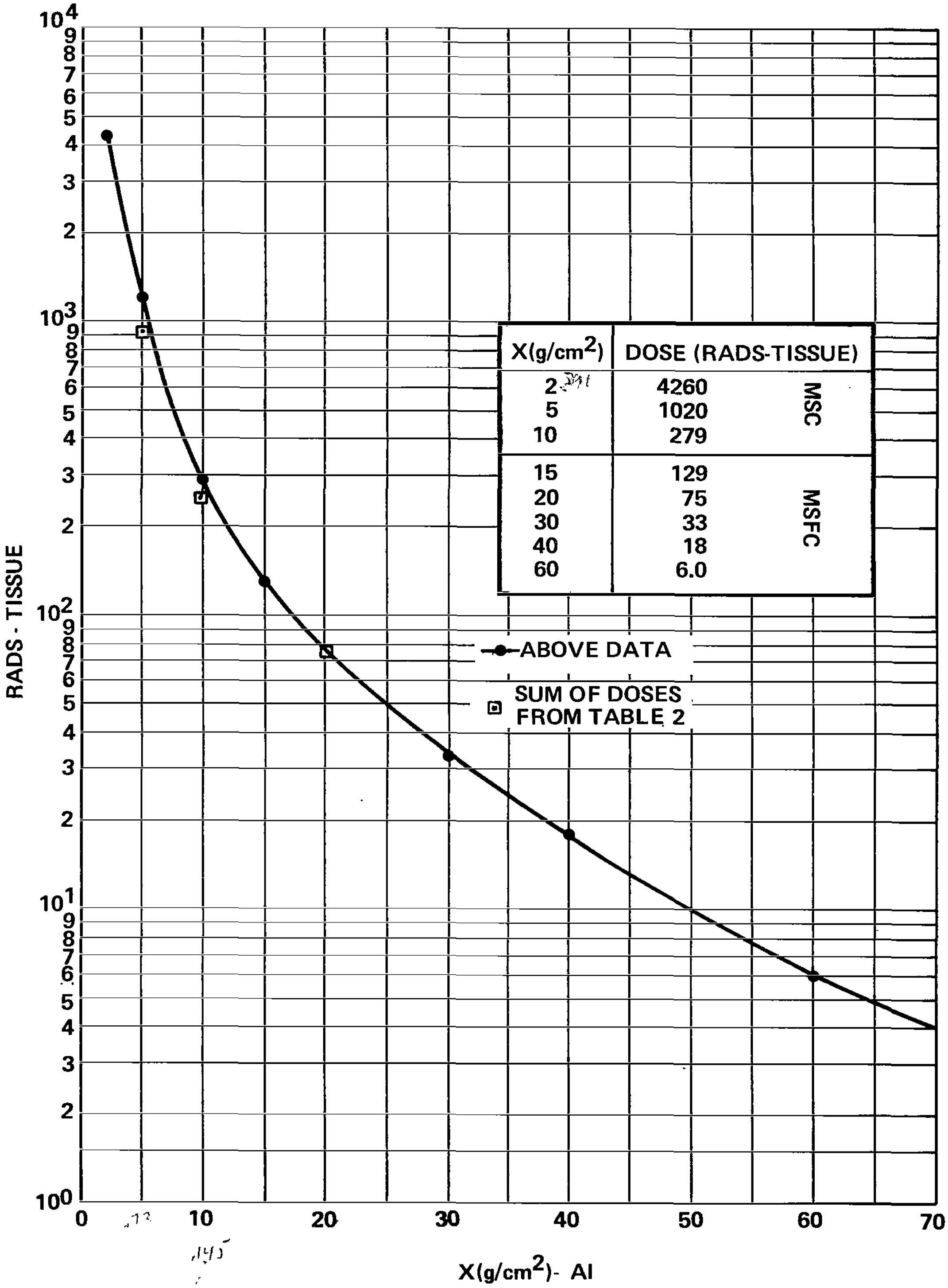

Figure 1. Total tissue dose at the center of a spherical shield from solar protons of the 19th cycle as a function of shield thickness X.

in 1 week is given by $\lambda_0 = 0.01$ (event/week). Model II will depict the possibility of a large, medium, or small dose in a week where the percentages of the total dose in Table 2 are used to determine the relative size of a dose week in the three groups. For the three categories of Table 2, the values $\lambda_1 = 0.01$, $\lambda_2 = 0.02$, and $\lambda_3 = 0.03$ are the mutually independent weekly expectations of large, medium, and small doses, respectively. Table 3 summarizes the dose models. It should be clear that the author has presented only the primitive elements of a probability model in Table 3. Thus, the table gives the small sample estimates of the weekly expectations (λ) and the consequences (rads/week) if an event occurs according to model I or model II. It should also be made clear that the values of λ are valid only for the 300 most active weeks of a solar cycle similar to the 19th. The major unknown factor is an estimate of the chance of obtaining a solar cycle that would give total proton doses as large or larger than the 19th cycle.

The values of λ should be considerably smaller for the so-called quiet sun (260 weeks) or else the magnitude of an event should be reduced to the level of an event of category 3. Perhaps one might assume that $\lambda = 0.02$ (one event per year) during the 260 weeks of the quiet sun, but an event should be depicted by the D_3 column (small event) of model II. A similar set of revisions might be in order if the solar activity is definitely smaller than the 19th cycle at the maximum point of the cycle. However, there is no justification for any of the above suggestions.

As a conclusion to this section, it is of interest to recall that Figure 1 represents the total dose versus spherical shell thickness for the 19th cycle. If for the data of Figure 1, an effort is made to correct for the self-shielding of an astronaut and the complex geometry of a spacecraft that has aluminum walls of 13.5 g/cm^2 (assume that a 5-cm depth dose corresponds to an additional aluminum thickness of 6.5 g/cm^2), then the 5-cm depth dose is estimated by dividing the dose at 20 g/cm^2 by a factor of three. Thus, one finds that the total dose from the 19th cycle was 25 rads at a 5-cm tissue depth. Of course, this value is for the solar proton dose and does not account for the galactic cosmic ray dose that may range from 5 to 12 rads/year behind 20 g/cm^2, depending on how much dose is contributed by the heavy cosmic ray particles ($Z > 2$). The implications of all the above are simply that since the 19th solar cycle was an example of a very active sun and if one believes that the high energy proton fluxes received during this cycle are associated with this activity, he must be cautious in drawing certain conclusions. For example, the same astronaut in the previous example may have received from 30 to 70 rads from galactic cosmic rays during a six-year trip, whereas the

TABLE 3. MODELS OF A 1-WEEK DOSE EVENT FOR THE ACTIVE 300 WEEKS OF THE 19TH SOLAR CYCLE

Model I		Model II				
	N = 3^{a} 100 Percent[b] $\lambda_0 = 0.01$[c]		N(Large) = 3 84 Percent $\lambda_1 = 0.01$	N(Med.) = 6 13 Percent $\lambda_2 = 0.02$	N(Small) = 9 3 Percent $\lambda_3 = 0.03$	$\sum_{i=1}^{3} D_i N_i$
X (g/cm^2)	$D_0\left(\frac{\text{rads}}{\text{week}}\right)$	X (g/cm^2)	$D_1\left(\frac{\text{rads}}{\text{week}}\right)$	$D_2\left(\frac{\text{rads}}{\text{week}}\right)$	$D_3\left(\frac{\text{rads}}{\text{week}}\right)$	Total Dose 300 Weeks
2	1420	2	1195	95	15	4290
5	340	5	286	24	3.4	1032.6
10	93	10	78	7.5	0.9	281.1
15	43	15	36	3.5	0.43	132.87
20	25	20	21	2.0	0.25	77.25
30	11	30	9.3	0.80	0.11	33.69
40	6	40	5.03	0.45	0.06	18.33
60	2	60	1.70	0.15	0.02	6.18

a. N indicates the number of weeks used in the dose category.

b. Percentage shown is the percent of total dose in the category.

c. λ gives the expectations (probability) of the category for 1 week.

previously discussed dose from solar protons at bone-marrow depth was 25 rads (13.5 g/cm^2 aluminum walls). Here, one assumes that the galactic cosmic ray spectrum is so energetic that geometry factors are negligible for dose reduction. Thus, the cosmic ray dose component may determine the limiting dose factor for long duration space travel. The purpose of the foregoing discussion is not to minimize the importance of solar proton events, but to illustrate that it may be quite feasible to shield against solar protons but probably impractical to consider shielding against galactic cosmic rays. This very fact may be more important in determining man's exposure time to space outside the earth's magnetic field than the solar protons.

SECTION III. A PROBABILITY MODEL IS DERIVED

In this section, the problem of choosing and using a probability density function in order to arrive at the probability of getting X events in t weeks and the associated problem of probability or percentile levels shall be addressed. The important question of establishing confidence intervals for the basic statistical parameters that are obtained from a small sample will be undertaken in the next section. In this section, the naive assumption that the basic statistics or population parameters (mean and variance) are well known, either by experience or a priori knowledge, will be made.

To derive the probability model that seems to be the most natural outgrowth of the solar proton dose week, the process will be described in terms of probabilities $P_n(t)$ that exactly n events occur during a time interval t (weeks in our case). Thus, $P_0(t)$ is the probability of no event in the interval t, and $1 - P_0(t)$ is the probability of one or more events. Next, λ is defined to be the mean or expectation of an event for a unit time interval. That is,

$$\hat{\lambda} = \frac{\text{number of events}}{\text{total weeks of observation}} \tag{2}$$

is a statistical estimate of λ. More precisely, λ is a constant that determines the density of points on the t axis. Thus, for a small interval of time Δt, the probability of one or more events is given by

$$1 - P_0(\Delta t) = \lambda \Delta t + \epsilon(\Delta t) \quad , \tag{3}$$

where $\epsilon(\Delta t)$ is an infinitesimal and small compared to $\lambda \Delta t$ such that

$$\lim_{\Delta t \to 0} \frac{\epsilon(\Delta t)}{\Delta t} = 0 \quad .$$

Now, the following postulate is made:

> Whatever the number of events during $(0, t)$, the probability during $(t, t + \Delta t)$ that one event occurs is given by $P_1(\Delta t) = \lambda \Delta t + \epsilon_0(\Delta t)$, and the probability that more than one event occurs is given by $P_{n > 1}(\Delta t) = \epsilon_1(\Delta t)$.

These conditions are the basic assumptions of the Poisson process [7]. It should be clear that for a small time interval Δt, the chance for one event is approximately $\lambda\Delta t$ and the chance for more than one event is very small compared to $\lambda\Delta t$. Since t is for a relative time scale, it is not contradictory that a Δt of 1 week can be small on our time scale. The above conditions lead to a system of differential equations for $P_n(t)$. Consider two time intervals $(0,t)$ and $(t, t + \Delta t)$, where Δt is small. If $n \geq 1$, n events can occur in the interval $(0, t + \Delta t)$ in three ways:

1. No events during $(t, t + \Delta t)$ and n events during $(0, t)$.
2. One event during $(t, t + \Delta t)$ and $n-1$ events during $(0, t)$.
3. $X \geq 2$ events during $(t, t + \Delta t)$ and $n-X$ events during $(0, t)$.

Following our assumptions, the probability for the first case is given by

$$P_n(t)\ P_0(\Delta t) = P_n(t)\ (1 - \lambda\Delta t) + \epsilon_1(\Delta t) \quad , \tag{4}$$

where $P_0(\Delta t) = 1 - \lambda\Delta t - \epsilon_0(\Delta t)$ from equation (3).

For the second case, one has similarly

$$P_{n-1}(t)\ P_1(\Delta t) = P_{n-1}(t)\, \lambda\Delta t + \epsilon_2(\Delta t) \quad . \tag{5}$$

and for the third case,

$$P_{X \geq 2}(t)\ P_{n-X}(\Delta t) = \epsilon_3(\Delta t) \quad . \tag{6}$$

The result of combining the above is

$$P_n(t + \Delta t) = P_n(t)\ (1 - \lambda\Delta t) + P_{n-1}(t)\ \lambda\Delta t + \epsilon'(\Delta t) \quad , \tag{7}$$

which can be expanded and rewritten as

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{\epsilon'(\Delta t)}{\Delta t} \quad . \tag{8}$$

As $\Delta t \to 0$ the last term tends to zero, hence one obtains in the limit

$$\frac{dP_n(t)}{dt} = -\lambda P_n(t) + \lambda P_{n-1}(t), \; n \geq 1 \quad . \tag{9}$$

For n = 0, the second and third possibilities above do not exist, and equation (7) becomes simply

$$P_0(t + \Delta t) = P_0(t)\,(1 - \lambda \Delta t) + \epsilon_0(\Delta t) \quad , \tag{10}$$

which leads to

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \frac{\epsilon_0(\Delta t)}{\Delta t} \tag{11}$$

and in the limit

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) \quad . \tag{12}$$

From equation (12) and $P_0(0) = 1$, one obtains

$$P_0(t) = e^{-\lambda t} \quad . \tag{13}$$

Using equation (13) and $P_1(0) = 0$, equation (9) can be solved for $P_1(t) = \lambda t e^{-\lambda t}$. Using the fact that $P_n(0) = 0$ $(n>0)$, equation (9) becomes a recursion equation and successive values of $P_n(t)$ can be found. The resulting solutions give the terms of the Poisson distribution:

$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad . \tag{14}$$

If the reader accepts the postulate following equation (3), the Poisson distribution is the natural outcome. The foregoing arguments have been presented to minimize the illusion that the author has "pulled a distribution function out of the sky." For a more rigorous treatment, Reference 7 is recommended. However, there is one condition necessary for a Poisson process that can be examined easily. This condition is that the occurrence of events on the time axis $(0, T)$ are random, independent, and uniformly distributed. That is, no time interval Δt in $(0, T)$ is more likely to have an event than any other time interval of the same length. To test this condition for the data of Table 2, the following development is given. First, the time interval $(0, T)$ of our solar proton events is considered to be 300 weeks. Next, the time from the starting point zero to the occurrence of the ith event is defined to be t_i and the sum of n such events in $(0, T)$ is defined to be

$$S_n = \sum_{i=1}^{n} t_i \quad . \tag{15}$$

Now, for moderately large values of n, the sum S_n of n independent random variables, each uniformly distributed on the interval 0 to T, may be considered to be normally distributed with mean,

$$E\left(S_n\right) = n\left(\frac{T}{2}\right) \quad , \tag{16}$$

and variance,

$$\mathrm{Var}\left(S_n\right) = n\left(\frac{T^2}{12}\right) \tag{17}$$

The standard deviation is then

$$\sigma\left(S_n\right) = \sqrt{\mathrm{Var}\left(S_n\right)} = \sqrt{\frac{n}{3}}\left(\frac{T}{2}\right) \tag{18}$$

(See Reference 8, Stochastic Processes by E. Parzen.) The reader may object to the use of this test since there is only one set of sums S_n to test, and the values of n are small for either large, medium, or small events $(n = 3, 6, 9)$. In any case, the test should have some merit in establishing a degree of credibility. To carry out the above test, the time scale will begin on January 1, 1956, and terminate on October 7, 1961. This time span corresponds to 300 weeks. To aid in showing any tendency for lumping, a further division of the event occurrence time into three consecutive groups of 100 weeks is made. Table 4 summarizes the results; the order of events is identical to that of Table 2. The results of Table 4 tend to imply that the assumption of random, uniformly distributed events on the time axis is valid at a reasonably high level of certainty. The assumptions of a normal distribution in S_n for moderately large n could allow a probability statement

TABLE 4. TEST FOR RANDOM AND UNIFORM OCCURRENCE OF EVENTS

Type Events	i	t_i (weeks)	Time		
			0 – 100	101 – 200	201 – 300
Large Events	1	255			X
	2	185		X	
	3	8	X		
n = 3	S_n	448	1	1	1
	$E(S_n)$[a]	450			
	$\sigma(S_n)$[b]	150			
Medium Events	1	175		X	
	2	289			X
	3	116		X	
	4	131		X	
	5	55	X		
	6	94	X		
n = 6	S_n	860	2	3	1
	$E(S_n)$	900			
	$\sigma(S_n)$	212			
Small Events	1	138		X	
	2	87	X		
	3	255			X
	4	31	X		
	5	245			X
	6	137		X	
	7	78	X		
	8	288			X
n = 9	9	226			X
	S_n	1485	3	2	4
	$E(S_n)$	1350			
	$\sigma(S_n)$	260			
Grand Totals	S_n	2793	6	6	6
	$E(S_n)$	2700			
	$\sigma(S_n)$	367			

a. $E(S_n) = n\left(\frac{T}{2}\right) = 150n$

b. $\sigma(S_n) = \sqrt{\frac{n}{3}}\left(\frac{T}{2}\right) = 150\sqrt{\frac{n}{3}}$

to be made concerning the degree of belief that the distribution is random and uniform; however, since n is rather small, it is believed that the value of such a statement is dubious. Note that all values of the calculated S_n fall well within a one-sigma bound of the theoretical mean.

There are many interesting uses of the Poisson distribution in addition to the occurrence of rare events in a continuum of time. The distribution is used to approximate the binomial distribution for the case of rare events ($p < 0.05$). The word rare means individually rare. In a large population several such events may occur, but the probability of occurrence of each individual event is small; for example, the number of people killed by horses in 1969. An important feature of the Poisson distribution is that for large values of the mean ($\lambda t >> 20$), the distribution approaches the normal (or Gaussian) distribution. There are many applications of the Poisson distribution given in any standard text on probability and statistics. The most common include such studies as the number born blind in a large city, radioactive disintegration, bacteria on plates, telephone traffic, etc. The remainder of this section will be devoted to a discussion of the Poisson distribution and how to apply it to our class of problems.

If a discrete variable X has a Poisson distribution, then the probability that (X = x) is given by the expression:

$$P\,(X = x) = \frac{e^{-m}\, m^{x}}{x!}, \quad x = 0, 1, 2, \ldots ; \tag{19}$$

and $P(X = 0) = e^{-m}$, since $0! = 1$. The mean (or expected value of the variable X) is given by $\mu = m$, and the variance σ^2 is given by $\sigma^2 = m$. The standard deviation is given simply by $\sigma = \sqrt{m}$. The summation equation, $\sum_{x=0}^{\infty} e^{-m}\, m^{x}/x! = 1$, is satisfied. Since the time-dependent form of the Poisson is primarily of interest, $m = \lambda t$ becomes the mean. Even though the Poisson is discrete in the variable X, it is continuous in t. For purposes herein, the Poisson distribution shall be written as:

$$P\,(X = x) = \frac{e^{-\lambda t}\, (\lambda t)^{x}}{x!}, \quad x = 0, 1, 2, \ldots , \tag{20}$$

where t = the number of weeks, x = the number of events in t weeks, λ = the mean number of events per week with λt becoming the expected or mean number of events in t weeks, and $P(X = x)$ is the probability of exactly x events in t weeks.

Thus,

$$P(X \geq x^*) = \sum_{x=x^*}^{\infty} \frac{e^{-\lambda t}(\lambda t)^x}{x!} \qquad (21)$$

and

$$P(X \leq x^*) = \sum_{x=0}^{x^*} \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad . \qquad (22)$$

Note that since one is dealing with a discrete variable x,

$$P(X > x^*) = P(X \geq x^*) - P(X = x^*) = 1 - P(X \leq x^*) \qquad (23)$$

and

$$P(X < x^*) = P(X \leq x^*) - P(X = x^*) = 1 - P(X \geq x^*) \quad . \qquad (24)$$

Tables of the Poisson distribution are available in most handbooks of statistics. They are usually given in the form of the discrete distribution of equation (19) and the summed Poisson $P(X \geq x^*)$. Other combinations can be obtained by using the relations shown in equations (23) and (24). Figure 2 depicts a family of summed Poisson curves as a function of λt, obtained by cross plotting from a standard table. For example, if $\lambda = 0.02$ and $t = 100$, then $\lambda t = 2$. From Figure 2, the probability of getting six or more events is approximately 0.0166 and the probability of getting less than six (five or less) events is 0.9834 [from equation (24)].

If one wished to find a value of X that would not be exceeded, for example, at least 99 percent of the time (a 1-percent chance at most of exceeding X) for a given λt, the cumulative distribution function $P(X \leq x^*)$ is utilized. The value of X that is sought here is the smallest x^* that satisfies

$$\Pr(X \leq x^*) = \sum_{x=0}^{x^*} \frac{e^{-\lambda t}(\lambda t)^x}{x!} \geq P \quad , \qquad (25)$$

where $P = 0.99$ for the case in question.

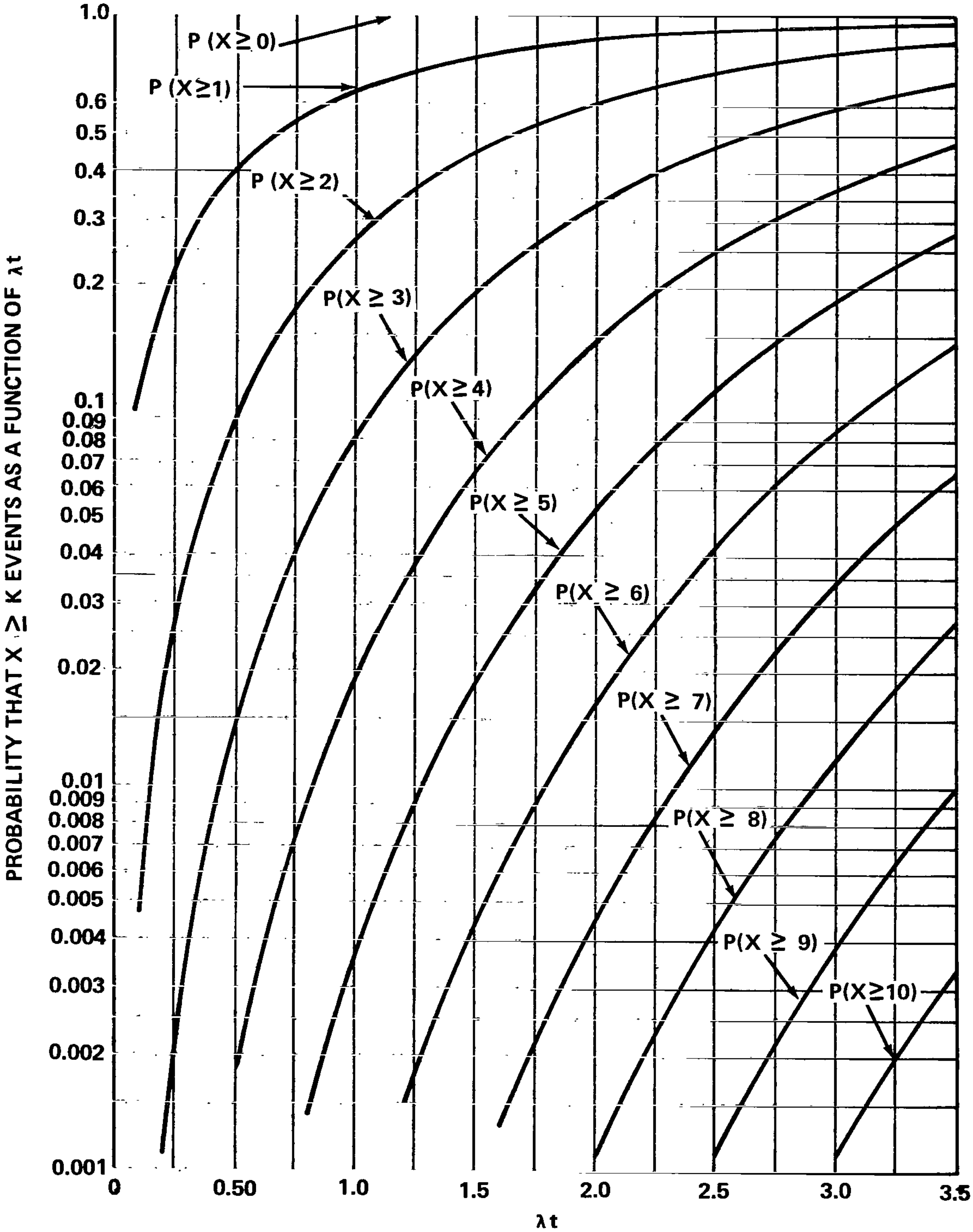

Figure 2. Summed Poisson distribution as a function of λt.

Since one is dealing with a discrete distribution function, the inequality is necessary and one usually obtains a value of x^* that corresponds to a value slightly above P(0.99). This discrepancy is usually circumvented by the proper use of the words "at least" and "at most". For example, if $\lambda t = 1$, a probability table for exactly X events and the cumulative distribution is given in Table 5. From Table 5 one can see that for 98.1 percent of the time there will be three or less events when $\lambda t = 1.0$, or for 1.9 percent of the time there will be four or more events.

TABLE 5. POISSON DISTRIBUTION FUNCTION ($\lambda t = 1$)

x	P(x)	Σ P(x)
0	0.3679	0.3679
1	0.3679	0.7358
2	0.1839	0.9197
3	0.0613	0.9810
4	0.0153	0.9963
5	0.0031	0.9994
6	0.0005	0.9999

Next, the application of the above information to our problem of predicting the solar proton dose that would be seen on a space mission at various percentile levels will be examined. For simplicity, assume that if an event occurs, it is depicted by the so-called "large" event model or model I of Table 3. Using the model and the sample statistical estimate of $\lambda_0 = 0.01$ event/week and for convenience assuming that the duration of exposure to large solar events is 100 weeks, then the value of λt is 1.0 for the Poisson distribution (Table 5). Now assume that one wishes to find the dose levels that would be exceeded 27 percent, 2 percent, and 0.1 percent of the time. Applying equation (25) with P taking on the values 0.73, 0.98, and 0.999, it is seen from Table 5 that the values of x are 1, 3, and 5 events. Therefore, to find the dose at the respective levels, the doses behind the various shield thicknesses of model I are multiplied by the values of x above. The results are shown in Table 6 and plotted in Figure 3. Note that the 73.6-percent level corresponds in our model to the mean or expected value of the Poisson distribution function. The percentile level at the expected value is found only when λt is an integer. It becomes smaller, approaching 50 percent, as λt increases to large values. For example, when $\lambda t = 20$, the percentile at $x^* = 20$ is 55.9 percent [from equation (22)].

A similar computation can be carried out using event model II. The values of the Poisson mean become respectively $\lambda_1 t = 1.0$, $\lambda_2 t = 2.0$, and $\lambda_3 t = 3.0$, where t = 100 weeks. Tables similar to Table 5 are constructed and the following results are found at the expected value, the 98-percent, and

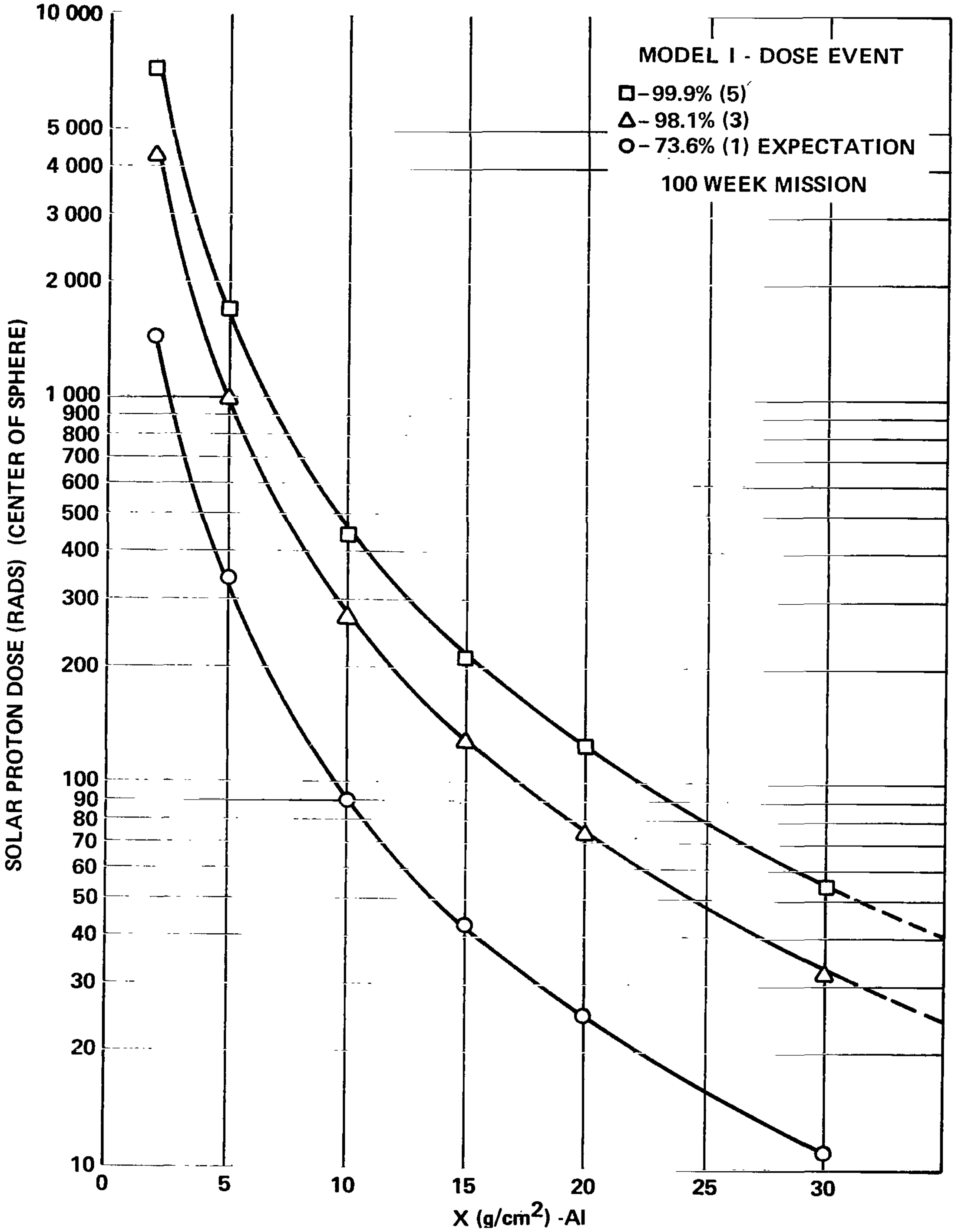

Figure 3. Solar proton dose curves as a function of shield thickness at various probability levels for 100-week missions.

TABLE 6. SOLAR PROTON DOSES (RADS) AT VARIOUS PROBABILITY LEVELS FOR 100-WEEK MISSIONS USING MODEL I

	x = 1	x = 3	x = 5
$Z(g/cm^2)$	73.6-Percent Level	98.1-Percent Level	99.9-Percent Level
2	1420	4260	7100
5	340	1020	1700
10	93	279	465
15	43	129	215
20	25	75	125
30	11	33	55

the 99.9-percent levels. For the large events, x = 1, 3, and 5; for the medium events, x = 2, 5, and 8; and for the small events, x = 3, 7, and 10. The percentile values at the expected or mean value levels ($x^* = \lambda t$) are 73.6 percent, 67.7 percent, and 64.7 percent, respectively. To reduce the necessity of repetition, this set of levels shall be denoted as the approximately 70-percent level. The results are shown in Table 7. The comparison of results from the total columns of Table 7 with the comparable percentile level of Table 6 indicates very good agreement. The larger values of Table 7 at the 70-percent level reflect the larger values of model II for 1 event week, shown in Table 3. However, at the 99.9-percent level, model II seems to underestimate the large model I event. This is because the five events of model I are larger than the five large events of model II, and the medium and small events of model II do not compensate in full for this difference. It is seen that for an exposure period of 100 weeks, model I is completely adequate for predicting the solar proton hazard at any reasonably high level of confidence (> 70 percent). In general, the writer believes that if a level of confidence is chosen that leads to at least one large event, then model I should be sufficient for predicting the solar proton hazard. An example where this does not occur is when t is only 10 weeks; and if one calculates the 90-percent level, the value X = 0 satisfies the limits of equation (25) for the large event case ($\lambda t = 0.1$). For the medium ($\lambda t = 0.20$) and small ($\lambda t = 0.30$) event case, the value X = 1 satisfies the limits of equation (25). Thus, if model I is used there are no events at the 90-percent level; whereas, if model II is used, one expects one medium event and one small event. Hence, for a shield of 5 g/cm^2 of aluminum, the point dose at the center of a spherical shell is 0.0 rads for model I and is 27.4 rads for model II at the 90-percent level.

TABLE 7. SOLAR PROTON DOSES AT VARIOUS PROBABILITY LEVELS FOR 100-WEEK MISSIONS USING MODEL II

a. Approximately 70-Percent Level

$Z (g/cm^2)$	$x = 1$ D_1	$x = 2$ D_2	$x = 3$ D_3	Total
2	1195	190	45	1430
5	286	48	10.2	344.2
10	78	15	2.7	95.7
15	36	7.0	1.29	44.3
20	21	4.0	0.75	25.75
30	9.3	1.6	0.33	11.23

b. 98-Percent Level

$Z (g/cm^2)$	$x = 3$ D_1	$x = 5$ D_2	$x = 7$ D_3	Total
2	3585	475	105	4165
5	858	120	23.8	1002
10	234	37.5	6.3	277.8
15	108	17.5	3.01	128.5
20	63	10	1.75	74.75
30	27.9	4	0.77	32.70

c. 99.9-Percent Level

$Z (g/cm^2)$	$x = 5$ D_1	$x = 8$ D_2	$x = 10$ D_3	Total
2	5975	760	150	6885
5	1430	192	34	1656
10	390	60	9	459
15	180	28	4.3	212
20	105	16	2.5	124
30	46.5	6.4	1.1	54

At the 99-percent level, one finds that model I predicts one large event, while model II predicts one large event, two medium events, and two small events. For the same shield as above, the model I dose is 340 rads and the model II dose is 341 rads, which is very good agreement at the 99-percent level. For the reasons given above, a useful rule would be:

> Empirical Rule: If at a given probability level there is at least one large event, model I is sufficient to describe the radiation hazard. If no large event is found at the given probability level, model II should be used.

At this point it might be useful to depict several curves similar to Figure 3 for various mission times t, with $\lambda = 0.01$ event/week, at given probability or percentile levels. To do this, arrays similar to Table 5 could be constructed for various values of λt, and the number of events, x, necessary to satisfy the inequality of equation (25) could be found for various values of $P \leq 1$.

Rather than provide a multitude of similar graphs, Table 8 will allow the reader to find the number of events for a range of anticipated values of λt at nine different percentile levels from 50 percent to 99.9 percent. Table 8 was constructed by choosing a probability level P, the number of events N, and then finding the value of $m(\lambda t)$ that satisfied the following equation:

$$\Pr(x \leq N) = \sum_{x=0}^{N} \frac{e^{-m} m^x}{x!} = P \quad , \quad (m = \lambda t) \quad . \tag{26}$$

For example, if $P = 0.99$ and $N = 2$, one has

$$f(m) = e^{-m} + me^{-m} + \frac{m^2 e^{-m}}{2} - 0.99 = 0 \quad . \tag{27}$$

Equation (27) has the solution of $m = 0.44$, which is found by Newton's method of successive iteration. Newton's method is simply,

$$m_{k+1} = m_k - \frac{f(m_k)}{f'(m_k)} \quad . \tag{28}$$

The derivative of equation (26) can be shown to be simply,

$$\frac{d}{dm}\left(\sum_{x=0}^{N} \frac{e^{-m} m^x}{x!} - P \right) = f'(m) = \frac{-m^N e^{-m}}{N!} \quad . \tag{29}$$

TABLE 8. THE VALUES OF λt TO GIVE N OR LESS EVENTS AT THE EXACT PERCENTILES

α / P^a / N^b	0.001 99.9 Percent	0.005 99.5 Percent	0.010 99.0 Percent	0.025 97.5 Percent	0.050 95.0 Percent	0.100 90.0 Percent	0.15 85.0 Percent	0.25 75.0 Percent	0.50 50.0 Percent
0	0.0010	0.0050	0.010	0.025	0.051	0.105	0.16	0.29	0.69
1	0.045	0.10	0.15	0.24	0.36	0.53	0.68	0.96	1.68
2	0.19	0.34	0.44	0.62	0.82	1.10	1.33	1.73	2.67
3	0.43	0.67	0.82	1.09	1.37	1.74	2.04	2.54	3.67
4	0.74	1.08	1.28	1.62	1.97	2.43	2.79	3.37	4.67
5	1.11	1.54	1.79	2.20	2.61	3.15	3.56	4.22	5.67
6	1.52	2.04	2.33	2.81	3.29	3.89	4.35	5.08	6.67
7	1.97	2.57	2.91	3.45	3.98	4.66	5.15	5.96	7.67
8	2.45	3.13	3.51	4.12	4.70	5.43	5.97	6.84	8.67
9	2.96	3.72	4.13	4.80	5.43	6.22	6.80	7.73	9.67
10	3.49	4.32	4.77	5.49	6.17	7.02	7.64	8.62	10.67
11	4.04	4.94	5.43	6.20	6.92	7.83	8.48	9.52	11.67
12	4.61	5.58	6.10	6.92	7.69	8.65	9.34	10.42	12.67
13	5.20	6.23	6.78	7.65	8.46	9.47	10.19	11.33	13.67
14	5.79	6.89	7.48	8.40	9.25	10.30	11.06	12.24	14.67
15	6.41	7.57	8.18	9.15	10.04	11.14	11.92	13.15	15.67
16	7.03	8.25	8.89	9.90	10.83	11.98	12.79	14.07	16.67
17	7.66	8.94	9.62	10.67	11.63	12.82	13.67	14.99	17.67
18	8.31	9.64	10.35	11.44	12.44	13.67	14.55	15.91	18.67
19	8.96	10.35	11.08	12.22	13.25	14.53	15.43	16.83	19.67
20	9.62	11.07	11.83	13.00	14.07	15.38	16.31	17.75	20.67
21	10.29	11.79	12.57	13.79	14.89	16.24	17.20	18.68	21.67
22	10.96	12.52	13.33	14.58	15.72	17.11	18.09	19.61	22.67
23	11.65	13.26	14.09	15.38	16.55	17.97	18.98	20.54	23.67
24	12.34	14.00	14.85	16.18	17.38	18.84	19.88	21.47	24.67
25	13.03	14.74	15.62	16.98	18.22	19.72	20.77	22.40	25.67
26	13.73	15.49	16.40	17.79	19.06	20.59	21.67	23.34	26.67
27	14.44	16.25	17.17	18.61	19.90	21.47	22.57	24.27	27.67
28	15.15	17.00	17.96	19.42	20.75	22.35	23.48	25.21	28.67
29	15.87	17.77	18.74	20.24	21.59	23.23	24.38	26.15	29.67
30	16.59	18.53	19.53	21.06	22.44	24.11	25.28	27.09	30.67
31	17.32	19.30	20.32	21.89	23.30	25.00	26.19	28.03	31.67
32	18.05	20.08	21.12	22.72	24.15	25.89	27.10	28.97	32.67
33	18.78	20.86	21.92	23.55	25.01	26.77	28.01	29.91	33.67
34	19.52	21.64	22.72	24.38	25.87	27.66	28.92	30.85	34.67
35	20.26	22.42	23.53	25.21	26.73	28.56	29.83	31.79	35.67
36	21.00	23.21	24.33	26.05	27.59	29.45	30.75	32.74	36.67
37	21.75	24.00	25.14	26.89	28.46	30.34	31.66	33.68	37.67
38	22.51	24.79	25.96	27.73	29.33	31.24	32.58	34.63	38.67
39	23.26	25.59	26.77	28.58	30.20	32.14	33.50	35.57	39.67
40	24.02	26.38	27.59	29.42	31.07	33.04	34.42	36.52	40.67

a. $P = 1 - \alpha$

b. If λt lies between two entries, use N corresponding to the largest entry. Thus, if $\lambda t = 0.5$, use N = 4 at the 99.9-percent level. If λt is less than or equal to the top entry, use N = 0. Thus, if $\lambda t = 0.02$, N = 0 at the 97.5-percent level, but N = 1 at the 99.0-percent level.

From equations (26) and (29), the iteration becomes

$$m_{k+1} = m_k + \frac{\sum_{x=0}^{N} e^{-m_k} m_k^{\,x} \Big/ x! \; - P}{m_k^{N} e^{-m_k} \Big/ N!} \quad . \tag{30}$$

The value of $m = \lambda t$ found in this manner is the expected value or mean of the Poisson distribution that has a percentile level of exactly $P \times 100$ percent for N or less events. For example, if $P = 0.99$, and $N = 2$, $\lambda t = 0.44$ satisfies equation (26). It should be noted that if $\lambda t = 0.70$, $N = 3$ must be chosen at the 99-percent level. This follows from equation (25), and P is no longer exactly 99 percent but becomes 99.42 percent. Hence, one says that for $\lambda t = 0.70$, the probability is at least 0.99 that the number of events is three or less. Another way of stating the condition is that the probability is no greater than $\alpha = 0.01$ that there will be more than three events when $\lambda t = 0.70$; i.e., $1 - P = 0.0058$.

The top of Table 8 is headed with a row of values labeled α, which denotes the probability of more than N events or simply $\alpha = 1 - P$. From equation (23) it can be seen that $Pr(X > N) = 1 - P(X \leq N) = \alpha$. The probability statements using α will be made as follows: the probability is no greater than α that more than N events will be observed in t weeks. The use of the α-probabilities will be derived in the next section. The range of values in Table 8 should provide for the refinements necessary in the following work.

If one uses the discrete Poisson distribution properly, it is not possible to get better values for N events at the preassigned probability levels than those obtained with the prescription used for Table 8. It should be remembered that a discrete distribution cannot have fractional events. The user should not be tempted to interpolate between probability values of the cumulative distribution function and obtain a fractional value of X that is used in the same manner as the writer is using discrete events in order to meet some special requirement. There are certain practical exceptions to this rule, but they occur only in that domain where the Poisson is well approximated by the Gaussian or normal distribution. This does not occur until $\lambda t > 20$.

It might be of value to examine the case where λt is sufficiently large that the Gaussian distribution becomes a reasonable approximation to the Poisson distribution. Where this becomes a safe assumption depends on the user's threshold of acceptable accuracy. For example, even at a value of $m = 40$, the probability of having a value of $X \leq 40$ is still 54 percent, whereas for a Gaussian distribution, the area to the left of the mean is exactly

50 percent. The Poisson distribution always has a slight skew to the right even for very large values of $m = \lambda t$. However, with symmetric distributions such as the Gaussian, it is common to use measures of central tendency about the mean for making probability statements. If the Poisson is to be approximated by a normal distribution, the value of the mean is $m = \lambda t$ and the standard deviation is $\sqrt{m}$ for the "equivalent" Gaussian distribution. The transformation to the normal distribution in the variable t is given by

$$\phi(t)dt = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \quad , \tag{31}$$

where

$$t = \frac{x - m}{\sqrt{m}} \quad .$$

Here, the mean value of the normal distribution is at $t = 0$, and the standard deviation $\sigma_t = 1$, x is the number of events for the Poisson, and m is the mean of the Poisson ($m = \lambda t$). The values of the integral,

$$\int_{-t}^{t} \phi(t)dt \tag{32}$$

are available in most handbooks, where $t = 1$ is one standard deviation and multiples of t are thought of as multiples of the standard deviation.

The reader will find that if probabilities are measured about the mean rather than from the tails, the Poisson distribution is fairly well approximated by the Gaussian even down to relatively low values of λt (e.g., 25). (See Figure 6 of Section IV for a heuristic comparison.)

It is of interest to recall that in dealing with a discrete distribution function, the difference between two cumulative distribution or sums [for example, $F(X'')$ and $F(X')$] denotes the probability:

$$P(X' < X \leq X'') = F(X'') - F(X') \quad . \tag{33}$$

As an example, the probability $f(x)$ of obtaining x dots when two dice are rolled is given in Table 9.

TABLE 9. PROBABILITIES FOR DICE PAIRS

x	2	3	4	5	6	7	8	9	10	11	12
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
F(x)	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	36/36

Therefore, from equation (33), $F(7) - F(4) = P(4 < X \leq 7) = 15/36$, or the probability of getting a 5, 6, or 7 on a pair of dice is 15/36.

SECTION IV. CONFIDENCE INTERVALS ARE ESTABLISHED

The results of the previous section depended strongly on the choice of the parameter λ or the probability of having a large event week. In this section the confidence that can be placed on the value of λ as calculated from observed sample data will be established. One of the virtues of the Poisson distribution is that the value of the mean $(m = \lambda t)$ completely determines the distribution function (i.e., $\sigma^2 = m$); whereas the mean and variance (σ^2) are needed for most distribution functions, and they are not simply related. Thus, in working with small sampling statistics from a normal distribution, one needs to establish confidence intervals for both the mean and the variance.

To more clearly explain what is being done in this section, an example will be given. The Poisson law arises very often in certain biological problems, such as with organisms distributed at random over the bottom of a lake. The number of such organisms found in a series of trial dredgings from separate small areas of the same size will follow this law. Statisticians calculate boundaries of possible outcomes from a given small sample, and these values are called confidence limits at a certain probability level for the assumed distribution function. If a biologist counted 21 organisms from one of his dredgings, he could assert that he is 95-percent confident that the mean or expected value lies between 13 and 32 organisms per unit area assuming a Poisson distribution. Thus, with only one sample and the assumption of a Poisson process, it is possible to set upper and lower bounds on possible outcomes at a given probability level. From the above example, it can be asserted that if many dredgings are made, it is expected that only 5 percent will contain a number of organisms outside the predicted range.

To be more specific for the class of problems herein, it will be assumed that the number of events observed over the sample space of T weeks is given by X_0 events. First, it should be made clear that a small sample estimate of $\hat{\lambda} = \frac{X_0}{T}$ exists. In the previous section it was assumed that m was the true mean for the Poisson distribution and that probability statements could be made;

$$\Pr\left(X > N_\alpha\right) \leq \alpha \qquad , \tag{34}$$

where X represents the total number of events that may occur in a time of T weeks, α is some small risk such as 0.01, or 0.001, and N_α is the smallest value of X that satisfies the inequality at the small risk. An upper bound on $m(\lambda t)$ is needed now to find a value N'_α so that the statement,

$$\left[\Pr\left(X > N'_\alpha\right) \leq \alpha\right] \qquad , \tag{35}$$

would be correct with a high probability; for example, 0.99 or, in general, P.

Thus, one finds a one-sided upper confidence interval on the true value of $m = \lambda t$. Suppose the probability is P that one would see more than X_0 events for the T weeks of observation. Then the true value of m would have to be a value such that

$$\Pr\left(X \leq X_0\right) = \sum_{x=0}^{X_0} \frac{e^{-m}\, m^x}{x!} = 1 - P \qquad , \tag{36}$$

where $1 - P$ is a small number.

Let m' be the solution to this equation. The N'_α will be that smallest integer for which

$$\Pr\left(x > N'_\alpha \,\middle|\, m = m'\right) \leq \alpha \qquad . \tag{37}$$

That is, N'_α is the smallest integer satisfying

$$\sum_{x=0}^{N'_\alpha} \frac{e^{-m'}\,(m')^x}{x!} \leq \alpha \qquad . \tag{38}$$

For example, if $X_0 = 3$, $P = 0.99$, and $\alpha = 0.001$, then m' is the solution to

$$\sum_{x=0}^{3} \frac{e^{-m'} (m')^x}{x!} = 1 - 0.99 = 0.01 \quad , \tag{39}$$

or

$$e^{-m'} \left[1 + m' + \frac{(m')^2}{2} + \frac{(m')^3}{6} \right] = 0.01 \quad . \tag{40}$$

Using a table of the cumulative Poisson distribution or Newton's method of finding the root of a transcendental equation, one finds that $m' = 10.05$ satisfies the equation. This means that one is 99-percent certain (confident) that the true mean for large events in the 300 weeks is 10 or less.

Next, one finds the smallest integer N'_α that satisfies

$$\sum_{x=0}^{N'_\alpha} \frac{e^{-10}\, 10^x}{x!} \geq 1 - \alpha = 0.999 \quad . \tag{41}$$

Again, by using tables of Poisson cumulative probabilities or Table 8, one finds that $N'_\alpha = N'_{0.001} = 21$ is the solution. Some other values for N'_α when $P = 0.99$ $(m' = 10)$ are readily found from Table 8 and are shown in Table 10.

TABLE 10. SUMMARY OF VALUES N'_α

$P = 0.99$	α	0.001	0.01	0.025	0.05	0.10
$m' = 10.05$	N'_α	21	18	17	15	14

As a summary of the above, one is 100P percent confident that the following statements are true[4]:

The true value of m is no greater than m', and the probability is no greater than α that more than N'_α event weeks will be observed in T weeks.

4. Most of these concepts grew out of a private communication with Dr. James E. Norman, Jr., Department of Statistics, University of Georgia.

Returning to the basic problem, one wishes to establish the confidence interval on m, hence to find the probability

$$\Pr(m'' \leq m \leq m') \geq (1 - 2\beta)\ 100\% \quad , \tag{42}$$

where $1 - P = \beta$. If $P = 0.99$ and $\beta = 0.01$, then

$$\Pr(m'' \leq m \leq m') \geq 98\% \quad .$$

and one is at least 98 percent certain that m lies between m' and m'', where m has only a 1-percent chance of being greater than m'. The general set of equations can now be written that will determine the value of m' and m'':

$$\sum_{x=0}^{X_0} \frac{e^{-m'}(m')^x}{x!} = 1 - P = \beta \tag{43}$$

and

$$\sum_{x=X_0}^{\infty} \frac{e^{-m''}(m'')^x}{x!} = \beta \quad ,$$

or

$$\sum_{x=0}^{X_0-1} \frac{e^{-m''}(m'')^x}{x!} = 1 - \beta \quad , \tag{44}$$

where X_0 is the observed number of events. The form of the above equation is an outgrowth of using a discrete distribution function and follows the common practice of textbooks on statistics. Solving equations (43) and (44) for m' and m'' by using Newton's method [see equation (30)], the upper and lower bounds on m are found for X_0 observed events. Table 11 summarizes these results for eight different values of β or P. Figures 4 and 5 depict typical results taken from Table 11.

Figure 6 presents comparable results when the normal law is assumed. For this calculation, the change of variable to the normal variate t is made using the relationship,

$$t = \frac{X_0 - x}{\sqrt{X_0}} \quad , \tag{45}$$

TABLE 11. UPPER AND LOWER BOUNDS FOR THE MEAN AT PROBABILITY OF $P = (1 - \beta)\,100\%$

Confidence Interval	99.8 Percent		99.0 Percent		98.0 Percent		95.0 Percent	
β	0.001		0.005		0.010		0.025	
Events, X_0	Lower Bound[a]	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
0	0	6.91	0	5.30	0	4.61	0	3.69
1	0.001	9.23	0.005	7.43	0.01	6.64	0.025	5.57
2	0.045	11.23	0.10	9.27	0.15	8.41	0.24	7.22
3	0.19	13.06	0.34	10.98	0.44	10.05	0.62	8.77
4	0.43	14.79	0.67	12.59	0.82	11.60	1.09	10.24
5	0.74	16.45	1.08	14.15	1.28	13.11	1.62	11.67
6	1.11	18.06	1.54	15.66	1.79	14.57	2.20	13.06
7	1.52	19.63	2.04	17.13	2.33	16.00	2.81	14.42
8	1.97	21.16	2.57	18.58	2.57	17.40	3.45	15.76
9	2.45	22.66	3.13	20.00	3.51	18.78	4.12	17.08
10	2.96	24.13	3.72	21.40	4.13	20.14	4.80	18.39
11	3.49	25.59	4.32	22.78	4.77	21.49	5.49	20.96
12	4.04	27.03	4.94	24.15	5.43	22.82	6.20	20.96
13	4.61	28.45	5.58	25.50	6.10	24.14	6.92	22.23
14	5.20	29.85	6.23	26.84	6.78	25.45	7.65	23.49
15	5.79	31.24	6.89	28.16	7.48	26.74	8.40	24.74
16	6.41	32.62	7.57	29.48	8.18	28.03	9.15	25.98
17	7.03	33.99	8.25	30.79	8.89	29.31	9.90	27.22
18	7.66	35.35	8.94	32.09	9.62	30.58	10.67	28.45
19	8.31	36.70	9.64	33.38	10.35	31.85	11.44	29.67
20	8.96	38.04	10.35	34.67	11.08	33.10	12.22	30.89
21	9.62	39.37	11.07	35.95	11.83	34.35	13.00	32.10
22	10.29	40.70	11.79	37.22	12.57	35.60	13.79	33.31
23	10.96	42.02	12.52	38.48	13.33	36.84	14.58	34.51
24	11.65	43.33	13.26	39.74	14.09	38.08	15.38	35.71
25	12.34	44.64	14.00	41.00	14.85	39.31	16.18	36.90

a. The lower bound is calculated so that β is less than the indicated value.

TABLE 11. (Concluded)

Confidence Interval	90.0 Percent		80.0 Percent		70.0 Percent		50.0 Percent	
β	0.050		0.100		0.15		0.25	
Events, X_0	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
0	0	3.00	0	2.30	0	1.90	0	1.39
1	0.051	4.74	0.105	3.89	0.16	3.37	0.29	2.69
2	0.36	6.30	0.53	5.32	0.68	4.72	0.96	3.92
3	0.82	7.75	1.10	6.68	1.33	6.01	1.73	5.11
4	1.37	9.15	1.74	7.99	2.04	7.27	2.54	6.27
5	1.97	10.51	2.43	9.27	2.79	8.49	3.37	7.42
6	2.61	11.84	3.15	10.53	3.56	9.70	4.22	8.56
7	3.29	13.15	3.89	11.77	4.35	10.90	5.08	9.68
8	3.98	14.43	4.66	12.99	5.15	12.08	5.96	10.80
9	4.70	15.71	5.43	14.21	5.97	13.25	6.84	11.91
10	5.43	16.96	6.22	15.41	6.80	14.41	7.72	13.02
11	6.17	18.21	7.02	16.60	7.64	15.57	8.62	14.12
12	6.92	19.44	7.83	17.78	8.48	16.71	9.52	15.22
13	7.69	20.67	8.65	18.96	9.34	17.86	10.42	16.31
14	8.46	21.89	9.47	20.13	10.19	19.00	11.33	17.40
15	9.25	23.10	10.30	21.29	11.06	20.13	12.24	18.49
16	10.04	24.30	11.14	22.45	11.92	21.26	13.15	19.57
17	10.83	25.50	11.98	23.61	12.79	22.38	14.07	20.65
18	11.63	26.69	12.82	24.76	13.67	23.50	14.99	21.73
19	12.44	27.88	13.67	25.90	14.55	24.62	15.91	22.81
20	13.25	29.06	14.53	27.05	15.43	25.74	16.83	23.88
21	14.07	30.24	15.38	28.18	16.31	26.85	17.75	24.96
22	14.89	31.41	16.24	29.32	17.20	27.96	18.68	26.03
23	15.72	32.59	17.11	30.45	18.09	29.07	19.61	27.10
24	16.55	33.75	17.97	31.58	18.98	30.17	20.54	28.17
25	17.38	34.92	18.84	32.71	19.88	31.28	21.47	29.23

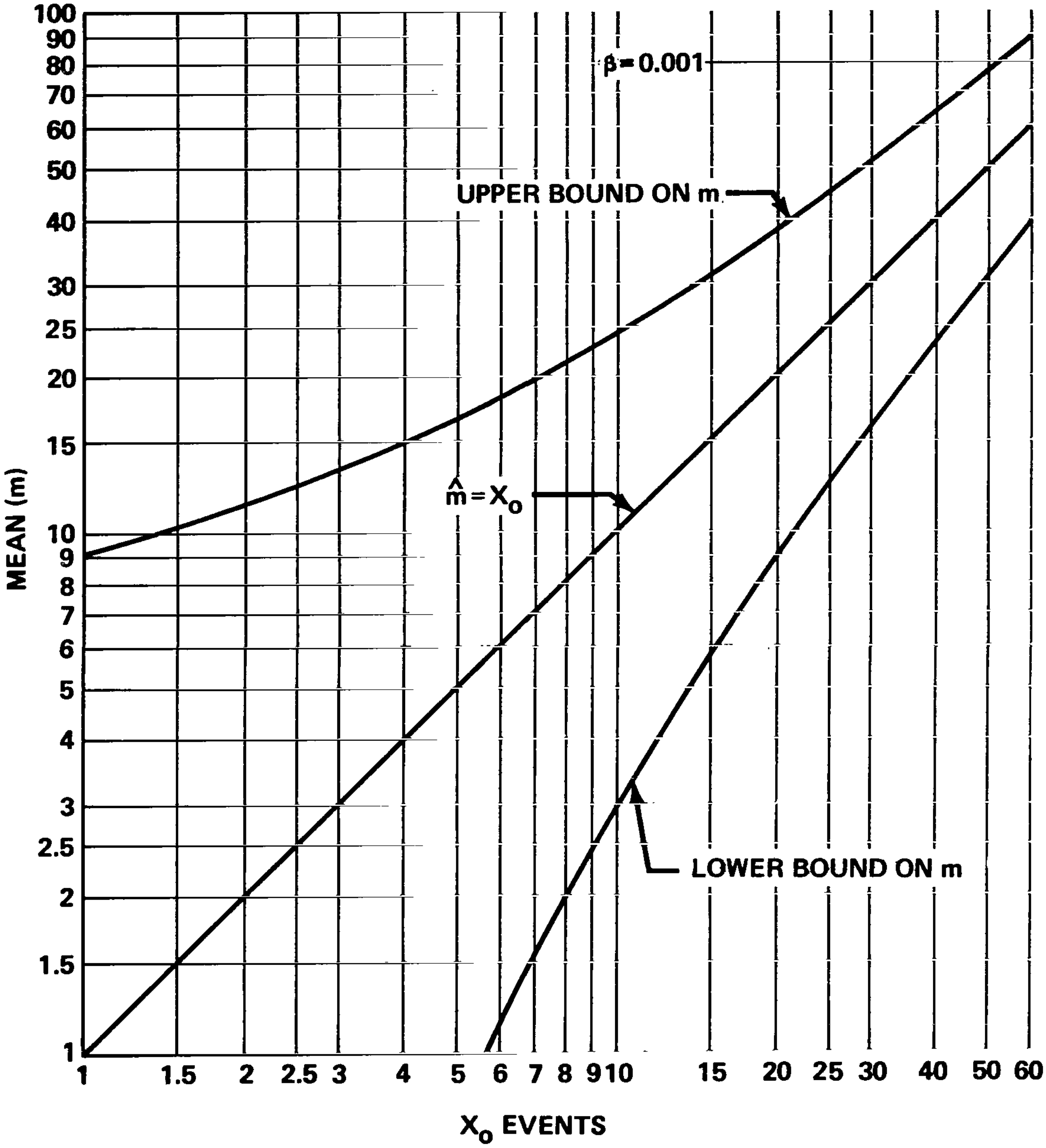

Figure 4. 99.8-percent confidence limits for Poisson law.

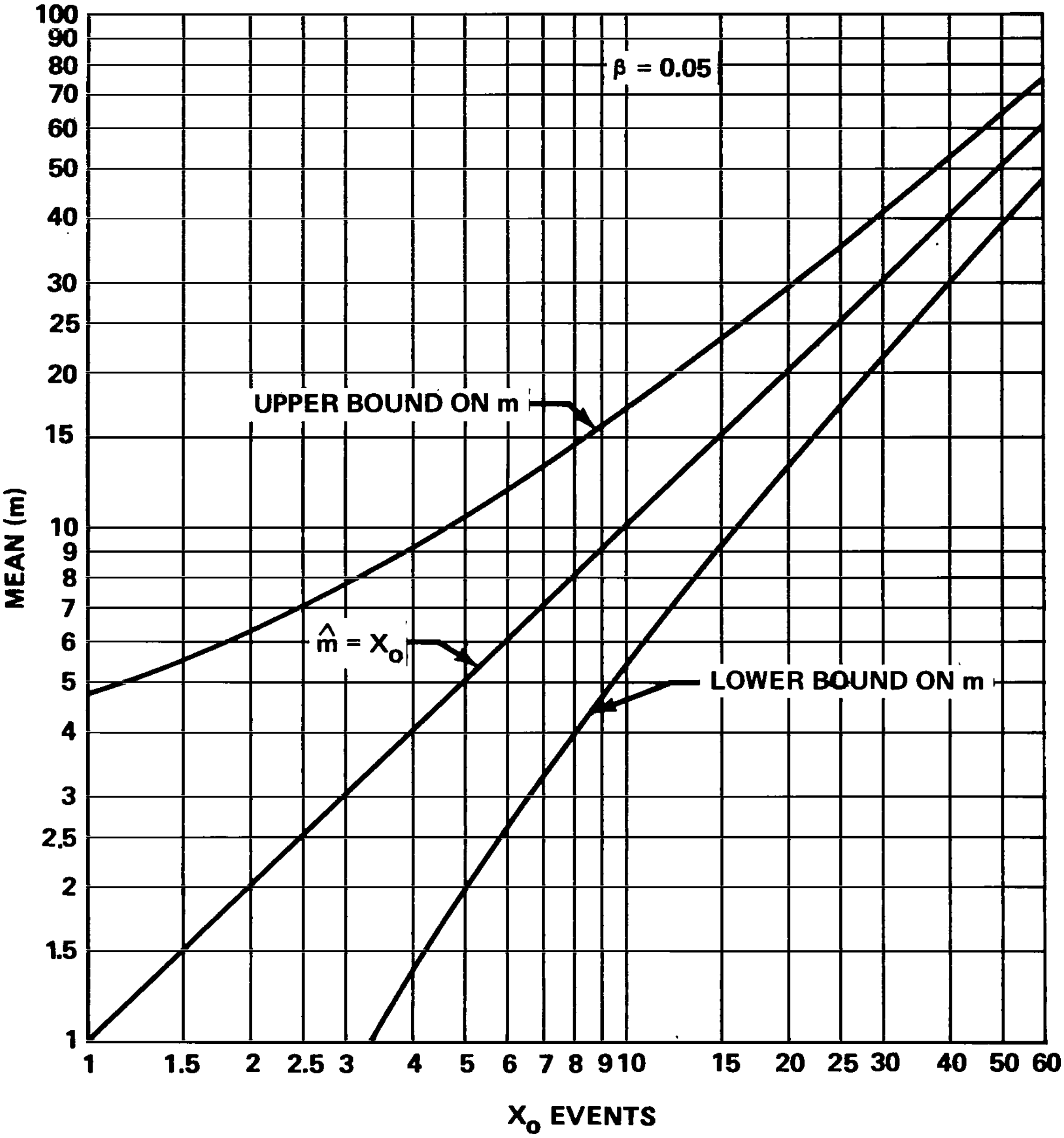

Figure 5. 90.0-percent confidence limits for Poisson law.

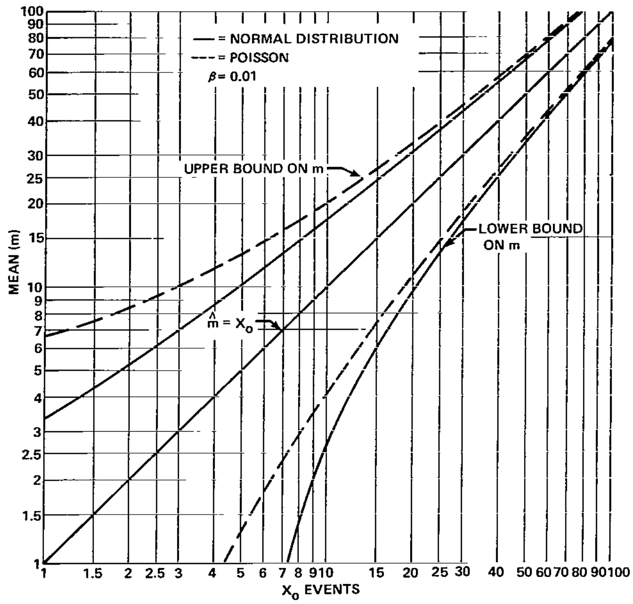

Figure 6. Comparison of 98.0-percent confidence limit for the Poisson and normal laws.

where X_0 is the observed number of events. (See equation (31) for an explanation.)

Even though confidence bounds on the estimate of $m = \lambda t$ have been obtained, the major interest is with the upper bounds that an investigator should use when only X_0 events have been observed over a T week time interval. Following this concept and the simple model of a large dose event week, it is recalled that only three such event weeks occurred over the 300 active weeks of the 19th solar cycle. Now, if one wishes to find the upper bounds on this observation, he may use Table 11, in conjunction with Table 8, and arrive at the 100P-percent confidence level that the probability that N'_α events will occur is no greater than some small value α. Thus, Table 12 is constructed. This table is a detailed extension of the results given in Table 10 with different values of P and m' for different values of α and N'_α.

The results shown in Table 12 are fairly clear; however, it may be useful to explain how Table 12 was constructed. First, the number of observed events $X_0 = 3$ is used as the starting point. Using Table 11 suppose the 95-percent upper bound on m is desired, then the value of $m' = 7.75$ is found opposite $X_0 = 3$ in the column $\beta = 0.05$. Next, one refers to Table 8 and under the column headed by $\alpha = 0.050$, the value of $m' = \lambda t = 7.75$ is found between 7.69 and 8.46 that corresponds to the N-numbers (first column) of 12 and 13. Since the value of N is discrete, one must choose $N'_\alpha = 13$ to guarantee that the probability is less than 0.05 that more than 13 events will occur. It should be clear to the reader that the exact value of α lies between 0.05 and 0.025, as can be seen from Table 8. However, because of the rather extensive computations necessary, it is sufficient to say that the probability is no greater than α (0.05) that more than N'_α (13) events will be observed in 300 solar active weeks.

Since the 20th solar cycle has produced no large events, it is of interest to ask, If zero large events occur for a period of 300 active weeks, what are the possible upper bounds of m' and the number of events N'_α that could be exceeded at some small probability? (This may be a nonsensical question and will be discussed in the next section.) Table 13 is constructed with this in mind. The methods used are the same as for Table 12. It is of interest to note that even though $X_0 = 0$ events are observed, the 95-percent upper bound for the Poisson mean is $m' = 3$, which was the actual observed number of large event weeks during the 19th cycle. Thus, it seems that for a reasonable level of confidence (95 percent), the analyst would be justified if he used $m' = 3$ ($\lambda = 0.01$ event/week) for large events even though the solar activity for a given cycle was considerably different from that of the 19th cycle. Perhaps, the only conclusion that can be drawn is that the observed 19th cycle dose events could be used for any near average solar cycle, and if a cycle is predicted to be similar to the 19th cycle, the results of Table 12 ($X_0 = 3$) should be seriously considered as a possible model.

To expand the events observed to cover both medium and small events as described in model II (Table 3), Tables 14 and 15 are presented for $X_0 = 6$ and 9 observed events. Finally, Table 16 is presented as a summary of values to use for λ (events/week) at various confidence levels. For large events, it seems that the $X_0 = 0$ column is probably reasonable to use if the solar cycle is not very active. If a cycle similar to the 19th is forecast, then the column under $X_0 = 3$ is preferred.

The value of λ and the dose event model to use during the remaining 270 less-active weeks of a solar cycle are not obtainable from the present analysis. However, until more data are available, the best one can do is use some criteria that are essentially arbitrary. For example, if one believes that the chance of a large dose event week is definitely dependent on some

TABLE 12. THE NUMBER OF EVENTS N'_{α} THAT WILL BE EXCEEDED AT A PROBABILITY NO GREATER THAN α FOR THE 100P-PERCENT UPPER BOUND VALUE OF THE POISSON MEAN, m', WHEN X_0 = 3 OBSERVED EVENTS

m'	100P Percent	α 0.001	0.005	0.010	0.025	0.050	0.100	0.150	0.250	0.500
13.06	99.9	25	23	22	20	19	18	17	15	13
10.98	99.5	22	20	19	18	17	15	14	13	11
10.05	99.0	21	19	18	17	15	14	13	12	10
8.77	97.5	19	17	16	15	14	13	12	11	9
7.75	95.0	18	16	15	14	13[a]	11	11	9	8
6.68	90.0	16	14	13	12	11	10	9	8	7
6.01	85.0	15	13	12	11	10	9	8	7	6
5.11	75.0	13	12	11	10	9	8	7	6	5
3.67	50.0	11	9	9	8	7	6	6	5	3

a. Example: One is 95 percent confident that the true mean does not exceed 7.75 (X_0=3), and using this value of the mean, one is certain that the probability of seeing more than 13 events during a 300 week active period is no greater than 0.050.

TABLE 13. THE NUMBER OF EVENTS N'_{α} THAT WILL BE EXCEEDED AT A PROBABILITY NO GREATER THAN α FOR THE 100P-PERCENT UPPER BOUND VALUE OF THE POISSON MEAN, m', WHEN X_0 = 0 OBSERVED EVENTS

m'	100P Percent	α 0.001	0.005	0.010	0.025	0.050	0.100	0.150	0.250	0.500
6.91	99.9	16	15	14	12	11	10	10	9	7
5.30	99.5	14	12	11	10	9	8	8	7	5
4.61	99.0	12	11	10	9	8	7	7	6	4
3.69	97.5	11	9	9	8	7	6	6	5	4
3.00	95.0	10	8	8	7	6	5	5	4	3
2.30	90.0	8	7	6	6	5	4	4	3	2
1.90	85.0	7	6	6	5	4	4	3	3	2
1.39	75.0	6	5	5	4	4	3	3	2	1
0.69	50.0	4	4	3	3	2	2	2	1	0

TABLE 14. THE NUMBER OF EVENTS N'_α THAT WILL BE EXCEEDED AT A PROBABILITY NO GREATER THAN α FOR THE 100P-PERCENT UPPER BOUND VALUE OF THE POISSON MEAN, m', WHEN X_0=6 OBSERVED EVENTS

m'	100P Percent	α 0.001	0.005	0.010	0.025	0.050	0.100	0.150	0.250	0.500
18.06	99.9	33	30	29	27	25	24	22	21	18
15.66	99.5	29	27	26	24	22	21	20	18	15
14.57	99.0	28	25	24	22	21	20	19	17	14
13.06	97.5	26	23	22	21	19	18	17	15	13
11.84	95.0	24	22	21	19	18	16	15	14	12
10.53	90.0	22	20	19	17	16	15	14	13	10
9.70	85.0	21	19	18	16	15	14	13	12	10
8.56	75.0	19	17	16	15	14	12	12	10	8
6.67	50.0	16	14	13	12	11	10	9	8	6

TABLE 15. THE NUMBER OF EVENTS N'_α THAT WILL BE EXCEEDED AT A PROBABILITY NO GREATER THAN α FOR THE 100P-PERCENT UPPER BOUND VALUE OF THE POISSON MEAN, m', WHEN X_0=9 OBSERVED EVENTS

m'	100P Percent	α 0.001	0.005	0.010	0.025	0.050	0.100	0.150	0.250	0.500
22.66	99.9	39	36	34	32	31	29	28	26	22
20.00	99.5	35	32	31	29	28	26	25	23	20
18.78	99.0	33	31	30	28	26	24	23	22	19
17.08	97.5	31	29	27	26	24	22	21	20	17
15.71	95.0	29	27	26	24	22	21	20	18	16
14.21	90.0	27	25	24	22	21	19	18	17	14
13.25	85.0	26	23	22	21	19	18	17	16	13
11.91	75.0	24	22	21	19	18	16	15	14	12
9.67	50.0	21	19	18	16	15	14	13	12	9

TABLE 16. VALUES FOR $\lambda' = m'/300$ (EVENTS/WEEK) FOR X_0 OBSERVED EVENTS AT THE 100P-PERCENT UPPER BOUND CONFIDENCE LEVEL

100P Percent \ X_0	0	3	6	9
99.9	0.0230	0.0435	0.0602	0.0755
99.5	0.0177	0.0366	0.0522	0.0667
99.0	0.0154	0.0335	0.0486	0.0626
97.5	0.0123	0.0292	0.0435	0.0569
95.0	0.0100	0.0258	0.0395	0.0524
90.0	0.0070	0.0223	0.0351	0.0474
85.0	0.0063	0.0200	0.0323	0.0442
75.0	0.0046	0.0170	0.0285	0.0392
50.0	0.0023	0.0122	0.0222	0.0322
OBS[a]	0.0000	0.0100	0.0200	0.0300

a. OBS denotes the value of $X_0/300$.

minimal level of solar activity and this level is not approached during the quiet periods of the cycle, it would be unreasonable to use the large event dose week even at a very low probability level, such as the values shown under the $X_0 = 0$ column of Table 16. Even if λ is only 0.002 event/week for the Poisson distribution, one sees that the chance of getting two or more large events is approximately 6 percent for T = 200 weeks. However, one cannot preclude the possibility of some smaller dose event occurring for the quiet period of a solar cycle. This writer suggests using $\lambda = 0.02$ event/week for the quiet period, but recommends that the dose model for the small dose event (D_3) of model II (Table 3) be used. This is equivalent to expecting about one such event per year, which is reasonably close to the actual observed number of events from October 1961 through July 1966.

To illustrate applications of the foregoing work, eight different trip lengths will be considered ranging from 13 weeks to 260 weeks (5 years), during the most active 6 years of a solar cycle that is similar to the 19th cycle ($X_0 = 3$). To simplify the possible combinations, four different probability levels for the upper bound mean (λt) are used at four probability levels for the Poisson distribution. A mean (0.01t) corresponding to the actual observed (OBS) large event weeks of the 19th cycle is also given. This corresponds to the 95-percent level of the mean for $X_0 = 0$ events, and this value is recommended for the average type cycle. The results of these combinations are shown in Table 17. The entries in Table 17 give the number of large events that will be exceeded at a probability equal to or less than α, corresponding to the Poisson mean (λt) that one is 100P percent certain will not be exceeded (see the footnote for Table 12). Table 17 is constructed in a manner similar to Table 12; however, one begins with Table 16 for values of λ at various levels of percent of confidence (100P).

The foregoing method is very convenient, and one can readily generate many such tables. However, it would be instructive to demonstrate to the reader that any more precise approach would be excessively tedious and the probabilities would be different for each combination of $m(\lambda t)$ and the number of events N. By using a Poisson distribution table, one can arrive at the exact probabilities of exceeding N events for a given value of $m = \lambda t$. These exact values were calculated for a 78-week mission using the 95-percent value of $m(\lambda t = 2.00)$. These exact results are shown in parentheses in Figure 7. The first quantities given, which identify the percent probabilities of Figure 7, are taken from the α values of Table 17. It should be noted that the methods used in forming Table 17 give a conservative estimate of the true probabilities shown in parentheses.

Figures 8, 9, and 10 illustrate values of α at 0.001, 0.01, and 0.1 for different upper bound values of the mean showing a range from the observed (approximately 50 percent) to the 99.9-percent confidence level of the mean (λt) for a 78-week mission during the active weeks of a very active cycle. Comparisons are also shown in Figures 8, 9, and 10 to the work of other authors [2, 9, 10] who have made similar computations but have used different models for predicting the solar proton dose. An interesting aspect of the above comparison is that at the 0.1-percent ($100 \times \alpha$) level, the present work is considerably lower even for the 99.9-percent upper bound value of λt. However, at the 10-percent level, the reverse situation seems to be the case. The major difference in the methods of most of the other authors and the present work is that the size of our large event is fixed but the number of events may be quite large, whereas in the other methods the size of a single event may be extremely large (several times larger than the large event used in this work). These differences will be discussed in the last section.

Figure 11 illustrates the conjectured proton event dose at the center of a spherical shell of aluminum during the quiet periods of a solar cycle. In this calculation, $\lambda = 0.02$ event/week and the product λt is found in the

TABLE 17. NUMBER OF LARGE EVENT WEEKS EXPECTED FOR VARIOUS MISSION LENGTHS AT GIVEN JOINT PROBABILITIES P AND α FOR $X_0 = 3$

a. t = 13 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
0.57	99.9	4	3	2	2
0.44	99.0	3	2	2	1
0.34	95.0	2	2	1	1
0.29	90.0	2	2	1	1
0.13	OBS	1	1	1	1

b. t = 26 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
1.13	99.9	6	4	3	3
0.87	99.0	5	4	3	2
0.67	95.0	4	3	2	2
0.58	90.0	4	3	2	2
0.26	OBS	3	2	1	1

c. t = 52 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
2.26	99.9	8	6	5	4
1.74	99.0	7	5	4	3
1.34	95.0	6	5	3	3
1.16	90.0	6	4	3	3
0.52	OBS	4	3	2	1

d. t = 78 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
3.39	99.9	10	8	7	6
2.61	99.0	9	7	5	5
2.01	95.0	8	6	5	4
1.74	90.0	7	5	4	3
0.78	OBS	5	3	2	2

e. t = 104 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
4.52	99.9	12	10	8	7
3.48	99.0	10	8	7	6
2.68	95.0	9	7	6	5
2.32	90.0	9	6	5	4
1.04	OBS	5	4	3	2

f. t = 156 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
6.79	99.9	16	16	11	10
5.23	99.0	14	11	9	8
4.02	95.0	11	9	8	7
3.48	90.0	10	8	7	6
1.56	OBS	7	5	4	3

g. t = 208 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
9.05	99.9	20	17	14	13
6.97	99.0	16	14	12	10
5.37	95.0	14	11	9	8
4.64	90.0	13	10	8	7
2.08	OBS	8	6	5	4

h. t = 260 Weeks

$m = \lambda t$	$P \times 100$ \ α	0.001	0.01	0.05	0.10
11.31	99.9	23	20	17	16
8.71	99.0	19	16	14	13
6.71	95.0	16	13	11	10
5.80	90.0	14	12	10	9
2.60	OBS	9	7	5	5

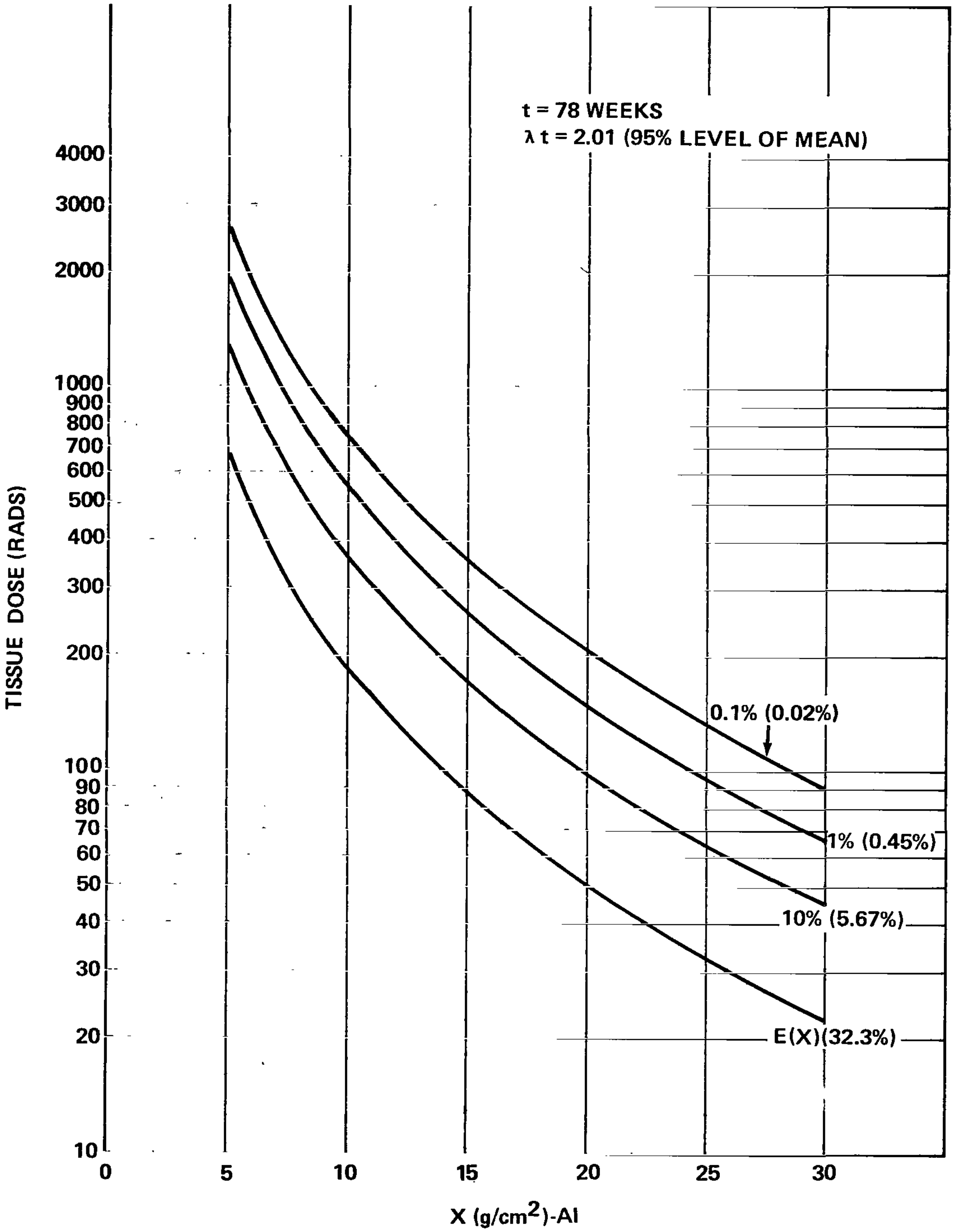

Figure 7. Solar proton dose curves as a function of shield thickness at various probability levels for a 78-week mission using a mean at the 95-percent level.

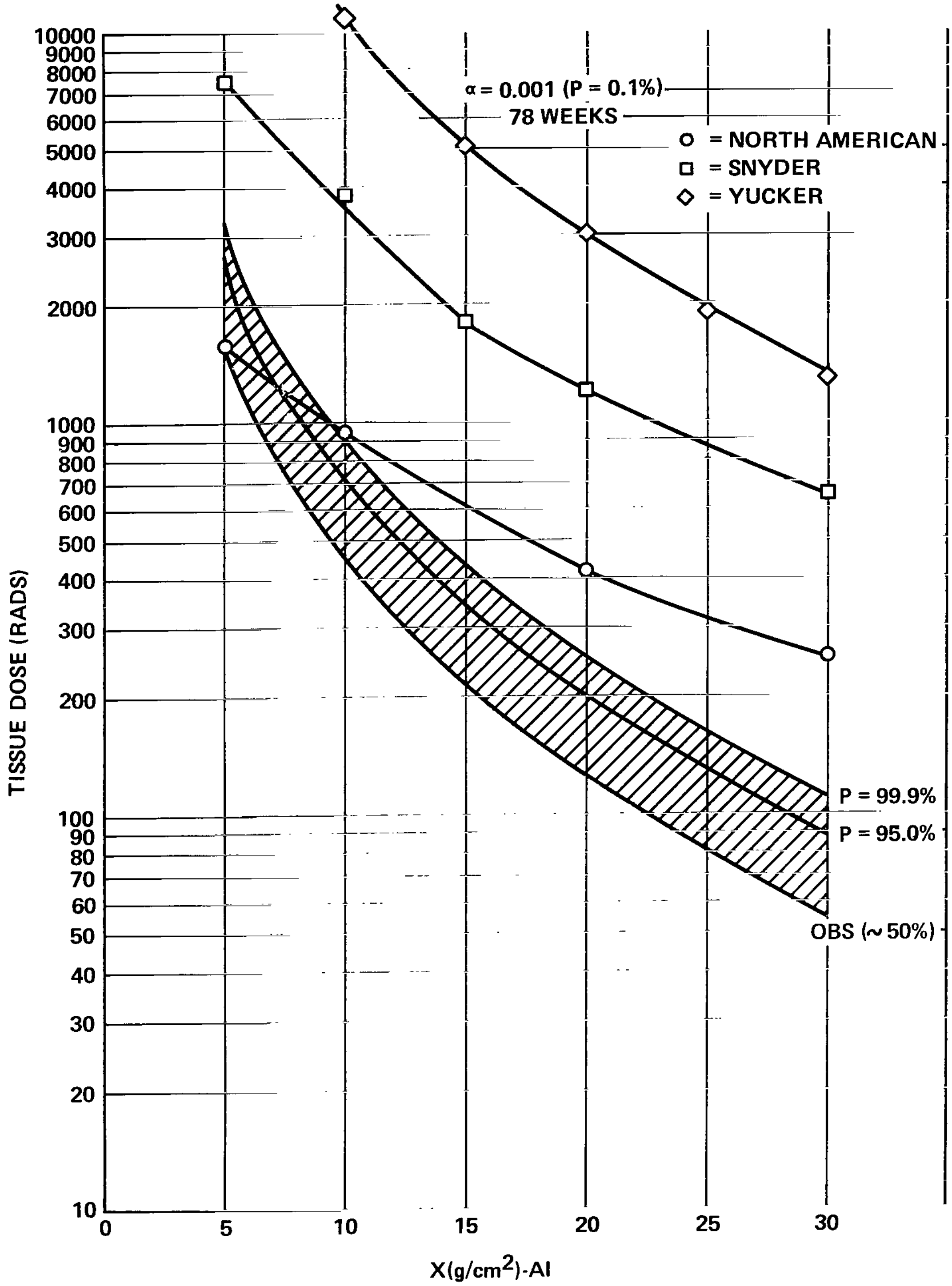

Figure 8. Solar proton dose curves as a function of shield thickness at probabilities $\alpha = 0.001$ for a spread of means from approximately 50 percent to the 99.9-percent level, as compared to other researchers at the same α.

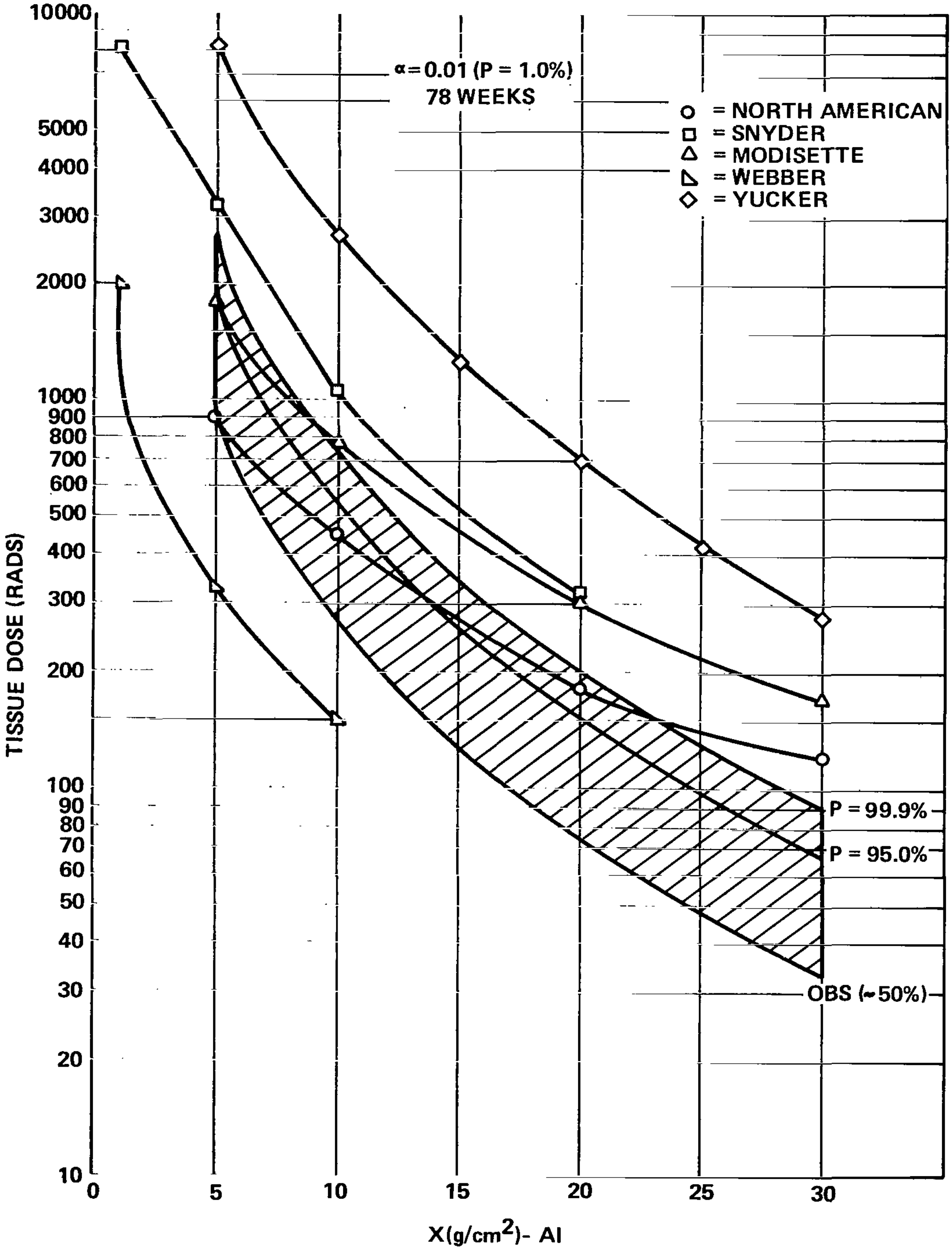

Figure 9. Solar proton dose curves as a function of shield thickness at probabilities α = 0.01 for a spread of means from approximately 50 percent to the 99.9-percent level, as compared to other researchers at the same α.

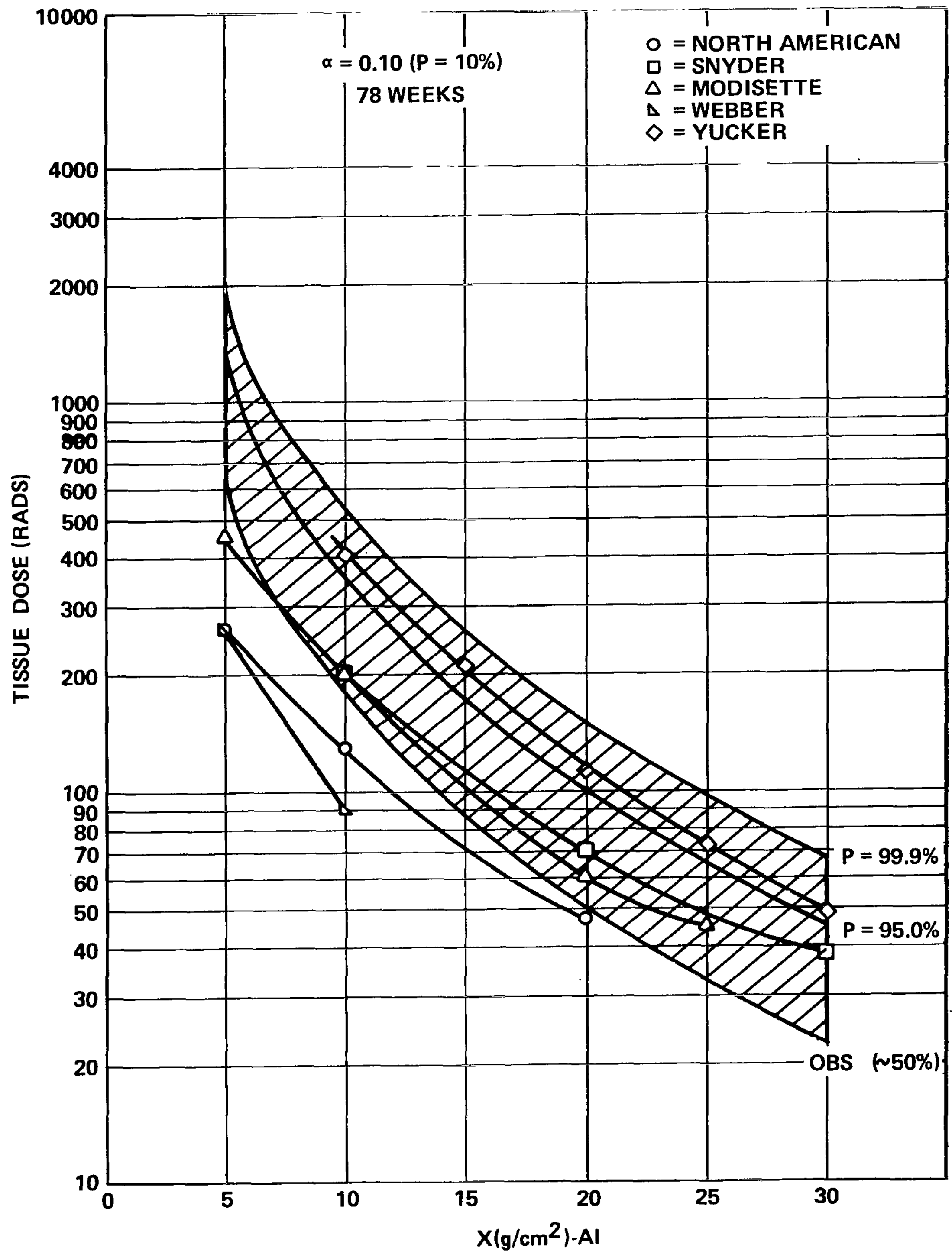

Figure 10. Solar proton dose curves as a function of shield thickness at probabilities $\alpha = 0.10$ for a spread of means from approximately 50 percent to the 99.9-percent level, as compared to other researchers at the same α.

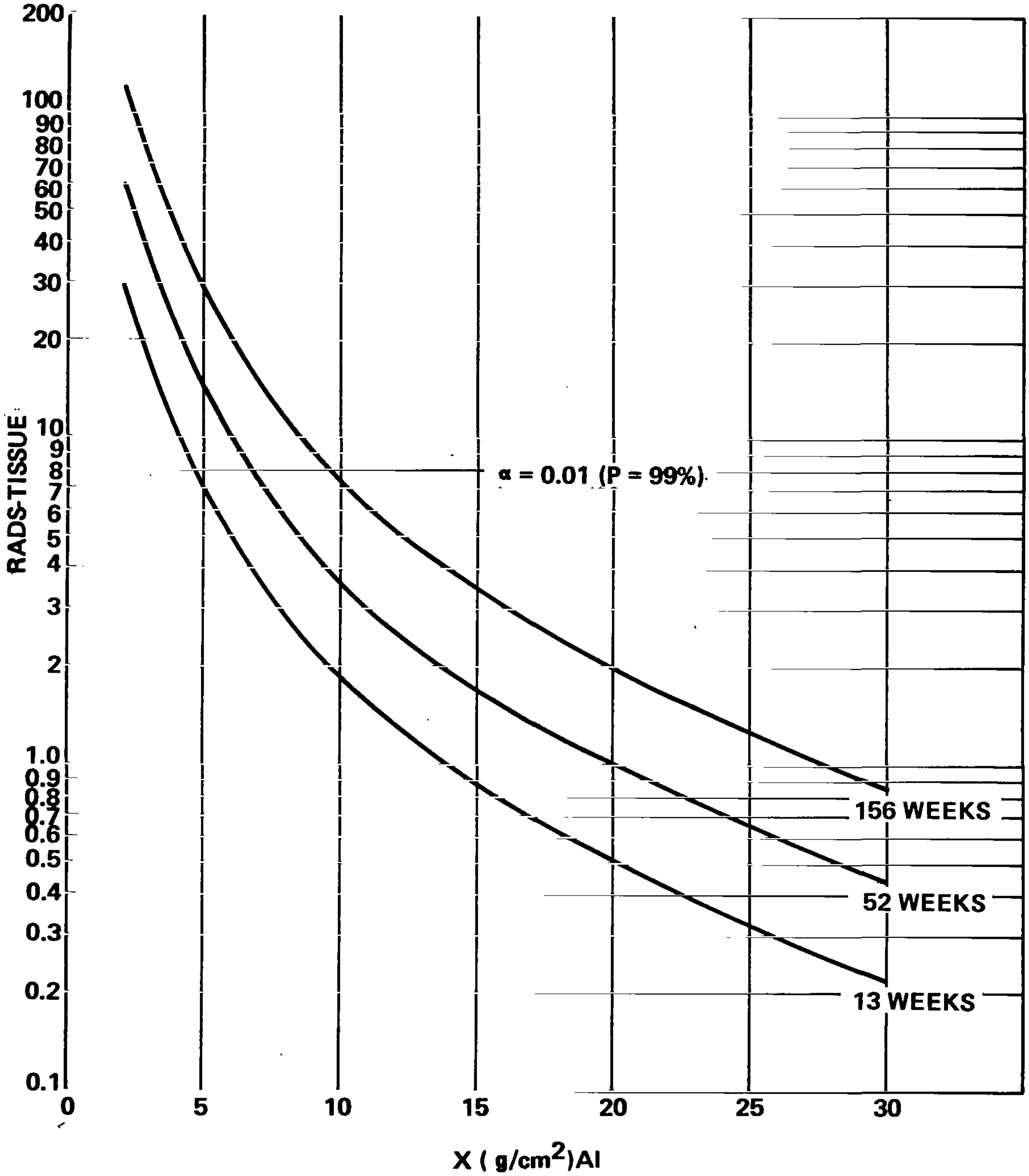

Figure 11. 99-percent confidence level doses as a function of shield thickness for the quiet period of the solar cycle using $\lambda = 0.02$ event/week.

99.0-percent column of Table 8 to ascertain the number of events for the time period depicted. The dose data, D_3, from model II of Table 3 is used. This corresponds to the small events of the 19th cycle, as shown in Table 2. It should be recalled that if the reader wishes to estimate the realistic dose to an astronaut in a complex spacecraft, then the doses of Figure 11 should be reduced by a factor of about three. This also applies to Figures 7 through 10.

SECTION V. A NEW PROBABILITY MODEL IS FOUND

The previous section seemingly provides some rationale for choosing a probability function for a given set of conditions: (1) the active 300 weeks of a very active cycle such as the 19th, (2) the active 300 weeks of an average cycle, and (3) the quiet 270 weeks of any solar cycle. However, if one examines Tables 12 through 15, a question arises. If a very high confidence level is chosen for the Poisson mean and then the number of events that would give a low probability α of getting a worse situation is determined, it seems that a very improbable situation is being discussed. That is, if the mean is at the 99.9 percent confidence level, what is the joint probability of seeing more than N'_α events at a probability no greater than 0.001? For example, in Table 12, one sees that $N'_\alpha = 25$ events when $P = 0.999$ and $\alpha = 0.001$. At first, one might suspect that the chance of both conditions occurring is on the order of 10^{-6}; however, care must be taken since one is dealing with cumulative distribution functions. Thus, the product $(1-P)\alpha$ does not correspond to a unique value of N'_α. This can be ascertained by examining the values of $(1-P)\alpha$ corresponding to $N'_\alpha = 13$ in Table 12. Now, the question is asked:

> "If X_0 events are observed in a given time T_0, what is the probability of seeing more than N events in any observation time t?"

With this knowledge, a unique value of the probability of seeing N events can be made for any period $t \leq T_0$.

This question is also important because it is proposed that the values of λ calculated for the total period of $T_0 = 300$ weeks be used and that this value be applied to periods of $t \leq T_0$. Thus, one might be suspicious of the true probability when one applies the value of λ to a time period of, for example, only 50 weeks. However, the reader surely agrees that the best estimate of λ corresponds to the total sample space of the 300 most active weeks during the 19th solar cycle.

To provide an answer to the above, one must first ask, what is the distribution of possible Poisson means if in a time T_0 there are only X_0 events observed? If equation (43), which gives the cumulative distribution of β as a function of the upper bound values of m for a given X_0, is examined, one sees that the derivative $\left(\frac{-d\beta}{dm}\right)$ yields the desired upper bound probability density function for m,

$$\frac{-d\beta}{dm} = f(m) = \frac{e^{-m} m^{X_0}}{X_0!} \quad . \tag{46}$$

This function is continuous in m and is the so-called gamma distribution in the variate m. The expected value of m is $E(m) = X_0 + 1$, the variance is $\sigma^2(m) = X_0 + 1$, and

$$\int_0^{\infty} \frac{e^{-m} m^{X_0} dm}{X_0!} = \frac{\Gamma(X_0 + 1)}{X_0!} = 1 \quad . \tag{47}$$

To summarize the meaning of equation (46), one can state that for a given observation of X_0 events, the probability of m being in the interval m to m + dm is given by the probability equation,

$$f(m)\,dm = \frac{e^{-m} m^{X_0} dm}{X_0!} \quad . \tag{48}$$

See Figure 12 for an illustration of equation (48) with $X_0 = 3$.

However, it is desirable to investigate the more subtle relationship that reflects the distribution of λ's for a given observation of X_0 events over a period of T_0 weeks. Therefore, one makes the change of variables denoted by $m = \lambda T_0$; $dm = T_0 d\lambda$, and our probability density function in the variate λ becomes

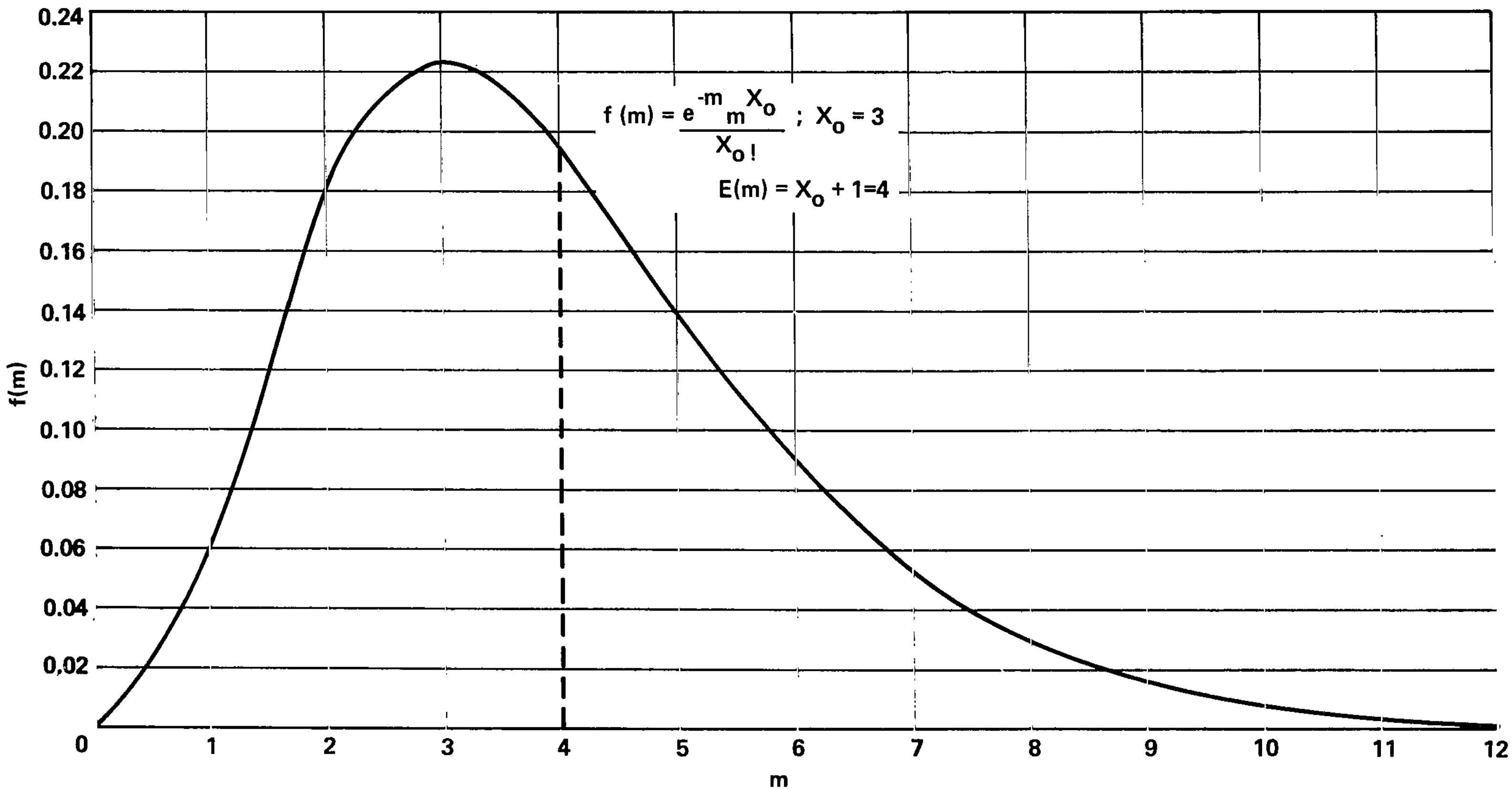

Figure 12. Plot of the continuous gamma probability density function when $X_0 = 3$.

$$f(\lambda)d\lambda = \frac{T_0^{X_0+1}}{X_0!} \lambda^{X_0} e^{-T_0\lambda} d\lambda \quad . \tag{49}$$

For a given value of λ, the probability of seeing exactly x events in time t is given by the discrete Poisson distribution function,

$$P(x) = Pr(x; \lambda t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!} \quad , \quad x = 0, 1, 2, \ldots \quad , \tag{50}$$

where the probability of having a value of λ in the interval $\lambda + d\lambda$ is given by the density function of equation (49) and $t \leq T_0$ (actually the case $t > T_0$ is equally valid).

Using the above definitions, the relationship that is sought is given by the following:

$$Pr\left(x, t \middle| X_0, T_0\right) = \int_0^\infty Pr(x; \lambda t) \cdot f(\lambda) d\lambda \quad . \tag{51}$$

Thus, it is being stated that for equation (50) of the Poisson distribution, the probability of each possible λ (a spectrum of possible means) be folded into the equation and the results integrated over all possible values of λ from zero to infinity. The result is a probability density function that is the expected value of the Poisson distribution over all possible means λ. Thus,

$$Pr\left(x, t \middle| X_0, T_0\right) = \frac{T_0^{X_0+1} t^x}{X_0! \, x!} \int_0^\infty \lambda^{X_0+x} e^{-(T_0+t)\lambda} d\lambda \, . \tag{52}$$

After integration the following is obtained:

$$Pr\left(x, t \middle| X_0, T_0\right) = \frac{T_0^{X_0+1} t^x}{X_0! \, x!} \left[\frac{(X_0 + x)!}{(T_0 + t)^{X_0+x+1}} \right] . \tag{53}$$

If the substitution $\theta = t/T_0$ is made and the resulting equation simplified, then the discrete distribution function in the variate x is,

$$\Pr\left(x \middle| X_0, \theta\right) = \frac{\left(x + X_0\right)!}{x!\, X_0!} \; \frac{\theta^x}{(1+\theta)^{x + X_0 + 1}} \quad . \tag{54}$$

It can be shown that for the discrete distribution function above

$$\sum_{x=0}^{\infty} \frac{\left(x + X_0\right)!}{x!\, X_0!} \; \frac{\theta^x}{(1+\theta)^{x + X_0 + 1}} = 1 \quad , \tag{55}$$

and the mean or expected value of x is given by

$$\bar{x} = E(x) = (X_0 + 1)\,\theta \quad . \tag{56}$$

The proof of equation (55) can be shown by using hypergeometric functions. The results of equation (56) can be found by multiplying the summand of equation (55) by x, and after simplifying, the value $(X_0 + 1)\theta$ can be factored out leaving a sum that is the equivalent to that shown in equation(55). An illustration of the density function [equation (54)] is shown in Figure 13 for $\theta = 1$ and $\theta = 0.52$[5].

It is of some interest to note that when $\theta = 1$ $(t = T_0)$, the function becomes simply,

$$\Pr\left(x \middle| X_0, 1\right) = \frac{\left(x + X_0\right)!}{x!\, X_0!\, 2^{x + X_0 + 1}} \tag{57}$$

and

$$\bar{x} = E(x) = X_0 + 1 \quad .$$

This report is not intended to be a study of probability theory, and the ramifications of the above distribution function and its possible parallels in other statistical work will not be pursued.

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5. The probability distribution of equation (54) is a special form of the so called "negative binomial."

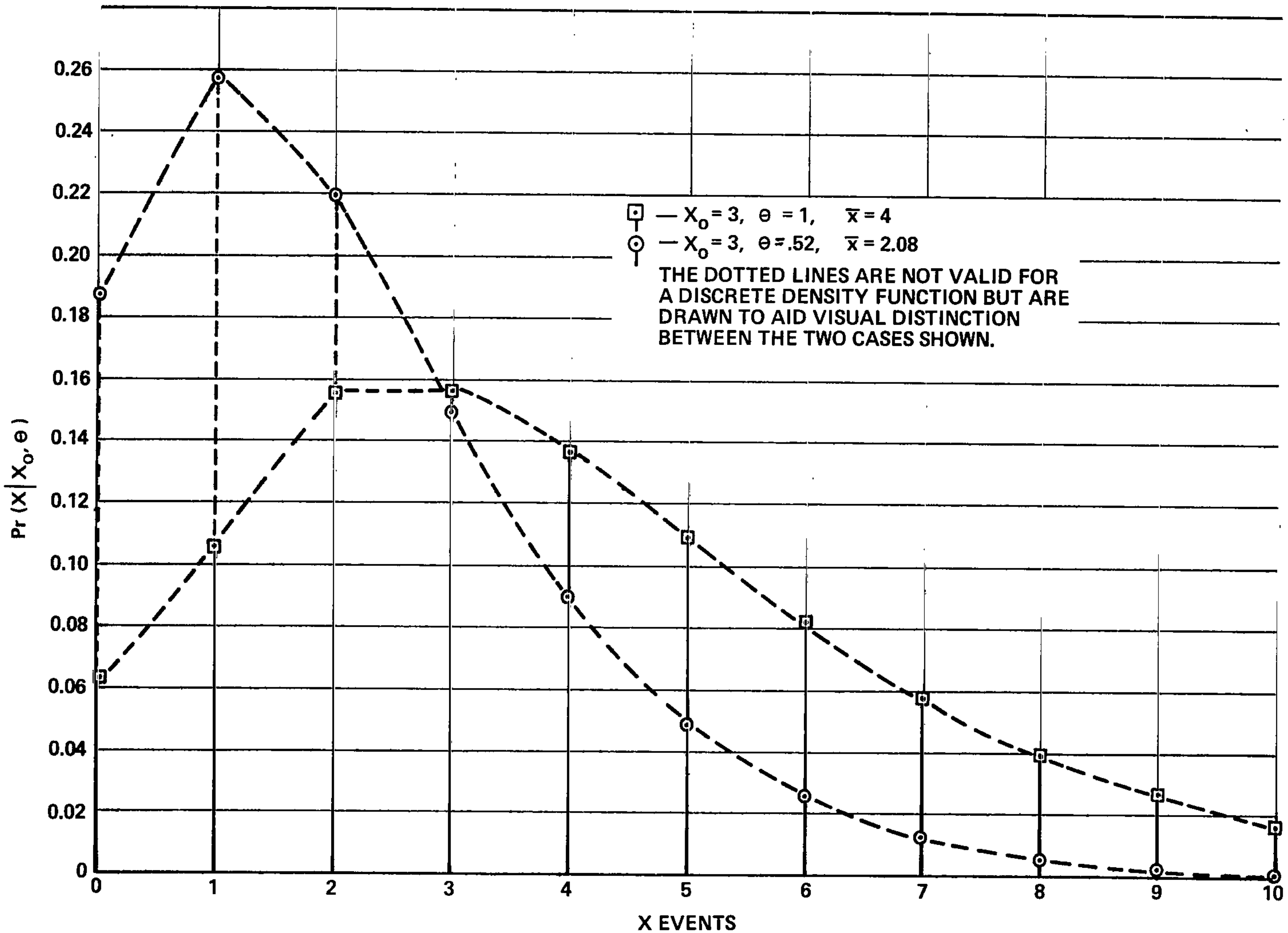

Figure 13. Plot of the discrete probability density function: $\Pr\left(x \middle| X_0, \theta\right)$.

Because of the format of Tables 12 through 17 and the manner in which probability statements have previously been made, it is desirable to find the probability that more than N events are seen, given X_0 and θ; or

$$Pr(x > N|X_0, \theta) = 1 - Pr(x \leq N|X_0, \theta) = 1 - \sum_{x=0}^{N} \frac{(x + X_0)!}{x!\, X_0!} \frac{\theta^x}{(1+\theta)^{x+X_0+1}}, \tag{58}$$

where $\theta = \frac{t}{T_0}$, X_0 = number of events observed in a time T_0 (300 weeks), and t is the observation time during which N events are seen.

Thus, the ultimate relation that is sought to answer the probability questions is given by equation (58) or the obvious variations associated with it. In fact, if the foregoing is valid, one may dispense with the difficulties of choosing an upper bound value of the Poisson mean at a given level and then determining the probability that N events will be exceeded at some probability α as shown in Tables 12 and 17, where the true probability is actually not known.

Using equation (58), the number of combinations of θ, X_0, and N can readily become excessive. For this reason, only the values of X_0 observed events from 0 through 9 will be used with nine different values of θ from 13 weeks to 300 ($\theta = 1$). The N-values will be extended to the point where $Pr(x>N|X_0, \theta) < 10^{-6}$. (A summary of these tabulations is shown in Table 20 at the end of this section.) There are many observations that can be made concerning the comparisons of Table 20 with those of Tables 12 through 15 and Table 17. For example, the probability of seeing more than 25 events as shown in Table 12 is actually 7.62×10^{-6} rather than the conjectured 1×10^{-6} given at the beginning of this section. Also, the probability of seeing more than 10 events when t = 78 weeks with X_0 = 3 is 5.61×10^{-6}, and the probability of seeing more than two events is actually 0.107, which one might guess to be less than 0.100 from Table 17.

It is very interesting to note that when $X_0 = 0$ and $\theta = 1$, the chance of seeing more than three events is as high as 0.0625. This infers that even though no events were observed during a given 300-week active cycle, one cannot be more than 93.75 percent confident that three or less events could occur in a similar cycle. Also, it can be seen from the same column under $X_0 = 0$ that one is 50 percent certain that more than zero events will be seen. This seems to infer a dilemma bordering on the naive statement that if you know nothing about the probability of an event you can only be 50 percent certain that it cannot happen, e. g., the probability of life on Mars. This last statement seems to cast doubt on the usefulness of the case when $X_0 = 0$.

However, it is of academic interest to investigate further the case when $X_0 = 0$. For example, equation (54) becomes

$$\Pr(x \mid 0, \theta) = \frac{\theta^x}{(1 + \theta)^{x+1}} \quad ; \tag{59}$$

and for $\theta > 0$, one sees that when $x = 0$,

$$\Pr(x = 0 \mid 0, \theta) = \frac{1}{1 + \theta} \quad , \tag{60}$$

and

$$\Pr(x > 0 \mid 0, \theta) = 1 - \frac{1}{1 + \theta} = \frac{\theta}{1 + \theta} \quad . \tag{61}$$

The probability of seeing an event as θ approaches zero becomes very small, as one would suspect for a very rare event. As θ increases to large values ($\theta >> 1$), the value of the probability approaches one. This infers that if an event can happen at a given small probability, then it is almost certain that the event will occur after a sufficiently long period. Thus, equations (60) and (61) do not defy intuition in an ordinary sense, but leave us with a rather insecure feeling since the number of actual observed events in time T_0 is zero. However, if one examines the density function of possible values of λ [equation (49)] when $X_0 = 0$, a plausible probability density function is found,

$$f(\lambda)\, d\lambda = T_0 e^{-T_0 \lambda}\, d\lambda \quad . \tag{62}$$

Table 18 is presented as a survey of equation (58) for $\Pr(x > N \mid 3, \theta) \leq \epsilon$ at several values of ϵ and θ. A point of interest concerns the values of the results in Table 17 (Section IV) as compared to those of Table 18. It should be clear that the results of Table 18 do not provide the same type of probability statements as do those of Table 17. In general, the author feels that the results of Table 18 are more useful and to the point; however, others may disagree.

TABLE 18. NUMBER OF EVENTS N THAT SATISFY $Pr(x > N | 3, \theta) \le \epsilon$

t (weeks)	θ \ ϵ	0.10	0.01	0.001	0.0001	0.00001
13	0.0433	1	2	3	3	4
26	0.0866	1	2	4	5	6
52	0.1733	2	4	5	7	8
78	0.2600	3	5	7	8	10
104	0.3467	3	6	8	10	12
156	0.5200	4	8	10	13	16
208	0.6933	6	9	13	16	19
260	0.8667	7	11	15	19	22
300	1.000	9	13	17	21	25

In any case, one may ascertain which probability level of the parameter λ is more credible when using the methods of Section IV and constructing tables similar to Tables 12 and 17. For example, one sees that the value of λ at the 90-percent level yields event numbers N at probabilities α in Table 17 that seem to be comparable to the probabilities ϵ in Table 18.

Table 19 is presented for the case when $X_0 = 0$ even though doubt has been cast on the validity of the meaning of this rather extreme case. However, it does signify a sort of boundary condition for those periods when the probability of a large event is very small, as perhaps exists for the active years of an average type solar cycle.

Table 20, which was discussed previously is presented on pages 58 through 63.

SECTION VI. REVIEW AND COMMENTS

In the previous sections, the author has attempted to convey a method of considering the radiation hazard associated with solar proton events. The method of approach is believed to be more important than the actual results presented. The methods and models as developed utilize very little solar

TABLE 19. NUMBER OF EVENTS N THAT SATISFY $Pr(x>N|0,\theta) \leq \epsilon$

t (weeks)	θ \ ϵ	0.1	0.01	0.001	0.0001	0.00001
13	0.0433	0	1	2	2	3
26	0.0866	0	1	2	3	4
52	0.1733	1	2	3	4	6
78	0.2600	1	2	4	5	7
104	0.3467	1	3	5	6	8
156	0.5200	2	4	6	8	10
208	0.6933	2	5	7	10	12
260	0.8667	2	5	8	11	14
300	1.0000	3	6	9	13	16

physics as such and, consequently, will be very unsatisfying to many physicists who have examined various aspects of the problem. There is no attempt to model or predict a solar proton spectrum, the time dependence of particle arrival, or angular distributions of the flux.

In summary, the total dose in rads-tissue from solar protons was tabulated for weekly time intervals, and the number of weeks that gave a dose above 25 rads behind 10 g/cm^2 of aluminum for the active six years of the 19th cycle were called dangerous or large event weeks. The number of such event weeks was found to be only three weeks during the past 20 years.

Even though the possiblility for smaller events was examined, it was found that for any reasonable high confidence level (95 percent), the smaller events could be ignored. Consequently, the total particle flux was taken for the 19th cycle, this spectrum was divided by a factor of three, and a single large event week was derived.

The spectrum for this large event is

$$J(>p) = 1.76 \times 10^{11}\, e^{-p/73} \quad (30 \leq E \leq 115 \text{ MeV}) \quad ,$$
$$J(>p) = 3.8 \times 10^{10}\, e^{-p/100} \quad (E > 115 \text{ MeV}) \qquad (63)$$

where p is in rigidity units of MV.

TABLE 20. VALUES OF $Pr(x > N|X_0, \theta)$[a] FOR A GIVEN $\theta = t/T_0$

$\theta = 1$ (300 Weeks)

$N \backslash X_0$	0	1	2	3	4	5	6	7	8	9
0	5.00-1[b]	7.50-1	8.75-1	9.37-1	9.69-1	9.84-1	9.92-1	9.96-1	9.98-1	9.99-1
1	2.50-1	5.00-1	6.87-1	8.12-1	8.91-1	9.37-1	9.65-1	9.80-1	9.89-1	9.94-1
2	1.25-1	3.12-1	5.00-1	6.56-1	7.73-1	8.55-1	9.10-1	9.45-1	9.67-1	9.81-1
3	6.25-2	1.87-1	3.44-1	5.00-1	6.37-1	7.46-1	8.28-1	8.87-1	9.27-1	9.54-1
4	3.12-2	1.09-1	2.27-1	3.63-1	5.00-1	6.23-1	7.26-1	8.06-1	8.67-1	9.10-1
5	1.56-2	6.25-2	1.45-1	2.54-1	3.77-1	5.00-1	6.13-1	7.09-1	7.88-1	8.49-1
6	7.81-3	3.52-2	8.98-2	1.72-1	2.74-1	3.87-1	5.00-1	6.05-1	6.96-1	7.73-1
7	3.91-3	1.95-2	5.47-2	1.13-1	1.94-1	2.91-1	3.95-1	5.00-1	5.98-1	6.85-1
8	1.95-3	1.07-2	3.27-2	7.30-2	1.33-1	2.12-1	3.04-1	4.02-1	5.00-1	5.93-1
9	9.77-4	5.86-3	1.93-2	4.61-2	8.98-2	1.51-1	2.27-1	3.15-1	4.07-1	5.00-1
10	4.88-4	3.17-3	1.12-2	2.87-2	5.92-2	1.05-1	1.66-1	2.40-1	3.24-1	4.12-1
11	2.44-4	1.71-3	6.47-3	1.76-2	3.84-2	7.17-2	1.19-1	1.80-1	2.52-1	3.32-1
12	1.22-4	9.16-4	3.69-3	1.06-2	2.45-2	4.81-2	8.35-2	1.32-1	1.92-1	2.62-1
13	6.10-5	4.88-4	2.09-3	6.36-3	1.54-2	3.18-2	5.77-2	9.46-2	1.43-1	2.02-1
14	3.05-5	2.59-4	1.17-3	3.77-3	9.61-3	2.07-2	3.92-2	6.69-2	1.05-1	1.54-1
15	1.53-5	1.37-4	6.56-4	2.21-3	5.91-3	1.33-2	2.62-2	4.66-2	7.58-2	1.15-1
16	7.63-6	7.25-5	3.64-4	1.29-3	3.60-3	8.45-3	1.73-2	3.20-2	5.39-2	8.43-2
17	3.81-6	3.81-5	2.01-4	7.45-4	2.17-3	5.31-3	1.13-2	2.16-2	3.78-2	6.10-2
18	1.91-6	2.00-5	1.11-4	4.28-4	1.30-3	3.31-3	7.32-3	1.45-2	2.61-2	4.36-2
19	9.54-7	1.05-5	6.06-5	2.44-4	7.72-4	2.04-3	4.68-3	9.58-3	1.78-2	3.07-2
20		5.48-6	3.30-5	1.39-4	4.55-4	1.25-3	2.96-3	6.27-3	1.21-2	2.14-2
21		2.86-6	1.79-5	7.83-5	2.67-4	7.57-4	1.86-3	4.07-3	8.06-3	1.47-2
22		1.49-6	9.72-6	4.40-5	1.55-4	4.56-4	1.16-3	2.61-3	5.34-3	1.00-2
23		7.75-7	5.25-6	2.46-5	9.00-5	2.73-4	7.15-4	1.66-3	3.50-3	6.77-3
24			2.82-6	1.37-5	5.19-5	1.62-4	4.39-4	1.05-3	2.28-3	4.52-3
25			1.52-6	7.62-6	2.97-5	9.61-5	2.68-4	6.59-4	1.47-3	2.99-3
26			8.12-7	4.22-6	1.70-5	5.65-5	1.62-4	4.11-4	9.39-4	1.97-3
27				2.32-6	9.65-6	3.31-5	9.76-5	2.54-4	5.97-4	1.28-3
28				1.28-6	5.47-6	1.93-5	5.84-5	1.56-4	3.76-4	8.29-4
29				7.00-7	3.08-6	1.12-5	3.48-5	9.55-5	2.36-4	5.33-4
30					1.74-6	6.46-6	2.06-5	5.81-5	1.47-4	3.40-4
31					9.70-7	3.71-6	1.22-5	3.51-5	9.11-5	2.15-4
32						2.13-6	7.15-6	2.11-5	5.61-5	1.36-4
33						1.21-6	4.19-6	1.27-5	3.44-5	8.51-5
34						6.91-7	2.44-6	7.55-6	2.10-5	5.30-5
35							1.42-6	4.48-6	1.27-5	3.29-5
36							8.24-7	2.65-6	7.69-6	2.03-5
37								1.56-6	4.63-6	1.25-5
38								9.14-7	2.78-6	7.62-6
39									1.66-6	4.63-6
40									9.85-7	2.81-6
41										1.70-6
42										1.02-6
43										6.16-7

a. $Pr(x > N|X_0, \theta) = 1 - Pr(x \leq N|X_0, \theta)$
$= Pr(x \geq N+1|X_0, \theta)$

b. The Notation:
$a.bc\text{-}y = a.bc \times 10^{-y}$

TABLE 20. (Continued)

$\theta = 0.8667$ (260 Weeks)

$N \backslash X_o$	0	1	2	3	4	5	6	7	8	9
0	4.64-1	7.13-1	8.46-1	9.18-1	9.56-1	9.76-1	9.87-1	9.93-1	9.96-1	9.98-1
1	2.16-1	4.47-1	6.32-1	7.65-1	8.53-1	9.11-1	9.46-1	9.68-1	9.81-1	9.89-1
2	1.00-1	2.61-1	4.33-1	5.87-1	7.11-1	8.04-1	8.70-1	9.15-1	9.46-1	9.66-1
3	4.65-2	1.46-1	2.79-1	4.22-1	5.56-1	6.71-1	7.63-1	8.34-1	8.86-1	9.23-1
4	2.16-2	7.94-2	1.72-1	2.88-1	4.13-1	5.33-1	6.40-1	7.30-1	8.02-1	8.58-1
5	1.00-2	4.22-2	1.03-1	1.89-1	2.93-1	4.04-1	5.14-1	6.14-1	7.01-1	7.74-1
6	4.65-3	2.21-2	5.95-2	1.20-1	2.00-1	2.95-1	3.96-1	4.97-1	5.92-1	6.77-1
7	2.16-3	1.14-2	3.37-2	7.36-2	1.32-1	2.08-1	2.95-1	3.89-1	4.83-1	5.73-1
8	1.00-3	5.84-3	1.88-2	4.42-2	8.51-2	1.42-1	2.13-1	2.95-1	3.82-1	4.71-1
9	4.65-4	2.96-3	1.03-2	2.60-2	5.35-2	9.46-2	1.50-1	2.17-1	2.94-1	3.76-1
10	2.16-4	1.49-3	5.58-3	1.51-2	3.29-2	6.15-2	1.02-1	1.56-1	2.20-1	2.92-1
11	1.00-4	7.45-4	2.99-3	8.61-3	1.99-2	3.92-2	6.86-2	1.09-1	1.60-1	2.22-1
12	4.66-5	3.71-4	1.59-3	4.85-3	1.18-2	2.45-2	4.50-2	7.47-2	1.15-1	1.64-1
13	2.16-5	1.84-4	8.36-4	2.70-3	6.94-3	1.51-2	2.90-2	5.02-2	8.01-2	1.19-1
14	1.00-5	9.07-5	4.37-4	1.49-3	4.02-3	9.17-3	1.84-2	3.32-2	5.49-2	8.48-2
15	4.66-6	4.46-5	2.27-4	8.11-4	2.30-3	5.49-3	1.15-2	2.15-2	3.70-2	5.92-2
16	2.16-6	2.19-5	1.17-4	4.39-4	1.30-3	3.25-3	7.06-3	1.38-2	2.46-2	4.07-2
17	1.00-6	1.07-5	6.00-5	2.36-4	7.32-4	1.90-3	4.30-3	8.70-3	1.61-2	2.75-2
18	4.66-7	5.22-6	3.07-5	1.26-4	4.07-4	1.10-3	2.58-3	5.42-3	1.04-2	1.83-2
19		2.54-6	1.56-5	6.69-5	2.25-4	6.31-4	1.54-3	3.34-3	6.60-3	1.20-2
20		1.23-6	7.90-6	3.53-5	1.23-4	3.59-4	9.06-4	2.04-3	4.16-3	7.82-3
21		5.97-7	3.99-6	1.85-5	6.72-5	2.03-4	5.29-4	1.23-3	2.59-3	5.02-3
22			2.01-6	9.67-6	3.64-5	1.14-4	3.07-4	7.35-4	1.60-3	3.18-3
23			1.01-6	5.03-6	1.96-5	6.32-5	1.76-4	4.36-4	9.74-4	2.00-3
24			5.03-7	2.60-6	1.05-5	3.50-5	1.01-4	2.56-4	5.90-4	1.24-3
25				1.34-6	5.59-6	1.92-5	5.70-5	1.49-4	3.54-4	7.67-4
26				6.90-7	2.97-6	1.05-5	3.21-5	8.66-5	2.11-4	4.69-4
27					1.57-6	5.72-6	1.80-5	4.98-5	1.25-4	2.85-4
28					8.25-7	3.10-6	1.00-5	2.85-5	7.31-5	1.71-4
29						1.67-6	5.54-6	1.62-5	4.26-5	1.02-4
30						8.96-7	3.05-6	9.15-6	2.47-5	6.07-5
31							1.67-6	5.14-6	1.42-5	3.58-5
32							9.16-7	2.88-6	8.14-6	2.10-5
33								1.60-6	4.64-6	1.22-5
34								8.88-7	2.63-6	7.09-6
35									1.49-6	4.08-6
36									8.35-7	2.34-6
37										1.34-6
38										7.60-7

TABLE 20. (Continued)

$\theta = 0.6933$ (208 Weeks)

N \ X_o	0	1	2	3	4	5	6	7	8	9
0	4.09-1	6.51-1	7.94-1	8.78-1	9.28-1	9.58-1	9.75-1	9.85-1	9.91-1	9.95-1
1	1.68-1	3.66-1	5.41-1	6.79-1	7.81-1	8.53-1	9.03-1	9.37-1	9.59-1	9.74-1
2	6.86-2	1.90-1	3.34-1	4.75-1	6.01-1	7.04-1	7.86-1	8.47-1	8.93-1	9.26-1
3	2.81-2	9.45-2	1.93-1	3.08-1	4.28-1	5.41-1	6.41-1	7.26-1	7.94-1	8.48-1
4	1.15-2	4.55-2	1.06-1	1.89-1	2.87-1	3.91-1	4.93-1	5.88-1	6.73-1	7.45-1
5	4.71-3	2.14-2	5.59-2	1.10-1	1.82-1	2.68-1	3.60-1	4.54-1	5.43-1	6.26-1
6	1.93-3	9.90-3	2.87-2	6.21-2	1.11-1	1.75-1	2.51-1	3.34-1	4.20-1	5.04-1
7	7.90-4	4.52-3	1.44-2	3.40-2	6.57-2	1.11-1	1.68-1	2.36-1	3.11-1	3.90-1
8	3.23-4	2.04-3	7.12-3	1.81-2	3.76-2	6.75-2	1.09-1	1.61-1	2.22-1	2.91-1
9	1.32-4	9.15-4	3.45-3	9.46-3	2.10-2	4.00-2	6.81-2	1.06-1	1.54-1	2.10-1
10	5.42-5	4.06-4	1.65-3	4.85-3	1.15-2	2.31-2	4.16-2	6.80-2	1.03-1	1.47-1
11	2.22-5	1.80-4	7.83-4	2.45-3	6.13-3	1.31-2	2.48-2	4.25-2	6.73-2	9.99-2
12	9.09-6	7.89-5	3.67-4	1.22-3	3.23-3	7.27-3	1.44-2	2.59-2	4.28-2	6.62-2
13	3.72-6	3.45-5	1.71-4	6.00-4	1.68-3	3.97-3	8.25-3	1.55-2	2.67-2	4.29-2
14	1.52-6	1.50-5	7.88-5	2.92-4	8.59-4	2.13-3	4.64-3	9.08-3	1.63-2	2.72-2
15	6.24-7	6.52-6	3.61-5	1.41-4	4.35-4	1.13-3	2.57-3	5.23-3	9.76-3	1.69-2
16		2.82-6	1.65-5	6.74-5	2.18-4	5.91-4	1.40-3	2.97-3	5.75-3	1.03-2
17		1.22-6	7.46-6	3.20-5	1.08-4	3.06-4	7.54-4	1.66-3	3.33-3	6.19-3
18		5.23-7	3.36-6	1.51-5	5.32-5	1.57-4	4.01-4	9.17-4	1.91-3	3.66-3
19			1.51-6	7.07-6	2.60-5	7.95-5	2.11-4	5.00-4	1.08-3	2.13-3
20			6.74-7	3.29-6	1.26-5	4.00-5	1.10-4	2.70-4	6.00-4	1.23-3
21				1.52-6	6.05-6	1.99-5	5.69-5	1.44-4	3.31-4	6.98-4
22				7.02-7	2.89-6	9.87-6	2.91-5	7.62-5	1.80-4	3.92-4
23					1.37-6	4.85-6	1.48-5	3.99-5	9.74-5	2.18-4
24					6.48-7	2.37-6	7.45-6	2.07-5	5.22-5	1.20-4
25						1.15-6	3.73-6	1.07-5	2.77-5	6.55-5
26						5.56-7	1.85-6	5.47-6	1.46-5	3.54-5
27							9.15-7	2.78-6	7.61-6	1.90-5
28								1.41-6	3.94-6	1.01-5
29								7.05-7	2.03-6	5.34-6
30									1.04-6	2.80-6
31									5.26-7	1.46-6
32										7.51-7

TABLE 20. (Continued)

$\theta = 0.5200$ (156 Weeks)

N \ X_0	0	1	2	3	4	5	6	7	8	9
0	3.42-1	5.67-1	7.15-1	8.13-1	8.77-1	9.19-1	9.47-1	9.65-1	9.77-1	9.85-1
1	1.17-1	2.71-1	4.23-1	5.56-1	6.66-1	7.52-1	8.19-1	8.69-1	9.06-1	9.33-1
2	4.00-2	1.19-1	2.23-1	3.37-1	4.50-1	5.53-1	6.44-1	7.21-1	7.84-1	8.35-1
3	1.37-2	4.97-2	1.09-1	1.87-1	2.77-1	3.71-1	4.65-1	5.52-1	6.32-1	7.01-1
4	4.69-3	2.01-2	5.05-2	9.72-2	1.59-1	2.31-1	3.11-1	3.94-1	4.75-1	5.52-1
5	1.60-3	7.93-3	2.25-2	4.81-2	8.59-2	1.36-1	1.96-1	2.63-1	3.36-1	4.10-1
6	5.48-4	3.07-3	9.72-3	2.28-2	4.44-2	7.56-2	1.17-1	1.67-1	2.25-1	2.88-1
7	1.88-4	1.18-3	4.10-3	1.05-2	2.21-2	4.04-2	6.65-2	1.01-1	1.43-1	1.93-1
8	6.42-5	4.44-4	1.69-3	4.71-3	1.07-2	2.08-2	3.65-2	5.85-2	8.75-2	1.24-1
9	2.20-5	1.66-4	6.89-4	2.06-3	5.01-3	1.04-2	1.93-2	3.27-2	5.15-2	7.61-2
10	7.51-6	6.19-5	2.76-4	8.88-4	2.30-3	5.08-3	9.96-3	1.77-2	2.93-2	4.53-2
11	2.57-6	2.29-5	1.10-4	3.76-4	1.03-3	2.42-3	5.00-3	9.36-3	1.62-2	2.61-2
12	8.79-7	8.40-6	4.30-5	1.57-4	4.57-4	1.13-3	2.45-3	4.81-3	8.70-3	1.47-2
13		3.07-6	1.67-5	6.47-5	1.99-4	5.16-4	1.18-3	2.42-3	4.57-3	8.02-3
14		1.12-6	6.46-6	2.64-5	8.54-5	2.33-4	5.56-4	1.19-3	2.35-3	4.29-3
15		4.05-7	2.48-6	1.07-5	3.62-5	1.03-4	2.58-4	5.79-4	1.18-3	2.25-3
16			9.43-7	4.26-6	1.52-5	4.54-5	1.18-4	2.76-4	5.87-4	1.15-3
17				1.69-6	6.31-6	1.97-5	5.34-5	1.29-4	2.86-4	5.83-4
18				6.66-7	2.60-6	8.44-6	2.38-5	6.00-5	1.37-4	2.90-4
19					1.06-6	3.59-6	1.05-5	2.74-5	6.50-5	1.42-4
20					4.27-7	1.51-6	4.58-6	1.24-5	3.04-5	6.85-5
21						6.28-7	1.98-6	5.54-6	1.40-5	3.27-5
22							8.47-7	2.45-6	6.42-6	1.54-5
23								1.07-6	2.91-6	7.18-6
24								4.65-7	1.30-6	3.31-6
25									5.76-7	1.51-6
26										6.82-7

$\theta = 0.3467$ (104 Weeks)

N \ X_0	0	1	2	3	4	5	6	7	8	9
0	2.57-1	4.49-1	5.91-1	6.96-1	7.74-1	8.32-1	8.75-1	9.08-1	9.31-1	9.49-1
1	6.63-2	1.65-1	2.74-1	3.83-1	4.84-1	5.73-1	6.51-1	7.17-1	7.72-1	8.18-1
2	1.71-2	5.51-2	1.12-1	1.81-1	2.59-1	3.40-1	4.20-1	4.97-1	5.68-1	6.32-1
3	4.39-3	1.74-2	4.17-2	7.76-2	1.24-1	1.80-1	2.42-1	3.07-1	3.74-1	4.41-1
4	1.13-3	5.33-3	1.47-2	3.09-2	5.49-2	8.71-2	1.27-1	1.73-1	2.25-1	2.81-1
5	2.91-4	1.59-3	4.96-3	1.16-2	2.28-2	3.93-2	6.19-2	9.06-2	1.25-1	1.65-1
6	7.49-5	4.64-4	1.62-3	4.20-3	8.98-3	1.68-2	2.84-2	4.44-2	6.52-2	9.09-2
7	1.93-5	1.34-4	5.17-4	1.46-3	3.40-3	6.85-3	1.24-2	2.06-2	3.21-2	4.73-2
8	4.96-6	3.81-5	1.61-4	4.97-4	1.24-3	2.69-3	5.19-3	9.16-3	1.51-2	2.34-2
9	1.28-6	1.08-5	4.95-5	1.65-4	4.42-4	1.02-3	2.09-3	3.91-3	6.79-3	1.11-2
10	3.28-7	3.01-6	1.50-5	5.35-5	1.54-4	3.77-4	8.18-4	1.61-3	2.95-3	5.03-3
11		8.38-7	4.48-6	1.71-5	5.22-5	1.36-4	3.11-4	6.47-4	1.24-3	2.22-3
12			1.32-6	5.38-6	1.74-5	4.79-5	1.16-4	2.53-4	5.06-4	9.46-4
13			3.86-7	1.67-6	5.73-6	1.66-5	4.21-5	9.63-5	2.02-4	3.94-4
14				5.10-7	1.86-6	5.65-6	1.50-5	3.59-5	7.87-5	1.60-4
15					5.92-7	1.89-6	5.27-6	1.32-5	3.00-5	6.34-5
16						6.23-7	1.82-6	4.74-6	1.12-5	2.47-5
17							6.17-7	1.68-6	4.14-6	9.43-6
18								5.80-7	1.49-6	3.54-6
19									5.28-7	1.31-6
20										4.72-7

TABLE 20. (Continued)

$\theta = 0.2600$ (78 Weeks)

$N \backslash X_o$	0	1	2	3	4	5	6	7	8	9
0	2.06-1	3.70-1	5.00-1	6.03-1	6.85-1	7.50-1	8.02-1	8.43-1	8.75-1	9.01-1
1	4.26-2	1.10-1	1.91-1	2.76-1	3.60-1	4.41-1	5.15-1	5.83-1	6.43-1	6.96-1
2	8.79-3	2.97-2	6.29-2	1.07-1	1.59-1	2.17-1	2.79-1	3.41-1	4.04-1	4.64-1
3	1.81-3	7.57-3	1.90-2	3.71-2	6.23-2	9.43-2	1.32-1	1.75-1	2.23-1	2.72-1
4	3.74-4	1.86-3	5.39-3	1.19-2	2.23-2	3.72-2	5.68-2	8.13-2	1.10-1	1.44-1
5	7.72-5	4.45-4	1.47-3	3.63-3	7.49-3	1.36-2	2.25-2	3.47-2	5.03-2	6.96-2
6	1.59-5	1.04-4	3.85-4	1.05-3	2.38-3	4.70-3	8.38-3	1.38-2	2.13-2	3.13-2
7	3.29-6	2.42-5	9.87-5	2.96-4	7.26-4	1.55-3	2.96-3	5.19-3	8.52-3	1.32-2
8	6.78-7	5.52-6	2.47-5	8.07-5	2.14-4	4.89-4	9.98-4	1.86-3	3.24-3	5.30-3
9		1.25-6	6.10-6	2.15-5	6.12-5	1.49-4	3.24-4	6.42-4	1.18-3	2.03-3
10		2.80-7	1.48-6	5.61-6	1.71-5	4.44-5	1.02-4	2.14-4	4.13-4	7.46-4
11			3.54-7	1.44-6	4.66-6	1.29-5	3.13-5	6.89-5	1.40-4	2.65-4
12				3.62-7	1.25-6	3.65-6	9.35-6	2.16-5	4.60-5	9.12-5
13					3.26-7	1.01-6	2.73-6	6.63-6	1.48-5	3.05-5
14						2.75-7	7.80-7	1.99-6	4.62-6	9.97-6
15								5.82-7	1.42-6	3.18-6
16									4.25-7	9.90-7

$\theta = 0.1733$ (52 Weeks)

$N \backslash X_o$	0	1	2	3	4	5	6	7	8	9
0	1.48-1	2.74-1	3.81-1	4.72-1	5.50-1	6.17-1	6.73-1	7.22-1	7.63-1	7.98-1
1	2.18-2	5.90-2	1.07-1	1.61-1	2.18-1	2.77-1	3.36-1	3.93-1	4.47-1	4.99-1
2	3.22-3	1.15-2	2.55-2	4.55-2	7.10-2	1.01-1	1.36-1	1.74-1	2.14-1	2.56-1
3	4.76-4	2.10-3	5.56-3	1.15-2	2.03-2	3.22-2	4.76-2	6.62-2	8.81-2	1.13-1
4	7.04-5	3.70-4	1.14-3	2.66-3	5.26-3	9.25-3	1.49-2	2.25-2	3.22-2	4.41-2
5	1.04-5	6.35-5	2.22-4	5.82-4	1.27-3	2.45-3	4.29-3	6.98-3	1.07-2	1.56-2
6	1.53-6	1.07-5	4.19-5	1.22-4	2.92-4	6.11-4	1.15-3	2.02-3	3.30-3	5.12-3
7	2.26-7	1.77-6	7.70-6	2.46-5	6.40-5	1.45-4	2.94-4	5.48-4	9.55-4	1.57-3
8		2.89-7	1.38-6	4.81-6	1.36-5	3.29-5	7.15-5	1.42-4	2.62-4	4.55-4
9			2.44-7	9.16-7	2.78-6	7.24-6	1.67-5	3.52-5	6.87-5	1.26-4
10					5.55-7	1.54-6	3.79-6	8.43-6	1.73-5	3.34-5
11						3.19-7	8.30-7	1.95-6	4.23-6	8.53-6
12								4.38-7	9.98-7	2.11-6
13										5.05-7

TABLE 20. (Concluded)

$\theta = 0.0867$ (26 Weeks)

N \ X_o	0	1	2	3	4	5	6	7	8	9
0	7.98-2	1.53-1	2.21-1	2.83-1	3.40-1	3.93-1	4.41-1	4.86-1	5.27-1	5.64-1
1	6.36-3	1.81-2	3.42-2	5.41-2	7.69-2	1.02-1	1.29-1	1.58-1	1.87-1	2.17-1
2	5.07-4	1.91-3	4.49-3	8.44-3	1.39-2	2.09-2	2.96-2	3.98-2	5.15-2	6.47-2
3	4.05-5	1.89-4	5.32-4	1.16-3	2.18-3	3.67-3	5.74-3	8.45-3	1.19-2	1.61-2
4	3.23-6	1.81-5	5.91-5	1.47-4	3.09-4	5.77-4	9.89-4	1.58-3	2.41-3	3.50-3
5	2.57-7	1.68-6	6.25-6	1.75-5	4.07-5	8.35-5	1.56-4	2.70-4	4.40-4	6.84-4
6		1.52-7	6.37-7	1.98-6	5.07-6	1.13-5	2.28-5	4.25-5	7.42-5	1.23-4
7				2.13-7	6.01-7	1.45-6	3.16-6	6.30-6	1.17-5	2.06-5
8						1.76-7	4.14-7	8.82-7	1.75-6	3.25-6
9									2.44-7	4.82-7

$\theta = 0.0433$ (13 Weeks)

N \ X_o	0	1	2	3	4	5	6	7	8	9
0	4.15-2	8.13-2	1.19-1	1.56-1	1.91-1	2.25-1	2.57-1	2.88-1	3.17-1	3.46-1
1	1.73-3	5.03-3	9.79-3	1.59-2	2.31-2	3.15-2	4.09-2	5.11-2	6.22-2	7.40-2
2	7.16-5	2.78-4	6.73-4	1.30-3	2.21-3	3.43-3	4.98-3	6.90-3	9.20-3	1.19-2
3	2.98-6	1.44-5	4.17-5	9.41-5	1.82-4	3.17-4	5.11-4	7.76-4	1.13-3	1.57-3
4	1.24-7	7.17-7	2.42-6	6.23-6	1.35-5	2.61-5	4.62-5	7.66-5	1.20-4	1.80-4
5			1.34-7	3.88-7	9.34-7	1.98-6	3.82-6	6.84-6	1.15-5	1.86-5
6						1.42-7	2.94-7	5.66-7	1.02-6	1.75-6
7										1.56-7

Using this spectrum, one can calculate the tissue dose in rads at the center of an aluminum spherical shell (see Table 3). To correct for geometric effects and self-shielding, this dose should be reduced by a factor of approximately three. If the space mission is planned during the quiet period of a cycle, then the small event dose curve (D3) of model II (Table 3) may be used with $\lambda = 0.02$ ($m \cong 1$ event/year).

To predict the probability of an event occurring, the Poisson distribution seemed to be the most logical choice. Section III was devoted to examining this conclusion and the methods of using the probability model. Section IV defined the confidence one could use in employing the Poisson process and specifically arriving at confidence levels for the experimental or observed value of the mean of the Poisson distribution function. Several examples were given for different mission lengths, and comparisons were made to the results of other authors.

Section V was an extension of the Poisson process to incorporate the concept of small sample theory and arrive at the expected distribution function that answers the following question, If X_0 events are observed in time T_0, what is the probability of seeing X events in any observation time t? The results are represented by the discrete probability density function in the variate x,

$$\Pr\left(x, t \mid X_0, T_0\right) = \frac{T_0^{X_0+1} \, t^x}{X_0! \, x!} \left[\frac{\left(X_0 + x\right)!}{\left(T_0 + t\right)^{x + X_0 + 1}} \right] \quad . \tag{64}$$

Table 20, which is presented on pages 58 through 63, was generated using the preceding function.

In the beginning of this work, the author intended to avoid commenting on the methods used by other investigators, but in order to explain the radical differences shown in Figures 8, 9, and 10, the following comments are pertinent. One of the more common procedures used is to obtain the logarithms to the base ten of the solar proton flux above 30 MeV for each of the events of the 19th cycle. These data are then plotted on normal probability graph paper, and a distribution called the log normal is obtained.

Thus, if $x = \log_{10}\phi$, one has the log normal probability density function,

$$f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-\bar{x})^2/2\sigma_x^2} \, dx \quad , \tag{65}$$

where $\bar{x}$ and σ_x^2 are the mean and variance respectively and are estimated from the sample variables $x_i = \log_{10}\phi_i$. This is a true normal distribution in the variate x and there is no virtue in examining the transformed distribution in the variable ϕ, which has a more complex representation. Also, the values of the mean and standard deviation between the two distributions are not simply antilog related. For example,

$$\log_{10}\bar{\phi} = \overline{\log_{10}\phi} + \frac{\sigma_x^2}{2\log_{10}e} \tag{66a}$$

and

$$\sigma_x^2 = \log_{10}e \log_{10}\left(1 + \frac{\sigma_\phi^2}{\bar{\phi}^2}\right) , \tag{66b}$$

where σ_ϕ^2 is the variance in the distribution ϕ. For more details, see Reference 11. The author has assumed that the users of this distribution keep their statistics in the variable x, which is certainly the simplest process.

From Reference 10, 60 events were taken that had fluxes measured above 30 MeV. The 60 events chosen are shown in Table 21. The logarithms of these 60 entries were ordered from the smallest to the largest, and the normalized cumulative sums were plotted on log normal paper as shown in Figure 14. The ungrouped data had a mean of 7. 39, and the standard deviation was 0. 97 as shown at bottom of Table 21. The straight line in Figure 14 depicts the cumulative normal distribution with a mean of 7. 39 and a variance of 0. 97. From an examination of this fit, one might conclude that the log normal gives a reasonable representation of the data. Note that the flux for the November 12, 1960, event is plotted as a square that falls exactly on the line of best fit. This occurs because when the cumulative distribution from a discrete set of data is obtained, the last point has the cumulative probability of 1. 00; but since this cannot theoretically occur, the best choice is to place this point on the best fit line. This point, which is $\log_{10}\phi = 9.9562$, corresponds to the cumulative probability of 99. 6 percent, or in terms of the standard deviation, the November 12, 1960, event is at 2. 63 standard deviations above the mean based on the values shown below Table 21. If one wished to go to the 99. 9-percent level, it would be necessary to take 3. 09 standard deviations above the mean, or a log flux of 10. 4016. If the antilog of this value is found, one sees that the flux at the 99. 9-percent level in the log normal distribution

TABLE 21. SOLAR FLARE EVENTS OF THE 19TH CYCLE

Event Date	Flux (E>30 MeV)	$Log_{10}\ \phi$ (E>30 MeV)	Event Date	Flux (E>30 MeV)	$Log_{10}\ \phi$ (E>30 MeV)
2-23-56	1.00(+9)	9.0000	6-13-59	8.50(+7)	7.9294
3-10-56	1.10(+8)	8.0414	7-10-59	1.00(+9)	9.0000
8-31-56	2.50(+7)	7.3979	7-14-59	1.30(+9)	9.1139
11-13-56	1.00(+8)	8.0000	7-16-59	9.10(+8)	8.9590
1-20-57	3.00(+8)	8.4771	8-18-59	1.80(+6)	6.2553
4- 3-57	5.00(+7)	7.6990	9- 2-59	1.20(+7)	7.0792
4- 6-57	3.80(+7)	7.5798	1-11-60	4.00(+5)	5.6021
6-21-57	1.50(+8)	8.1761	3-29-60	6.00(+6)	6.7782
7- 3-57	2.00(+7)	7.3010	3-30-60	6.00(+6)	6.7782
7-24-57	7.50(+6)	6.8751	4- 1-60	5.00(+6)	6.6990
8- 9-57	1.50(+6)	6.1761	4- 5-60	1.10(+6)	6.0414
8-29-57	1.20(+8)	8.0792	4-28-60	5.00(+6)	6.6990
8-31-57	8.00(+7)	7.9031	4-29-60	7.00(+6)	6.8451
9- 2-57	5.00(+7)	7.6990	5- 4-60	6.00(+6)	6.7782
9-12-57	6.00(+6)	6.7782	5- 6-60	4.00(+6)	6.6021
9-21-57	1.50(+6)	6.1761	5-13-60	4.00(+6)	6.6021
10-20-57	5.00(+7)	7.6990	6- 1-60	4.00(+5)	5.6021
11- 4-57	9.00(+6)	6.9542	8-12-60	6.00(+5)	5.7782
2- 9-58	5.00(+6)	6.6990	9- 3-60	3.50(+7)	7.5441
3-23-58	2.50(+8)	8.3979	9-26-60	2.00(+6)	6.3010
3-25-58	6.00(+8)	8.7782	11-12-60	9.04(+9)	9.9562
4-10-58	5.00(+6)	6.6990	11-15-60	7.20(+8)	8.8573
7- 7-58	2.50(+8)	8.3979	11-20-60	4.50(+7)	7.6532
7-29-58	8.50(+6)	6.9294	7-11-61	3.00(+6)	6.4771
8-16-58	4.00(+7)	7.6021	7-12-61	4.00(+7)	7.6021
8-22-58	7.00(+7)	7.8451	7-15-61	1.30(+7)	7.1139
8-26-58	1.10(+8)	8.0414	7-18-61	3.00(+8)	8.4771
9-22-58	6.00(+6)	6.7782	7-20-61	5.00(+6)	6.6990
2-13-59	2.80(+7)	7.4472	7-28-61	4.40(+6)	6.6435
5-10-59	9.60(+8)	8.9823	9- 8-61	3.00(+6)	6.4771

Summary of Statistics:

$$\sum_{i=1}^{60} x_i = 443.5541$$

$$\sum_{i=1}^{60} x_i^2 = 3335.90112251$$

$$\overline{x} = 7.3926$$

$$\sigma_x^2 = 0.94828 \ ; \ \sigma_x = 0.9738$$

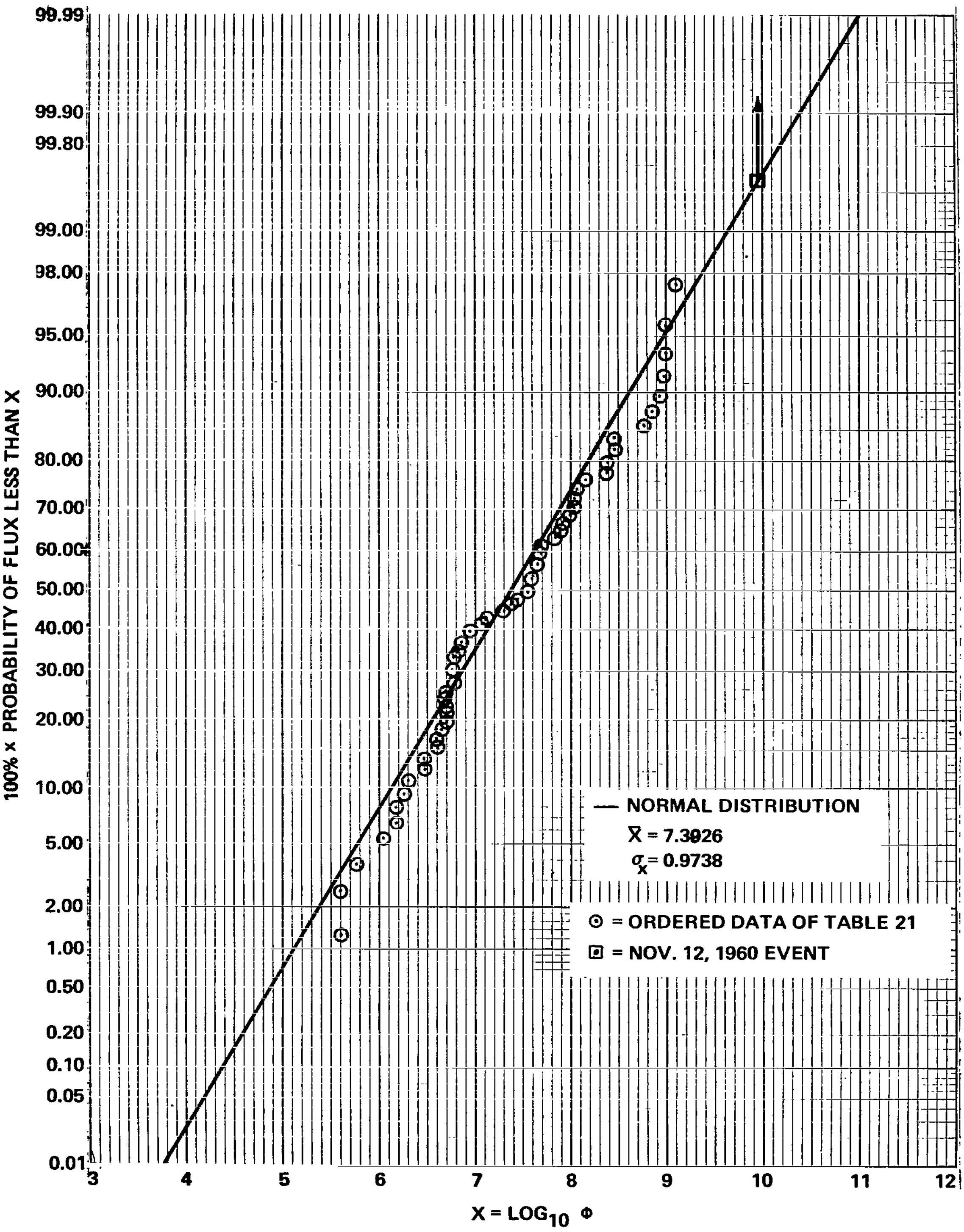

Figure 14. Cumulative distribution of solar proton event log fluxes during solar maximum.

is $2.52 \times 10^{10} \frac{\text{proton}}{\text{cm}^2}$ ($E > 30$ MeV), which is a factor of about 2.8 larger than the November 12, 1960, event. Hence, it would seem that a flux above 30 MeV, which is three times larger than the November 12, 1960, event, would have less than a 0.001 probability of occurring. For example, the probability of an event that is ten times larger than the November 12, 1960, event is about 10^{-4}, or 99.99 percent of all events would be less than such an event.

From the above analysis, one may be convinced that a reasonable upper bound value for a single event is at most a factor of three larger than the November 12, 1960, event. For this reason, the author feels that the results obtained by some investigators for the extreme probability tails must depict a smaller probability than the estimates given. These investigators have possibly used joint probabilities of flux and rigidity parameters, which may be a factor of ten or more smaller than those indicated by the 0.1-percent probability tail shown in the various reports at the disposal of the author. For example, from Table 18 of this report, one observes that the actual probability of seeing more than seven large events during 78 active weeks of the 19th cycle is less than 10^{-3}. However, in Figure 8, the work of Snyder at 20 g/cm^2 indicates a dose that requires 48 large events [as defined by equation (63)] at the 0.001 probability level. This seems high unless Mr. Snyder uses a large event that is about seven times larger than that used by the author. There is no intention here of isolating Mr. Snyder's work [9], since the values of Mr. Yucker of McDonnell Douglas [10] are obviously higher than any other previous analysis and would require 120 large events, or else each event is 17 times larger than the largest event of the 19th cycle if only 7 large events are seen according to Table 18. (If the reader wishes to use the 99.9-percent probability event in the present work, he may ignore self-shielding and geometry factors, a factor of three.)

Of course, all cumulative distributions have the same general appearance and tend to mask any special structure of the basic frequency distribution. The author, being somewhat perplexed by this state of affairs, has attempted to examine the data of Table 21 from a somewhat more fundamental approach. These data are grouped into intervals and frequency tables as shown in Table 22. The mean and the standard deviation of this grouped data are not exactly the same as in the ungrouped data, and for simplicity of plotting, the frequency histograms are transformed to units of $t = (x - \bar{x})/\sigma$. The value of $\sigma = 1.0$ is used for convenience. This would lead to an error of from 1 percent to approximately 4 percent in the correct value of the abscissa, but this would not be detected in the plots shown in Figures 15 through 17. The ordinates or relative frequencies are shown in the fifth column of the tables composing Table 22. The smooth curve drawn on Figures 15 through 17 is that of the Gaussian density function,

TABLE 22. ANALYSIS OF THE DATA IN TABLE 21

Class Interval, Log ϕ	Mid x	Frequency f_i	$t=x-7.4$	$\frac{f_i}{\Delta t \times 60}$	End t	Cumulative Relative Frequency
5.6 — 6.0	5.8	3	-1.6	0.12500	-1.4	0.0500
6.0 — 6.4	6.2	5	-1.2	0.20825	-1.0	0.1333
6.4 — 6.8	6.6	15	-0.8	0.62500	-0.6	0.3833
6.8 — 7.2	7.0	6	-0.4	0.2500	-0.2	0.4833
7.2 — 7.6	7.4	5	0	0.20825	+0.2	0.5666
7.6 — 8.0	7.8	10	+0.4	0.41667	+0.6	0.7333
8.0 — 8.4	8.2	6	+0.8	0.2500	+1.0	0.8333
8.4 — 8.8	8.6	3	+1.2	0.1250	+1.4	0.8833
8.8 — 9.2	9.0	6	+1.6	0.2500	+1.8	0.9833
9.2 — 9.6	9.4	0	+2.0	0	+2.2	0.9833
9.6 — 10.0	9.8	1	+2.4	0.041667	+2.6	1.0000
	Σ	60				

Summary of Statistics:

$\overline{x} = 7.39, \quad \sigma = 0.97$

Class Interval, Log ϕ	Mid x	Frequency f_i	$t=x-7.4$	$\frac{f_i}{\Delta t \times 60}$	End t	Cumulative Relative Frequency
5.6 — 6.4	6.0	8	-1.4	0.1667	-1.0	0.1333
6.4 — 7.2	6.8	21	-0.6	0.4375	-0.2	0.4833
7.2 — 8.0	7.6	15	+0.2	0.3125	+0.6	0.7333
8.0 — 8.8	8.4	9	+1.0	0.1875	+1.4	0.8833
8.8 — 9.6	9.2	6	+1.8	0.1250	+2.2	0.9833
9.6 — 10.4	10.0	1	+2.6	0.0208	+3.0	1.0000
	Σ	60				

Summary of Statistics:

$\overline{x} = 7.42, \quad \sigma = 0.99$

Class Interval, Log ϕ	Mid x	Frequency f_i	$t=x-7.5$	$\frac{f_i}{\Delta t \times 60}$	End t	Cumulative Relative Frequency
5.6 — 6.6	6.1	10	-1.4	0.1666	-0.9	0.1667
6.6 — 7.6	7.1	24	-0.4	0.4000	+0.1	0.5667
7.6 — 8.6	8.1	18	+0.6	0.3000	+1.1	0.8667
8.6 — 9.6	9.1	7	+1.6	0.1167	+2.1	0.9834
9.6 — 10.6	10.1	1	+2.6	0.0167	+3.1	1.0000
	Σ	60				

Summary of Statistics:

$\overline{x} = 7.51, \quad \sigma = 0.96$

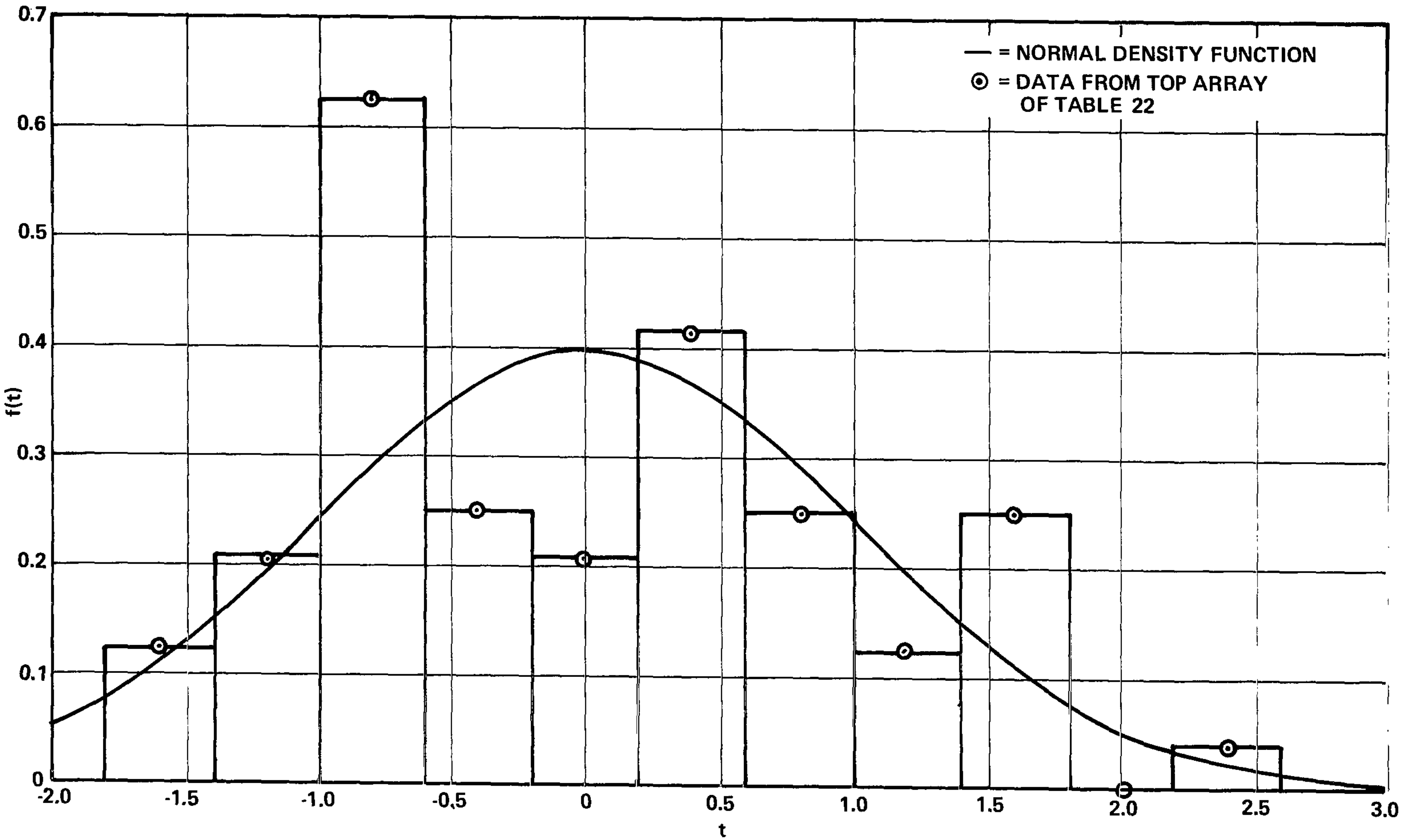

Figure 15. Normal density function and grouped frequency data from top array of Table 22.

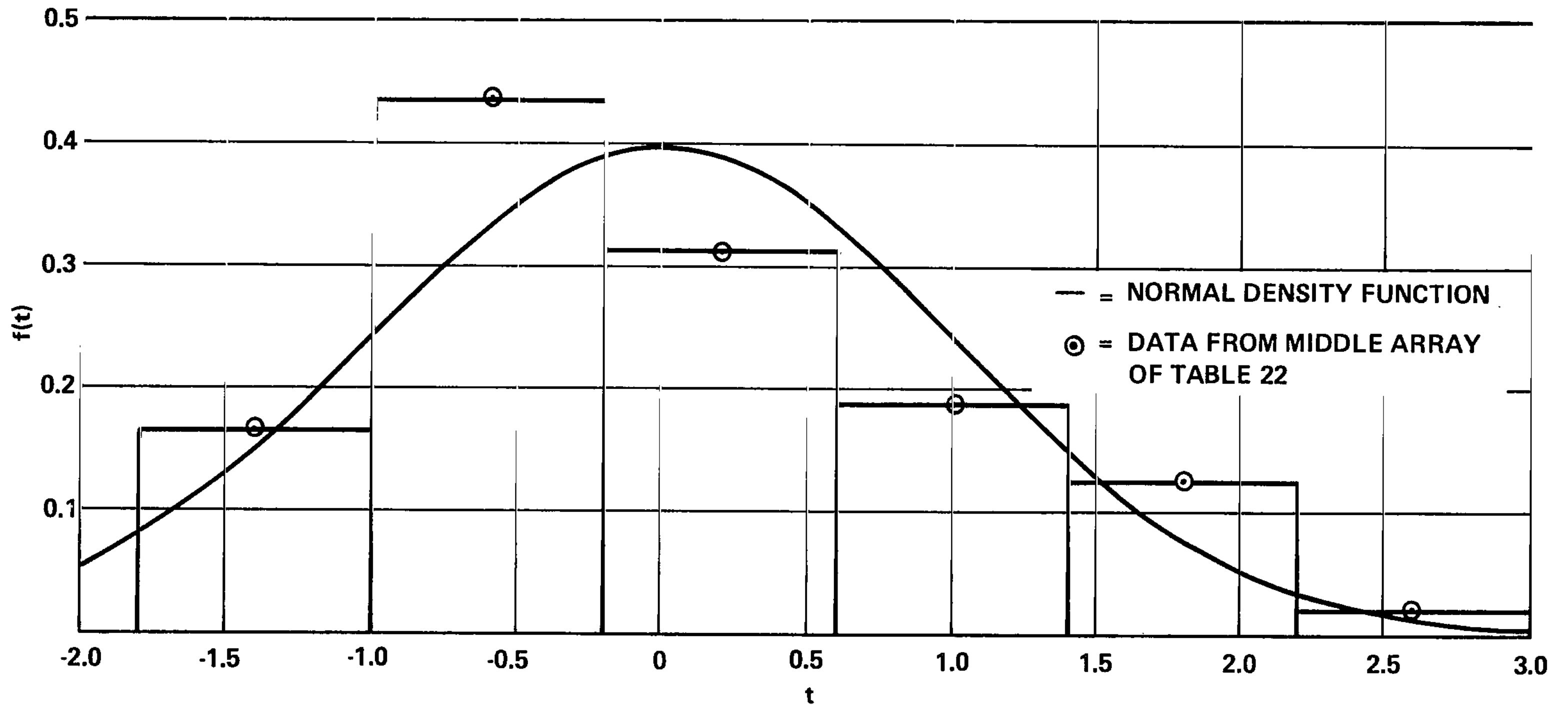

Figure 16. Normal density function and grouped frequency data from middle array of Table 22.

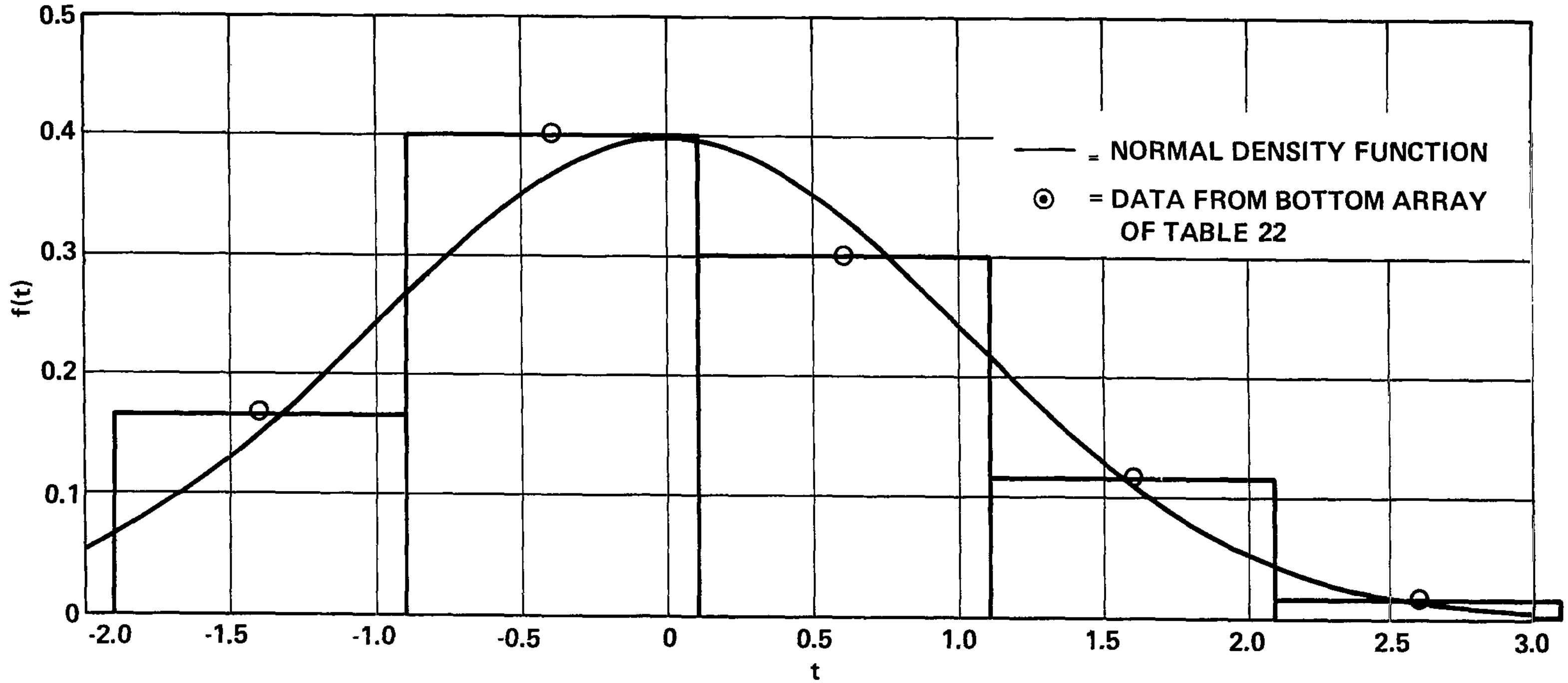

Figure 17. Normal density function and grouped frequency data from bottom array of Table 22.

$$f(t)\,dt = \frac{1}{\sqrt{2\pi}} \quad e^{-t^2/2}\,dt \quad . \qquad (67)$$

The midpoint of the frequency histograms is plotted at the value of t shown in Table 22. The width of these histograms is Δt, or the simple difference in the value of the tabulated t's. By choosing a value of $\sigma = 1.0$, Δt becomes equal to the grouping intervals in the variate $\log_{10}\phi$ as shown in Table 22. This small inaccuracy allows a simple direct comparison between the three curves shown in Figures 15 through 17 and the true Gaussian frequency distributions. The results shown in Figures 15 through 17 indicate a reasonable resemblance to a Gaussian distribution. However, from observation, one sees that the distribution has a skew to the right. A measure of skewness could be calculated from the data in Tables 21 and 22 but would be of no more value than the above observation. The purpose of the preceding is not to find the best probability density function to fit the data but to demonstrate that the use of the log normal for the particle flux above 30 MeV is reasonable. However, a reasonable use of the above log normal distribution would lead one to choose a maximum flux (above 30 MeV) that does not exceed the largest observed event by more than a factor of three to arrive at the 99.9-percentile level.

From Table 22 (first data group), Figure 18 is presented to compare the cumulative distribution of the grouped data to the true Gaussian drawn as a solid line. It should be noted how well the data follow the Gaussian distribution where the cumulative area of the rectangles in Figure 15 is plotted at the end points, as they should be to obtain the comparable values on the Gaussian ogive. The reader can recognize the difference in implications between Figure 18 and Figure 15. Stated simply, a cumulative distribution hides the true nature of the goodness of fit of basic data to a given distribution function.

In conclusion, one graph will be presented for the dose expected on the skin of an astronaut in a typical spacecraft for a 78-week mission during the active weeks of an average cycle at various probabilities, ϵ, as shown in Table 19 of Section V. The results are shown in Figure 19. The doses in Figure 19 have been made a factor of three lower than a point dose at the center of a sphere. Figure 19 presents what the author believes to be a realistic evaluation of the risk of solar proton events to space travel. Thus, if one assumes that $6.5\ g/cm^2$ of aluminum is equal to 5 cm of tissue and the spacecraft shield is $13.5\ g/cm^2$, then the 5-cm depth dose is about 17 rads at

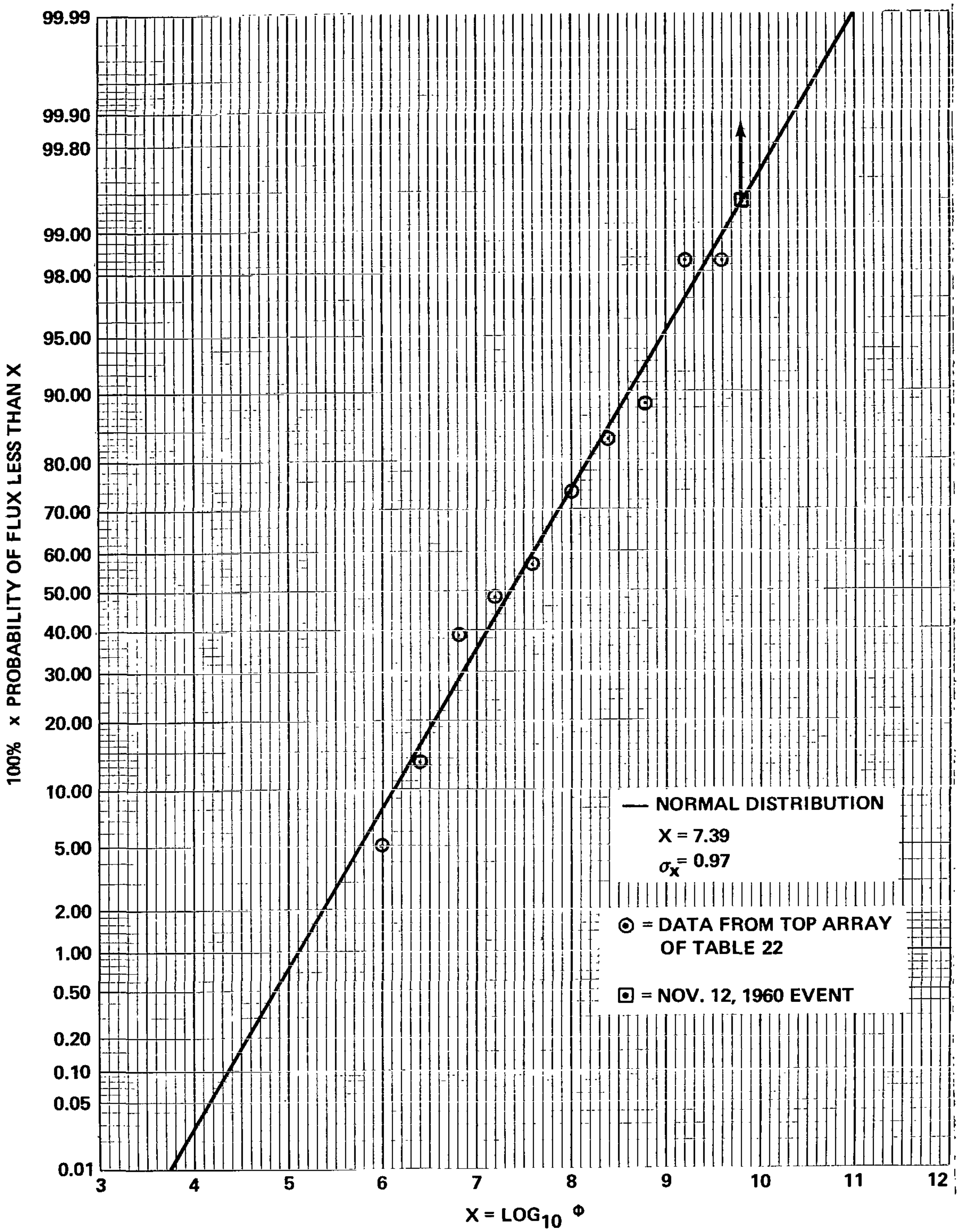

Figure 18. Cumulative distribution of grouped data from top array of Table 22.

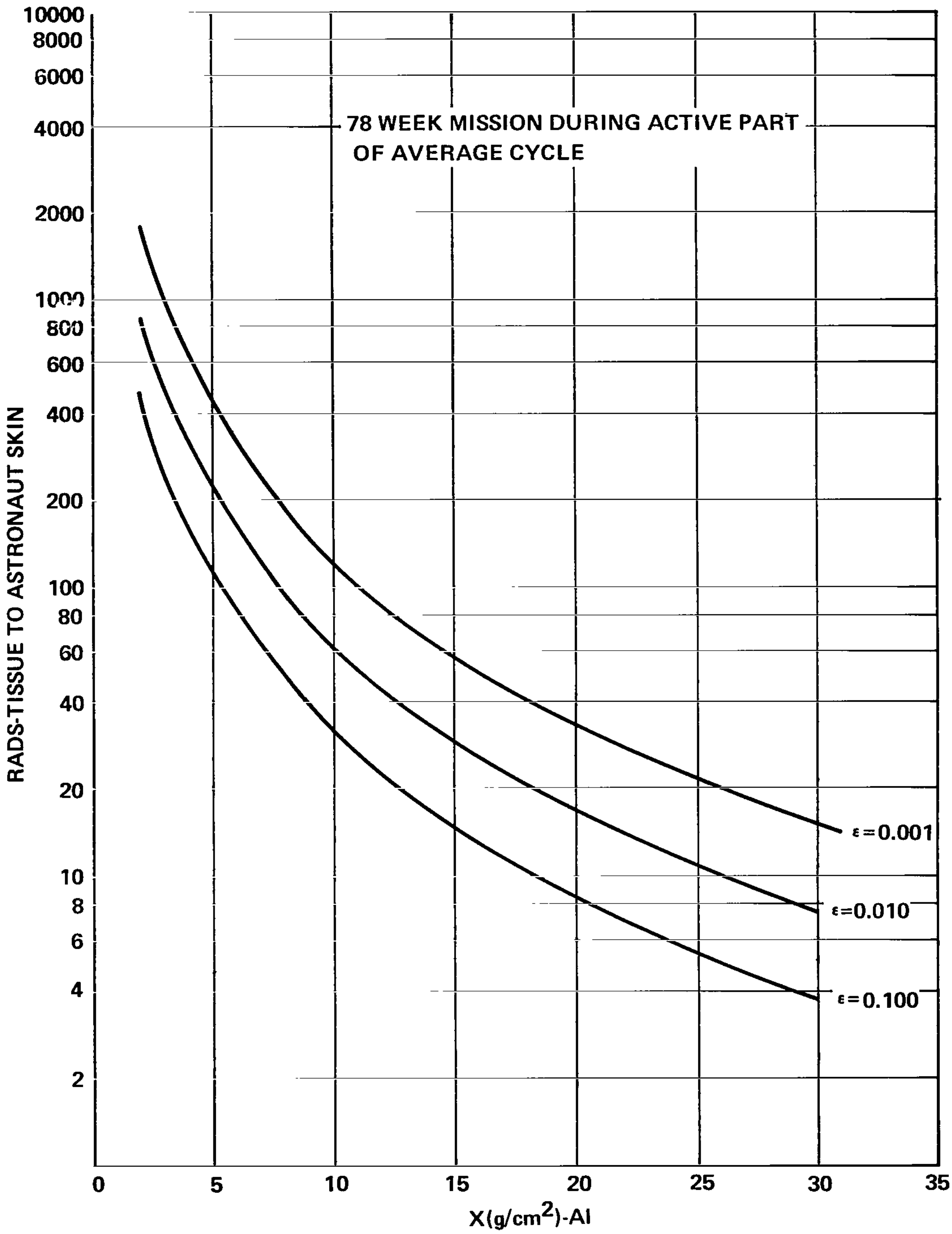

Figure 19. Expected dose at probabilities ϵ for a 78-week mission on astronaut skin inside typical spacecraft of uniform wall thickness, x g/cm^2, during the active part of an average cycle.

the 99-percent confidence level during the active part of an average cycle. If one believes that the galactic cosmic ray (GCR) dose is about 12 rads/year and is unattenuated by the shield (this is in doubt for the high Z component), one sees that in 18 months the GCR dose is 18 rads. Using the above assumptions, the GCR depth dose is about the same as the solar proton depth dose for an 18-month trip at a 99-percent confidence level during an average solar cycle (not the 19th cycle).

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November 30, 1970 124-09-21-00-62

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